Lecture7

Saturday, January 26, 2019 2:07 PM

Fundamental Concepts of GR

- Recall, local inertial (Lorentz) frame:
  1. a latticework moving freely w/ no forces acting on it.
  2. measuring rods are orthogonal and uniformly marked
  3. clocks are densely packed & tick uniformly & are synchronized (Einstein)

- In GR, Lorentz frames must be small enough to neglect changes in gravity!
- Modern Einstein equivalence principle: specific forms that local, nongravitational laws take in GR local Lorentz frames are the same as those in global Lorentz frames of SR.

Gravity as curvature of spacetime

- Equivalence principle guarantees that non-

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gravitational forces in local Lorentz frame are described by the metric, which gives the invariant interval between neighboring events, $\vec{\xi} = \Delta x^\alpha \, \partial/\partial x^\alpha$

- special relativistic interval:

$$\vec{\xi}^{\,2} = g_{\alpha\beta}\, \xi^\alpha \xi^\beta = (\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

- also, in local Lorentz frame (LLF), components of metric,

$$g_{\alpha\beta} = \eta_{\alpha\beta} \equiv \left\{ -1 \text{ if } \alpha = \beta = 0,\ +1 \text{ if } \alpha = \beta \neq 0 \right\}$$

- since two adjacent LLF's will fall together if dropped from high enough, gravity prevents the meshing of LLFs into global LLF's!
- by analogy with local Cartesian meshes on Earth's surface (Einstein, 1912) gravity must be a manifestation of spacetime curvature!
- for a freely falling observer anywhere then for a LLF centered on their world line,

$$g_{\alpha\beta} = \left\{ \begin{array}{l} \eta_{\alpha\beta} + \mathcal{O}\left(\dfrac{\delta_{jk} x^j x^k}{R^2} \right) \\ \eta_{\alpha\beta} \quad \text{at spatial origin} \end{array} \right\}$$

where R is radius of curvature of spacetime.

$\hookrightarrow$ $g_{\alpha\beta,\mu} = \mathcal{O}(x^j/R^2)$

$$\therefore \Gamma^{\alpha}{}_{\beta\gamma} = \begin{cases} \mathcal{O}\left(\dfrac{\sqrt{\delta_{jk}\, x^j x^k}}{R^2}\right) \\ 0 \quad \text{at spatial origin} \end{cases}$$

.... connection coefficients

$\rightarrow$ "Fermi coordinates"

Free-fall Motion & Geodesics

- in SR, free particle moves on straight world line,

$$(t, x, y, z) = (t_0, x_0, y_0, z_0) + (p^0, p^x, p^y, p^z)$$

or, $$x^\alpha = x^\alpha_0 + p^\alpha \zeta.$$

p^α : Lorentz-frame 4-momentum

ζ : affine parameter, $\vec{p} = d/d\zeta$ or $p^\alpha = dx^\alpha/d\zeta$

- in other words, components of 4-momentum are constant,

$$\frac{dp^\alpha}{d\zeta} = 0$$

- Or particle parallel transports its tangent vector $\vec{p}$ along its world line vector $\vec{p}$ along its world-line,

$$\nabla_{\vec{p}}\,\vec{P} = 0 \;, \text{ or } \; P^{\alpha}{}_{;\beta}\, P^{\beta} = 0$$

- for non-zero rest mass particles,

  $\vec{p} = m\vec{u}$, $\lambda = \tau/m$, $\vec{u} = d/d\tau$

  $\to$ $\nabla_{\vec{p}}\,\vec{P} = 0 \iff \nabla_{\vec{u}}\,\vec{u} = 0$ .... geodesic law of motion

- Via the equivalence principle, and using Fermi coordinates above, law of motion for freely-falling particle is the same! (since choice of origin along world line is arbitrary)

  $\to$ in curved spacetime free fall also along geodesics.

- curve to which $\vec{u}$ is tangent is a geodesic
- if geodesic is spacelike, tangent vector can be normalized so that $\vec{u} = d/ds$ w/ s the proper distance along geodesic
- if geodesic timelike, $\vec{u} = d/d\tau$
- since $\nabla_{\vec{p}}\,\vec{P} = 0$ means square of 4-momentum is conserved along world line,

$$(g_{\alpha\beta}p^\alpha p^\beta)_{;\gamma}\, p^\gamma = 2 g_{\alpha\beta} p^\alpha p^\beta{}_{;\gamma}\, p^\gamma = 0$$

- recall, $\vec{p}\cdot\vec{p} = -m^2$, so it better be conserved!
  ↳ geodesic is timelike if $m > 0$.
  if $m=0$ (photons) it is null, $\vec{p}\cdot\vec{p} = 0$

Properties of free-fall geodesics:

1. in a coordinate basis, $\nabla_{\vec{p}}\,\vec{p} = 0$ becomes,

$$\frac{d^2 x^\alpha}{d\zeta^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\zeta}\frac{dx^\nu}{d\zeta} = 0$$

where $x^\alpha(\zeta)$ are the coordinates of the world line.

2. for spacetime w/ a symmetry that makes metric independent of one of the coords. x^A Then associated w/ that symmetry will be a Killing vector field $\vec{\xi} = \partial/\partial x^A$ and a conserved quantity $p_A \equiv \vec{p}\cdot\partial/\partial x^A$ for free-particle motion.

3. Timelike geodesics $\mathcal{P}(\lambda)$ satisfy,

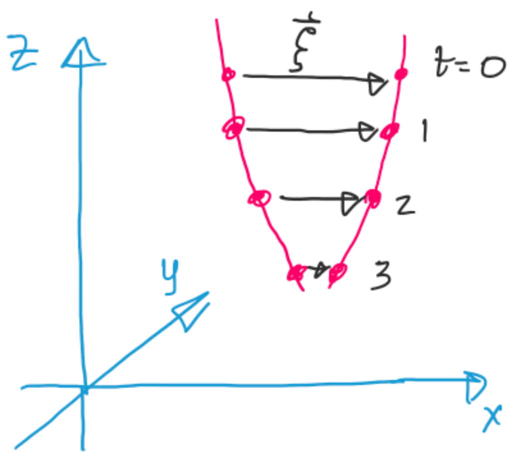
$$\delta\int_{\mathcal{P}_0}^{\mathcal{P}_1} d\tau = \delta\int_0^1\left(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right)d\lambda = 0$$

Relative Acceleration, Tidal Gravity, Curvature

Newtonian tidal gravity

- consider two particles in non-uniform external gravitational field w/ potential Φ. Particle masses are small & do not affect Φ. At (Newtonian) time $t=0$, particles have the same 3-velocity $\vec{v}_A = \vec{v}_B$. Differences in Φ at locations of A & B will cause particles to drift relative to each other since acceleration

$$\vec{g} = -\nabla \Phi$$



- $\vec{\xi}$ is 3-vector separation w/ components

$$\xi^j = x_B^j - x_A^j$$

and

$$\frac{d\xi^j}{dt} = v_B^j - v_A^j$$

- relative acceleration of two particles,

$$\frac{d^2\xi^j}{dt^2} = \frac{d^2 x_B^j}{dt^2} - \frac{d^2 x_A^j}{dt^2} = -\left(\frac{\partial\Phi}{\partial x^j}\right)_B + \left(\frac{\partial\Phi}{\partial x^j}\right)_A$$

$$= -\frac{\partial^2\Phi}{\quad}\,\xi^k \quad + \text{1st} \ldots \xi^k$$

$$\ldots \frac{\partial^2 \Phi}{\partial x^j \partial x^k}\xi \quad \ldots \text{ to 1}^{\text{st}} \text{ order in } \xi$$

- use this to define *Newtonian tidal field*,

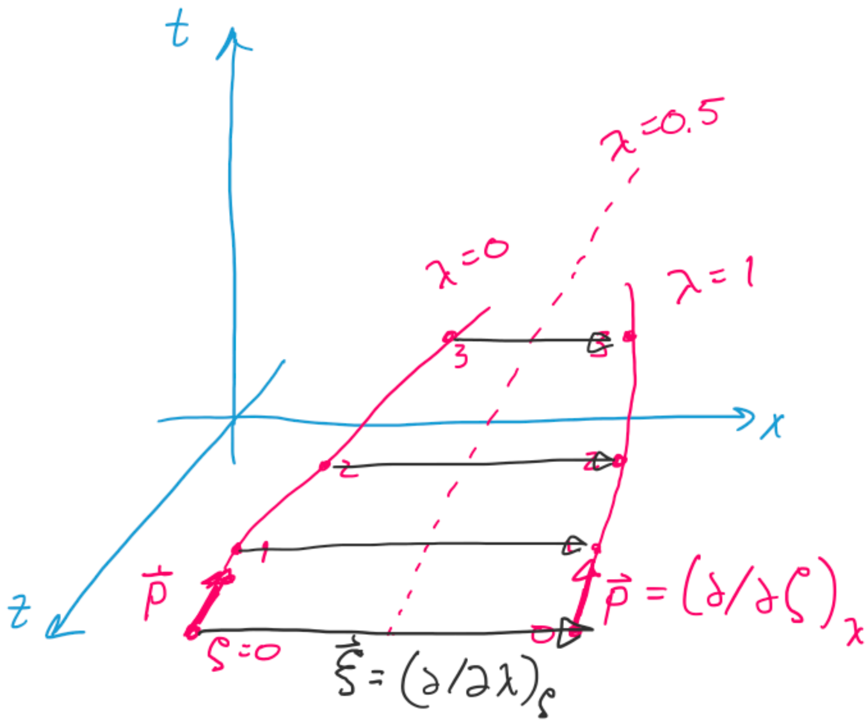
$$\frac{d^2\vec{\xi}}{dt} = -\overleftrightarrow{\mathcal{E}}(\_,\vec{\xi}\,); \quad \text{ie. } \frac{d^2\xi^j}{dt^2} = -\mathcal{E}^j{}_k\,\xi^k$$

where,
$$\overleftrightarrow{\mathcal{E}} \equiv \nabla\nabla\Phi = -\nabla\vec{g}$$

or
$$\mathcal{E}_{jk} = \frac{\partial^2\Phi}{\partial x^j \partial x^k} \quad \ldots\ \text{Euclidean}$$

↳ quantifies variation in Newtonian gravity

Relativistic Tidal Gravity



- particles are in free fall and so move on geodesics w/ affine parameters ζ

- 4-momentum $p = d/d\zeta$
- 4-separation $\vec{\xi}(\zeta) = \mathcal{P}_B(\zeta) - \mathcal{P}_A(\zeta)$
- Assume initially, $\nabla_{\vec{p}} \vec{\xi} = 0$
- tidal gravity leads to $\nabla_{\vec{p}} \nabla_{\vec{p}} \vec{\xi} \neq 0$
- $\mathcal{P}_A(\zeta)$ + $\mathcal{P}_B(\zeta)$ must be close enough that power series expansion in the separation are accurate.
- embed a 2D surface in spacetime that contains the geodesics of the two particles (and an infinite number of other geodesics). Label them w/ λ.

↳ $\vec{p} = (\partial/\partial\zeta)_{\lambda=\text{const}}$, $\vec{\xi} \equiv (\partial/\partial\lambda)_{\zeta=\text{const}}$

- we need $\nabla_{\vec{p}} \nabla_{\vec{p}} \vec{\xi}$, which we can get at from the commutator of $\vec{p}$ + $\vec{\xi}$:

$$[\vec{p}, \vec{\xi}] = \nabla_{\vec{p}} \vec{\xi} - \nabla_{\vec{\xi}} \vec{p}$$

- from the definitions of $\vec{p}$ + $\vec{\xi}$ above, we know they commute (commutator is zero).

↳ $\nabla_{\vec{p}} \vec{\xi} = \nabla_{\vec{\xi}} \vec{p}$

- taking the second gradient along $\vec{p}$,

$$\nabla_{\vec{p}} \nabla_{\vec{p}} \vec{\xi} = \nabla_{\vec{p}} \nabla_{\vec{\xi}} \vec{p} = (\nabla_{\vec{p}} \nabla_{\vec{\xi}} - \nabla_{\vec{\xi}} \nabla_{\vec{p}}) \vec{p}$$

where we added on zero for convenience in what follows ($\nabla_{\vec{p}}\vec{p} = 0$ on geodesic).

- in slot-naming,

$$(\xi^\alpha{}_{;\beta}\, p^\beta)_{;\gamma}\, p^\gamma = (p^\alpha{}_{;\gamma}\, \xi^\gamma)_{;\delta}\, p^\delta - (p^\alpha{}_{;\gamma}\, p^\gamma)_{;\delta}\, \xi^\delta$$

- expanding using product rule & collecting terms,

$$(\xi^\alpha{}_{;\beta}\, p^\beta)_{;\gamma}\, p^\gamma = (p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma})\, \xi^\gamma p^\delta + \underbrace{p^\alpha{}_{;\gamma}\, (\xi^\gamma{}_{;\delta}\, p^\delta - p^\gamma{}_{;\delta}\, \xi^\delta)}_{\text{commutator of } \vec{p} \,\&\, \vec{\xi} \ (=0)}$$

$$\rightarrow (\xi^\alpha{}_{;\beta}\, p^\beta)_{;\gamma}\, p^\gamma = (p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma})\, \xi^\gamma p^\delta$$

$\therefore$ relative acceleration is result of non-commutation of the two slots of double gradient (γ & δ)

- different from SR where it would commute since $\Gamma^\alpha{}_{\beta\gamma} = 0$ everywhere

- spacetime curvature (which makes the geodesics non-parallel) prevents the double gradient from commuting, causing relative acceleration.
- since $p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma}$ is linear in p^α, there

must be a rank-4 tensor such that

$$p^{\alpha}{}_{;\gamma\delta} - p^{\alpha}{}_{;\delta\gamma} = -R^{\alpha}{}_{\beta\gamma\delta}\, p^{\beta}$$

where $\overleftrightarrow{R}$ is Riemann curvature tensor and is responsible for gradients not commuting. $\vec{p}$ is any vector field

$$\hookrightarrow \left(\xi^{\alpha}{}_{;\beta}\, p^{\beta}\right)_{;\gamma} p^{\gamma} = -R^{\alpha}{}_{\beta\gamma\delta}\, p^{\beta} \xi^{\gamma} p^{\delta}$$

$$\nabla_{\vec{p}} \nabla_{\vec{p}}\, \vec{\xi} = -\overleftrightarrow{R}\left(\_, \vec{p}, \vec{\xi}, \vec{p}\right)$$

.... eqn. of geodesic deviation

- Riemann curvature tensor embodies that gravity is caused by curved spacetime

$$\boxed{\text{Properties of Riemann Curvature Tensor}}$$

- Consider evaluating components of $\overleftrightarrow{R}$ at the spatial origin of an LLF, there $g_{\alpha\beta} = \eta_{\alpha\beta}$ & $\Gamma^{\alpha}{}_{\beta\gamma} = 0$ (but $\Gamma^{\alpha}{}_{\beta\gamma,\mu} \neq 0$)
- For any vector $\vec{p}$, definition of gradient says

$$p^{\alpha}{}_{;\gamma\delta} - p^{\alpha}{}_{;\delta\gamma} = \left(\Gamma^{\alpha}{}_{\beta\gamma,\delta} - \Gamma^{\alpha}{}_{\beta\delta,\gamma}\right) p^{\beta}$$

- recall also

$$p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma} = -R^\alpha{}_{\beta\gamma\delta}\, p^\beta$$

so we can deduce the components of $\overleftrightarrow{R}$,

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} \quad \text{-- at spatial origin}$$

- at spatial distance $\sqrt{\delta_{ij} x^i x^j}$,

$$\Gamma^\alpha{}_{\beta\gamma} = \mathcal{O}\left(R^\mu{}_{\nu\lambda\rho} \sqrt{\delta_{ij} x^i x^j}\right)$$

$$g_{\alpha\beta} - \eta_{\alpha\beta} = \mathcal{O}\left(R^\mu{}_{\nu\lambda\rho}\, \delta_{ij} x^i x^j\right)$$

- Radius of curvature of spacetime is

$$\mathcal{R} = \mathcal{O}\left(|R^\alpha{}_{\beta\gamma\delta}|^{-1/2}\right)$$

↳ for outside weakly gravitating body

$$\mathcal{R} \sim \left(\frac{r^3}{GM}\right)^{1/2} = \left(\frac{c^2 r^3}{GM}\right)^{1/2}$$

$\boxed{\text{Ex. 25-10}}$

a) Earth's surface, $r_\oplus = 6.4 \times 10^8$ cm

$M_\oplus \sim 6 \times 10^{27}$ g, $\mathcal{R} \sim \left[\frac{10^{21} \cdot 260 \times 10^{24}}{7 \times 10^{-8} \cdot 6 \times 10^{27}}\right]^{1/2} \sim 2.5 \times 10^{13}$ cm ~ 1 AU

b) Sun, $r_\odot \sim 7 \times 10^{10}$ cm, $M_\odot \sim 2 \times 10^{33}$ g

$\mathcal{R}_\odot \sim \left[\frac{10^{21} \cdot 343 \times 10^{30}}{7 \times 10^{-8} \cdot 2 \times 10^{33}}\right]^{1/2} \sim 5 \times 10^{13}$ cm ~ 3 AU

c) WD, $r_{WD} \sim 5 \times 10^8$ cm, $M_{WD} \sim M_\odot \sim 1.5 \times 10^5$ cm

$\mathcal{R}_{WD} \sim \left[\frac{\cancel{1.25} \times 10^{26}}{\cancel{1.5} \times 10^5}\right]^{1/2} \sim 3 \times 10^{10}$ cm

d) NS, $r_{NS} \sim 10^6$ cm, $M_{NS} \sim 3$ km

$R_{NS} \sim \left[\frac{\quad}{3 \times 10^{8}} \right] \sim 0.5 \times 10 \quad cm \sim 5 \times 10' \, cm$

e) BH, $R_s \sim 2M$, $M \sim M_{\odot} \sim 1.5\,km$

$$R_{BH} \sim \left[\frac{8 M^3}{M} \right]^{1/2} \sim \left[8 M^2 \right]^{1/2} \sim 3 \times 1.5 \sim 5\,km$$

f) space, Big ---

- recall the connection coeffs,

$$\Gamma^{\mu}{}_{\beta\gamma} = g^{\mu\alpha} \Gamma_{\alpha\beta\gamma}$$

&

$$\Gamma_{\alpha\beta\gamma} \equiv \frac{1}{2} \Big[g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + \overbrace{C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha}}^{0} \Big]$$

$\hookrightarrow$ $$R_{\alpha\beta\gamma\delta} = \frac{1}{2} \left[g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma} \right]$$

- since $g_{\alpha\gamma,\beta\delta} = g_{\alpha\gamma,\delta\beta}$, Riemann tensor has following properties,

$$R_{\alpha\beta\gamma\delta} = - R_{\beta\alpha\gamma\delta} \,, \quad R_{\alpha\beta\gamma\delta} = - R_{\alpha\beta\delta\gamma}$$

$$R_{\alpha\beta\gamma\delta} = + R_{\gamma\delta\alpha\beta}$$

and

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$$

- these symmetries reduce the number of

independent components of $\overleftarrow{R}$ from $4^4 = 256$ to 20

- Contraction of 1^{st} & 3^{nd} slot gives the Ricci tensor,

$$R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta}$$

&

$$R_{\alpha\beta} = R_{\beta\alpha} \quad \text{--- 10 components of } R_{\alpha\beta\gamma\delta}$$

- The other 10 components in Weyl curvature tensor

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - 2g^{[\mu}{}_{[\rho} R^{\nu]}{}_{\sigma]} + \frac{1}{3} g^{[\mu}{}_{[\rho}\, g^{\nu]}{}_{\sigma]} R$$

where antisymmetrization

$$A_{[\alpha\beta]} \equiv \frac{1}{2}\left(A_{\alpha\beta} - A_{\beta\alpha}\right)$$

&

$$R \equiv R^{\alpha}{}_{\alpha}$$

- In a completely arbitrary basis (non LLF),

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\mu}\, c_{\gamma\delta}{}^{\mu}$$

→ this is tedious to compute. Use software.