Lecture19

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Black Hole Horizons

- BH is region of spacetime from which no null geodesic can escape to infinity

- Event horizon is boundary of causal past of future null infinity

  → spanned by light rays that do not reach future null infinity nor hit the BH singularity

Oppenheimer - Snyder Collapse

- collapse of pressureless fluid to BH
- each mass element follows radial geodesic
- interior metric is closed Friedmann:

$$ds^2 = -d\tau^2 + a^2\left(d\chi^2 + \sin^2\chi \, d\Omega^2\right)$$

τ : time coord / proper time of elements

χ : Lagrangian/comoving radial coord.

- scale factor a related to τ via conformal time parameter η :

$$a = \frac{1}{2} a_m (1 + \cos\eta)$$

$$\tau = \frac{1}{2} a_m (\eta + \sin\eta)$$

$$0 \leq \eta \leq \pi$$

- comoving coords χ, θ, ϕ stay fixed
- surface of "star" initially at χ_0
- exterior metric is Schwarzschild:

$$ds^2 = -\left(1 - \frac{2M}{r_s}\right) dt^2 + \left(1 - \frac{2M}{r_s}\right)^{-1} dr_s^2 + r_s^2 d\Omega^2$$

$\rightarrow$ interior metric must match this at χ_0, or $r_s = R(\tau)$ .... stellar surface

- surface follows radial geodesic,

$$R = \frac{1}{2} R_0 (1 - \cos\eta)$$

$$\tau = \left(\frac{R_0^3}{8M}\right)^{1/2} (\eta + \sin\eta)$$

- matching interior & exterior at surface:

- matching interior & exterior at surface:

$$a_m = \left(\frac{R_0^3}{2M}\right)^{1/2}$$

$$\sin \chi_0 = \left(\frac{2M}{R_0}\right)^{1/2}$$

$\therefore \; 0 \leq \chi_0 \leq \pi/2$

- 4-velocity $u^a = \partial_\tau$ satisfies geodesic eqns,

$$0 = u^b \nabla_b u^a = \frac{d^2 x^a}{d\lambda^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda}$$

- rest mass density goes like the comoving volume,

$$\frac{\rho_0(\tau)}{\rho_0(0)} = Q^{-3}(\tau)$$

$$Q(\tau) = \frac{a}{a_m} = \frac{1}{2}(1 + \cos\eta)$$

$\rightarrow$ star remains constant density throughout collapse

- collapse complete when

$$R = \frac{1}{2} R_0 (1 + \cos\eta) = 0$$

$$\rightarrow \; \eta = \pi$$

proper time is then:

$$\tau_{coll} = \left(\frac{R_0^3}{8M}\right)^{1/2}(\pi + \cancel{\sin\pi}) = \pi\left(\frac{R_0^3}{8M}\right)^{1/2}$$

→ singularity forms at center since density is infinite

- Event horizon can be found by locating where outgoing null rays in the interior satisfy $ds^2=0$

↳ $d\tau = a(\tau)d\chi$

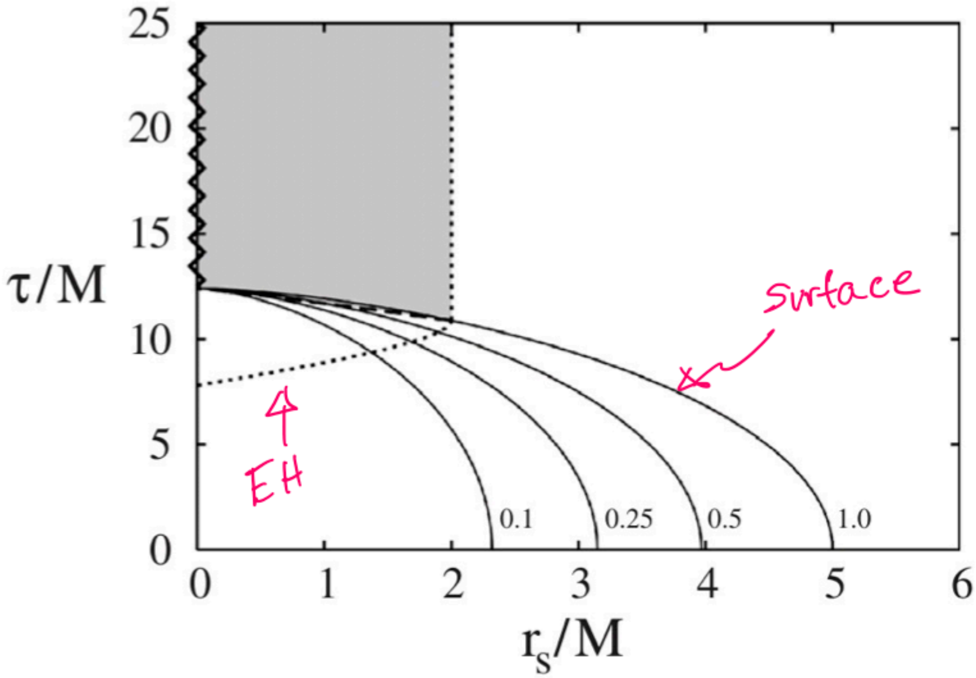
from above, $\frac{d\tau}{d\eta} = \frac{1}{2}a_m(1+\cos\eta) = a(\tau)$

$\therefore d\chi = d\eta$

- integrating this from some event at which a ray is emitted gives

$$\chi = \chi_e + (\eta + \eta_e)$$

- for Schwarz., radius of EH is $2M$, so in OS collapse the EH is trajectory of outgoing null ray that starts at center & hits surface of star when surface reaches $R = 2M$



- solving for η:

$$R = \tfrac{1}{2} R_0 (1+\cos\eta) = 2M$$

$$\to \eta \equiv \eta_{AH} = 2\cos^{-1}(2M/R_0)^{1/2}$$

- subbing into trajectory of EH yields

$$\chi = \chi_0 + (\eta - \eta_{AH})$$

- for all $\eta \geq \eta_{AH}$ $(\tau \geq \tau(\eta_{AH}))$ the entire star is inside EH

- singularity that is formed at center is <u>always</u> "clothed" by EH

$\to$ Penrose Cosmic Censureship Conjecture "there are no naked singularities" from well-behaved initial conditions

- region of trapped surfaces that from whence rays emitted converge at singularity is bounded by an apparent horizon

- finding EH requires <u>global</u> knowledge of spacetime; finding AH requires only <u>local</u> knowledge

Locating EHs

- once spacetime has settled to "steady-state", can evolve null geodesics to locate EH:

$$\frac{d^2 x^a}{d\lambda^2} = {}^{(4)}\Gamma^a_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = 0$$

using definition of 4-momentum & splitting into 3+1 (w/ ADM metric):

$$p^i = \frac{dx^i}{d\lambda} \quad , \quad p^0 = (\gamma^{ij} p_i p_j)^{1/2} / \alpha$$

$$(\text{such that } g^{ab} p_a p_b = 0)$$

$\rightarrow$

$$\frac{dp_i}{d\lambda} = -\alpha \partial_i \alpha (p^0)^2 + \partial_i \beta^k p_k p^0 - \frac{1}{2} \partial_i \gamma^{lm} p_l p_m$$

$$\frac{dx^i}{d\lambda} = \gamma^{ij} p_j - \beta^i p^0$$

.... first-order ODEs

- ejection & tracking of light rays from an event x^a in many directions p^i can be used to numerically find EH
  - if ray escapes, not inside EH
  - if ray does not leave → inside EH

- in practice, easier to integrate backward in time since rays will then <u>converge</u> at EH

- rather than integrate individual rays, can construct 2+1 "null hypersurface"
  - define this as level surface of some function, $f(t, x^i) = 0$
  - normal vector to surface, $\partial_a f$, must also be null

$\longmapsto \quad g^{ab}\partial_a f \partial_b f = 0$

- solving for evolution of function,

$$\partial_t f = \frac{-g^{ti}\partial_i f + \sqrt{(g^{ti}\partial_i f)^2 - g^{tt}g^{ij}\partial_i f \partial_j f}}{g^{tt}}$$

- using ADM metric,

$$\boxed{\partial_t f = \beta^i \partial_i f - (\alpha^2 \gamma^{ij}\partial_i f \partial_j f)^{1/2}}$$

- solution easy in spherical symmetry
  - f is constant along outgoing light rays
  - in Kerr-Schild coords, such rays travel on characteristics,

$$t - r = 4M \ln|r/2M - 1| + \text{const.}$$

$$\therefore \quad f = f(t - r - 4M \ln|r/2M - 1|)$$

must be a solution to above eqn.

Apparent Horizons

- easier to find than EH (local)
- always interior to EH

- let S be a spatial 2-surface in Σ
  - s^a : outward normal to S in Σ

  $\rightarrow$ $s_a s^a = 1$ & $s^a n_a = 0$ (spatial)

  - spatial metric γ_{ab} induces a 2-metric:

$$m_{ab} = \gamma_{ab} - s_a s_b = g_{ab} + n_a n_b - s_a s_b$$

  - tangents to two future-pointing null geodesics:

$$k^a \equiv \frac{1}{\sqrt{2}}(n^a + s^a) \;\ldots\; \text{"outgoing"}$$

$$l^a \equiv \frac{1}{\sqrt{2}}(n^a - s^a) \;\ldots\; \text{"ingoing"}$$

  such that $k_a k^a = 0$ & $m_{ab} k^a = 0$

  $l_a l^a = 0$ & $m_{ab} l^a = 0$

  $k^a l_a = -1$

  - solving for m_{ab} :

$$m_{ab} = g_{ab} + k_a l_b + l_a k_b$$

- expansion of outgoing null geodesics orthogonal to S :

$$\Theta = m^{ab} \nabla_a k_b$$

- "outer-trapped" surface has $\Theta < 0$ everywhere

- boundary of these trapped surfaces is apparent horizon where

$$\Theta = 0$$

• now assume spherical symmetry & line element:

$$ds^2 = -(\alpha^2 - A^2\beta^2)dt^2 + 2A^2\beta\, dr\, dt + A^2 dr^2 + B^2 r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

- only $\beta^i \neq 0$ is $\beta^r \equiv \beta$

→ α, β, A, B functions of t & r only

• for surface S centred on origin:

$$s^a = (0, A^{-1}, 0, 0) \quad \ldots \text{ spatial norm}$$

↳ $s_a = (A\beta, A, 0, 0)$

- outgoing null norm,

$$k_a = \frac{1}{\sqrt{2}}(A\beta - \alpha, A, 0, 0)$$

$$k^a = \frac{1}{\sqrt{2}}\left(\alpha^{-1}, A^{-1} - \alpha^{-1}\beta, 0, 0\right)$$

and
$$m_{ab} = \mathrm{diag}\left(0, 0, B^2 r^2, B^2 r^2 \sin^2\theta\right)$$

- plugging into expansion:

$$\Theta = m^{ab}\nabla_a k_b$$

$$\rightarrow \quad \Theta = \frac{\sqrt{2}}{rB}\left(\frac{1}{\alpha}\partial_t(Br) + \left(\frac{1}{A} - \frac{\beta}{\alpha}\right)\partial_r(Br)\right)$$

↳ proportional to rate of change of "areal" radius, $R_{area} = Br$

$$\therefore \; k^a \nabla_a R_{area} = k^t \partial_t(Br) + k^r \partial_r(Br)$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\alpha}\partial_t(Br) + \left(\frac{1}{A} - \frac{\beta}{\alpha}\right)\partial_r(Br)\right)$$

- combining these expressions:

$$\Theta = \frac{2}{R_{area}} k^a \nabla_a R_{area} = \frac{1}{4\pi R_{area}^2} k^a \nabla_a\left(4\pi R_{area}^2\right)$$

$\therefore$ Θ measures fractional change in area of an outward spherical flash of light

- in 3+1, can re-express expansion:

$$\sqrt{2}\, m^{ab} \nabla_a k_b = m^{ab} \nabla_a (n_b + s_b)$$
$$= m^{ij} (D_i s_j - K_{ij})$$

now, $m^{ij} K_{ij} = K - s^i s^j K_{ij}$ via 2-metric

$$\rightarrow \quad \sqrt{2}\, m^{ab} \nabla_a k_b = D_i s^i - K + s^i s^j K_{ij}$$

(recall, $K_{ab} \equiv -\gamma_a{}^c \gamma_b{}^d \nabla_{(c} n_{d)} = -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d$)

- apparent horizon fullfills,

$$0 = \sqrt{2}\, \Theta = D_i s^i - K + s^i s^j K_{ij}$$