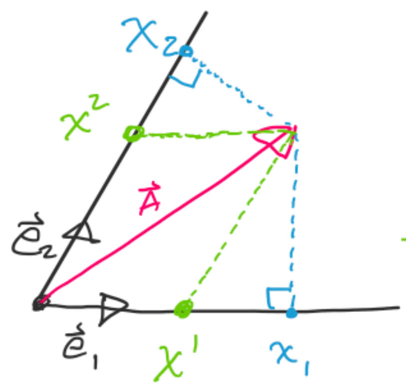
Lecture4

Wednesday, January 16, 2019 3:44 PM

Another look at contravariant vs. covariant

Consider general, non-orthogonal basis:



Covariant: perpendicular projection

Contravariant: parallel projection

$\vec{A} \equiv A^\alpha \vec{e}_\alpha \rightarrow A^\alpha = \vec{A} \cdot \frac{1}{\vec{e}_\alpha}$, $A_\alpha \equiv \vec{A}(\vec{e}_\alpha) = \vec{A} \cdot \vec{e}_\alpha$

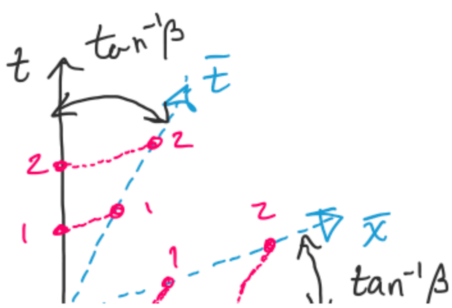
Spacetime Diagrams for Boosts

- for a pure boost in the $\vec{e}_1$ direction:

$t = \gamma(\bar{t} + \beta\bar{x})$, $x = \gamma(\bar{x} + \beta\bar{t})$

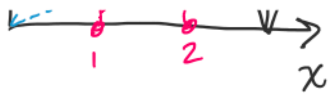
$\bar{t} = \gamma(t - \beta\bar{x})$, $\bar{x} = \gamma(x - \beta t)$

where $|\beta| < 1$, $\gamma \equiv (1 - \beta^2)^{-1/2}$

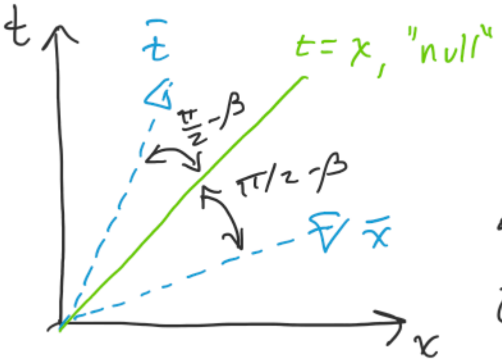


invariance of interval $(a = 1, 2)$

$\hookrightarrow$ event $\bar{x} = a$ on $\bar{x}$-axis lies at intersection of

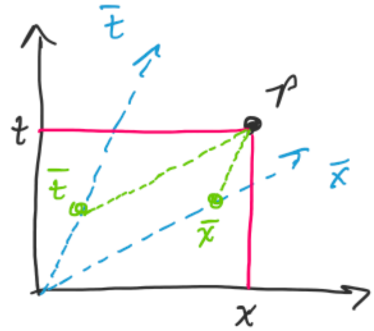


$\bar{x}$-axis with $x^2 - t^2 = a^2$.
and $\bar{t} = a$, where $\bar{t}$-axis intersects $t^2 - x^2 = a^2$

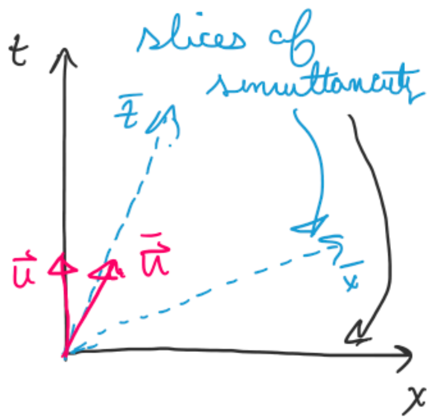


- barred basis is orthogonal

$$\vec{e}_{\bar{t}} \cdot \vec{e}_{\bar{x}} = \frac{1}{4}\left[(\vec{e}_{\bar{t}} + \vec{e}_{\bar{x}})^2 - (\vec{e}_{\bar{t}} - \vec{e}_{\bar{x}})^2\right] = 0$$



- t, x from perpendicular projection in $\vec{e}_\alpha$
- $\bar{t}, \bar{x}$ from parallel projection in $\vec{e}_{\bar{\alpha}}$



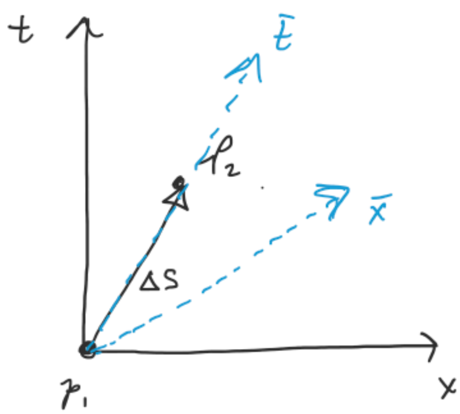
- observer at rest in unbarred frame: 4-velocity $\vec{u} = \{1, 0, 0, 0\}$
- observer at rest in barred frame: $\vec{\bar{u}}$

$\boxed{\text{Exercise 2.14}}$ Prove the following

(a) Two events simultaneous in one inertial frame are not simultaneous in boosted frame.

(b) Two events at same spatial location in one frame are not at same location in boosted frame location in boosted frame.

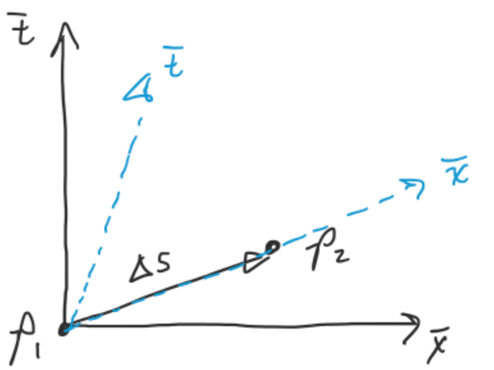
(c) if P_1 & P_2 have a timelike separation then there is an inertial frame in which they occur at the same spatial location. In that frame the time lapse between them is $\Delta t = \Delta\tau \equiv \sqrt{-(\Delta s)^2}$.



$$(\Delta s)^2 < 0$$

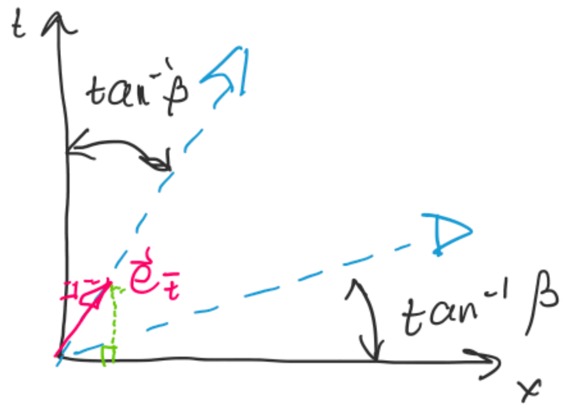
(d) P_1 & P_2 have spacelike separation, then there is a frame in which they are simultaneous. In that frame the spatial distance between them is

$$\sqrt{g_{ij}\,\Delta x^i \Delta x^j} = \Delta s = \sqrt{(\Delta s)^2}$$



$$(\Delta s)^2 > 0$$

(e) if frame $\bar{F}$ moves w/ speed β relative to frame F then clock at rest in $\bar{F}$ ticks more slowly as viewed from F than $\bar{F}$ by a factor $\gamma^{-1} = (1-\beta^2)^{1/2}$.



$$\frac{\vec{e}_t}{\vec{e}_{\bar{t}}} = \cos(\tan^{-1}\beta) = (1+\beta^2)^{-1/2} < 1$$

↳ "time dilation"

(f) if frame moves w/ $\vec{v} = \beta\vec{e}_x$ relative to unbarred frame, object at rest in barred frame appears shortened by a factor $\gamma^{-1} = (1-\beta^2)^{1/2}$.

$$\to \frac{\vec{e}_{\bar{t}} \cdot \vec{e}_x}{\vec{e}_{\bar{t}}} = \sin(\tan^{-1}\beta) = \beta(1+\beta^2)^{-1/2}$$

Exercise 2.16 ... Maybe

$\boxed{\text{Directional derivatives, gradients, Levi-Civita}}$

- Defined in Minkowski spacetime precisely as for Euclidean space

- "gradient": $\vec{\nabla} \overleftrightarrow{T}$ produces directional deriv.

$$\nabla_{\vec{A}} \overleftrightarrow{T} = \vec{\nabla} \overleftrightarrow{T}(\_, \_, \_, \vec{A})$$

can write gradient $T_{\alpha\beta\gamma;\mu}$

- in Lorentz basis, gradient is partial derivs:

$$T_{\alpha\beta\gamma;\mu} = \frac{\partial T_{\alpha\beta\gamma}}{\partial x^\mu} \equiv T_{\alpha\beta\gamma,\mu}$$

where "comma" means partial deriv

- divergence of the gradient of tensor in Minkowski is wave operator:

$$T_{\alpha\beta\gamma;\mu\nu}\, g^{\mu\nu} = T_{\alpha\beta\gamma,\mu\nu}\, g^{\mu\nu}$$
$$= -\frac{\partial^2 T_{\alpha\beta\gamma}}{\partial t^2} + \frac{\partial^2 T_{\alpha\beta\gamma}}{\partial x^j \partial x^k}\, \delta^{jk}$$
$$= \Box\, T_{\alpha\beta\gamma}$$

- metric tensor in 4D still connotes distance
- Levi-Civita still gives volume

$\epsilon_{\alpha\beta\gamma\delta} = +1$ if $\alpha, \beta, \gamma, \delta$ even permutation

$\epsilon_{\alpha\beta\gamma\delta} = -1$ if $\alpha, \beta, \gamma, \delta$ odd permutation

$\epsilon_{\alpha\beta\gamma\delta} = 0$ if

sign-flip if temporal, $\epsilon_{0123} = +1 \rightarrow \epsilon^{0123} = -1$

$\boxed{\text{Maxwell's Equation}}$

• Electromagnetic 4-force on particle of charge q:

$$\frac{dp^\alpha}{d\tau} = q\, F^{\alpha\beta} u_\beta$$

where $\overleftrightarrow{F}(-,-)$ is EM field tensor.

• if particle ordinary velocity is $v^j = v_j$ and 4-velocity components $u^0 = \gamma$, $u^j = \gamma v^j$, contravariant components of antisymmetric EM tensor:

$$F^{0j} = -F^{j0} = +F_{j0} = -F_{0j} = E_j$$

.... Electric field

$$F^{ij} = F_{ij} = \epsilon_{ijk} B_k$$ .... Magnetic field

• since $\frac{dt}{d\tau} = u^0 = \gamma$, using the above in force law:

$$\frac{dp_j}{d\tau} = \gamma \frac{dp_j}{dt} = q\left(F_{j0} u^0 + F_{jk} u^k\right)$$

$$= q u^0 \left(F_{j0} + F_{jk} v^k\right)$$

$$= q\gamma\left(E_j + \epsilon_{ijk} v_j B_k\right)$$

and

$$\frac{dp_0}{d\tau} = \gamma \frac{dp^0}{dt} = q\, F^{0j} u_j = q\gamma E_j v_j$$

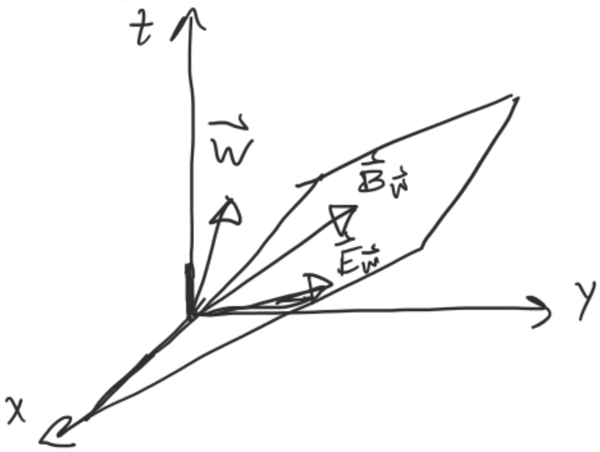
• with particle energy $\mathcal{E} = p^0$, divide by γ:

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right); \quad \frac{d\mathcal{E}}{dt} = q\, \vec{v}\cdot\vec{E}$$

... in 3-space

Note! $\vec{E}$ & $\vec{B}$ tied to chosen frame!

- assume 3-vectors $\vec{E}$ & $\vec{B}$ are spatial parts of 4-vectors in slice of simultaneity t



$\vec{w}$: 4-velocity of frame's observers.

$\vec{E}_{\vec{w}}$ & $\vec{B}_{\vec{w}}$: 4-vectors orthogonal to $\vec{w}$.

- in rest frame of observer $\vec{w}$ $(w^0 = 1, w^i = 0)$:

$$E^0_{\vec{w}} = 0, \quad E^j_{\vec{w}} = E_j = F_{j0}$$

$$\therefore \quad E^\alpha_{\vec{w}} = F^{\alpha\beta} w_\beta, \quad B^\beta_{\vec{w}} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} w_\alpha$$

- since this is true for any observer $\vec{w}$, these equations can be regarded as frame-independent. $\therefore$ the Electric & magnetic fields can be found solely from an observer's 4-velocity and the EM field tensor.

→ $\overleftrightarrow{F}$ is geometric, $\vec{E}$ & $\vec{B}$ require the choice of an observer. (inertial frame)

→ only after a 3+1 split are $\vec{E}$ & $\vec{B}$ even separate entities.

EM fields:

- EM field tensor:

$$F^{\alpha\beta} = w^\alpha E^\beta_{\vec{w}} - E^\alpha_{\vec{w}} w^\beta + \epsilon^{\alpha\beta}{}_{\gamma\delta} w^\gamma B^\delta_{\vec{w}}$$

- Maxwell's Equations:

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha$$

$$\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta;\beta} = 0$$

or

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0$$

J^α is charge-current:

$J^0 = \rho_e$ ... charge density

$J^i = J_i$ ... current density

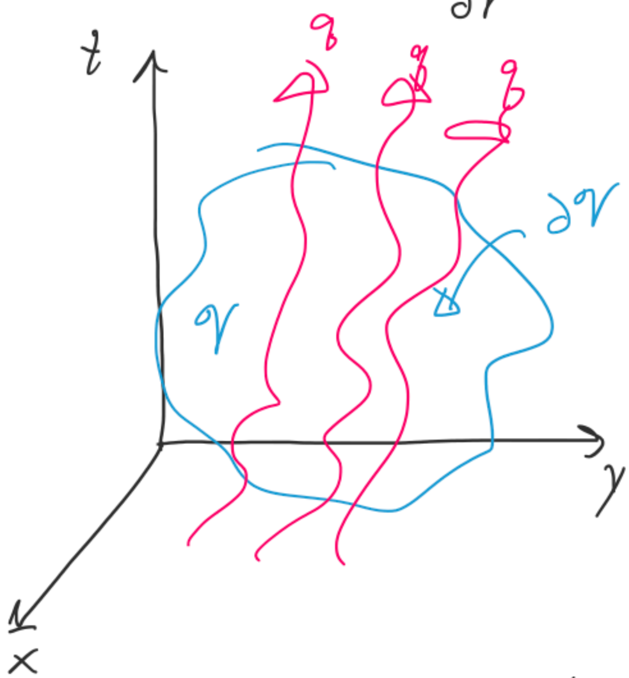
Charge Conservation in Spacetime

- for unit 3-surface $\vec{\Sigma} = \vec{e}_x$,
  $\vec{J}(\vec{\Sigma}) = \vec{J}(\vec{e}_x)$ is total charge Q that flows across $\vec{\Sigma}$ in positive sense
- Charge density ρ_e is temporal component
  $J^0 = -J_0 = \vec{J}(-\vec{e}_0) = \rho_e$
  $\hookrightarrow$ $\vec{J}(\vec{\Sigma}) = \vec{J}(-\vec{e}_0)$ is total charge Q that flows through $\vec{\Sigma} = -\vec{e}_0$ in positive sense
  $\rightarrow$ $\vec{J}(\vec{\Sigma}) \equiv J^\alpha \Sigma_\alpha$ for any small 3-surface
- Law of charge conservation:
  all charge that enters 4-volume $\mathcal{V}$ through the past part of bounding surface $\partial\mathcal{V}$ must exit through

the future part of $\partial\mathcal{V}$.

- globally,

$$\int_{\partial\mathcal{V}} J^{\alpha} d\Sigma_{\alpha} = 0$$



- 4D Gauss theorem:

$$\int_{\mathcal{V}_4} (\vec{\nabla}\cdot\vec{A})\, d\Sigma = \int_{\partial\mathcal{V}_4} \vec{A}\cdot d\vec{\Sigma}$$

or

$$\int_{\partial\mathcal{V}} J^{\alpha} d\Sigma_{\alpha} = \int_{\mathcal{V}} J^{\alpha}{}_{;\alpha}\, d\Sigma$$

- As in 3-space, this requires that the integrand must vanish

$\rightarrow \quad J^{\alpha}{}_{;\alpha} = 0 \quad (\text{or } \vec{\nabla}\cdot\vec{J} = 0)$

- similar conservation laws can be derived for other conserved scalars

Stress-Energy Tensor & 4-Momentum Conservation

- in a medium flowing through spacetime, net 4-momentum $\overleftrightarrow{T}(\_, \vec{\Sigma})$ is transported from negative to positive:

$$\overleftrightarrow{T}(\_, \vec{\Sigma}) \equiv (\text{total 4-momentum } \vec{P} \text{ flowing thru } \vec{\Sigma})$$

or $\quad T^{\alpha\beta}\Sigma_\beta = p^\alpha$

where $\overleftrightarrow{T}(-,-)$ is the stress-energy tensor

- components of stress-energy:

$T^{00} = $ (energy density in chosen Lorentz frame)

$T^{j0} = $ (density of j-component of momentum in frame)

$T_{\alpha 1} \equiv T_{\alpha x} \equiv \overleftrightarrow{T}(\vec{e}_\alpha, \vec{e}_x) = $ (α-component of 4-momentum that crosses area $\Delta y \Delta z = 1$ lying in surface of constant x during time Δt, crossing from $-x$ to $+x$)

$=$ (α component of flux of 4-momentum across surface perpendicular to $\vec{e}_x$)

$\Rightarrow$ $T^{00} = $ (energy density)

$T^{j0} = $ (momentum density)

$T^{0i} = $ (energy flux)

$T^{jk} = $ (stress)

- stress-energy is symmetric: $T^{\alpha\beta} = T^{\beta\alpha}$

$$\overleftrightarrow{T}(\vec{A},\vec{B}) = \overleftrightarrow{T}(\vec{B},\vec{A})$$

- Conservation of 4-momentum:

$$\int_{\partial \mathcal{V}} T^{\alpha\beta} d\Sigma_\beta^{\,0} = 0$$

Gauss's law $\rightarrow$ $\int_{\mathcal{V}} T^{\alpha\beta}{}_{;\beta}\, d\Sigma = \int_{\partial\mathcal{V}} T^{\alpha\beta} d\Sigma_\beta = 0$

$\hookrightarrow$ $T^{\alpha\beta}{}_{;\beta} = 0$ (or $\vec{\nabla} \cdot \overleftrightarrow{T} = 0$)

- Consider a perfect fluid in its local rest frame ($T^{j0} = T^{0j} = 0$):

$$T^{00} = \rho \quad , \quad T^{jk} = P\,\delta^{jk}$$

where P is isotropic pressure & ρ is energy density

these assumptions lead to

$$T^{\alpha\beta} = (\rho + P)\, u^\alpha u^\beta + P\, g^{\alpha\beta}$$

or $\overleftrightarrow{T} = (\rho + P)\, \vec{u} \otimes \vec{u} + P \overleftrightarrow{g}$