Lecture20

Tuesday, April 9, 2019 9:14 AM

$\boxed{\text{Spherical spacetimes}}$

- reduce 3+1 to 1+1 where everything depends only on t & r
  - can't really treat, e.g., rotation
- general 3-metric:

$$dl^2 = A dr^2 + B r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

A & B: functions of t & r only

→ full 4-metric then requires gauge functions $\alpha(t,r)$ & $\beta^r(t,r)$

- radial gauge: set $B=1$

→ r is the areal radius:

$$r = r_s = (\mathcal{A}/4\pi)^{1/2} = (\mathcal{C}/2\pi)$$

$\mathcal{A}$: proper area

$\mathcal{C}$: circumference

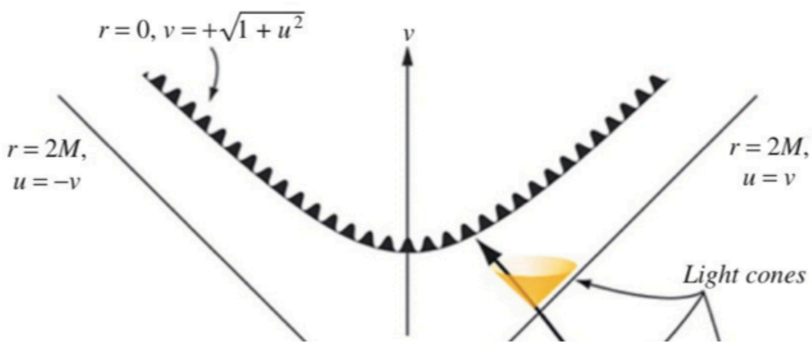
- isotropic gauge : $A = B \to r = \bar{r}$
- comoving gauge : set $\beta^r(t,r)$ such that fluid is always at rest w.r.t. coords

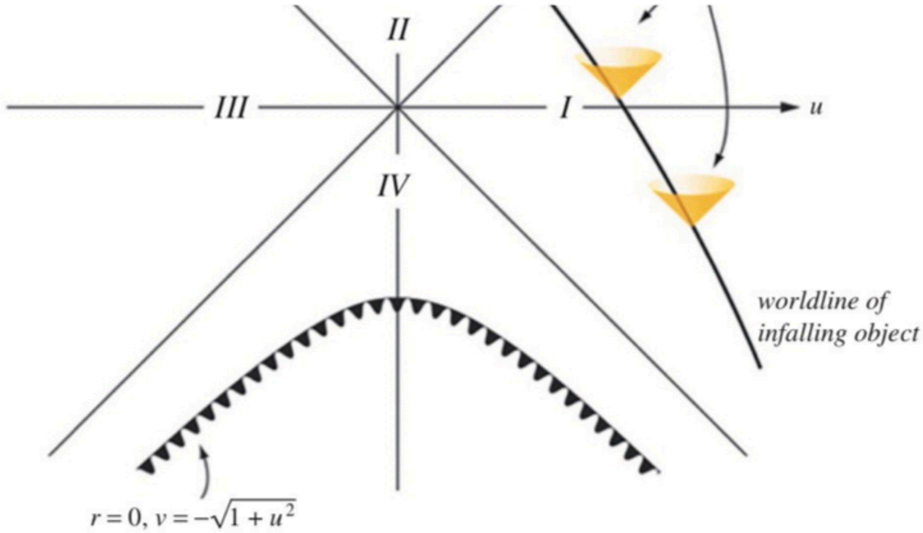
__Slicing__

- must then specify $\alpha(t,r)$
- Geodesic slicing : $\alpha = 1 \to$ coord. singularities
- alter choices : maximal or polar slicing

$\boxed{\text{Black Holes}}$

- start by adopting analytic $\alpha + \beta^r$ that lead to Killing lapse & shift
  ↳ metric is time independent
- take initial data as spacelike $v = 0$ surface from $0 \leq u \leq \infty$ in u-v plane:





Kruskal-Szekeres diagram

→ this is equivalent to $t_s = 0$.
from $2M \leq r_s \leq \infty$ in Schwarzschild coords.

- gauge choice:

$$\alpha(r_s) = (1 - 2M/r_s)^{1/2} \quad \beta^r(r_s) = 0$$

$$\rightarrow A(r_s) = \frac{1}{1-2M/r_s}, \quad B(r_s) = 1$$

.... static solution

- asymptotic limit reached at $t_s = \infty$ when $r_s = 2M$
  → no time slice ever penetrates horizon

- coordinate transformation to isotropic $\bar{r}$:

$$\alpha(\bar{r}) = \frac{1 - M/2\bar{r}}{1 + M/2\bar{r}} \quad , \quad \beta^{\bar{r}}(\bar{r}) = 0$$

$$\rightarrow \quad A(\bar{r}) = B(\bar{r}) = \left(1 + \frac{M}{2\bar{r}}\right)^4$$

- same time slicing as Schwarzs.
- still not horizon penetrating

Kerr - Schild

$$\alpha(r_s) = \left(\frac{r_s}{r_s + 2M}\right)^{1/2} \quad , \quad \beta^{r_s}(r_s) = \frac{2M}{r_s + 2M}$$

$$\rightarrow \quad A(r_s) = 1 + \frac{2M}{r_s} \quad , \quad B(r_s) = 1$$

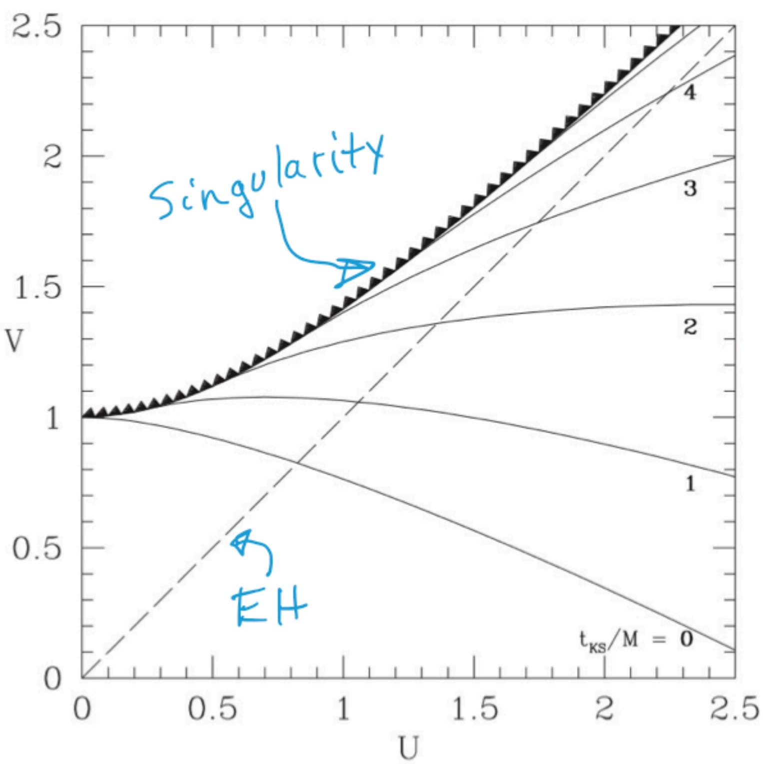
- horizon penetrating, but not singularity avoiding
- must locate at $t_{KS} = $ const. slice to set initial data

- solving to u + v in terms of t_{KS}:

$$u = \frac{1}{2}\left[e^{(t_{KS} + r_s)/4M} + e^{(t_{KS} - r_s)/4M}(r_s/2M - 1)\right]$$

$$v = \frac{1}{2}\left[e^{(t_{KS} + r_s)/4M} - e^{(t_{KS} - r_s)/4M}(r_s/2M - 1)\right]$$

where $t_{KS} = 4M \ln(u+v) - r_s$



- all slices penetrate horizon + hit singularity

Maximal slicing

- horizon penetrating and singularity avoiding
- usually no exact solutions (but see chapter 4.2 in BS)

• line element:

$$ds^2 = -\left(\alpha^2 - \beta^2/A\right) d\bar{t}^2 + 2\beta\, d\bar{t}\, dr + A\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

r: Schwarzschild radial coord.

$\beta = \beta_r = A\beta^r$

$\bar{t}$: maximal time coord.

α, β, A depend on $\bar{t}$ & r only

• for 3+1 ADM eqns., need connection coeffs:

$$\Gamma^r_{rr} = \partial_r A/(2A), \quad \Gamma^r_{\theta\theta} = -r/A, \quad \Gamma^r_{\phi\phi} = -r\sin^2\theta/A$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta, \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = 1/r$$

• Use these to evaluate Ricci tensor:

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^k_{il}\Gamma^l_{jk}$$

$\hookrightarrow$ $R_{rr} = \partial_r A/(rA)$

$$R_{\theta\theta} = R_{\phi\phi}/\sin^2\theta = 1 - A^{-1} + r\partial_r A/(2A^2)$$

• curvature scalar:

$$R = R^i{}_i = 2\partial_r A/(rA^2) + 2(1-A^{-1})/r^2$$

• find extrinsic curvature from spatial metric evolution eqn:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\rightarrow \quad K_{rr} = -(\partial_t A + \beta \partial_r A/A - 2\partial_r \beta)/(2\alpha)$$

$$K_{\theta\theta} = K_{\phi\phi}/\sin^2\theta = r\beta/(\alpha A)$$

• maximal slicing requires $K = K^i{}_i = 0$
– combine with above to find,

$$K_{rr} = -2\beta/(\alpha r), \quad K_{ij} K^{ij} = 6(\beta/\alpha A r)^2$$

$$-\partial_t \ln A + (\beta/A)\partial_r \ln(\beta^2 r^4/A) = 0$$

• then Hamiltonian contraint,

$$R + K^2 - K_{ij} K^{ij} = 16\pi\rho$$

is

$$R = K_{ij} K^{ij} \qquad \text{(assume vacuum)}$$

– insert into above,

$$3\beta^2/(\alpha^2 A) = A - 1 + r\partial_r A/A$$

this plus K_{rr} above give for the momentum constraint,

$$\nabla_j(K^{ij} - \gamma^{ij}K) = 0$$

$$\longrightarrow \quad \partial_r \ln(\beta r^2/A\alpha) = 0$$

- max. slicing requires $\partial_t K = 0$, so contraction of extrinsic curvature evolution eqn. gives

$$\nabla^2\alpha = \alpha R$$

- using curvature scalar above,

$$\partial_r\partial_r\alpha + 2\partial_r\alpha/r - (\partial_r \ln A)\partial_r\alpha/2$$
$$= 2\alpha(A - 1 + r\partial_r \ln A)/r^2$$

... lapse eqn.

- full extrinsic curvature evolution eqn is

$$\partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k_j) - \nabla_i\nabla_j\alpha$$
$$+ \beta^k\partial_k K_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k$$

which for K_{rr} becomes,

$$\partial_{\bar{t}} \ln(\beta/\alpha) = (3\beta/A + \alpha^2 A/\beta - \alpha^2/\beta)/r$$
$$+ 3(\partial_r \beta)/A + (\alpha^2/\beta - 4\beta/A)(\partial_r \ln A)/2$$
$$- (\beta/A + \alpha^2/\beta)\,\partial_r \ln\alpha$$

- integrating $\partial_r \ln(\beta r^2/A\alpha) = 0$ gives

$$\beta = \alpha A T/r^2$$

$$T = T(\bar{t}) \quad \text{... constant of int.}$$

- using this in

$$3\beta^2/(\alpha^2 A) = A - 1 + r\partial_r A/A$$

$\hookrightarrow$ $$A = \frac{1}{1 - 2M/r + T^2/r^4}$$

M function of $\bar{t}$ only

- using this to compute $\partial_{\bar{t}} A$ & inserting into above gives

$$\partial_r(\alpha A^{1/2}) = A^{3/2}(r\,\partial_{\bar{t}} M/T - \partial_{\bar{t}} T/r^2)$$

– can be used to show $\partial_{\bar{t}} M = 0$

- integrating this gives

$$\alpha = \left(1 - 2M/r + T^2/r^4\right)^{1/2}\left[1 + \frac{d_E T}{M}\int_0^{M/r} dx \left(1 - 2x + T^2 x^4/M^4\right)^{-3/2}\right]$$

- $T = T(\bar{t})$ undetermined
  choose $T = 0 \to$ zero shift
  $T = C \to$ time-independent solns. from Ch. 4.2

Stellar Collapse

- see May + White (1966, 1967)
- treating singularity formation tricky. but collapse to NS is ok.

Misner-Sharp formalism

- Lagrangian, spherical symmetry
- see MTW ex. 32.7
- diagonal line element:

$$ds^2 = -e^{2\phi(t,A)}dt^2 + e^{\lambda(t,A)}dA^2$$
$$+ R^2(t,A)\, d\Omega^2$$

R : circumferential radius

→ e^{ϕ} is the lapse & $\beta = 0$

- each mass shell labeled by A & its worldline is $R(t,A)$

  - can choose A to be enclosed rest mass

- 4-velocity:

$$u^a = (e^{-\phi}, 0, 0, 0)$$

- define the following,

$$m = 4\pi \int_0^A \rho_0 (1+\epsilon) R^2 (\partial_A R)\, dA$$

.... gravitational mass enclosed

$$U = e^{-\phi}\partial_t R$$ ... coord. velocity

$$\Gamma = e^{-\lambda/2}\partial_A R$$ ... radial metric function

- Einstein field eqns become,

$$\partial_t U = -e^{\phi}\left(\frac{4\pi \Gamma R^2}{h}\partial_A P + \frac{m + 4\pi R^3 P}{R^2}\right)$$

$$\partial_t m = -e^{\phi}\, 4\pi R^2 P U$$

$$\partial_A \phi = -\frac{1}{\rho_0 h}\partial_A P$$

$$\Gamma = \left(1 + U^2 - 2m/R\right)^{1/2}$$

$$\rho_0 = \frac{\Gamma}{4\pi R^2 \partial_A R}$$

- P pressure & h enthalpy of a perfect gas

- differentiating mass eqn.,

$$\partial_A m = (1+\epsilon)\Gamma$$

- using rest mass in above for Γ,

$$e^{-\lambda/2} = 4\pi \rho_0 R^2$$

- need an EOS:

$$\partial_t \epsilon = -P\partial_t\left(\frac{1}{\rho_0}\right) \quad \ldots \text{ 1st law of thermo}$$

& $P = P(\rho_0, \epsilon)$

- for γ-law EOS: $P = (\gamma - 1)\rho_0 \epsilon$

- Boundary conditions:

$R = 0$, $U = 0$, $\Gamma = 1$, $m = 0$ ... origin, $A = 0$
$P = 0$, $e^{\phi} = 1$ ... surface, $A = A_{total}$

→ similar to Newtonian, but crashes if a singularity forms

Eulerian 1D GR Hydro

- see O'Connor & Ott (2010)
- use "radial gauge polar slicing" RGPS
  → shift vanishes & metric is diagonal + almost Schwarzschild

- line element:

$$ds^2 = -\alpha(r,t)^2 dt^2 + X(r,t)^2 dr^2 + r^2 d\Omega^2$$

$$\alpha(r,t) = e^{\phi(r,t)} \quad X(r,t) = \left(1 - \frac{2m(r,t)}{r}\right)^{-1/2}$$

where ϕ is metric potential &

$$M_{grav}(r,t) = m(r,t) \text{ ... enclosed grav. mass}$$

- for matter:

$$T^{\mu\nu} = \rho_0 h u^\mu u^\nu + P g^{\mu\nu} \quad \text{... S-E}$$

$$J^\mu = \rho_0 u^\mu \quad \text{... mass current}$$

$$h = 1 + \epsilon + \frac{P}{\rho_0} \quad \text{... enthalpy}$$

$$u^\mu = [W/\alpha, W v^r, 0, 0] \quad \text{... 4-velocity}$$

where

$$W = (1 - v^2)^{-1/2} \quad \text{... Lorentz factor}$$

& $v = X v^r$

- from the Hamiltonian constraint, can derive mass eqn.

$$m(r,t) = 4\pi \int_0^r (\rho_0 h W^2 - P + \tau_m^\nu) r'^2 dr'$$

τ_m^ν due to trapped neutrinos

- metric potential $\phi(r,t)$ found from momentum constraint w/ polar slicing:

$$\mathrm{tr}\, K = K_r{}^r$$

$$\hookrightarrow \phi(r,t) = \int_0^{r'} X^2 \left[\frac{m(r',t)}{r'^2} + 4\pi r' \left(\rho_0 h W^2 v^2 + P + \tau_\phi^\nu \right) \right] dr' + \phi_0$$

- constant offset ϕ_0 determined by matching to Schwarzschild at surface ($r = R_*$):

$$\phi(R_*, t) = \ln\left[\alpha(R_*, t)\right] = \frac{1}{2} \ln\left[1 - \frac{2m(R_*, t)}{R_*}\right]$$

Matter evolution

- as usual, obtained from:

$\nabla_\mu T^{\mu\nu} = 0$ ... 4-momentum conservation

$\nabla_\mu J^\mu = 0$ ... baryon number conservation

- Valencia formulation in flux-conservative:

$$\partial_t \vec{U} + \frac{1}{r^2} \partial_r \left[\frac{\alpha r^2}{X} \vec{F} \right] = \vec{S}$$

- conserved variables:

$$\left[D = \alpha X T^t = X \rho W \right.$$

$$\vec{U} = \begin{bmatrix} \ldots \\ D Y_e = \alpha X Y_e J^t = X \rho_0 W Y_e \\ S^r = \alpha X T^{tr} = \rho_0 h W^2 v \\ \tau = \alpha^2 T^{tt} - D = \rho_0 h W^2 - P - D \end{bmatrix}$$

$$\vec{\mathcal{F}} = \begin{bmatrix} D v \\ D Y_e v \\ S^r v + P \\ S^r - D v \end{bmatrix}$$

$$\vec{S} = \begin{bmatrix} 0 \\ 0 \\ (S^r v - \tau - D)\alpha X\left(8\pi r P + \frac{m}{r^2}\right) + \alpha P X \frac{m}{r^2} + \frac{2\alpha P}{X r} \\ 0 \end{bmatrix}$$