Lecture18

Sunday, March 24, 2019 7:29 AM

MHD Equations

- ideal limit of MHD gives:

$$F^{ab} = \epsilon^{abcd} u_c B^{(u)}_d$$
$$B^a_{(u)} = \frac{1}{2} \epsilon^{abcd} u_b F_{dc}$$

- auxilliary magnetic field variables:

$$b^a = B^a_{(u)} / \sqrt{4\pi}$$

$$\mathcal{B}^i = \sqrt{4\pi}\, \gamma^{1/2} W (b^i - v^i b^t)$$

- Constraint equation:

$$\partial_i \mathcal{B}^i = 0 \quad (\text{or } D_i B^i = 0)$$

- induction equation:

$$\partial_t \mathcal{B}^i - \partial_j (v^i \mathcal{B}^j - v^j \mathcal{B}^i) = 0$$

Beyond MHD limit

- in MHD: field is "frozen" into (infinitely conducting) fluid."

- for, e.g., propagation in vacuum, need more general solutions to Maxwell's eqns.
- EM current 4-vector:

$$J^a = n^a \rho_e + j^a$$

ρ_e: charge density, j^a: spatial current for a normal observer ($j^a n_a = 0$)

→ Maxwell's eqns:

$$\nabla_b F^{ab} = 4\pi J^a$$

$$\nabla_{[a} F_{bc]} = 0$$

- in 3+1:

$$D_i E^i = 4\pi \rho_e$$

$$\partial_t E^i = \epsilon^{ijk} D_j (\alpha B_k) - 4\pi \alpha j^i + \alpha K E^i + \mathcal{L}_{\vec{\beta}} E^i$$

$$D_i B^i = 0$$

$$\partial_t B^i = -\epsilon^{ijk} D_j (\alpha E_k) + \alpha K B^i + \mathcal{L}_{\vec{\beta}} B^i$$

- charge conservation: $\nabla_a J^a = 0$, becomes

$$\partial_t \rho_e = -D_i(\alpha j^i) + \alpha K \rho_e + \mathcal{L}_{\vec{\beta}} \rho_e$$

$$\partial_t \rho_e = -\partial_i(\alpha J^i) + \alpha K \rho_e + \mathcal{L}_\beta \rho_e$$

∴ in non-MHD situations, must solve for E^i and B^i

MHD Equations for baryon, energy, & momentum

- S-E in terms of b^a:

$$T^{ab}_{em} = b^2 u^a u^b + \tfrac{1}{2} b^2 g^{ab} - b^a b^b$$

$$b^a = B^a_{(u)}/\sqrt{4\pi} \quad \& \quad b^2 = b^a b_a$$

- EM contribution to 3+1 source terms:

$$\rho_{em} = n_a n_b T^{ab}_{em}$$

$$S_i^{em} = -\gamma_{ia} n_b T^{ab}_{em} \quad \ldots \text{ Poynting flux}$$

$$S_{ij}^{em} = \gamma_{ia}\gamma_{jb} T^{ab}_{em}$$

$$S_{em} = \gamma^{ij} S^{em}_{ij}$$

using above S-E,

$$\rho_{em} = b^2(W^2 - \tfrac{1}{2}) - (\alpha b^t)^2$$

$$S_i^{em} = b^2 u_i W - \alpha b^t b_i$$

$$S_{ij}^{em} = b^2(u_i u_j + \tfrac{1}{2}\gamma_{ij}) - b_i b_j$$

$$S_{em} = b^2(\gamma^{ij} u_i u_j + \tfrac{3}{2}) - \gamma^{ij} b_i b_j$$

$S_{em} = \ldots (\ldots b_i b_j \ldots \frac{\ }{2}) - \ldots b_i b_j$

- <u>Baryon conservation</u> (same as hydro):

$$\boxed{\partial_t(\gamma^{1/2} D) + \partial_j(\gamma^{1/2} D v^j) = 0}$$

$$D \equiv \rho_0 W \quad , \quad W \equiv -n_a u^a = \alpha u^t$$

<u>energy conservation:</u>

- EM S-E obeys: $4\pi T^{ab}_{em} = F^{ac}F^b{}_c - \frac{1}{4} g^{ab} F_{cd} F^{cd}$

$\rightarrow \nabla_a T^{ab}_{em} = -F^{bc} J_c$ w/ Maxwell's eqns.

- this results in additional term on RHS of energy eqn:

$$\boxed{\partial_t(\gamma^{1/2} E) + \partial_j(\gamma^{1/2} E v^j) = -P\left(\partial_t(\gamma^{1/2} W) + \partial_i(\gamma^{1/2} W v^i)\right)}$$

$$-(\alpha\gamma^{1/2})\, \underbrace{u_b F^{bc}}_{0} J_c$$

where, $\qquad E \equiv \rho_0 \epsilon W$

- for perfect conductor this term vanishes, leaving the energy eqn. unchanged from hydro case, since in ideal MHD:

$$u_a F^{ab} = 0 \quad \text{... no "divergence" from fluid}$$

<u>Momentum conservation:</u>

- generalized 4-momentum density:

$$S^*_a = (\rho_0 h + b^2) W u_a$$

- momentum conservation from $\nabla_a T^a{}_b = 0$:

$$\boxed{\begin{aligned} &\partial_t\left(\gamma^{1/2}(S^*_i - \alpha b_i b^t)\right) + \partial_j\left(\gamma^{1/2}(S^*_i v^j - \alpha b_i b^j)\right) \\ &= -\alpha\gamma^{1/2}\left(\partial_i\left(P + \frac{b^2}{2}\right) + \frac{1}{2}\left(\frac{S^*_a S^*_c}{\alpha S^{*t}} - \alpha b_a b_c\right)\partial_i g^{ac}\right) \end{aligned}}$$

The Valencia Formulation

- see, e.g., Mösta et al (2014)

- ADM 3+1 metric:

$$ds^2 = g_{ab}\,dx^a dx^b \equiv (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i\,dt\,dx^i + \gamma_{ij}\,dx^i dx^j$$

- dual to Faraday:

$$^*F^{ab} \equiv \frac{1}{2}\epsilon^{abcd} F_{cd}$$

- Note! F^{ab} & $^*F^{ab}$ magnitudes rescaled by $1/\sqrt{4\pi}$ to simplify eqns.

- Hydro S-E tensor:

$$T_H^{ab} = \rho h u^a u^b + P g^{ab} = (\rho + \rho\epsilon + P)u^a u^b + P g^{ab}$$

- EM S-E tensor:

$$T^{ab}_{EM} = F^{ac} F^{b}{}_{c} - \frac{1}{4} g^{ab} F^{cd} F_{cd}$$
$$= b^2 u^a u^b - b^a b^b + \frac{b^2}{2} g^{ab}$$

where $\quad b^a = u_b {}^{*}F^{ab}$

& $\quad b^2 = b^a b_a = 2 P_m$ --- magnetic pressure

- Total S-E:

$$T^{ab} = (\rho + \rho\epsilon + P + b^2) u^a u^b + \left(P + \frac{b^2}{2}\right) g^{ab} - b^a b^b$$
$$\equiv \rho h^* u^a u^b + P^* g^{ab} - b^a b^b$$

$$h^* \equiv 1 + \epsilon + (P + b^2)/\rho$$
$$P^* \equiv P + P_m = P + b^2/2$$

- spatial magnetic field (Eulerian component of Maxwell tensor):

$$B^i = n_a {}^{*}F^{ib} = -\alpha {}^{*}F^{i0}$$

$$n_a = [-\alpha, 0, 0, 0] \text{ in } 3+1$$

<u>Evolution eqns:</u>

- derived from:
  - mass conservation: $\nabla_a J^a = 0$

$J^a = \rho u^a$

$J = \rho u$ .... mass current

- energy-momentum conservation: $\nabla_a T^{ab} = 0$
- Maxwell's eqns: $\nabla_a {}^*F^{ab} = 0$

- conserved variables:

$$D = \sqrt{\gamma}\,\rho W$$

$$S_j = \sqrt{\gamma}\left(\rho h^* W^2 v_j - \alpha b^t b_j\right)$$

$$\tau = \sqrt{\gamma}\left(\rho h^* W^2 - p^* - (\alpha b^t)^2\right) - D$$

$$\mathcal{B}^k = \sqrt{\gamma}\, B^k$$

- 3-velocity of Eulerian observer at rest in Σ:

$$v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}\,, \quad W \equiv (1 - v^i v_i)^{-1/2}$$

- evolution eqns. in flux-conservative form:

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}^i}{\partial x^i} = \vec{S} \quad \ldots \text{ 1st order PDE}$$

$$\vec{U} = [D, S_j, \tau, \mathcal{B}^k]^T$$

$$\vec{F} = \alpha \begin{bmatrix} D\tilde{v}^i \\ S_j \tilde{v}^i + \sqrt{\gamma}\, p^* \delta^i_j - b_j \mathcal{B}^i / W \\ \ldots \end{bmatrix}$$

$$
\begin{bmatrix}
\tau v^{i} + \sqrt{\gamma}\, P^{*} v^{i} - \alpha b^{0} B^{i}/W \\
B^{k} \tilde{v}^{i} - B^{i} v^{k}
\end{bmatrix}
$$

$$
\vec{S} = \alpha\sqrt{\gamma}
\begin{bmatrix}
0 \\
T^{ab}\left(\partial_a g_{bj} - \Gamma^{c}_{ab}\, g_{cj}\right) \\
\alpha\left(T^{a0}\,\partial_a \ln\alpha - T^{ab}\,\Gamma^{0}_{ab}\right) \\
\vec{0}
\end{bmatrix}
$$

where $\tilde{v}^{i} = v^{i} - \beta^{i}/\alpha$

- solenoidal constraint:

$$\nabla_b {}^{*}F^{0b} = 0$$

$$\hookrightarrow \quad \frac{1}{\sqrt{\gamma}}\,\partial_i\left(\sqrt{\gamma}\, B^{i}\right) = 0$$

and $\boxed{\partial_i B^{i} = 0}$

- convenient to define magnetic field in fluid rest frame:

$$b^{0} = \frac{W B^{k} v_{k}}{\alpha}, \quad b^{i} = \frac{B^{i}}{W} + W\left(B^{k} v_{k}\right)\left(v^{i} - \frac{\beta^{i}}{\alpha}\right)$$

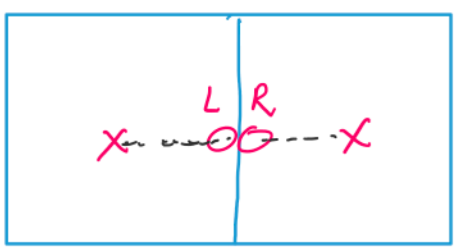
$$b^{2} = \frac{B^{i} B_{i}}{W^{2}} + \left(B^{i} v_{i}\right)^{2}$$

- two common approaches to satisfy:

- divergence cleaning: advects away & damps divergence errors
- constrained transport: induction eqn. solved in such a way to maintain $\partial_i B^i \approx 0$ (note the approx....)

Solving the Riemann Problem

- see Toro
- First step is to reconstruct primitive values from cell averages to pointwise interface values:



→ gives "left" L & "right" R states at interface

- must construct numerical flux at interface based on solution to Riemann problem
- Harten-Lax-van Leer-Einfeldt (HLLE):
  - "two wave" approximation:
    ξ_- : most negative wave speed eigenvalue
    ξ_+ : most positive wave speed eigenvalue

- solution vector:

$$\vec{U} = \begin{cases} \vec{U}^L & \text{if } 0 < \xi_- \\ \vec{U}_* & \text{if } \xi_- < 0 < \xi_+ \\ \vec{U}^R & \text{if } 0 > \xi_+ \end{cases}$$

- intermediate state:

$$\vec{U}_* = \frac{\xi_+ \vec{U}^R - \xi_- \vec{U}^L - \vec{F}(\vec{U}^R) + \vec{F}(\vec{U}^L)}{\xi_+ - \xi_-}$$

- numerical flux at interface:

$$\vec{F}(\vec{U}) = \frac{\hat{\xi}_+ \vec{F}(\vec{U}^L) - \hat{\xi}_- \vec{F}(\vec{U}^R) + \hat{\xi}_+ \hat{\xi}_- (\vec{U}^R - \vec{U}^L)}{\hat{\xi}_+ - \hat{\xi}_-}$$

$$\hat{\xi}_- = \min(0, \xi_-) \quad , \quad \hat{\xi}_+ = \max(0, \xi_+)$$

- requires approximation of wave speeds.
  - detailed analysis costly
  - quick analysis overestimates speeds (diffusive)

- approx. MHD dispersion relation:

$$\omega_d^2 = k_d^2 \left[v_A^2 + c_s^2 \left(1 - \frac{v_A^2}{c^2}\right) \right]$$

$$k_a \equiv (-\omega, k_i) \quad \ldots \text{ wave vector}$$

c_s : fluid sound speed

$$v_A \equiv \sqrt{\frac{b^2}{\rho h + b^2}} = \sqrt{\frac{b^2}{\rho h^*}}$$ ... alfvén speed

- projected wave vector:

$$K_a \equiv \left(\delta_a^b \; u_a u^b\right) k_b$$

↳ $$\omega_d = k_a u^a = -\omega u^t + k_i u^i$$

$$k_A^2 = K_a K^a = \omega_d^2 + g^{bc} k_b k_c$$

- results in quadratic eqn. to solve:

$$\xi^2\left[W^2(V^2-1) - V^2\right] - 2\xi^2\left[\alpha W \tilde{v}^i (V^2-1) + V^2\beta^i\right]$$
$$+\left[(\alpha W \tilde{v}^i)^2 (V^2-1) + V^2(\alpha^2 \gamma^{ii} - \beta^i\beta^i)\right]$$

$$V^2 \equiv v_A^2 + c_s^2(1 - v_A^2)$$

ξ : resulting wave speed

- "i" index NOT summed

  → different speeds in different directions

Conservative to Primitive transform

- complex in GR

- see Mösta et al (2014) or Siegel et al (2018)

Satisfying divergence-free Constraint

Divergence cleaning:

- advect out of domain & damp $\nabla\cdot\vec{B}$ error
- see Dedner
- introduce new scalar field ψ such that:

$$\nabla_a\left({}^*F^{ab}+g^{ab}\psi\right)=\kappa n^b\psi$$

  - reduces to Maxwell's eqns. if $\psi\to 0$
  - κ determines damping rate
- using ${}^*F^{ab}=\frac{1}{W}\left(u^a B^b-u^b B^a\right)$

and $B^0=-\alpha F^{00}=0$ (purely spatial)

LHS becomes,

$$\nabla_a{}^*F^{a0}=-\frac{1}{\alpha\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}B^i\right)$$

and

$$\nabla_a g^{a0}\psi=g^{a0}\partial_a\psi=\frac{1}{\alpha^2}\left[-\partial_t\psi+\beta^i\partial_i\psi\right]$$

- combining these two we have,

$$\boxed{\partial_t \psi + \partial_i(\alpha B^i - \psi\beta^i) = \psi(-K\alpha - \partial_i\beta^i) + \sqrt{\gamma}\, B^i \partial_i\left(\frac{\alpha}{\sqrt{\gamma}}\right)}$$

- spatial part of modified Maxwell's eqns. gives new evolution eqn. for mag. field:

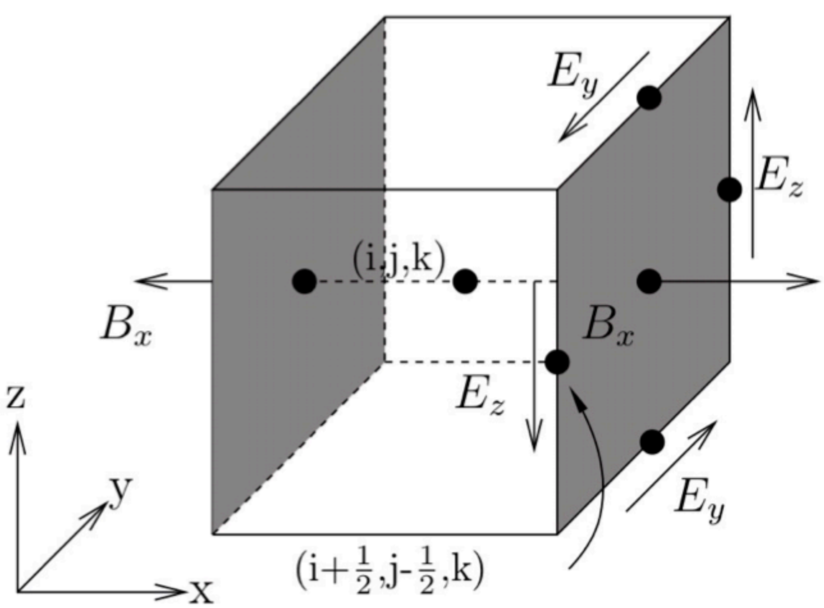
$$\partial_t B^j + \partial_i\left[(\alpha v^i - \beta^i)B^j - \alpha v^j B^i + \alpha\sqrt{\gamma}\,\gamma^{ij}\psi\right]$$

$$= -B^i\partial_i\beta^j + \psi\,\partial_i(\alpha\sqrt{\gamma}\,\gamma^{ij})$$

- reduces to usual eqn. as $\psi \to 0$ & $\partial_i B^i \to 0$

- care must be taken in constructing numerical flux for the auxiliary field ψ!

Constrained transport:



- use Ampere's law to evolve $\vec{B}$ in a way that preserves $\nabla \cdot \vec{B} = 0$

↳ Time derivative of face-averaged magnetic field component $\hat{B}^x$ is

$$
\begin{aligned}
\frac{\partial \hat{B}^x_{i+\frac{1}{2},j,k}}{\partial t} \Delta y \Delta z = & - \mathcal{E}^y_{i+\frac{1}{2},j,k-\frac{1}{2}} \Delta y \\
& - \mathcal{E}^z_{i+\frac{1}{2},j+\frac{1}{2},k} \Delta z \\
& + \mathcal{E}^y_{i+\frac{1}{2},j,k-\frac{1}{2}} \Delta y \\
& + \mathcal{E}^z_{i+\frac{1}{2},j-\frac{1}{2},k} \Delta z
\end{aligned}
$$

where $\vec{\mathcal{E}}$ are the electric fields at edges of cell face

- likewise for other components

- $\vec{\mathcal{E}}$ obtain from numerical fluxes of induction equation

- change in cell-average $\vec{B}$:

$$
\frac{\partial B^x_{i,j,k}}{\partial t} = \frac{1}{2}\left(\frac{\partial \hat{B}^x_{i-\frac{1}{2},j,k}}{\partial t} + \frac{\partial \hat{B}_{i+\frac{1}{2},j,k}}{\partial t} \right)
$$