Lecture14

Sunday, March 3, 2019 7:33 AM

Lapse and Shift

- in order to solve evolution eqns, must specify lapse α & shift β^i.
- these are freely-specifiable gauge conditions
- no general choice; strongly impacts long-term stability.
- picking coords referred to as time slicing and spatial gauge
- lapse α ("time slicing") relates advance of proper time to coordinate time along normal vector n^a

Geodesic slicing

- simple choice: $\alpha = 1$, $\beta^i = 0$

↳ "geodesic slicing" or Gaussian-normal coordinates

- coordinate observers: $u^a = t^a = e^a_{(0)}$ $(u^i = 0)$

∴ w/ $\beta^i = 0$ coordinate observer = normal observer

and $\alpha = 1$ means proper & coord. time agree.

- acceleration of normal vector field:

$$a_b = D_b \ln \alpha = 0$$

→ normal observers are in free-fall (world line is a geodesic)

- recall evolution eqn,

$$\partial_t K = -D^2\alpha + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho + S)\right) + \beta^i D_i K$$

becomes,

$$\partial_t K = K_{ij}K^{ij} + 4\pi(\rho + 3P) \geq 0$$

... for a perfect fluid

- consider expansion of normal observers:

$$\nabla_a n^a = g^{ab}\nabla_a n_b = (g^{ab} + n^a n^b)\nabla_a n_b$$
$$= \gamma^{ab}\nabla_a n_b = -K$$

∴ normal observers (geodesics) converge with time.

- evolution of spatial metric:

$$\partial_t \ln \gamma^{1/2} = -\alpha K + D_i \beta^i = -K \quad \text{... geodesic slicing}$$

→ coordinate volume element goes to zero & K grows w/o bound!

$\therefore$ coordinate singularity

Maximal slicing

- singularities in coordinates can be avoided by suitably specifying K.

$\rightarrow$ easiest choice:

$$K = 0 \quad \text{.... maximal slicing}$$

- then eqn for lapse:

$$\Delta^2 \alpha = -\partial_t K + \alpha \left(K_{ij} K^{ij} + 4\pi(\rho + S) \right) + \beta^i \Delta_i K$$
$$= \alpha \left(K_{ij} K^{ij} + 4\pi(\rho + S) \right)$$

where we assume maximal slicing on all Σ_i,

$$K = 0 = \partial_t K$$

- w/ Hamiltonian constraint,

$$R + \cancelto{0}{K^2} - K_{ij} K^{ij} = 16\pi\rho$$

lapse eqn becomes,

$$\Delta^2 \alpha = \alpha \left(R - 4\pi(3\rho - S) \right)$$

- conformally-related lapse eqn.:

$$\bar{\Delta}^2 (\alpha\psi) = \alpha\psi \left(\frac{7}{8} \psi^{-8} \bar{A}_{ij} \bar{A}^{ij} + \frac{1}{8} \bar{R} + 2\pi \psi^4 (\rho + 2S) \right)$$

- maximal slicing is analogous to soap bubble on a wire loop stretched tight

on a wire loop · stretched tight.
(in Euclidean, this is minimal, in pseudo-Riemannian, maximal)

- normal observers are incompressible and irrotational
- above eqns. are spatial, 2^{nd}-order PDEs, requiring two boundary conditions.

Schwarzschild & Max. slicing

- isotropic coords. & maximal slicing for Schwarz. But an entire family is possible!

- metric in Schwarz. coords:

$$ds^2 = -f_0\,dt^2 + f_0^{-1}\,dr_s^{\,2} + r_s^2\,d\Omega^2$$

where

$$f_0(r_s) = 1 - 2M/r_s$$

- define new time : $\bar{t} = t + h(r_s)$
where $h(r_s)$ is the "height" function, measures how far $\bar{t}$ = const. surfaces "lift off" t = const. surfaces
- since $dt = d\bar{t} - h'\,dr_s$ $[h' \equiv dh/dr_s]$,

$$ds^2 = -f_0\,d\bar{t}^2 + 2f_0 h'\,d\bar{t}\,dr_s + (f_0^{-1} - f_0 h'^2)\,dr_s^2 + r_s^2\,d\Omega^2$$

- recall 3+1 metric,

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$\hookrightarrow$

$$\gamma_{ij} = \mathrm{diag}\left[(1-f_0^2 h'^2)/f_0 \,,\; r_s^2 \,,\; r_s^2 \sin^2\theta\right]$$

$$\beta^{r_s} = \frac{f_0^2 h'}{1-f_0^2 h'^2} \,, \quad \alpha^2 = \frac{f_0}{1-f_0^2 h'^2}$$

- if $h = \text{const}$, then $t = \text{const}$, as expected
- normal vector: $n^a = \alpha^{-1}(1, -\beta^i)$
- requiring max. slicing for $\bar{t} = \text{const}$,

$$K = -\nabla_a n^a = -|g|^{-1/2}\partial_a(|g|^{1/2} n^a) = 0$$

- using $|g|^{1/2} = \alpha\,\gamma^{1/2} = r_s^2 \sin\theta$ and all time derivatives must vanish,

$$\rightarrow \quad \frac{d}{dr_s}\left(r_s^2 \left(\frac{f_0}{1-f_0^2 h'^2}\right)^{1/2} f_0 h' \right) = 0$$

- integrating once,

$$r_s^2 \left(\frac{f_0}{1-f_0^2 h'^2}\right)^{1/2} f_0 h' = C \quad \text{(const. of integration)}$$

$$\rightarrow \quad f_0^2 h'^2 = \frac{C^2}{f_0 r_s^4 + C^2}$$

- using this,

$$dl^2 = f^{-2}(r_s; C)\, dr_s^2 + r_s^2 (d^2\theta + \sin^2\theta\, d^2\phi)$$

... spatial metric

$$\alpha = f(r_s; C) \quad \text{--- lapse}$$

$$\beta^{r_s} = \frac{C f(r_s; C)}{r_s^2} \quad \text{---- shift}$$

where, $$f(r_s; C) = \left(1 - \frac{2M}{r_s} + \frac{C^2}{r_s^4}\right)^{1/2}$$

∴ a family of solutions in C

- Schwarzschild coords recovered w/ $C=0$

- choice of BCs & coordinates can severely impact evolution and stability.
  - natural to specify asymptotic-flatness:

    $\alpha \to 1$ as $r_s \to \infty$

- for $C=0$, $\alpha=0$ at $r_s = 2M$: "static slicing"
- for $C = 3\sqrt{3}\,M^2/4$, $\partial_{r_s}\alpha = 0$ at $r_s = 3M/2$

  ↳ "dynamical slicing" as $t \to \infty$
- these solutions avoid the singularity at $r_s = 0$
  - it takes πM time for a timelike observer in free-fall to reach singularity from just outside the horizon
  - advance of proper time is $d\tau = \alpha\, dt$

→ in order to avoid singularity, lapse

must collapse and go to zero at $r_s = 0$

- in vacuum, maximal slicing condition,

$$D^2 \alpha - \alpha R = 0$$

- Maximal slicing has geometric advantages, but results in (first-order) elliptic equations for lapse (hard to solve).
- but is only a gauge condition: can modify condition w/o changing physical solution.
  - such choice would satisfy $K = 0$ only approximately. $\rightarrow \partial_t K \neq 0$
  - $\rightarrow$ drive K back toward zero:

$$\partial_t K = -cK \quad , \quad c > 0$$

- lapse eqn. becomes,

$$D^2 \alpha = \alpha\left(K_{ij} K^{ij} + 4\pi(\rho + S)\right) + \beta^i D_i K + cK$$

$\rightarrow$ still elliptic

  - make this parabolic by adding a "time" derivative:

$$\partial_\lambda \alpha = D^2 \alpha - \alpha\left(K_{ij} K^{ij} + 4\pi(\rho + S)\right) - \beta^i D_i K - cK$$

where λ is a time parameter.

  - w/ $\lambda = \epsilon t$,

$$\partial_t \alpha = \epsilon \Delta \alpha - \epsilon \alpha (K_{ij} K + 4\pi(\rho + S)) - \epsilon \beta \partial_i K - \epsilon c K$$

→ ϵ is a diffusion constant

or

$$\partial_t \alpha = -\epsilon (\partial_t K + cK)$$

.... K-driver condition

- K decays exponentially to zero w/ time as $\epsilon \to \infty$

  → but this would violated parabolic Courant condition

Harmonic Coordinates

$$^{(4)}\Gamma^a \equiv g^{bc\,(4)}\Gamma^a_{bc} = -\frac{1}{|g|^{1/2}} \partial_b \left(|g|^{1/2} g^{ab} \right)$$

... contraction of connection coeffs.

- can choose gauge conditions such that,

$$^{(4)}\Gamma^a = 0$$

  - for this, the coordinates x^a satisfy

$$\nabla^2 x^a \equiv \nabla^b \nabla_b x^a = 0$$

- inserting 3+1 metric into contraction eqn,

$$(\partial_t - \beta^j \partial_j)\alpha = -\alpha^2 K$$

$$(\partial_t - \beta^j \partial_j)\beta^i = -\alpha^2 (\gamma^{ij} \partial_j \ln\alpha + \gamma^{jk} \Gamma_{jk})$$

.... hyperbolic eqns. for lapse & shift

- in Harmonic coords, 4D Ricci tensor :

$$\ldots K_{ab} = -2g \;\partial_d \partial_c g_{ab} + g_{c(a}\partial_{b)} \Gamma^{c}$$
$$+ {}^{(4)}\Gamma^{c}\,{}^{(4)}\Gamma_{(ab)c} + 2g^{ed}\,{}^{(4)}\Gamma^{c}{}_{e(a}\,{}^{(4)}\Gamma_{b)cd}$$
$$+ g^{cd}\,{}^{(4)}\Gamma^{e}{}_{ad}\,{}^{(4)}\Gamma_{ecb}$$

- w/ this, field equations reduce to (hyperbolic) wave equations

- More common in 3+1 simulations is **harmonic slicing**: ${}^{(4)}\Gamma^{0} = 0$ only
  - w/ $\beta^i = 0 \rightarrow \partial_t \alpha = -\alpha^2 K$
  - w/ contraction of evolution eqn:
    $$\partial_t \ln \gamma^{1/2} = -\alpha K + D_i \beta^i$$
  $\rightarrow \alpha = C(x^i)\,\gamma^{1/2}$ ... after integrating
  where $C(x^i)$ is integration constant that depends on coord. x^i (not time)

- minor modification: $\alpha = 1 + \ln \gamma$
  .... **"1+log" slicing**

- if $\beta^i \neq 0$
  $\hookrightarrow (\partial_t - \beta^j \partial_j)\alpha = -2\alpha K$ (hyperbolic)
  .... "moving puncture"

$\boxed{\text{Minimal distortion}}$

- seek to minimize time-rate-of-change of conformally-related spatial metric $\bar{\gamma}_{ij}$

- recall traceless part of spatial metric time derivative:

$$u_{ij} \equiv \gamma^{1/2} \partial_t \left(\gamma^{-1/3} \gamma_{ij} \right)$$

- can decompose into transverse-traceless and longitudinal part:

$$u_{ij} = u^{TT}_{ij} + u^{L}_{ij}$$

- by definition, $D^j u^{TT}_{ij} = 0$

and

$$u^{L}_{ij} = D_i X_j + D_j X_i - \frac{2}{3} \gamma_{ij} D^k X_k = (LX)^{ij}$$

.... vector gradient of X^i

- RHS can be written in terms of Lie deriv.

$$u^{L}_{ij} = \gamma^{1/3} \mathcal{L}_{\vec{X}} \bar{\gamma}_{ij}$$

where $\bar{\gamma}_{ij} = \gamma^{-1/3} \gamma_{ij}$

- this implies that u^{L}_{ij} arises from changing coords in time, generated by X^i

$\therefore$ we can <u>choose</u> to eliminate it:

$$u^{L}_{ij} = 0$$

leaving only u^{TT}_{ij}, or

$$D^j u_{ij} = 0$$

- recall the conformally-related, traceless part of extrinsic curvature:

$$\bar{A}_{ij} = \frac{\psi^6}{2\alpha}\left((\bar{L}\beta)^{ij} - \bar{u}^{ij}\right)$$

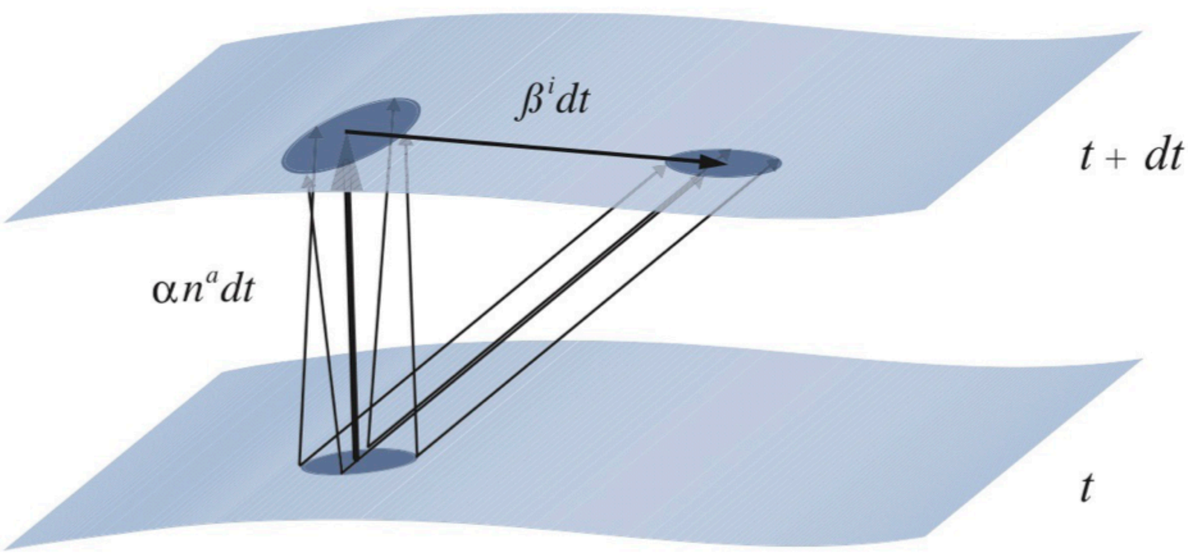
$$\rightarrow D^j(L\beta)_{ij} = 2D^j(\alpha A_{ij})$$

- using momentum constraint to eliminate divergence of A^{ij},

$$\underbrace{(\Delta_L \beta)^i}_{\text{"vector Laplacian"}} = 2A^{ij}D_j\alpha + \frac{4}{3}\alpha\gamma^{ij}D_jK + 16\pi\alpha S^i$$

"vector Laplacian": ···· minimal distortion condition

$$(\Delta_L W)^i \equiv D^2 W^i + \frac{1}{3}D^i(D_j W^j) + R^i_{\ j}W^j$$



- this slicing minimizes shear in coords.

→ a sphere transported in time will remain as close to spherical as allowed by the spacetime.

- Can extend minimal distortion using a related spatial gauge based on "connection functions":

$$\bar{\Gamma}^i \equiv \bar{\gamma}^{kl} \bar{\Gamma}^i_{kl}$$

- assuming $\bar{\gamma} = \det(\bar{\gamma}_{ij}) = 1$ in Cartesian,

$$\bar{\Gamma}^i = -\partial_j \bar{\gamma}^{ij}$$

- rather than set to zero (harmonic),

$$\partial_t \bar{\Gamma}^i = 0$$

- using spatial metric evolution eqn.:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

we will find,

$$\bar{A}_{ij} = e^{-4\phi} A_{ij}$$

where $\phi = \frac{1}{12} \ln \gamma$

- using this in $\partial_t \bar{\Gamma}^i = 0$ gives,

$$\bar{\gamma}^{lj} \partial_j \partial_l \beta^i + \frac{1}{3} \bar{\gamma}^{li} \beta^j{}_{,jl} + \beta^j \partial_j \bar{\Gamma}^i - \bar{\Gamma}^j \partial_j \beta^i + \frac{2}{3} \bar{\Gamma}^i \partial_j \beta^j = 2 \tilde{A}^{ij} \partial_j \alpha - 2\alpha \left(\bar{\Gamma}^i_{jk} \tilde{A}^{jk} - \frac{2}{3} \bar{\gamma}^{ij} \partial_j K - \bar{\gamma}^{ij} S_j + 6 \tilde{A}^{ij} \partial_j \ln \psi \right)$$

where $\tilde{A}^{ij} = \psi^4 A^{ij}$

.... Gamma-freezing condition

- used in BSSN formulation
- related to minimal distortion since

$$\partial_j(\bar{\gamma}^{ij}) = 0$$

- forms complex system of elliptic eqns. for shift.

→ make parabolic by approximating condition,

$$\partial_t \beta^i = k(\partial_t \bar{\Gamma}^i + \eta \bar{\Gamma}^i) \quad \text{... Gamma-driver}$$

w/ k & η positive constants

- can make hyperbolic using,

$$\partial_t \beta^i = \tfrac{3}{4} B^i \,, \quad \partial_t B^i = \partial_t \bar{\Gamma}^i - \eta B^i$$

w/ $\eta \sim 1/2M$, M mass of system