Lecture8

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Curvature coupling, Equivalence, Laws of Physics

- consider stress-energy tensor for perfect fluid,

$$\overleftrightarrow{T} = (\rho + P)\vec{u} \otimes \vec{u} + P\overleftrightarrow{g}$$

→ a local, SR, non-gravitational law of physics

- Equivalence principle tells us the form is the same in GR!

→ if we can write SR laws in frame-independent way, then they are equivalent to the GR versions. But they must be local!

- technically, equivalence principle only valid at the origin of a LLF (that is, along timelike geodesics)
- global law of 4-momentum conservation,

$$\int T^{\alpha\beta} d\Sigma_\beta = 0$$

runs afoul of these requirements.

↳ equivalence principle does not hold!

- Any not-completely local law will be in trouble. Consider spin angular momentum of some particle,

$$S_i = \int_{\text{interior}} \epsilon_{ijk} x^j T^{k0} \, dx \, dy \, dz$$

(T^{k0} are components of momentum density)

- in SR, $\nabla_{\vec{u}} \vec{S} = 0$

↳ spin angular momentum is parallel transported along world line

- analogous to 4-momentum conservation, derivation of this is not totally local
- if particle is small (i.e., always & only at origin of LLF) then we are fine
- for finite size particle, curvature & quadrupole moments couple to produce a torque on particle,

$$S^{\alpha}{}_{;\mu} u^{\mu} = \epsilon^{\alpha\beta\sigma\delta} I_{\beta\mu} R^{\mu}{}_{\nu\delta\rho} u_{\delta} u^{\nu} u^{\rho}$$

- Curvature at Earth ($R^{\alpha}{}_{\beta\gamma\delta}$) from sun-Earth-moon system couples to slightly flattened earth (non-zero quadrupole moment).
  ↳ precession of poles

- When does curvature-coupling matter? When the physics involves double gradients.
  - since connection coeffs. vanish at origin of LLF, gradient is the same in LLF (GR) as in global LF (SR),
    $$A^{\alpha}{}_{;\beta} = \partial A^{\alpha} / \partial x^{\beta}$$
  - but recall <u>derivatives</u> of connection coeffs do NOT vanish if spacetime is curved. So double gradient will NOT vanish.
  - Example: wave equation for EM vector 4-potential
    in SR: $A^{\alpha;\mu}{}_{\mu} = 0$

in GR: $A^{\alpha;\mu}{}_{\mu} = R^{\alpha\mu} A_{\mu}$

[all indices after semicolon differentiate,
$A^{\alpha;\mu}{}_{\mu} \equiv A^{\alpha;\mu}{}_{;\mu}$]

$\boxed{\text{Einstein Field Equations}}$

- Have seen how curvature leads to motion, but what causes spacetime curvature?
- Can use the Newtonian limit for guidance.

$$\nabla^2 \Phi = 4\pi G \rho$$

- Laplacian can be re-written in terms of the tidal field tensor,

$$\mathcal{E}_{jk} = \frac{\partial^2 \Phi}{\partial x^j \partial x^k}$$

$\hookrightarrow \quad \mathcal{E}^j{}_j = 4\pi G \rho$ ... trace of tidal field

- so Newtonian gravity is completely determined by potential Φ, as embodied in tidal field

- by analogy, GR gravity must be determined by the metric $\overleftrightarrow{g}$, as embodied by some piece of the curvature tensor $\overleftrightarrow{R}$
[recall, this is the GR equivalent of the tidal field],
plus some GR equivalent of mass density
- in LLF,

$$\mathcal{E}_{jk} = R_{j0k0}$$

now take trace, $\mathcal{E}^j{}_j = R^\alpha{}_{0\alpha 0} = R_{00}$

since $R^0{}_{000} = 0$ by symmetry

then arrive at a guess for GR gravity,

$$R_{00} = 4\pi G \rho$$

- Must generalize this to a frame-independent law of the correct Newtonian limit.
generalized mass density must be stress-energy tensor so,

$$R_{\alpha\beta} = 4\pi G T_{\alpha\beta}$$

→ looks good, but ... too many independent equations (10) for the number of unknowns!
∴ over-specified! Only 6 unknowns.

- Einstein (1915) & Hilbert (1915) both realized a trick to resolve this,
  - replace Ricci tensor w/ some new curvature tensor times a constant,

$$G^{\alpha\beta} = \kappa T^{\alpha\beta}$$

where we require new tensor to be divergenceless, $G^{\alpha\beta}{}_{;\beta} \equiv 0$.

  - since $T^{\alpha\beta}{}_{;\beta} = 0$ due to local 4-momentum conservation,

$$(G^{\alpha\beta} - \kappa T^{\alpha\beta})_{;} = 0$$

is automatically satisfied by the properties of these two tensors, ie. these four eqns. do NOT constrain the metric!

∴ 10 equations become 6. Huzzah.

- form of the Einstein curvature tensor,

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}$$

curvature scalar

- independent of metric,

$$\vec{\nabla} \cdot \overleftrightarrow{G} \equiv 0$$

- so what is constant κ in new

field equations? Can determine by making connection w/ Newtonian limit field eqns. in terms of Ricci tensor:

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = \kappa T^{\alpha\beta}$$

- take the trace of this: $R = -\kappa g_{\mu\nu} T^{\mu\nu}$ which determines curvature scalar

↳ $R^{\alpha\beta} = \kappa \left(T^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} g_{\mu\nu} T^{\mu\nu} \right)$

- in Newtonian limit, mass-energy density

$$T^{00} \cong \rho \gg T^{j0} \quad \text{(momentum density)}$$

and $T^{00} \gg T^{jk}$ (the stress)

so consider only time-time component,

$$R^{00} = \kappa \left(T^{00} - \frac{1}{2} \eta^{00} \eta_{00} T^{00} \right) = \frac{1}{2} \kappa T^{00} = \frac{1}{2} \kappa \rho$$

where ρ is <u>mass</u> density and η is flat metric.

- since $R^{00} = R_{00}$ in LLF & the earlier Newtonian limit was $R_{00} = 4\pi G \rho$,

$$\kappa = 8\pi G$$

$$\therefore \quad G^{\alpha\beta} = 8\pi G \, T^{\alpha\beta} \quad \ldots \text{Einstein field equations}$$

• Can re-express this in geometrized units,

$G = c = 1$

$\hookrightarrow 1 = \frac{G}{c^2} = 7.42 \times 10^{-28} \frac{m}{kg}$

or, $1 \text{ kg} = 7.42 \times 10^{-28} \text{ m}$

- mass is now length

$\longrightarrow$ field eqns. become,

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta} \quad (\text{frame-independent})$$

And now you've taken your first steps into a larger world.

$\boxed{\text{Weak Gravitational Fields}}$

The Newtonian Limit

- by "weak" we mean there's nearly a global Lorentz frame,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad \ldots \quad |h_{\alpha\beta}| \ll 1$$

- To use this to obtain Newtonian limit, have following requirements:

1. $|h_{\alpha\beta,t}| \ll |h_{\alpha\beta,j}|$
   gravity changes slowly in time

2. $|T^{i0}| \ll T^{00} \equiv \rho$ .... energy density

stuff is moving slowly

3. $|u^i| \ll u^0$

particles influenced by gravity are moving slowly.

4. $|T^{ik}| \ll T^{00}$

stresses in gravitating bodies are small.

- if the above are true then, to leading order, GR reduces to Newtonian.

Linearized Theory

- Let's relax all of the above Newtonian constraints except for weak gravity,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \ , \ |h_{\alpha\beta}| \ll 1$$

then we can develop of theory that is linear in $h_{\alpha\beta}$.

- $h_{\mu\nu}$ is regarded as gravitational field living in flat spacetime; essentially SR

- Riemann tensor is

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\right)$$

- Einstein tensor: $G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}$

- use above metric & Riemann:

$$2 G_{\mu\nu} = h_{\mu\alpha,\nu}{}^{\alpha} + h_{\nu\alpha,\mu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{\beta}\right) = 16\pi T_{\mu\nu}$$

where $h \equiv \eta^{\alpha\beta} h_{\alpha\beta}$ is trace of $h_{\alpha\beta}$

- define trace-reversed metric perturbation:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$$

such that $\bar{h} \equiv \bar{h}_{\alpha\beta}\eta^{\alpha\beta} = -h$

- substitute into field eqn.:

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\mu\alpha,\nu}{}^{\alpha} + \bar{h}_{\nu\alpha,\mu}{}^{\alpha} = 16\pi T_{\mu\nu}$$

- now introduce new coords. for nearly global Lorentz frame:

$$x^{\alpha}_{new}(\mathcal{P}) = x^{\alpha}_{old}(\mathcal{P}) + \xi^{\alpha}(\mathcal{P})$$

$\hookrightarrow h^{new}_{\mu\nu} = h^{old}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$

$\rightarrow$ this is a gauge transformation

- recall from E+M, for scalar field ψ can gauge transform vector potential,

$$A_\mu^{new} = A_\mu^{old} - \psi_{,\mu}$$

  - under such transformation, EM field tensor $F_{\mu\nu}$ is <u>invariant</u>.

- turns out, gauge transformations leave Riemann tensor unchanged, too
- Choosing ξ^α appropriately, can impose gauge conditions:

$$\bar{h}_{\mu\nu,}{}^{\nu} = 0 \quad \ldots\ldots$$ gravitational Lorentz gauge

[just like Lorentz gauge, $A_{\mu,}{}^{\mu} = 0$]

- now field eqns. become,

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 16\pi T_{\mu\nu} \quad \ldots\ldots$$ gravitational wave eqn.

- can solve for $\bar{h}_{\mu\nu}$ via

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \int \frac{4T_{\mu\nu}(t-|\vec{x}-\vec{x}'|, \vec{x}'])}{|\vec{x}-\vec{x}'|}\, dV_{x'}$$

where $t' = t - |\vec{x}-\vec{x}'|$ is "retarded" time

→ these equations imply that changing

$T_{\mu\nu}$ can generate waves in spacetime

↳ gravitational waves