Lecture10

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$\boxed{\text{3+1 decomposition}}$

- solving Einstein's equations is analogous to classical IVP.
  - there, specify $\vec{x}$ & $\vec{v}$ of stuff then use to integrate forward in time
  - in GR, must specify g_{ab} & $g_{ab,t}$ on some hypersurface w/ constant $x^0 = t$.
  - then compute $g_{ab,tt}$ to integrate forward to a new hypersurface at $t + \delta t$
- finding $g_{ab,tt}$ is not easy, however
- need 10 second derivatives and there are 10 field equations $G_{ab} = 8\pi T_{ab}$

But Bianchi identities $\nabla_b G^{ab} = 0$

$$\rightarrow \partial_t G^{a0} = -\partial_i G^{ai} - G^{bc}\Gamma^a{}_{bc} - G^{ab}\Gamma^c{}_{bc}$$

? ... term on r.h.s. contains third time

• no work on r.h.s. contains third time derivatives, G^{a0} cannot contain second time derivatives.

$\therefore$ $G^{a0} = 8\pi T^{a0}$ cannot provide any second time derivatives we need

• instead, these are constraints between g_{ab} & $\partial_t g_{ab}$ on hypersurface $x^0 = t$

• only dynamical equations (that can be used)

$$G^{ij} = 8\pi T^{ij} \quad \ldots \quad 6 \text{ equations}$$

[• recall symmetry of tensors involved:
$G_{ab} = G_{ba}$ & $T_{ab} = T_{ba}$]

• the appearance of this ambiguity [10 vs. 6] is to be expected since we have complete freedom to choose our (4D) coordinate system. Thus 4 d.o.f. <u>must</u> be non-physical.

• if the constraints, $G^{a0} = 8\pi T^{a0}$, are satisfied initially (at $x^0 = t = 0$), then it is satisfied at all later times. I.e.,

$$\left(G^{a0} - 8\pi T^{a0}\right)_{,t} = 0$$

- "3+1" decomposition :
  - more natural way to solve Cauchy problem
  - 4 constraints w/ no time derivatives that must be satisfied on every time slice
  - 12 evolution eqns. (coupled, 1st order, ODEs)
  - 4 freely specifiable functions that result from freedom to choose coords.

$\boxed{\text{Maxwell's Eqns. in Minkowski}}$

$$C_E \equiv D_i E^i - 4\pi \rho = 0$$

$$C_B \equiv D_i B^i = 0$$

where,

ρ : charge density
E^i : electric field
B^i : magnetic field
D_i : spatial covariant derivative (gradient in x^i)

→ these are constraint eqns. (contain no time derivatives)

- Evolution eqns. :

$$\partial_t E_i = \epsilon_{ijk} D^j B^k - 4\pi j_i$$

$$\partial_t B_i = -\epsilon_{ijk} D^j E^k$$

where, $\quad j^i$: charge 3-current

$$D_i \partial_t E^i = \cancel{D_i \epsilon^{ijk} D_j B_k}^{\,0} - D_i 4\pi j^i$$

$$\partial_t D_i E^i = \partial_t (4\pi \rho) = -4\pi D_i j^i$$

$$\rightarrow \quad \partial_t \rho + D_i j^i = 0 \quad \text{--- continuity eqn.}$$

- can recast eqns. into "3+1" like form
- introduce vector potential $A^a = (\Phi, A^i)$

then, $\quad B_i = \epsilon_{ijk} D^j A^k$

(automatically obeys $D_i B^i = 0$)

- now, evolution eqns:

$$\partial_t A_i = -E_i - D_i \Phi$$

$$\partial_t E_i = D_i D^j A_j - D^j D_j A_i - 4\pi j_i$$

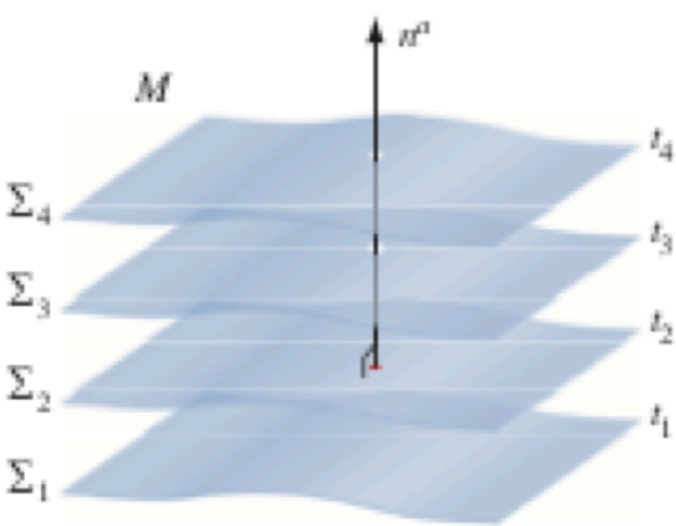
$\rightarrow$ still satisfies constraint:

$$C_E \equiv D_i E^i - 4\pi \rho = 0$$

$$\partial_t C_E = \partial_t D_i E^i - 4\pi \partial_t \rho$$
$$= D_i \left(D^i D_j A_j - D_j D^j A^i - 4\pi j^i \right)$$
$$+ 4\pi D_i j^i$$
$$= D_i D^i D_j A_j - D_i D_j D^j A^i$$

$$= D_i \cancel{D D_j} A_j - D_i \cancel{D_j} D A$$
$$- 4\pi \cancel{D_i} j^i + 4\pi \cancel{D_i} j^i = 0 \quad \text{q.e.d.}$$

- use of vector potential form introduces **gauge freedom**: can freely specify Φ
- solving EM equations then involves specifying IC's $[A_i, E_i, \rho, j^i]$ that obey constraints, then using vector potential evolution equations to integrate forward in time.
  - This is analogous to 3+1 eqns.



$\boxed{\text{Foliations of spacetime}}$

- will **foliate** the spacetime manifold M, described by metric g_{ab}, into family of non-intersecting spacelike 3-surfaces Σ which are level functions of scalar function t (a pseudo global time).

- can define 1-form

$$\Omega_a = \nabla_a t$$

which is closed by

$$\nabla_{[a}\Omega_{b]} = \nabla_{[a}\nabla_{b]} t = 0$$

[recall notation of symmetric & anti-symmetric tensors:

$$T_{(ab)} \equiv \frac{1}{2}\left(T_{ab} + T_{ba}\right), \quad T_{[ab]} \equiv \frac{1}{2}\left(T_{ab} - T_{ba}\right)$$]

- use the metric to compute norm of Ω_a: $\quad \|\Omega\|^2 = g_{ab}\nabla_a t \nabla_b t \equiv -\frac{1}{\alpha^2}$

- α measures lapses in proper time between neighbouring time slices along normal vector Ω^a, called the lapse
  - will assume $\alpha > 0 \rightarrow \Omega^a$ is timelike, making Σ spacelike

- the normalized 1-form $\omega_a \equiv \alpha\,\Omega_a$ is rotation free, $\omega_{[a}\nabla_b\,\omega_{c]} = 0$

$$\omega_{[a}\nabla_b\omega_{c]} = \alpha\,\Omega_{[a}\nabla_b\alpha\,\Omega_{c]}$$

- unit normal to "time" slices,

$$n^a \equiv -g^{ab}\omega_b$$

→ points in direction of increasing time

- normalized & timelike,

$$n^a n_a = g^{ab}\omega_a\omega_b = -1$$

↳ like a 4-velocity of a "normal" observer w/ world line always normal to Σ

- now construct purely spatial metric,

$$\gamma_{ab} = g_{ab} + n_a n_b$$

- a projection tensor that projects out all geometric objects lying along n^a
- computes distances on Σ

- now,

$$n^a\gamma_{ab} = n^a g_{ab} + n^a n_a n_b = n_b - n_b = 0$$

∴ γ_{ab} is purely spatial & resides entirely in Σ

- inverse spatial metric,

$$\gamma^{ab} = g^{ac}g^{bd}\gamma_{cd} = g^{ab} + n^a n^b$$

- need to break up 4D tensors into purely spatial & purely temporal parts. first needed projection operator:

$$\gamma^a{}_b = g^a{}_b + n^a n_b = \delta^a{}_b + n^a n_b$$

→ projects 4D tensors into spatial slice

$\boxed{\text{Ex. 2.6}}$ for arbitrary spacetime vector v^a,

$$n^c g_{ac} \gamma^a{}_b v^b = n^c g_{ac} (\delta^a{}_b v^b + n^a n_b v^b)$$

$$= n^c g_{ac} \delta^a{}_b v^b + g_{ac} n^c n^a n_b v^b$$

$$= n_c v_a + n_c n_a n_b v^b$$

$$= n_c v_a - n_c v_a = 0 \quad ?$$

∴ $\gamma^a{}_b v^b$ is purely spatial.

- Projecting higher rank tensors spatially,

$$\perp T_{ab} = \gamma_a{}^c \gamma_b{}^d T_{cd}$$

- normal projection,

$$N^a{}_b \equiv -n^a n_b = \delta^a{}_b - \gamma^a{}_b$$

- any vector in terms of its spatial + temporal parts:

$$v^a = \delta^a{}_b v^b = (\gamma^a{}_b + N^a{}_b) v^b = \perp v^a - n^a n_b v^b$$

- Example: Schwarzschild metric in isotropic spherical coordinates,

$$ds^2 = -\left(\frac{1 - M/(2r)}{1 + M/(2r)}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2)$$

- for spatial slices Σ w/ constant t, then components of 1-form $\Omega_a = (1, 0, 0, 0)$

- recall $\|\Omega\|^2 = g^{ab} \nabla_a t \nabla_b t \equiv -\alpha^{-2}$

↳ $$\alpha = \frac{1 - M/(2r)}{1 + M/(2r)}$$ ... the lapse

→ $$n_a = -g^{ab}\omega_b = \frac{1 + M/(2r)}{1 - M/(2r)}(1, 0, 0, 0)$$ .... normal vector

spatial metric:

$$\gamma_{ab} = \left(1 + \frac{M}{2r}\right)^4 \mathrm{diag}\left(0, 1, r^2, r^2 \sin^2\theta\right)$$

→ no time components

- covariant derivative in 3+1:
  - project 4D gradient onto Σ,

$$D_a f \equiv \gamma_a{}^b \nabla_b f$$

... for scalar field f

and $$D_a T^b{}_c \equiv \gamma_a{}^d \gamma_e{}^b \gamma_c{}^f \nabla_d T^e{}_f$$

... for mixed-rank-2 tensor $T^b{}_c$

- w/ this definition, $D_a \gamma_{bc} = 0$
- 3D connection coeffs. in coord. basis:

$$\Gamma^a{}_{bc} = \frac{1}{2}\gamma^{ad}\left(\partial_c \gamma_{db} + \partial_b \gamma_{dc} - \partial_d \gamma_{bc}\right)$$

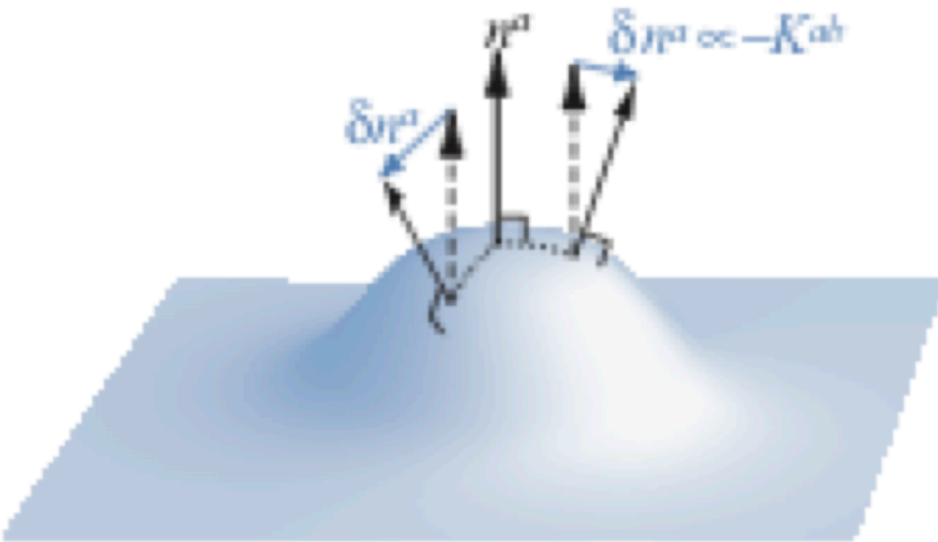
- then the 3D Riemann tensor is,

$$R_{abc}{}^{d} = \partial_b \Gamma^{a}_{ac} - \partial_a \Gamma^{a}_{bc} + \Gamma^{e}_{ac}\Gamma^{a}_{eb} - \Gamma^{e}_{bc}\Gamma^{a}_{ea}$$

obeys $\quad 2D_{[a}D_{b]}w_c = R^{d}{}_{cba}\, w_d$

$$R^{d}{}_{cba}\, n_d = 0$$

- $R_{abc}{}^{d}$ is a 3D, purely spatial object
  - lacks some information in ${}^{(4)}R^{d}_{abc}$
  - has information only about curvature intrinsic to Σ
  - to get at shape of Σ in M, need another tensor: extrinsic curvature



$\boxed{\text{Extrinsic Curvature}}$

- K_{ab} : from projection of gradients of normal vector onto Σ; also, first time derivative

of γ_{ab}

- together (γ_{ab}, K_{ab}) are analogous to position & velocity in classical mechanics but for gravitational field
- projection of gradient of normal vector:
  $$\gamma_a{}^c \gamma_b{}^d \nabla_c n_d$$
  - split into symmetric part, expansion tensor,
  $$\theta_{ab} = \gamma_a{}^c \gamma_b{}^d \nabla_{(c} n_{d)}$$
  and antisymmetric twist,
  $$\omega_{ab} = \gamma_a{}^c \gamma_b{}^d \nabla_{[c} n_{d]}$$
- the twist must vanish since the normal vector is rotation free:

∴ extrinsic curvature is
$$K_{ab} \equiv -\gamma_a{}^c \gamma_b{}^d \nabla_{(c} n_{d)} = -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d$$
.... since $\nabla_c n_d$ is symmetric

→ $K_{ab} = K_{ba}$ & is purely spatial

- n^a are normalized & vary only in direction so K_{ab} says how much the direction changes between different points in Σ

$\to$ how much Σ deforms as it is carried forward along n^a

- Can also express K_{ab} in terms of acceleration of the normal vector:

$$a_a \equiv n^b \nabla_b n_a$$

$\hookrightarrow$

$$K_{ab} = -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d$$

$$= -(\delta_a{}^c + n_a n^c)(\delta_b{}^d + n_b n^d)\nabla_c n_d$$

$$= -(\delta_a{}^c + n_a n^c)\delta_b{}^d \nabla_c n_d$$

$$= -\nabla_a n_b - n_a a_b$$

- can also write in terms of the Lie derivative $\mathcal{L}_{\vec{n}}$ along n^a

$$K_{ab} = -\frac{1}{2}\mathcal{L}_{\vec{n}} \gamma_{ab}$$

- for scalar f,

$$\mathcal{L}_{\vec{X}} f = X^b D_b f = X^b \partial_b f \quad \text{.... partial}$$

- for vector v^a:

$$\mathcal{L}_{\vec{X}} v^a = X^b \partial_b v^a - v^b \partial_b X^a = [X, v]^a$$

... commutator

- 1-form:

$$\mathcal{L}_{\vec{X}} \omega_a = X^b \partial_b \omega_a + \omega_b \partial_a X^b$$

- tensor:

$$\mathcal{L}_{\vec{X}} T^a{}_b = X^c \partial_c T^a{}_b - T^c{}_b \partial_c X^a + T^a{}_c \partial_b X^c$$

→ measures change of tensor along a vector relative to a simple coordinate transform

- if two hypersurfaces differed by only a coordinate transform (steady-state),

$$\mathcal{L}_{\vec{n}} \gamma_{ab} = 0$$

- thus, this definition of K_{ab} makes apparent its connection to a time derivative of γ_{ab} (since n^a is timelike)

↳ time derivative of γ_{ab} is proportional to K_{ab}

- derive it: write γ_{ab} in terms of g_{ab} & n_a:

$$\begin{aligned}\mathcal{L}_{\vec{n}} \gamma_{ab} &= \mathcal{L}_{\vec{n}} (g_{ab} + n_a n_b) \\ &= 2\nabla_{(a} n_{b)} + n_a \mathcal{L}_{\vec{n}} n_b + n_b \mathcal{L}_{\vec{n}} n_a \\ &= 2(\nabla_{(a} n_{b)} + n_{(a} a_{b)}) = -2K_{ab}\end{aligned}$$

where $\mathcal{L}_{\vec{X}} g_{ab} = \nabla_a X_b + \nabla_b X_a$

.... symmetries of metric

- trace of extrinsic curvature is mean curvature,

$$K = g^{ab} K_{ab} = \gamma^{ab} K_{ab} \quad \ldots \left(\begin{array}{c}\text{purely}\\ \text{spatial}\end{array}\right)$$

$$= -\frac{1}{2} \gamma^{ab} \mathcal{L}_{\vec{n}} \gamma_{ab}$$

$$= -\frac{1}{2\gamma} \mathcal{L}_{\vec{n}} \gamma = -\frac{1}{\gamma^{1/2}} \mathcal{L}_{\vec{n}} \gamma^{1/2} \quad \left(\begin{array}{c}\text{chain}\\ \text{rule}\end{array}\right)$$

$$= -\mathcal{L}_{\vec{n}} \ln \gamma^{1/2}$$

→ $\gamma^{1/2} d^3x$ is the proper volume element in Σ, so $-K$ measures change in 3-volume element along n^a