Static Spherically-Symmetric Stellar Structure in General Relativity

Christian D. Ott
TAPIR, California Institute of Technology
cott@tapir.caltech.edu

July 24, 2013

1 Introduction

Neutron stars and, to some extent, also white dwarfs are relativistic objects and computations of their structure should be carried out in a general-relativistic (GR) framework. Given the complexity and non-linearity of Einstein's equations, this, at first, seems like a daunting task. However, when making the assumption of spherical symmetry, zero velocities (i.e., a static spacetime), and an ideal fluid model, computing stellar structure in GR is only slightly more involved than doing so in Newtonian gravity.

This write-up / exercise guide introduces the Tolman-Oppenheimer-Volkoff (TOV) equations and delineates numerical methods for their solution.

At `http://www.tapir.caltech.edu/ cott/CGWAS2013/tov.tar.gz` a code skeleton (in Fortran 90) is available that can be used to easily implement a TOV solver with a polytropic equation of state. The solver can be extended to include more complicated physics.

2 The Tolman-Oppenheimer-Volkoff Equations

We begin with the Einstein equations,

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \ , \tag{1}$$

where $G^{\mu\nu}$ is the Einstein tensor, describing the curvature of spacetime and $T^{\mu\nu}$ is the stress energy tensor, describing matter/energy sources of spacetime curvature. Here, we won't worry further about the details of $G^{\mu\nu}$ (see, e.g., [1–3]), but give the stress-energy tensor for an ideal fluid:

$$T^{\mu\nu} = (\rho(1+\epsilon/c^2) + p/c^2)u^\mu u^\nu + pg^{\mu\nu} \ . \tag{2}$$

Here, ρ is the baryon rest mass density, ϵ is the specific energy density, p is the fluid pressure, u^α is the 4-velocity, and $g^{\mu\nu}$ is the 4-metric which we use to measure distances in spacetime ($ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, note the Einstein sum convention!). See [1–3] for more details, but note that in the case of vanishing space velocity (3-velocity), $u^\alpha = (c, 0, 0, 0)$, and $T^{\mu\nu}$ simplifies quite a bit.

Now, let's consider the interior Schwarzschild metric which gives the following line element ds^2:

$$ds^2 = -e^{2\phi(r)}c^2dt^2 + \left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \ . \tag{3}$$

$m(r)$ is the gravitational mass inside radius r, where r is defined such that the circumference of a circle about the origin at that space location is $2\pi r$. $e^\phi(r)$ is the lapse function and $\phi(r)$ will be called the metric potential in the following. In vacuum, $T^{\mu\nu} = 0$, $m = M_{\text{total}}$, and $\phi = \frac{1}{2}\ln(1 - 2GM/(rc^2))$.

Using eq. (3), assuming zero space velocity, spherical symmetry, and the stress energy tensor of eq. (2), one can solve or, rather, reformulate eq. (1) (which consists, primarily, of derivatives of the metric). Tolman, Oppenheimer, and Volkoff worked this out [4–6] for barotropic stars (stars, in which the pressure is a function of density alone), arriving at a system of ordinary differential equations (ODEs), the so-called TOV equations:

$$\begin{aligned}
\frac{dp}{dr} &= -G(\rho(1+\epsilon/c^2)+p/c^2)\frac{m+4\pi r^3 p/c^2}{r(r-2Gm/c^2)} \ , &(4)\\
\frac{dm}{dr} &= 4\pi r^2\rho(1+\epsilon/c^2) \ , &(5)\\
\frac{d\phi}{dr} &= \frac{m+4\pi r^3 p/c^2}{r(r-2Gm/c^2)} \ . &(6)
\end{aligned}$$

We can solve these equations by numerical integration from $r = 0$, but we need a set of boundary conditions to fix the integration constants and an *equation of state* (EOS) that provides a relationship between density (and possibly temperature and composition) and pressure. Also, note that eqs. (4) and (6) are singular at $r = 0$. They also decouple, which means we can, if we are just interested in stellar structure, neglect eq. (6).

For the boundary conditions, we'll make the sensible choices $p(r = 0) = p_c(\rho_c)$, $m(r = 0) = 0$, and $\phi(r = r_*) = \frac{1}{2}ln(1 - 2GM_\star/(rc^2)$, where $r_* = r(p = 0)$ is the surface of the star and $M_\star$ is the mass of the star enclosed by its surface.

For the EOS, we will start with a polytropic EOS,

$$p = K\rho^\gamma \ , \tag{7}$$

where K is the polytropic constant and $\gamma = d\ln p/d\ln\rho|_s$ is the adiabatic exponent derived at constant entropy (which is implied by the polytropic EOS). Sensible choices for a neutron star may be $\gamma = 2.75$ and $K = 1.982\times10^{-6}$[cgs].

Using the first law of thermodynamics,

$$d\epsilon = -p\,d\left(\frac{1}{\rho}\right) \ , \tag{8}$$

we can solve for the specific internal energy ϵ of a polytropic fluid. This yields

$$\epsilon = \frac{P}{(\gamma-1)\rho} \ , \tag{9}$$

where we have chosen to set the integration constant to zero.

Real neutron stars are, of course, not polytropes. The real neutron star EOS is not known, but many possibilities exist (e.g., [?]) and are available in tabulated form for use in constructing neutron star models and using them in dynamical simulations.

3 Gravitational Mass vs. Baryonic Mass

A note on the meaning of the *mass* variable m: The integral,

$$m = \int_0^{r_\star} 4\pi r^2\rho(1+\epsilon/c^2)dr \ , \tag{10}$$

is the *gravitational mass* of the star and includes all contributions to the relativistic mass (baryonic mass, internal energy, and the negative gravitational binding energy). To compute the baryonic mass of the star, i.e., the mass obtained by 'counting' baryons and multiplying them with the atomic mass unit, we must take into account curvature:

$$m_{\text{bary}} = \int_0^{r_\star} 4\pi r^2 \rho \sqrt{\det(g)}\, dr = \int_0^{r_\star} \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}} 4\pi r^2 \rho\, dr \ , \qquad (11)$$

where we have used the *proper volume element* $dV = 4\pi r^2 \sqrt{\det(g)} dr$, accounting for the curvature of 3-space by including the square root of the determinant of the spatial part of the Schwarzschild metric (see [1–3] for details).

4 Making your life easier and harder: Units

Working out equations in GR is considerably simplified if one adopts a system of units in which $G = c = 1$, so-called *geometrized units* or *geometric units*. For numerical calculations, life can get even easier by measuring mass in units of solar masses $M_\odot$ and setting $G = c = M_\odot = 1$. The hard and tedious part is to work out the appropriate conversion factors. In Table 1, we provide numbers for the fundamental physical constants needed to compute conversion factors. In Table 2, we list useful conversions.

Table 1: Fundamental Physical Constants.

Constant	Symbol	Value	Rel. Uncertainty
Speed of light	c	$2.99792458 \times 10^{10}$ cm s^{-1}	0
Gravitational constant	G	$6.6742(10) \times 10^{-8}$ cm^3 g^{-1} s^{-2}	1.5×10^{-4}
Elementary charge	e	$1.60317653(14) \times 10^{-19}$ C	8.5×10^{-8}
Boltzmann constant	k_B	$1.3806505(24) \times 10^{-16}$ erg K^{-1}	1.8×10^{-6}
Avogadro constant	N_A	$6.0221415(10) \times 10^{23}$ mol^{-1}	1.7×10^{-7}
Planck constant	h	$6.6260693(11) \times 10^{-27}$ erg s	1.7×10^{-7}
Atomic mass unit	m_u	$1.66053886(28) \times 10^{-24}$ g	1.7×10^{-7}

Comment: Physical constants used in this work in cgs units. The numbers are taken from the NIST table of fundamental physical constants [7]. The number of significant decimal places is determined by the relative uncertainty of each constant as given by NIST.

5 Solving the TOV Equations

We will now numerically integrate the TOV equations (eq. 4 and 5) and solve for the structure of a neutron star. We begin with setting up a computational grid with outer radius `Rmax` and `nzones` computational zones such that each individual zone has a radial extent `dr` = `Rmax` / (`nzones` - 1). Each zone is assigned an inner radius r_n, where n is the zone index.

We begin at the origin $r = 0$ (zone 1) where we set

$$\begin{aligned} \rho_c &= 5.0 \times 10^{14}\,\mathrm{g\,cm^{-3}} \text{ (or any other sensible value above nuclear density)}\,, && (12)\\ p_c &= K\rho_c^\gamma && (13)\\ m_c &= 0\,. && (14) \end{aligned}$$

The simplest method for integrating ODEs like eqs. (4) and (5) is the forward Euler method. It integrates a function f from zone n to zone n+1 according to

$$f_{n+1} = (\Delta x) \cdot f'_n \ , \qquad (15)$$

Table 2: Physical Constants and units in cgs, geometric, and $c = G = M_\odot = 1$ units

Dimension / Quantity	cgs	Geometric	$c = G = M_\odot = 1$
Time	1 s	$3.33564095198 \times 10^{-11}$ cm	2.0296×10^{5}
Length	1 cm	1 cm	6.7706×10^{-6}
Mass	1 g	7.4261×10^{-29} cm	5.0279×10^{-34}
Density	1 g cm^{-3}	7.4261×10^{-29} cm^{-2}	1.6199×10^{-18}
Energy	1 erg = 1 g cm^2 s^{-2}	8.2627×10^{-50} cm	5.5953×10^{-55}
Specific internal energy	1 erg g^{-1}	$1.11265005605 \times 10^{-21}$	$1.11265005605 \times 10^{-21}$
Solar mass	1.9891×10^{33} g	1.4772×10^{5} cm	1.0
Speed of light	$2.99792458 \times 10^{10}$ cm s^{-1}	1.0	1.0
Pressure	1 dyn cm^{-2}	6.6742×10^{-8}	1.8063×10^{-39}

Comment: The number of significant decimal places is determined by the relative uncertainty of the values of the physical constants as given by NIST [7].

where f_n is the function value in zone n and f'_n is its derivative with respect to x. This Euler method is of first order, that is, the error is proportional to the step size (Δx).

In the specific case of eqs. (4) and (5), we have

$$p_{n+1} = p_n + \Delta r \frac{dp}{dr}(r_n) \tag{16}$$

$$m_{n+1} = m_n + \Delta r \frac{dm}{dr}(r_n) \; . \tag{17}$$

For computing $\frac{dp}{dr}(r_n)$, we have to *invert* the EOS to obtain ρ_n from p_n. This is trivial for the polytropic EOS: $\rho_n = (p_n/K)^{1/\gamma}$. Once everything is in place, we loop with eqs. (16) and (17) over the entire grid, then go back and find the surface of the neutron star where the pressure crosses zero and determine M_* and $r_\star$.

The Euler method has the disadvantage that its error converges away only linearly and very high resolution (i.e., small step sizes Δx) is necessary to obtain acceptable results. Hence, one uses higher-order integration methods, such as the common Runge-Kutta algorithms (e.g., [**?**]).

A 2nd-order Runge-Kutta (RK2) is specified by

$$k_1 = f'_n \tag{18}$$

$$k_2 = f'(x_n + \frac{\Delta x}{2}, f_n + \frac{\Delta x}{2} k_1) \tag{19}$$

$$f_{n+1} = f_n + \Delta x k_2 \; . \tag{20}$$

In the same notation, we give expressions for RK3 and RK4 below.

RK3:

$$k_1 = f'_n \tag{21}$$

$$k_2 = f'(x_n + \frac{\Delta x}{2}, f_n + \frac{\Delta x}{2} k_1) \tag{22}$$

$$k_3 = f'(x_n + \Delta x, f_n - \Delta x k_1 + 3\Delta x k_2) \tag{23}$$

$$f_{n+1} = f_n + \frac{\Delta x}{6}(k_1 + 4k_2 + k_3) \tag{24}$$

RK4:

$$
\begin{aligned}
k_1 &= f'_n && (25)\\
k_2 &= f'(x_n + \frac{\Delta x}{2}, f_n + \frac{\Delta x}{2} k_1) && (26)\\
k_3 &= f'(x_n + \frac{\Delta x}{2}, f_n + \frac{\Delta x}{2} k_2) && (27)\\
k_4 &= f'(x_n + \Delta x, f_n + \Delta x k_3) && (28)\\
f_{n+1} &= f_n + \frac{\Delta x}{6}(k_1 + 2k_2 + 2k_3 + k_4) && (29)
\end{aligned}
$$

6 Computer Exercise

6.1 Build a Neutron Star.

Download the TOV code skeleton from `http://www.tapir.caltech.edu/ cott/CGWAS2013/tov.tar.gz` . The tar ball contains a `Makefile` that you will use to compile the code and the file `tov.F90` containing the code. Bring up `tov.F90` in an editor (e.g., `emacs`). For compiling the code, type `make`. Once compiled, you can execute it by typing `./tov`. However, without your help, nothing will happen.

In the following, you will be guided through the steps necessary to accomplish the goal of building a neutron star with $\rho_c = 5.0 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, $K = 30000$ (in $c = G = M_\odot = 1$) and $\gamma = 2.75$. The question we want to answer is: What is the gravitational mass and the circumferential radius of this neutron star model?

The following steps will be necessary:

1. Choose your system of units and convert the constants into the right units. Hint: You'll have to take special care of K: If f_ρ and f_p are the density and pressure conversion factors to go between units A and B, the following relations hold:

$$
\begin{aligned}
p_B &= f_p\, p_A && (30)\\
\rho_B &= f_\rho\, \rho_A && (31)\\
K_B(\rho_B)^\gamma &= f_p K_A(\rho_A)^\gamma\,. && (32)
\end{aligned}
$$

   Now solve for K_B as a function of K_A and γ.

   Easy choice: Do everything in cgs. In these units, the above value of K is 1.98183×10^{-6} [cgs].

2. Choose an outer radius (should be at least a bit more than the expected stellar radius, so, e.g. 50 km is a very safe number) and a reasonable number of zones (e.g., 2000) to set up your grid in `program tov`.

3. In `subroutine make_a_tov_star`, set the EOS parameters and central density.

4. In `subroutine tov_RHS`, define the right-hand-side (RHS) of eqs. (4) and (5). You'll need to invert the EOS to find ρ. Data of zone n are stored in `old_data` (index 1 for eq. 4 and index 2 for eq. 5, indices 3 to 5 may be used later). The results of the computation must be stored in `source_data`. Handle the case $r \approx 0$ with special care and remember that it corresponds to $m = 0$!

5. In `subroutine tov_RK3`, we have prepared for you a RK3 integrator. You can just call it from `subroutine tov_integrate`. Subroutine `subroutine tov_RHS` is called from by the integrator.

6. In `subroutine tov_integrate`, set up the pressure in the first zone based on the central density. Also, you'll need to add code to find the surface of the star (use `isurf` to store the zone number of the surface!). Use a write statement to print out neutron star mass (in solar masses) and circumferential radius. What values do you get? How do they depend on the number of zones you use? How many zones do you need to get converged results?

Once you are done, write the neutron star structure into a file (see commented example code in `tov.F90`) and plot $\rho(r)$, $m(r)$, and $rho(m)$ in `gnuplot`. Hint: `plot ''star.dat'' u 2:3 w l` will make `gnuplot` plot column 3 as a function of column 2 in file `star.dat`.

7 Advanced Exercises

7.1 Try other integrators.

Implement the Euler, RK2, and RK4 integrators in separate subroutines and run the code at various resolutions and compare results.

7.2 Find the baryonic mass.

Extend the integrator to also integrate eq. 9. What is the baryonic mass of the neutron star?

7.3 Try out the Maximally Compact EOS.

Introducing $e = \rho + \rho\epsilon/c^2$ as the total energy density, we adopt the simple EOS of Koranda, Stergioulas, and Friedman [8] (see also Jim Lattimer's lecture notes, slide 19!):

$$\begin{aligned} p(e) &= 0\,, & e \leq e_0 \\ p(e) &= e - e_0\,, & e \geq e_0\,, \end{aligned} \tag{33}$$

where the parameter e_0 corresponds to the surface energy density. Below e_0, the EOS is maximally soft ($dp/de = 0$) and above e_0, it is maximally stiff ($dp/de = 1 \rightarrow c_s^2 = c^2$). With this choice of EOS and replacing the baryonic rest-mass density with the total energy density, we can rewrite the TOV equations in dimensionless form. Using the new variables

$$y = m\, e_0^{1/2} \qquad x = r\, e_0^{1/2} \qquad q = p\, e_0^{-1}\,, \tag{34}$$

we rewrite eqs. 4 and 5 to

$$\frac{dq}{dx} = -\frac{(y + 4\pi q x^3)(1+2q)}{x(x-2y)}\,, \tag{35}$$

$$\frac{dy}{dx} = 4\pi x^2 (1+q)\,. \tag{36}$$

Rewrite your code to solve these equations (save your current version!). Try out the maximally compact configuration with $y_{\max} = 0.0851$, $q_{\max} = 2.026$, and $x_{\min}/y_{\max} = 2.825$. What mass do you get?

7.4 Find the maximum neutron star mass.

Go back to your original code version. See how changes in ρ_c affect the neutron star mass. Write code that varies ρ_c to find the maximum mass for a fixed choice of K and γ. Then vary K or γ and see how this affects the maximum mass. Make a plot that shows $m_\star$ as a function of ρ_c.

7.5 Compute the metric potential ϕ

Implement the equation for ϕ. Remember that you have to match ϕ to the exterior Schwarzschild solution. Plot ϕ as a function of r for our $\rho_c = 5 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, $\gamma = 2.75$, $K = 30000$ star.

7.6 Newtonian Stars

Go to the Newtonian limit and drop GR terms in eqs. (4-6). How does the computed stellar structure/mass/radius change?

References

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation.* San Francisco: W.H. Freeman and Co., 1973.

[2] B. F. Schutz. *A First Course in General Relativity.* Cambridge, UK: Cambridge University Press, 1985.

[3] L. S. Shapiro and S. A. Teukolsky. *Black Holes, White Dwarfs and Neutron Stars.* John Wiley & Sons, New York U. S. A., 1983.

[4] R. C. Tolman. *Proc. Nat. Acad. Sc.*, **20**, 169, 1934.

[5] R. C. Tolman. *Phys. Rev.*, **55**, 364, 1939.

[6] J. R. Oppenheimer and G. M. Volkoff. *Phys. Rev.*, **55**, 374, 1939.

[7] P. J. Mohr and B. N. Taylor. *Rev. Mod. Phys.*, **77**, 1, 2005.

[8] S. Koranda, N. Stergioulas, and J. L. Friedman. *Astrophys. J.*, **488**, 799, 1997.