AST 900 Homework 4

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1. TB Exercise 25.4 *Constant of Geodesic Motion in a Spacetime with Symmetry*

(a) *Suppose that in some coordinate system the metric coefficients are independent of some specific coordinate x^A: $g_{\alpha\beta,A} = 0$ (e.g. in spherical polar coordinates $\{t, r, \theta\phi\}$ in flat spacetime $g_{\alpha\beta,\phi} = 0$, so we could set $x^A = \phi$). Show that $p_A \equiv \vec{p} \cdot \partial/\partial x^A$ is a constant of motion for a freely moving particle.*

Solution

Starting with the geodesic equation:

$$\begin{aligned} 0 &= \vec{\nabla}_{\vec{P}}\vec{p} \\ &= p^{\alpha}{}_{;\nu} p^{\nu} \\ &= p^{\alpha}{}_{,\nu} p^{\nu} + \Gamma^{\alpha}{}_{\mu\nu} p^{\mu} p^{\nu} \end{aligned}$$

Using the definition of the 4-momentum as $p^{\nu} \equiv \mathrm{d}x^{\nu}/\mathrm{d}\zeta$, where $\zeta = \tau/m$ is the affine parameter, then gives the covariant components as:

$$\begin{aligned} 0 &= \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\zeta}\frac{\partial p_{\alpha}}{\partial x^{\nu}} - \Gamma^{\mu}{}_{\alpha\nu} p_{\mu} p^{\nu} \\ &= \frac{\mathrm{d}p_{\alpha}}{\mathrm{d}\zeta} - \Gamma^{\mu}{}_{\alpha\nu} p_{\mu} p^{\nu} \\ &= \frac{\mathrm{d}p_{\alpha}}{\mathrm{d}\zeta} - \Gamma^{\mu}{}_{\alpha\nu} (g_{\mu\mu} p^{\mu}) p^{\nu} \\ &= \frac{\mathrm{d}p_{\alpha}}{\mathrm{d}\zeta} - \Gamma_{\mu\alpha\nu} p^{\mu} p^{\nu} \end{aligned} \tag{1}$$

Solving Eq. 1 for p_A:

$$\begin{aligned} 0 &= \frac{\mathrm{d}p_A}{\mathrm{d}\zeta} - \Gamma_{\mu A\nu} p^{\mu} p^{\nu} \\ &= \frac{\mathrm{d}p_A}{\mathrm{d}\zeta} - \frac{1}{2}(g_{\mu A,\nu} + g_{\mu\nu,A} - g_{A\nu,\mu}) p^{\mu} p^{\nu} \\ &= \frac{\mathrm{d}p_A}{\mathrm{d}\zeta} - \frac{1}{2} g_{\mu A,\nu} p^{\mu} p^{\nu} - \frac{1}{2} g_{\mu\nu,A} p^{\mu} p^{\nu} + \frac{1}{2} g_{A\nu,\mu} p^{\mu} p^{\nu} \end{aligned}$$

Re-arranging and renaming contracted indices:

$$\frac{\mathrm{d}p_A}{\mathrm{d}\zeta} = \cancel{\frac{1}{2}g_{\mu A,\nu}p^\mu p^\nu} + \frac{1}{2}g_{\mu\nu,A}p^\mu p^\nu - \cancel{\frac{1}{2}g_{A\mu,\nu}p^\nu p^\mu}$$

where the two terms cancel due to the symmetry of the metric $g_{\mu A,\nu} = g_{A\mu,\nu}$. Then, using $g_{\alpha\beta,A} = 0$, this reduces to:

$$\boxed{\frac{\mathrm{d}p_A}{\mathrm{d}\zeta} = 0} \tag{2}$$

giving p_A as a constant of motion for a freely moving particle.

(b) *Consider a particle moving freely through a time-independent, Newtonian gravitational field that can be described in the language of general relativity by the spacetime metric:*

$$ds^2 = -(1+2\Phi)dt^2 + (\delta_{jk} + h_{jk})dx^j dx^k \tag{3}$$

where $\phi(x,y,z)$ is a time-independent Newtonian potential, and h_{jk} are contributions to the metric that are independent of the time coordinate t and have magnitude of order $|\Phi|$. For a weak gravitational field, $|\Phi| \ll 1$, suppose that the particle has a velocity $v^j \equiv \mathrm{d}x^j/\mathrm{d}t \leq |\Phi|^{\frac{1}{2}}$. Since the metric is independent of t, the component p_t must be conserved along the particle's worldline. Show that $p_t \equiv E = m\Phi + \frac{1}{2}mv^j v^k \delta_{jk}$ (aside from some multiplicative and additive constants) when evaluated accurate to first order in $|\Phi|$.

Solution

A particle's 4-momentum can be defined as:

$$p^\alpha = mu^\alpha = m\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$$

where τ is the particle's proper time defined by:

$$\begin{aligned} d\tau &= \sqrt{-ds^2} \\ &= \left[(1+2\Phi)dt^2 - (\delta_{jk}+h_{jk})dx^j dx^k\right]^{1/2} \\ &= dt\left[1+2\Phi - (\delta_{jk}+h_{jk})v^j v^k\right]^{1/2} \\ &\approx dt\left(1+2\Phi - \delta_{jk}v^j v^k\right)^{1/2} \end{aligned}$$

where only terms up to linear in $|\Phi|$ are kept in the last line. Using this $d\tau$ in the 4-momentum gives:

$$p^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{m}{(1+2\Phi - \delta_{jk}v^j v^k)^{1/2}}$$

which since $|\Phi| \sim v^2 \ll 1$, this can be expanded as (keeping only terms up to linear in $|\Phi|$):

$$p^\alpha = m\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\left(1 - \Phi + \frac{1}{2}\delta_{jk}v^j v^k\right)$$

Applying the metric $g_{0\alpha}$ to both sides to obtain the p_t component ($\alpha = 0$ on the right-hand side since only g_{00} is nonzero):

$$\begin{aligned}
g_{0\alpha}p^\alpha &= mg_{0\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\left(1 - \Phi + \frac{1}{2}\delta_{jk}v^j v^k\right)\\
\rightarrow \quad p_t &= mg_{00}\frac{\mathrm{d}x^0}{\mathrm{d}t}\left(1 - \Phi + \frac{1}{2}\delta_{jk}v^j v^k\right)\\
&= -m(1+2\Phi)\frac{\mathrm{d}t}{\mathrm{d}t}\left(1 - \Phi + \frac{1}{2}\delta_{jk}v^j v^k\right)\\
&= -m\left(1 - \Phi + \frac{1}{2}\delta_{jk}v^j v^k + 2\Phi - 2\Phi^2 + \delta_{jk}v^j v^k\Phi\right)
\end{aligned}$$

Simplifying and keeping only terms linear in $|\Phi|$, this then gives:

$$\boxed{E = p_t = -\left(m + m\Phi + \frac{1}{2}mv^j v^k \delta_{jk}\right)} \tag{4}$$

This agrees with the expected value, with a multiplicative constant of -1 and additive constant of m.

2. TB Exercise 25.11 *Components of Riemann Tensor in an Arbitrary Basis*

By evaluating:

$$p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma} = -R^\alpha{}_{\beta\gamma\delta}p^\beta \tag{5}$$

in an arbitrary basis (which might not even be a coordinate basis), derive:

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\mu}c_{\gamma\delta}{}^\mu \tag{6}$$

for the components of the Riemann tensor.

Solution

First, apply the first gradient to each term:

$$R^\alpha{}_{\beta\gamma\delta}p^\beta = \left(p^\alpha{}_{,\delta} + \Gamma^\alpha{}_{\mu\delta}p^\mu\right)_{;\gamma} - \left(p^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\mu\gamma}p^\mu\right)_{;\delta}$$

Each term in parentheses on the right-hand side is a rank-two tensor in α and δ or γ, so the next gradient can be applied to the whole term generating a partial derivative and two connection coefficients (for each):

$$\begin{aligned}
R^\alpha{}_{\beta\gamma\delta}p^\beta &= \left(p^\alpha{}_{,\delta} + \Gamma^\alpha{}_{\mu\delta}p^\mu\right)_{,\gamma} + \Gamma^\alpha{}_{\nu\gamma}\left(p^\nu{}_{,\delta} + \Gamma^\nu{}_{\mu\delta}p^\mu\right) - \Gamma^\nu{}_{\delta\gamma}\left(p^\alpha{}_{,\nu} + \Gamma^\alpha{}_{\mu\nu}p^\mu\right) \\
&\quad - \left(p^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\mu\gamma}p^\mu\right)_{,\delta} - \Gamma^\alpha{}_{\nu\delta}\left(p^\nu{}_{,\gamma} + \Gamma^\nu{}_{\mu\gamma}p^\mu\right) + \Gamma^\nu{}_{\gamma\delta}\left(p^\alpha{}_{,\nu} + \Gamma^\alpha{}_{\mu\nu}p^\mu\right) \\
&= p^\alpha{}_{,\delta\gamma} + \Gamma^\alpha{}_{\mu\delta,\gamma}p^\mu + \cancel{\Gamma^\alpha{}_{\mu\delta}p^\mu{}_{,\gamma}} + \cancel{\Gamma^\alpha{}_{\nu\gamma}p^\nu{}_{,\delta}} + \Gamma^\alpha{}_{\nu\gamma}\Gamma^\nu{}_{\mu\delta}p^\mu - \Gamma^\nu{}_{\delta\gamma}p^\alpha{}_{,\nu} - \Gamma^\nu{}_{\delta\gamma}\Gamma^\alpha{}_{\mu\nu}p^\mu \\
&\quad - p^\alpha{}_{,\gamma\delta} - \Gamma^\alpha{}_{\mu\gamma,\delta}p^\mu - \cancel{\Gamma^\alpha{}_{\mu\gamma}p^\mu{}_{,\delta}} - \cancel{\Gamma^\alpha{}_{\nu\delta}p^\nu{}_{,\gamma}} - \Gamma^\alpha{}_{\nu\delta}\Gamma^\nu{}_{\mu\gamma}p^\mu + \Gamma^\nu{}_{\gamma\delta}p^\alpha{}_{,\nu} + \Gamma^\nu{}_{\gamma\delta}\Gamma^\alpha{}_{\mu\nu}p^\mu \\
&= \left(\Gamma^\alpha{}_{\mu\delta,\gamma} - \Gamma^\alpha{}_{\mu\gamma,\delta} + \Gamma^\alpha{}_{\nu\gamma}\Gamma^\nu{}_{\mu\delta} - \Gamma^\alpha{}_{\nu\delta}\Gamma^\nu{}_{\mu\gamma} + \Gamma^\nu{}_{\gamma\delta}\Gamma^\alpha{}_{\mu\nu} - \Gamma^\nu{}_{\delta\gamma}\Gamma^\alpha{}_{\mu\nu}\right)p^\mu \\
&\quad + \left(p^\alpha{}_{,\delta\gamma} - p^\alpha{}_{,\gamma\delta}\right) - \left(\Gamma^\nu{}_{\delta\gamma}p^\alpha{}_{,\nu} - \Gamma^\nu{}_{\gamma\delta}p^\alpha{}_{,\nu}\right)
\end{aligned}$$

Using the relation for the commutator coefficients in terms of the connection coefficients from last homework, $\Gamma^\gamma{}_{\beta\alpha} - \Gamma^\gamma{}_{\alpha\beta} = c_{\alpha\beta}{}^\gamma$, this reduces to:

$$\begin{aligned}
R^\alpha{}_{\beta\gamma\delta}p^\beta &= \left(\Gamma^\alpha{}_{\mu\delta,\gamma} - \Gamma^\alpha{}_{\mu\gamma,\delta} + \Gamma^\alpha{}_{\nu\gamma}\Gamma^\nu{}_{\mu\delta} - \Gamma^\alpha{}_{\nu\delta}\Gamma^\nu{}_{\mu\gamma} - c_{\gamma\delta}{}^\nu\Gamma^\alpha{}_{\mu\nu}\right)p^\mu \\
&\quad + \left(p^\alpha{}_{,\delta\gamma} - p^\alpha{}_{,\gamma\delta}\right) - c_{\gamma\delta}{}^\nu p^\alpha{}_{,\nu}
\end{aligned} \tag{7}$$

For the term $\left(p^\alpha{}_{,\delta\gamma} - p^\alpha{}_{,\gamma\delta}\right)$, since this is an arbitrary basis, the comma operators denote the result of letting a basis vector act as a differential operator, so this becomes the commutator:

$$\begin{aligned}
p^\alpha{}_{,\delta\gamma} - p^\alpha{}_{,\gamma\delta} &= \left(\vec{\nabla}_\gamma\vec{\nabla}_\delta - \vec{\nabla}_\delta\vec{\nabla}_\gamma\right)p^\alpha \\
&= \left[\vec{\nabla}_\gamma, \vec{\nabla}_\delta\right]p^\alpha \\
&= c_{\gamma\delta}{}^\nu\vec{\nabla}_\nu p^\alpha \\
&= c_{\gamma\delta}{}^\nu p^\alpha{}_{,\nu}
\end{aligned}$$

Plugging this in Eq. 7, all terms on the last line cancel out. After renaming contracted indices $\mu \to \beta$ and $\nu \to \mu$, this then gives:

$$R^\alpha{}_{\beta\gamma\delta}\cancel{p^\beta} = \left(\Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\mu}c_{\gamma\delta}{}^\mu\right)\cancel{p^\beta}$$

$$\rightarrow \quad \boxed{R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\mu}c_{\gamma\delta}{}^\mu} \tag{8}$$

3. TB Exercise 25.14 *Geodesic Deviation on a Sphere*

Consider two neighboring geodesics (great circles) on a sphere of radius a, one the equator and the other a geodesic slightly displaced from the equator (by $\Delta\theta = b$) and parallel to it *at $\phi = 0$. Let $\vec{\xi}$ be the separation vector between the two geodesics, and note that at $\phi = 0$, $\vec{\xi} = b\,\partial/\partial\theta$. Let l be the proper distance along the equatorial geodesic, so $\mathrm{d}/\mathrm{d}l = \vec{u}$ is its tangent vector.*

(a) *Show that $l = a\phi$ along the equatorial geodesic.*

Solution

The line element on the surface of a sphere of radius a is:

$$\mathrm{d}s^2 = a^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{9}$$

and the proper distance is related to the line element as:

$$\mathrm{d}l = \sqrt{\mathrm{d}s^2} = a\sqrt{\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2}$$

On the equator, $\theta = \pi/2$ and $\mathrm{d}\theta = 0$, so the proper distance element becomes:

$$\mathrm{d}l = a\,\mathrm{d}\phi \tag{10}$$

which gives the proper distance as:

$$\boxed{l = a\phi} \tag{11}$$

(b) *Show that the equation of geodesic deviation:*

$$\left(\xi^\alpha{}_{;\beta}p^\beta\right)_{;\gamma}p^\gamma = -R^\alpha{}_{\beta\gamma\delta}p^\beta\xi^\gamma p^\delta \tag{12}$$

reduces to:

$$\frac{\mathrm{d}^2\xi^\theta}{\mathrm{d}\phi^2} = -\xi^\theta \quad , \quad \frac{\mathrm{d}^2\xi^\phi}{\mathrm{d}\phi^2} = 0 \tag{13}$$

Solution

Evaluating the left-hand side of Eq. 12 on the equator gives (and replacing $\vec{p}$ with $\vec{u}$ since both are tangent and any test particle mass occurs in equal amounts on each side so cancels leaving only the 4-velocities):

$$\begin{aligned}
\left(\xi^\alpha{}_{;\beta}u^\beta\right)_{;\gamma}u^\gamma &= \left(\xi^\alpha{}_{,\beta}u^\beta + \Gamma^\alpha{}_{\mu\beta}\xi^\mu u^\beta\right)_{;\gamma}u^\gamma \\
&= \left(\xi^\alpha{}_{,\beta}u^\beta + \Gamma^\alpha{}_{\mu\beta}\xi^\mu u^\beta\right)_{,\gamma}u^\gamma + \Gamma^\alpha{}_{\nu\gamma}\left(\xi^\nu{}_{,\beta}u^\beta + \Gamma^\nu{}_{\mu\beta}\xi^\mu u^\beta\right)u^\gamma \\
\\
&= \xi^\alpha{}_{,\beta\gamma}u^\beta u^\gamma + \xi^\alpha{}_{,\beta}u^\beta{}_{,\gamma}u^\gamma + \Gamma^\alpha{}_{\mu\beta,\gamma}\xi^\mu u^\beta u^\gamma + \Gamma^\alpha{}_{\mu\beta}\xi^\mu{}_{,\gamma}u^\beta u^\gamma + \Gamma^\alpha{}_{\mu\beta}\xi^\mu u^\beta{}_{,\gamma}u^\gamma \\
&\quad + \Gamma^\alpha{}_{\nu\gamma}\xi^\nu{}_{,\beta}u^\beta u^\gamma + \Gamma^\alpha{}_{\nu\gamma}\Gamma^\nu{}_{\mu\beta}\xi^\mu u^\beta u^\gamma
\end{aligned} \tag{14}$$

Since $\vec{u} = \mathrm{d}/\mathrm{d}l = (1/a)\,\mathrm{d}/\mathrm{d}\phi$, the components of $\vec{u}$ are then $u^\theta = 0$ and $u^\phi = 1/a$. Further, since this is a coordinate basis, commas represent partial derivatives, all partial derivates of $\vec{u}$ vanish as its components are either zero or constant: ${p^\beta}_{,\gamma} = 0$. From this, only $\beta = \gamma = \phi$ components in Eq. 14 can be non-zero. The only non-zero connection coefficients (derived in homework three and given in TB Eq. 25.52a) on the surface of the sphere are:

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \quad , \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta \tag{15}$$

From this, any ϕ derivatives of the connection coefficients also vanish. On the equator, $\theta = \pi/2$, both connection coefficients are also zero. Chosing $\alpha = \theta$, $\beta = \gamma = \phi$, and using Eq. 15 in Eq. 14 yields (on the equator):

$$\left(\xi^\theta{}_{;\phi} u^\phi\right)_{;\phi} u^\phi = \xi^\theta{}_{,\phi\phi} u^\phi u^\phi = \frac{1}{a^2}\frac{\partial^2 \xi^\theta}{\partial \phi^2} \tag{16}$$

and for $\alpha = \phi$, $\beta = \gamma = \phi$, and using Eq. 15 in Eq. 14 yields (on the equator):

$$\left(\xi^\phi{}_{;\phi} u^\phi\right)_{;\phi} u^\phi = \xi^\phi{}_{,\phi\phi} u^\phi u^\phi = \frac{1}{a^2}\frac{\partial^2 \xi^\phi}{\partial \phi^2} \tag{17}$$

For the right-hand side of Eq. 12, the only non-zero components of the Riemann tensor are (TB Eq. 25.52b,c):

$$R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi} = a^2 \sin^2\theta \tag{18}$$

which at $\theta = \pi/2$, this gives the only non-zero component as $R_{\theta\phi\theta\phi} = a^2$. For the $\alpha = \theta$, $\beta = \delta = \phi$ components, this gives:

$$\begin{aligned}
-R^\theta{}_{\phi\gamma\delta} u^\phi \xi^\gamma u^\phi &= -R^\theta{}_{\phi\theta\phi} u^\phi \xi^\theta u^\phi - \cancel{R^\theta{}_{\phi\phi\phi} u^\phi \xi^\phi u^\phi} \\
&= -g^{\theta\theta} R_{\theta\phi\theta\phi} u^\phi \xi^\theta u^\phi \\
&= -\frac{1}{\cancel{a^2}} \cancel{a^2} \frac{1}{a^2} \xi^\theta \\
&= -\frac{1}{a^2} \xi^\theta
\end{aligned} \tag{19}$$

and for the $\alpha = \phi$, $\beta = \delta = \phi$ components:

$$-R^\phi{}_{\phi\gamma\delta} u^\phi \xi^\gamma u^\phi = -\cancel{R^\phi{}_{\phi\theta\phi} u^\phi \xi^\theta u^\phi} - \cancel{R^\phi{}_{\phi\phi\phi} u^\phi \xi^\phi u^\phi} = 0 \tag{20}$$

Then equating Eq. 16 with Eq. 19, and Eq. 17 with Eq. 20, and cancelling common factors:

$$\boxed{\frac{\mathrm{d}^2 \xi^\theta}{\mathrm{d}\phi^2} = -\xi^\theta \quad , \quad \frac{\mathrm{d}^2 \xi^\phi}{\mathrm{d}\phi^2} = 0} \tag{21}$$

(c) *Solve Eq. 13 subject to the above initial conditions to obtain:*

$$\xi^\theta(\phi) = b\cos\phi \quad , \quad \xi^\phi(\phi) = 0 \tag{22}$$

Solution

The initial conditions given in the problem are a separation between the geodesics at $\phi = 0$ is $\Delta\theta = b$, and that at $\phi = 0$, the geodesics are parallel, i.e. their tangent vectors $\vec{u}$ are parallel, and thus the separation vector $\vec{\xi}$ does not change in the ϕ direction:

$$\xi^\theta(\phi = 0) = b \quad , \quad \left.\frac{\mathrm{d}\xi^\theta}{\mathrm{d}\phi}\right|_{\phi=0} = 0 \tag{23a}$$

$$\xi^\phi(\phi = 0) = 0 \quad , \quad \left.\frac{\mathrm{d}\xi^\phi}{\mathrm{d}\phi}\right|_{\phi=0} = 0 \tag{23b}$$

Solving for ξ^θ in Eq. 21, this is a simple 2nd-order ordinary differential equation with a general solution of:

$$\xi^\theta(\phi) = A\cos\phi + B\sin\phi$$

Applying Eq. 23, this gives the two coefficients A and B as:

$$\begin{aligned} \xi^\theta(0) = b &= A\cos(0) + B\sin(0) && \rightarrow && A = b \\ \left.\frac{\mathrm{d}\xi^\theta}{\mathrm{d}\phi}\right|_{\phi=0} = 0 &= -A\sin(0) + B\cos(0) && \rightarrow && B = 0 \end{aligned}$$

giving the result:

$$\boxed{\xi^\theta(\phi) = b\cos\phi} \tag{24}$$

Likewise, solving for ξ^ϕ in Eq. 21, the general solution is linear in ϕ and obtained by integrating twice:

$$\xi^\phi(\phi) = C\phi + D$$

and since both inital conditions from Eq. 23 are zero, this implies that $C = D = 0$, giving the final result for:

$$\boxed{\xi^\phi(\phi) = 0} \tag{25}$$

4. TB Exercise 25.18 *Newtonain Limit of General Relativity*

Consider a system that can be covered by nearly globally Lorentz coordinates in which the Newtonian-limit contstraints of TB Eq. 25.75 are satisfied:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad , \quad |h_{\alpha\beta} \ll 1| \tag{26a}$$

$$|h_{\alpha\beta,t}| \ll |h_{\alpha\beta,j}| \tag{26b}$$
$$\left|T^{j0}\right| \ll T^{00} \equiv \rho \tag{26c}$$
$$\left|u^{j}\right| \ll u^{0} \tag{26d}$$

(a) *Derive the components of $\vec{u}$.*

Solution

The 4-velocity is defined as:

$$\vec{u} \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \tag{27}$$

The proper time $\mathrm{d}\tau$ can be obtained from the line element defined in terms of the metric in Eq. 26 (in the low-velocity limit):

$$\begin{aligned}
\mathrm{d}\tau^2 &= -\,\mathrm{d}s^2 \\
&= -g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \\
&= -(\eta_{\alpha\beta} + h_{\alpha\beta})\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta) \\
&\approx -\eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \\
&= \mathrm{d}t^2 - \eta_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k \\
&= \mathrm{d}t^2\left(1 - \eta_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}\right) \\
&\approx \mathrm{d}t^2
\end{aligned}$$

Using this in Eq. 27 then gives:

$$\boxed{u^0 \approx \frac{\mathrm{d}t}{\mathrm{d}t} = 1 \quad , \quad u^j \approx \frac{\mathrm{d}x^j}{\mathrm{d}t} = v^j} \tag{28}$$

(b) *Show that the geodesic equation reduces to:*

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v^j \approx \frac{1}{2}\frac{\partial h_{00}}{\partial x^j} \tag{29}$$

Solution

In the Newtonian limit, using Eq. 28 and $\frac{\partial u^0}{\partial x^\beta} = 0$, the geodesic equation becomes:

$$\begin{aligned}
0 &= u^\alpha{}_{,\beta}u^\beta + \Gamma^\alpha{}_{\mu\beta}u^\mu u^\beta \\
&= u^j{}_{,\beta}u^\beta + \Gamma^j{}_{\mu\beta}u^\mu u^\beta \\
&= v_{j,0} + v_{j,k}v_k + \Gamma_{j00} + \Gamma_{jk0}v_k + \Gamma_{j0k}v_k + \Gamma_{jik}v_iv_k
\end{aligned}$$

where the spatial indices can be lowered in the low velocity limit without changing the equation. Since $\Gamma_{j0k} = -\Gamma_{jk0}$ and $v^2 \ll 1$ in the Newtonian limit, as well as $|h_{\alpha\beta,t}| \ll |h_{\alpha\beta,j}|$, this further reduces to:

$$\begin{aligned}
v_{j,0} + v_k v_{j,k} &= -\Gamma_{j00} \\
&= -\frac{1}{2}(g_{j0,0} + g_{j0,0} - g_{00,j}) \\
&= -\frac{1}{2}(2h_{j0,0} - h_{00,j}) \\
&\approx \frac{1}{2} h_{00,j}
\end{aligned}$$

Re-writing this in terms of partial derivatives and gradients then gives:

$$\boxed{\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) v^j \approx \frac{1}{2}\frac{\partial h_{00}}{\partial x^j}} \tag{30}$$

(c) *Show that to linear order in the metric perturbation $h_{\alpha\beta}$, the components of the Riemann tensor take the form:*

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}) \tag{31}$$

Solution

Starting with the definition of the Riemann tensor in terms of the derivatives of the connection coefficients (TB Eq. 25.40; this is valid since the weak field limit considers a nearly global Lorentz frame), and using the metric in Eq. 26:

$$\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta} &= \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} \\
&= \left(g^{\alpha\mu}\Gamma_{\mu\beta\delta}\right)_{,\gamma} - \left(g^{\alpha\mu}\Gamma_{\mu\beta\gamma}\right)_{,\delta} \\
&= \frac{1}{2}[g^{\alpha\mu}(g_{\mu\beta,\delta} + g_{\mu\delta,\beta} - g_{\beta\delta,\mu})]_{,\gamma} - \frac{1}{2}[g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu})]_{,\delta} \\
&= \frac{1}{2}\left[(\eta^{\alpha\mu} + h^{\alpha\mu})\left(\cancelto{0}{\eta_{\mu\beta,\delta}} + h_{\mu\beta,\delta} + \cancelto{0}{\eta_{\mu\delta,\beta}} + h_{\mu\delta,\beta} - \cancelto{0}{\eta_{\beta\delta,\mu}} - h_{\beta\delta,\mu}\right)\right]_{,\gamma} \\
&\quad - \frac{1}{2}\left[(\eta^{\alpha\mu} + h^{\alpha\mu})\left(\cancelto{0}{\eta_{\mu\beta,\gamma}} + h_{\mu\beta,\gamma} + \cancelto{0}{\eta_{\mu\gamma,\beta}} + h_{\mu\gamma,\beta} - \cancelto{0}{\eta_{\beta\gamma,\mu}} - h_{\beta\gamma,\mu}\right)\right]_{,\delta}
\end{aligned}$$

Keeping only terms linear in $h_{\alpha\beta}$ removes the leading factor of $h^{\alpha\mu}$ in each term, and since $\eta^{\alpha\mu}$ is constant, it can be pulled outside the derivatives. This results in:

$$\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta} &= \frac{1}{2}\eta^{\alpha\mu}\left(\cancel{h_{\mu\beta,\delta\gamma}} + h_{\mu\delta,\beta\gamma} - h_{\beta\delta,\mu\gamma} - \cancel{h_{\mu\beta,\gamma\delta}} - h_{\mu\gamma,\beta\delta} + h_{\beta\gamma,\mu\delta}\right) \\
&= \frac{1}{2}\eta^{\alpha\mu}(h_{\mu\delta,\beta\gamma} - h_{\beta\delta,\mu\gamma} - h_{\mu\gamma,\beta\delta} + h_{\beta\gamma,\mu\delta})
\end{aligned}$$

where the two cancelled terms result from the fact that partial derivatives commute. Multiplying each side by $\eta_{\mu\alpha}$:

$$\eta_{\mu\alpha}R^{\alpha}{}_{\beta\gamma\delta} = \frac{1}{2}\eta_{\mu\alpha}\eta^{\alpha\mu}(h_{\mu\delta,\beta\gamma} - h_{\beta\delta,\mu\gamma} - h_{\mu\gamma,\beta\delta} + h_{\beta\gamma,\mu\delta})$$
$$\rightarrow \quad R_{\mu\beta\gamma\delta} = \frac{1}{2}(h_{\mu\delta,\beta\gamma} - h_{\beta\delta,\mu\gamma} - h_{\mu\gamma,\beta\delta} + h_{\beta\gamma,\mu\delta})$$

Renaming the free index μ to α and re-arranging the terms:

$$\boxed{R_{\alpha\beta\gamma\delta} = \frac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma})} \tag{32}$$

(d) *Show that in the slow-motion limit the space-time-space-time components of the Riemann tensor take the form:*

$$R_{j0k0} = -\frac{1}{2}h_{00,jk} = \Phi_{,jk} = \mathcal{E}_{jk} \tag{33}$$

Solution

Setting $\alpha = j$, $\beta = 0$, $\gamma = k$, and $\delta = 0$ in Eq. 32 gives:

$$R_{j0k0} = \frac{1}{2}(h_{j0,0k} + h_{0k,j0} - h_{jk,00} - h_{00,jk})$$

Using $|h_{\alpha\beta,t}| \ll |h_{\alpha\beta,j}|$, only the double spatial derivative term significantly contributes to the Riemann tensor:

$$R_{j0k0} = -\frac{1}{2}h_{00,jk}$$

From TB Eq. 25.78, $h_{00} = -2\Phi$ (this is obtained from comparing $\mathbf{g} = -\boldsymbol{\nabla}\Phi$ in Newtonian gravity to Eq. 30), so this becomes:

$$\boxed{R_{j0k0} = -\frac{1}{2}h_{00,jk} = \Phi_{,jk} = \mathcal{E}_{jk}}$$

where the last equality is just from the definition of the Newtonian tidal field $\mathcal{E}_{jk} = \Phi_{,jk}$.