AST 900 Homework 3

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1. TB Exercise 24.2 *Invariance of a Null Interval*

You have measured the intervals between a number of adjacent events in spacetime and thereby have deduced the metric $\boldsymbol{g}$. Your friend claims that the metric is some other frame-independent tensor $\tilde{\boldsymbol{g}}$ that differs from $\boldsymbol{g}$. Suppose that your correct metric $\boldsymbol{g}$ and his wrong one $\tilde{\boldsymbol{g}}$ agree on the forms of the light cones in spacetime (i.e. they agree as to which intervals are null, which are spacelike, and which are timelike), but they give different answers for the value of the interval in the spacelike and timelike cases: $\boldsymbol{g}(\Delta\vec{x}, \Delta\vec{x}) \neq \tilde{\boldsymbol{g}}(\Delta\vec{x}, \Delta\vec{x})$. Prove that $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ differ solely by a scalar multiplicative factor: $\tilde{\boldsymbol{g}} = a\boldsymbol{g}$ for some scalar a.

Solution

For both metrics $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ to agree on whether an interval is timelike, spacelike, or null, both must have the same basic structure, i.e. if $g_{\alpha\beta} \equiv \eta_{\alpha\beta}$, then $\tilde{g}_{\alpha\beta} \propto \eta_{\alpha\beta}$. If it is not, then off diagonal elements would produce cross terms in the interval $\tilde{g}_{\alpha\beta}\Delta x^\alpha \Delta x^\beta$ which could cause the characteristic of the computer interval to change. Thus only the components $\tilde{g}_{\alpha\alpha}$ are non-zero.

Assuming a local Lorentz frame and an interval composed of components of unit lengths, i.e. $\Delta x^\alpha \vec{e}_\alpha \equiv \vec{e}_\alpha$, and the basis vectors are orthonormal, intervals as separations from the origin can easily be constructed as timelike, spacelike, or null by linear combinations of the basis vectors (using the correct metric $\boldsymbol{g}$):

$$\text{(Timelike)} \quad (\Delta\vec{x})^2 = (\vec{e}_0)^2 = \vec{e}_0 \cdot \vec{e}_0 = -1 < 0$$

$$\begin{aligned} \text{(Spacelike)} \quad (\Delta\vec{x})^2 &= (\vec{e}_j)^2 = \vec{e}_i \cdot \vec{e}_i = 1 > 0 \\ (\Delta\vec{x})^2 &= (\vec{e}_i + \vec{e}_j)^2 = \vec{e}_i \cdot \vec{e}_i + \vec{e}_j \cdot \vec{e}_j = 1 + 1 > 0 \end{aligned}$$

$$\text{(Null)} \quad (\Delta\vec{x})^2 = (\vec{e}_0 \pm \vec{e}_i) = \vec{e}_0 \cdot \vec{e}_0 + \vec{e}_i \cdot \vec{e}_i = -1 + 1 = 0$$

In these cases, the components of the incorrect metric can be computed by first assigning a value to the timelike interval such that $\boldsymbol{g}(\vec{e}_0, \vec{e}_0) \neq \tilde{\boldsymbol{g}}(\vec{e}_0, \vec{e}_0)$. Based on the hints in the problem, assigning a value of:

$$\begin{aligned} &\tilde{\boldsymbol{g}}(\vec{e}_0, \vec{e}_0) = a(\vec{e}_0 \cdot \vec{e}_0) \\ \rightarrow \quad &\tilde{g}_{00} = -a \end{aligned}$$

For a null interval in terms of a linear combination of $\vec{e}_0$ and a spatial basis vector, another arbitrary coefficient can be used to scale the spatial dot products:

$$\begin{aligned} 0 = \tilde{\boldsymbol{g}}(\vec{e}_0 \pm \vec{e}_i, \vec{e}_0 \pm \vec{e}_i) &= \tilde{\boldsymbol{g}}(\vec{e}_0, \vec{e}_0) + \cancelto{0}{\tilde{\boldsymbol{g}}(\vec{e}_0, \vec{e}_i)} + \cancelto{0}{\tilde{\boldsymbol{g}}(\vec{e}_i, \vec{e}_0)} + \tilde{\boldsymbol{g}}(\vec{e}_i, \vec{e}_i) \\ &= a(\vec{e}_0 \cdot \vec{e}_0) + b(\vec{e}_i \cdot \vec{e}_i) \\ &= -a + b = 0 \\ \rightarrow \quad b &= a \\ \rightarrow \quad \tilde{g}_{ii} &= a \end{aligned}$$

Since the real metric $g_{\alpha\beta} \equiv \eta_{\alpha\beta}$, and from the above the incorrect metric $\tilde{g}_{\alpha\beta} \equiv a\eta_{\alpha\beta}$, then:

$$\boxed{\tilde{\boldsymbol{g}} = a\boldsymbol{g}}$$

Additionally, a must be a scalar, as if a took the form $a(\Delta\vec{x})$, different vectors could potentially change the characteristic of the interval calculated by metric $\tilde{\boldsymbol{g}}$.

Additionally, testing these results on a spacelike interval:

$$\begin{aligned} \tilde{\boldsymbol{g}}(\vec{e}_i + \vec{e}_j, \vec{e}_i + \vec{e}_j) &= \tilde{\boldsymbol{g}}(\vec{e}_i, \vec{e}_i) + \tilde{\boldsymbol{g}}(\vec{e}_j, \vec{e}_j) \\ &= a(\vec{e}_i \cdot \vec{e}_i) + a(\vec{e}_j \cdot \vec{e}_j) \\ &= 2a = a\boldsymbol{g}(\vec{e}_i + \vec{e}_j, \vec{e}_i + \vec{e}_j) \end{aligned}$$

Which is the expected result.

2. TB Exercise 24.6 *Properties of the Gradient* $\vec{\nabla}$

(a) *Derive* $\vec{\nabla}\boldsymbol{g} = 0$.

Solution

For some arbitrary vector $\vec{A}$, its gradient can be written as:

$$\begin{aligned} \vec{\nabla}\vec{A} &= A^{\alpha}{}_{;\beta} \\ &= g^{\alpha\mu} A_{\mu;\beta} \end{aligned} \tag{1}$$

Likewise, the same gradient can be written as:

$$\begin{aligned} \vec{\nabla}\vec{A} &= (g^{\alpha\mu} A_{\mu})_{;\beta} \\ &= g^{\alpha\mu}{}_{;\beta} A_{\mu} + g^{\alpha\mu} A_{\mu;\beta} \end{aligned} \tag{2}$$

Equating Eq. 1 and Eq. 2 then gives:

$$g^{\alpha\mu}{}_{;\beta} A_{\mu} + \cancel{g^{\alpha\mu} A_{\mu;\beta}} = \cancel{g^{\alpha\mu} A_{\mu;\beta}}$$

$$\rightarrow \quad g^{\alpha\mu}{}_{;\beta} A_\mu = 0$$

Since $\vec{A}$ is just some arbitrary vector, and this equation must always hold, this implies that:

$$\boxed{\vec{\nabla} \boldsymbol{g} = 0} \tag{3}$$

(b) *Derive* $\vec{\nabla}_{\vec{A}} \vec{B} - \vec{\nabla}_{\vec{B}} \vec{A} = \left[\vec{A}, \vec{B}\right]$.

Solution

For two arbitrary vectors $\vec{A}$ and $\vec{B}$, the difference of their directional derivatives along the each other is:

$$\begin{aligned}\vec{\nabla}_{\vec{A}} \vec{B} - \vec{\nabla}_{\vec{B}} \vec{A} &= \left(B^\alpha{}_{;\beta} A^\beta - A^\alpha{}_{;\beta} B^\beta\right) \vec{e}_\alpha \\ &= \left(A^\beta B^\alpha{}_{,\beta} + A^\beta B^\mu \Gamma^\alpha{}_{\mu\beta} - B^\beta A^\alpha{}_{,\beta} - B^\beta A^\mu \Gamma^\alpha{}_{\mu\beta}\right) \vec{e}_\alpha\end{aligned}$$

Assuming a locally Cartesian orthonormal basis, such that $\vec{e}_\alpha \equiv \partial/\partial x^\alpha$, the terms with the connection coefficients disappear, and after relabeling sets of contracted indices:

$$\begin{aligned}\vec{\nabla}_{\vec{A}} \vec{B} - \vec{\nabla}_{\vec{B}} \vec{A} &= \left(A^\alpha B^\beta{}_{,\alpha} - B^\alpha A^\beta{}_{,\alpha}\right) \frac{\partial}{\partial x^\beta} \\ &= \left(A^\alpha \frac{\partial B^\beta}{\partial x^\alpha} - B^\alpha \frac{\partial A^\beta}{\partial x^\alpha}\right) \frac{\partial}{\partial x^\beta}\end{aligned}$$

which according to TB Eq. 24.28 the right-hand side above is just the commutator of $\vec{A}$ and $\vec{B}$, so this gives:

$$\boxed{\vec{\nabla}_{\vec{A}} \vec{B} - \vec{\nabla}_{\vec{B}} \vec{A} = \left[\vec{A}, \vec{B}\right]} \tag{4}$$

3. TB Exercise 24.7 *Prescription for Computing Connection Coefficients*

Derive the prescription for computing connection coefficients in any basis:

$$[\vec{e}_\alpha, \vec{e}_\beta] \equiv c_{\alpha\beta}{}^\rho \vec{e}_\rho, \quad c_{\alpha\beta}{}^\rho \equiv \vec{e}^{\,\rho} \cdot [\vec{e}_\alpha, \vec{e}_\beta] \tag{5a}$$

$$c_{\alpha\beta\gamma} \equiv c_{\alpha\beta}{}^\rho g_{\rho\gamma} \tag{5b}$$

$$\Gamma_{\alpha\beta\gamma} \equiv \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha}) \tag{5c}$$

$$\Gamma^\mu{}_{\beta\gamma} = g^{\mu\alpha} \Gamma_{\alpha\beta\gamma} \tag{5d}$$

Solution

The gradient of the metric can be expanded as:

$$\begin{aligned}\vec{\nabla}\boldsymbol{g} &= g_{\alpha\beta;\gamma} \\ &= g_{\alpha\beta,\gamma} - \Gamma^{\mu}{}_{\alpha\gamma} g_{\mu\beta} - \Gamma^{\mu}{}_{\beta\gamma} g_{\alpha\mu} \\ &= g_{\alpha\beta,\gamma} - \Gamma_{\beta\alpha\gamma} - \Gamma_{\alpha\beta\gamma}\end{aligned}$$

but since according to Eq. 3 this is also equal to zero:

$$\boxed{\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = g_{\alpha\beta,\gamma}} \tag{6}$$

Since the metric, and thus its gradient, is symmetric in its first two indices α and β, this also implies that the covariant connection coefficients in Eq. 6 are symmetric in their first two indices as well. In a n-dimensional spacetime, where each index can take on values $0, 1, 2, n-1$, this gives the number of independent components of the tensor as n components where $\alpha = \beta$, and $(n^2 - n)/2$ independent components from the indices $\alpha\beta = \beta\alpha$. All of these occur n times due to the third index γ, so in total this gives the number of independent symmetric components as:

$$n\left[\frac{1}{2}\left(n^2 - n\right) + n\right] = \boxed{\frac{1}{2}n^2(n+1)} \tag{7}$$

Taking the difference of the gradients of two basis vectors along each other (where $\vec{\nabla}_{\beta} \equiv \vec{\nabla}_{\vec{e}_{\beta}}$):

$$\vec{\nabla}_{\alpha}\vec{e}_{\beta} - \vec{\nabla}_{\beta}\vec{e}_{\alpha} = \Gamma^{\mu}{}_{\beta\alpha}\vec{e}_{\mu} - \Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu} \tag{8}$$

but according to Eq. 4 this is also equal to the commutator of these basis vectors:

$$\Gamma^{\mu}{}_{\beta\alpha}\vec{e}_{\mu} - \Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu} = [\vec{e}_{\alpha}, \vec{e}_{\beta}] \tag{9}$$

From Eq. 5, this commutator can be written in terms of the commutator coefficients, giving:

$$\begin{aligned} &\Gamma^{\mu}{}_{\beta\alpha}\vec{e}_{\mu} - \Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu} = c_{\alpha\beta}{}^{\rho}\vec{e}_{\rho} \\ \rightarrow \quad &\left(\Gamma^{\mu}{}_{\beta\alpha}\vec{e}_{\mu} - \Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu}\right)\cdot\vec{e}^{\,\rho} = c_{\alpha\beta}{}^{\rho}\vec{e}_{\rho}\cdot\vec{e}^{\,\rho} \\ \rightarrow \quad &\left(\Gamma^{\mu}{}_{\beta\alpha} - \Gamma^{\mu}{}_{\alpha\beta}\right)\delta^{\rho}{}_{\mu} = c_{\alpha\beta}{}^{\rho} \\ \rightarrow \quad &\Gamma^{\rho}{}_{\beta\alpha} - \Gamma^{\rho}{}_{\alpha\beta} = c_{\alpha\beta}{}^{\rho}\end{aligned} \tag{10}$$

and applying the metric to both sides of Eq. 10 gives:

$$\begin{aligned} &\left(\Gamma^{\rho}{}_{\beta\alpha} - \Gamma^{\rho}{}_{\alpha\beta}\right)g_{\rho\gamma} = c_{\alpha\beta}{}^{\rho}g_{\rho\gamma} \\ \rightarrow \quad &\boxed{\Gamma_{\gamma\beta\alpha} - \Gamma_{\gamma\alpha\beta} = c_{\alpha\beta\gamma}}\end{aligned} \tag{11}$$

This implies that the covariant connection coefficient is antisymmetric in its last two indices. From this, the number of independent components is now just determined from $\alpha\beta = \beta\alpha$ where they differ only by a minus sign, and again, multiplied by n:

$$\frac{1}{2}n^2(n-1) \tag{12}$$

which when added to Eq. 7 gives the correct total of:

$$\frac{1}{2}n^2(n+1) + \frac{1}{2}n^2(n-1) = \frac{1}{2}n^2(n+1+n-1) = n^3 \tag{13}$$

To derive $\Gamma_{\alpha\beta\gamma}$ a combination of variations of Eq. 6 and Eq. 11 can be used. To start with, $g_{\alpha\beta,\gamma}$ and $c_{\beta\gamma\alpha}$ each contain a $\Gamma_{\alpha\beta\gamma}$ term, so adding these together and subtracting off the other terms gives:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\big(g_{\alpha\beta,\gamma} - c_{\beta\gamma\alpha} - \Gamma_{\beta\alpha\gamma} + \Gamma_{\alpha\gamma\beta}\big)$$

Similarly, the two extra connection coefficients here are also part of $g_{\alpha\gamma,\beta}$ and $c_{\alpha\gamma\beta}$, so replacing the connection coefficients and subtracting off the remainders gives:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\big(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha} - \Gamma_{\gamma\alpha\beta} - \Gamma_{\beta\gamma\alpha}\big)$$

Next, $\Gamma_{\beta\gamma\alpha}$ is a term in $g_{\beta\gamma,\alpha}$:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\big(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha} + \Gamma_{\gamma\beta\alpha} - \Gamma_{\gamma\alpha\beta}\big)$$

and the last two remaining connection coefficient terms are just Eq. 11, so this gives a final result of:

$$\boxed{\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha})} \tag{14}$$

Finally, applying the metric to raise the first index of the connection coefficient then gives:

$$\boxed{\Gamma^{\mu}{}_{\beta\gamma} = g^{\mu\alpha}\Gamma_{\alpha\beta\gamma}} \tag{15}$$

In terms of the qauntities easily computed by the formulas above, a plug-and-play formula for the connection coefficient is:

$$\Gamma^{\mu}{}_{\beta\gamma} = \frac{1}{2}g^{\mu\alpha}\big[g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\beta}{}^{\rho}g_{\rho\gamma} + c_{\alpha\gamma}{}^{\rho}g_{\rho\beta} - c_{\beta\gamma}{}^{\rho}g_{\rho\alpha}\big] \tag{16}$$

4. TB Exercise 24.9 *Connection Coefficients for Spherical Polar Coordinates*

(a) *Consider spherical polar coordinates in 3-dimensional space, and verify that the non-zero connection coefficients, assuming an orthonormal basis, are given by:*

$$\Gamma_{\theta r\theta} = \Gamma_{\phi r\phi} = -\Gamma_{r\theta\theta} = -\Gamma_{r\phi\phi} = \frac{1}{r} \quad , \quad \Gamma_{\phi\theta\phi} = -\Gamma_{\theta\phi\phi} = \frac{\cot\theta}{r} \tag{17}$$

Solution

In an orthonormal basis, the metric is $g_{ij} = \delta_{ij}$, so all derivatives of the metric in Eq. 14 vanish, so the formula to compute the connection coefficients for three-dimensional spherical polar coordinates reduces to:

$$\Gamma_{ijk} = \frac{1}{2}\big(c_{ijk} + c_{ikj} - c_{jki}\big) \tag{18}$$

The orthonormal basis vectors in spherical polar coordinates are:

$$\mathbf{e}_{\hat{r}} \equiv \frac{\partial}{\partial r} \quad , \quad \mathbf{e}_{\hat{\theta}} \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \quad , \quad \mathbf{e}_{\hat{\phi}} \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{19}$$

Using Eq. 5, all c_{iik} are zero since the commutator of a basis vector with itself is zero. Additionally, based on the properties of the commutator, $c_{ijk} = -c_{jik}$. Also, since the final index represents the orthonormal basis vector the commutator is projected along, the final index must be the same as one of the first two, i.e. only terms of the form c_{iji} or c_{ijj} are non-zero. For the basis vectors in Eq. 19, the only possible c_{ijk} are then:

$$\begin{aligned}
c_{r\theta\theta} &= [\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}}] \cdot \mathbf{e}_{\hat{\theta}} \\
&= \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial}{\partial r}\right]\frac{1}{r}\frac{\partial}{\partial \theta} \\
&= \left[-\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial}{\partial r}\right]\frac{1}{r}\frac{\partial}{\partial \theta} \\
&= -\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) \\
&= -\frac{1}{r}\mathbf{e}_{\hat{\theta}} \cdot \mathbf{e}_{\hat{\theta}} = -\frac{1}{r}
\end{aligned} \tag{20}$$

$$\begin{aligned}
c_{r\phi\phi} &= \Big[\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\phi}}\Big] \cdot \mathbf{e}_{\hat{\phi}} \\
&= \left[\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{\partial}{\partial r}\right]\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\
&= \left[-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \phi} + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\frac{\partial}{\partial \phi} - \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{\partial}{\partial r}\right]\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\
&= -\frac{1}{r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \\
&= -\frac{1}{r}\mathbf{e}_{\hat{\phi}} \cdot \mathbf{e}_{\hat{\phi}} = -\frac{1}{r}
\end{aligned} \tag{21}$$

$$c_{r\theta r} = [\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}}] \cdot \mathbf{e}_{\hat{r}}$$

$$
\begin{aligned}
&= \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial}{\partial r}\right]\frac{\partial}{\partial r} \\
&= \left[-\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial}{\partial r}\right]\frac{\partial}{\partial r} \\
&= -\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\partial}{\partial r}\right) \\
&= -\frac{1}{r}\mathbf{e}_{\hat{\theta}} \cdot \mathbf{e}_{\hat{r}} = 0
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
c_{r\phi r} &= \left[\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\phi}}\right] \cdot \mathbf{e}_{\hat{r}} \\
&= \left[\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{\partial}{\partial r}\right]\frac{\partial}{\partial r} \\
&= \left[-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \phi} + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\frac{\partial}{\partial \phi} - \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{\partial}{\partial r}\right]\frac{\partial}{\partial r} \\
&= -\frac{1}{r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\left(\frac{\partial}{\partial r}\right) \\
&= -\frac{1}{r}\mathbf{e}_{\hat{\phi}} \cdot \mathbf{e}_{\hat{r}} = 0
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
c_{\theta\phi\phi} &= \left[\mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}}\right] \cdot \mathbf{e}_{\hat{\phi}} \\
&= \left[\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)\right]\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\
&= \left[-\frac{\cot\theta}{r^2\sin\theta}\frac{\partial}{\partial \phi} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\frac{\partial}{\partial \phi} - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \phi}\frac{\partial}{\partial \theta}\right]\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\
&= -\frac{\cot\theta}{r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \\
&= -\frac{\cot\theta}{r}\mathbf{e}_{\hat{\phi}} \cdot \mathbf{e}_{\hat{\phi}} = -\frac{\cot\theta}{r}
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
c_{\theta\phi\theta} &= \left[\mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}}\right] \cdot \mathbf{e}_{\hat{\theta}} \\
&= \left[\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)\right]\frac{1}{r}\frac{\partial}{\partial \theta} \\
&= \left[-\frac{\cot\theta}{r^2\sin\theta}\frac{\partial}{\partial \phi} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\frac{\partial}{\partial \phi} - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \phi}\frac{\partial}{\partial \theta}\right]\frac{1}{r}\frac{\partial}{\partial \theta} \\
&= -\frac{\cot\theta}{r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) \\
&= -\frac{\cot\theta}{r}\mathbf{e}_{\hat{\phi}} \cdot \mathbf{e}_{\hat{\theta}} = 0
\end{aligned} \tag{25}
$$

For reference, the only non-zero commutator coefficients are then:

$$c_{\theta r\theta} = c_{\phi r\phi} = -c_{r\theta\theta} = -c_{r\phi\phi} = \frac{1}{r} \quad , \quad c_{\phi\theta\phi} = -c_{\theta\phi\phi} = \frac{\cot\theta}{r} \tag{26}$$

From Eq. 18 and the commutator coefficients above, any connection coefficient with two r indices or any three indices the same is zero. The remaining non-zero connection coefficients are then:

$$\begin{aligned}\Gamma_{\theta r\theta} &= \frac{1}{2}(c_{\theta r\theta} + c_{\theta\theta r} - c_{r\theta\theta}) \\ &= \frac{1}{2}\left(\frac{1}{r} + \frac{1}{r}\right) = \frac{1}{r} = -\Gamma_{r\theta\theta}\end{aligned} \tag{27}$$

$$\begin{aligned}\Gamma_{\phi r\phi} &= \frac{1}{2}(c_{\phi r\phi} + c_{\phi\phi r} - c_{r\phi\phi}) \\ &= \frac{1}{2}\left(\frac{1}{r} + \frac{1}{r}\right) = \frac{1}{r} = -\Gamma_{r\phi\phi}\end{aligned} \tag{28}$$

$$\begin{aligned}\Gamma_{\phi\theta\phi} &= \frac{1}{2}(c_{\phi\theta\phi} + c_{\phi\phi\theta} - c_{\theta\phi\phi}) \\ &= \frac{1}{2}\left(\frac{\cot\theta}{r} + \frac{\cot\theta}{r}\right) = \frac{\cot\theta}{r} = -\Gamma_{\theta\phi\phi}\end{aligned} \tag{29}$$

In total, the non-zero connection coefficients are:

$$\boxed{\Gamma_{\theta r\theta} = \Gamma_{\phi r\phi} = -\Gamma_{r\theta\theta} = -\Gamma_{r\phi\phi} = \frac{1}{r} \quad , \quad \Gamma_{\phi\theta\phi} = -\Gamma_{\theta\phi\phi} = \frac{\cot\theta}{r}} \tag{30}$$

These connection coefficients agree with the expected values in Eq. 17.

(b) *Repeat the exercise in part (a) assuming a coordinate basis with:*

$$\mathbf{e}_r \equiv \frac{\partial}{\partial r} \quad , \quad \mathbf{e}_\theta \equiv \frac{\partial}{\partial\theta} \quad , \quad \mathbf{e}_\phi \equiv \frac{\partial}{\partial\phi} \tag{31}$$

Solution

For a coordinate basis, all of the basis vector commutators, and thus, commutator coefficients, are zero, so only the derivatives of the metric remain in the connection coefficient formula, yielding:

$$\Gamma_{ijk} = \frac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i}) \tag{32}$$

In a polar spherical coordinate basis, the only non-zero terms in the metric are $g_{rr} = 1$, $g_{\theta\theta} = r^2$, and $g_{\phi\phi} = r^2 \sin^2\theta$. From this, the only non-zero derivatives of the metric are:

$$g_{\theta\theta,r} = 2r \quad , \quad g_{\phi\phi,r} = 2r\sin^2\theta \quad , \quad g_{\phi\phi,\theta} = 2r^2 \sin\theta\cos\theta \tag{33}$$

From this, the non-zero connection coefficients can be read off from Eq. 32 as only one term can ever have the same first two indices:

$$\boxed{\begin{aligned} &\Gamma_{\theta\theta r} = \Gamma_{\theta r\theta} = -\Gamma_{r\theta\theta} = r && (34a)\\ &\Gamma_{\phi\phi r} = \Gamma_{\phi r\phi} = -\Gamma_{r\phi\phi} = r\sin^2\theta && (34b)\\ &\Gamma_{\phi\phi\theta} = \Gamma_{\phi\theta\phi} = -\Gamma_{\theta\phi\phi} = r^2\sin\theta\cos\theta && (34c)\end{aligned}}$$

or in the form $\Gamma^i{}_{jk} = g^{i\ell}\Gamma_{\ell jk}$, where the inverse metric components are $g^{rr} = 1$, $g^{\theta\theta} = r^{-2}$, and $g^{\phi\phi} = r^{-2}\sin^{-2}\theta$:

$$\boxed{\begin{aligned} &\Gamma^r{}_{\theta\theta} = -r && (35a)\\ &\Gamma^r{}_{\phi\phi} = -r\sin^2\theta && (35b)\\ &\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta && (35c)\\ &\Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta && (35d)\\ &\Gamma^\theta{}_{\theta r} = \Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{\phi r} = \Gamma^\phi{}_{r\phi} = \frac{1}{r} && (35e)\end{aligned}}$$

(c) *Repeat both computations in parts (a) and (b) using symbolic manipulation software on a computer.*

Solution

See attached *Mathematica* notebook `conn_coeff_calc.nb`.

5. TB Exercise 24.13 *Stress-Energy Tensor for a Viscous Fluid*

Using the gradient of the fluid's 4-velocity broken down into irreducible tensorial parts as:

$$u_{\alpha;\beta} = -a_\alpha u_\beta + \frac{1}{3}\theta P_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{36}$$

where $P_{\alpha\beta}$ is the orthogonal projection tensor (that can also be regarded as the metric of the 3-space it projects into):

$$P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \tag{37}$$

and $\sigma_{\alpha\beta}$ is a symmetric, trace-free tensor orthogonal to $\vec{u}$ and $\omega_{\alpha\beta}$ is an antisymmetric tensor orthogonal to $\vec{u}$:

(a) *Show that the rate of change of $\vec{u}$ along itself, $\vec{\nabla}_{\vec{u}}\vec{u}$ (i.e. the fluid 4-acceleration) is equal to the vector $\vec{a}$ that appears in the decomposition Eq. 36. Show, further, that $\vec{a}\cdot\vec{u}=0$.*

Solution

Using Eq. 36 in the directional derivative of the 4-velocity along itself yields (and using $\vec{u}\cdot\vec{u}=-1$):

$$\begin{aligned}
\vec{\nabla}_{\vec{u}}\vec{u} &= \vec{\nabla}\vec{u}(_,\vec{u}) \\
&= u_{\alpha;\beta}u^\beta \\
&= -a_\alpha u_\beta u^\beta + \frac{1}{3}\theta P_{\alpha\beta}u^\beta + \cancelto{0}{\sigma_{\alpha\beta}u^\beta} + \cancelto{0}{\omega_{\alpha\beta}u^\beta} \\
&= -a_\alpha u_\beta u^\beta + \frac{1}{3}\theta\big(g_{\alpha\beta}u^\beta + u_\alpha u_\beta u^\beta\big) \\
&= a_\alpha + \cancel{\frac{1}{3}\theta u^\alpha} - \cancel{\frac{1}{3}\theta u^\alpha} \\
&= a_\alpha
\end{aligned}$$

$$\rightarrow \quad \boxed{\vec{\nabla}_{\vec{u}}\vec{u} = \vec{a}} \tag{38}$$

To show that the 4-acceleration and 4-velocity are orthogonal, the directional gradient of $\vec{u}\cdot\vec{u}=-1$ along $\vec{u}$ gives:

$$\begin{aligned}
\vec{\nabla}_{\vec{u}}(\vec{u}\cdot\vec{u}) &= \vec{\nabla}_{\vec{u}}(-1) \\
\rightarrow \quad \left(\vec{\nabla}_{\vec{u}}\vec{u}\right)\cdot\vec{u} + \vec{u}\cdot\left(\vec{\nabla}_{\vec{u}}\vec{u}\right) &= 0 \\
\cancel{2}\vec{u}\cdot\vec{\nabla}_{\vec{u}}\vec{u} &= 0
\end{aligned}$$

From Eq. 38, this is equivalent to:

$$\vec{a}\cdot\vec{u} = 0 \tag{39}$$

(b) *Show that the divergence of the 4-velocity, $\vec{\nabla}\cdot\vec{u}$, is equal to the scalar field θ that appears in the decomposition Eq. 36.*

Solution

To show that the divergence of the 4-velocity is equal to the scalar field θ, Eq. 36 and Eq. 39, along with the traces $g^\alpha{}_\alpha = 4$, $\sigma^\alpha{}_\alpha = 0$, and $\omega^\alpha{}_\alpha = 0$, give the result:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{u} &= u^\alpha{}_\alpha \\
&= -\cancelto{0}{a^\alpha u_\alpha} + \frac{1}{3}\theta P^\alpha{}_\alpha + \cancelto{0}{\sigma^\alpha{}_\alpha} + \cancelto{0}{\omega^\alpha{}_\alpha}
\end{aligned}$$

$$= \frac{1}{3}\theta(g^{\alpha}{}_{\alpha} + u^{\alpha}u_{\alpha})$$
$$= \frac{1}{3}\theta(4 - 1)$$

$$\rightarrow \quad \boxed{\vec{\nabla} \cdot \vec{u} = \theta} \tag{40}$$

(c) *The quantities $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are the relativistic versions of a Newtonians fluid's shear and rotation tensors. Derive equations for these tensors in terms of $u_{\alpha;\beta}$ and $P_{\alpha\beta}$.*

Solution

Using Eq. 39, the relation of the tensor product of the 4-velocity and 4-acceleration can be calculated as:

$$\vec{a} \cdot \vec{u} = 0$$
$$\rightarrow \quad a^{\beta}u_{\beta} = 0$$
$$\rightarrow \quad g_{\alpha\beta}a^{\beta}u_{\beta} = g_{\alpha\beta}(0) \tag{41}$$
$$\rightarrow \quad a_{\alpha}u_{\beta} = 0 \tag{42}$$

Solving for $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ in Eq. 36:

$$\sigma_{\alpha\beta} = u_{\alpha;\beta} + \cancelto{0}{a_{\alpha}u_{\beta}} - \frac{1}{3}\theta P_{\alpha\beta} - \omega_{\alpha\beta} \tag{43a}$$
$$\omega_{\alpha\beta} = u_{\alpha;\beta} + \cancelto{0}{a_{\alpha}u_{\beta}} - \frac{1}{3}\theta P_{\alpha\beta} - \sigma_{\alpha\beta} \tag{43b}$$

To eliminate $\omega_{\alpha\beta}$ from the first part of Eq. 43, the symmetry of $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$, along with the symmetry of $P_{\alpha\beta} = P_{\beta\alpha}$, and the antisymmetry of $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, can be used to write:

$$\sigma_{\alpha\beta} = \frac{1}{2}(\sigma_{\alpha\beta} + \sigma_{\beta\alpha})$$
$$= \frac{1}{2}\left(u_{\alpha;\beta} + u_{\beta;\alpha} - \frac{1}{3}\theta P_{\alpha\beta} - \frac{1}{3}\theta P_{\beta\alpha} - \omega_{\alpha\beta} - \omega_{\beta\alpha}\right)$$
$$= \frac{1}{2}\left(u_{\alpha;\beta} + u_{\beta;\alpha} - \frac{2}{3}\theta P_{\alpha\beta} - \cancel{\omega_{\alpha\beta}} + \cancel{\omega_{\alpha\beta}}\right)$$

$$\boxed{\sigma_{\alpha\beta} = \frac{1}{2}(u_{\alpha;\beta} + u_{\beta;\alpha}) - \frac{1}{3}\theta P_{\alpha\beta}} \tag{44}$$

Similarly for $\omega_{\alpha\beta}$:

$$\omega_{\alpha\beta} = \frac{1}{2}(\omega_{\alpha\beta} - \omega_{\beta\alpha})$$

$$
\begin{aligned}
&= \frac{1}{2}\left(u_{\alpha;\beta} - u_{\beta;\alpha} - \frac{1}{3}\theta P_{\alpha\beta} + \frac{1}{3}\theta P_{\beta\alpha} - \sigma_{\alpha\beta} + \sigma_{\beta\alpha}\right) \\
&= \frac{1}{2}\left(u_{\alpha;\beta} - u_{\beta;\alpha} - \cancel{\frac{1}{3}\theta P_{\alpha\beta}} + \cancel{\frac{1}{3}\theta P_{\alpha\beta}} - \cancel{\sigma_{\alpha\beta}} + \cancel{\sigma_{\alpha\beta}}\right)
\end{aligned}
$$

$$
\boxed{\omega_{\alpha\beta} = \frac{1}{2}(u_{\alpha;\beta} - u_{\beta;\alpha})} \tag{45}
$$

(d) *Show that, as viewed in a Lorentz frame where the fluid is moving a speed small compared to the speed of light, to first order in the fluid's ordinary velocity* $v^j = \mathrm{d}x^j/\mathrm{d}t$*, the following statements are true: (i)* $u^0 = 1$*,* $u^j = v^j$*; (ii)* θ *is the nonrelativistic rate of expansion of the fluid,* $\theta = \boldsymbol{\nabla} \cdot \mathbf{v} \equiv v^j{}_{,j}$*; (iii)* σ_{jk} *is the fluid's nonrelativistic shear; and (iv)* ω_{jk} *is the fluid's nonrelativistic rotation tensor.*

Solution

In a Lorentz frame with ordinary velocity $\mathbf{v}$ and Lorentz factor $\gamma = (1 - v^2)^{-1/2}$, the 4-velocity has components $u^0 = \gamma$ and $u^j = \gamma v^j$. When the velocity is small compared to the speed of light, $v \ll 1$, these components can be expanded as:

$$
\begin{aligned}
u^0 &= \gamma = \frac{1}{(1 - v^2)^{1/2}} = 1 + \frac{1}{2}v^2 + \mathcal{O}(v^4) \\
u^j &= \gamma v^j = \frac{v^j}{(1 - v^2)^{1/2}} = v^j\left(1 + \frac{1}{2}v^2 + \mathcal{O}(v^4)\right)
\end{aligned}
$$

To first order this gives the components of the 4-velocity as:

$$
\boxed{u^0 = 1 \quad , \quad u^j = v^j} \tag{46}
$$

In this limit of $v \ll 1$, the divergence of the 4-velocity is equal to the divergence of the ordinary velocity, so using the spatial components of Eq. 36 gives:

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{v} &= v^j{}_{,j} \\
&= -\cancel{a^j v_j} + \frac{1}{3}\theta\left(g^j{}_j + v^j v_j\right) + \cancel{\sigma^j{}_j} + \cancel{\omega^j{}_j} \\
&= \frac{1}{3}\theta\left(3 + v^2\right)
\end{aligned}
$$

but since $v^2 \ll 1$, this is approximately:

$$
\boxed{\boldsymbol{\nabla} \cdot \mathbf{v} = \theta} \tag{47}
$$

Additionally, in this limit, the spatial components of Eq. 44 reduce to:

$$
\sigma_{jk} = \frac{1}{2}(v_{j;k} + v_{k;j}) - \frac{1}{3}\theta(g_{jk} + v_j v_k)
$$

$$= \frac{1}{2}(v_{j;k} + v_{k;j}) - \frac{1}{3}\theta\big(g_{jk} + v^2\big)$$

which since $v^2 \ll 1$, this is approximately:

$$\boxed{\sigma_{jk} = \frac{1}{2}(v_{j;k} + v_{k;j}) - \frac{1}{3}\theta g_{jk}} \tag{48}$$

Finally, in this limit, the spatial components of Eq. 45 reduce to:

$$\boxed{\omega_{jk} = \frac{1}{2}(v_{j;k} - v_{k;j})} \tag{49}$$

(e) *At some event $\mathcal{P}$ where we want to know the influence of viscosity on the fluid's stress-energy tensor, introduce the fluid's local rest frame. Explain why, in that frame, the only contributions of visosity to the components of the stress-energy tensor are $T^{jk}_{visc} = -\zeta\theta g^{jk} - 2\mu\sigma^{jk}$, where ζ and μ are the coefficients of bulk and shear viscosity, respectively; the constributions T^{00} and $T^{j0} = T^{0j}$ vanish.*

Solution

In the fluid's rest frame, the 4-velocity has components $u^0 = 1$ and $u^j = 0$, so since the ordinary velocity in this frame is zero, only the spatial components of the shear and rate of expansion contribute to the viscosity. The term $-\zeta\theta g^{jk}$ shows properly that the bulk modulus ζ multiplied by the rate of expansion θ resists the change in volume of the fluid (since only the isotropic components of the stress energy tensor gain a contribution from g^{jj}). The term $-2\mu\sigma^{jk}$ only affects the symmetric terms of the stress energy tensor picked out by the symmetric, trace-free σ^{jk}, and the shear viscosity coefficient then contributes only to the terms $T^{jk} = T^{kj}$ for $j \neq k$, effectively resisting changes to the shape of the fluid.

(f) *From nonrelativistic fluid mechanics, infer that, in the fluid's rest frame at $\mathcal{P}$, the only contributions of diffusive heat conductivity to the stress-energy tensor are $T^{0j}_{cond} = T^{j0}_{cond} = -\kappa\, \partial T/\partial x^j$, where κ is the fliuid's thermal conductivity and T is its temperature. If the fluid is accelerating, there is a correction term: $\partial T/\partial x^j$ gets replaced by $\partial T/\partial x^j + a^j T$ where a^j is the acceleration.*

Solution

Heat conduction is caused by the flux of energy through the fluid. Since the energy density flux is zero in the fluid's rest frame, the factor of the temperature gradient $-\kappa\, \partial T/\partial x^j$ describes how heat (energy) will flow through the fluid, proportional to the fluid's thermal conductivity κ. Due to the stress-energy tensors symmetry, this also means that the momentum density flux is also equal to $-\kappa\, \partial T/\partial x^j$.

The corrected term $\partial T/\partial x^j + a^j T$ for an accelerating fluid includes the accerlation term to ensure that the temperature gradient is transported along the accelerating fluid's worldline.

(g) *Using the results of part (e) and (f), deduce that the following geometric, frame-invariant form of the fluid's stress-energy tensor:*

$$T_{\alpha\beta} = (\rho + P)u_\alpha u_\beta + P g_{\alpha\beta} - \zeta\theta g_{\alpha\beta} - 2\mu\sigma_{\alpha\beta} - 2\kappa u_{(\alpha} P_{\beta)}{}^{\mu}(T_{;\mu} + a_\mu T) \tag{50}$$

The first two terms of Eq. 50 are the standard terms for the stress-energy tensor of a perfect fluid. The addition of $-\zeta\theta g_{\alpha\beta}$ and $-2\mu\sigma_{\alpha\beta}$ with full α and β indices does not add anything to the T_{00} or $T_{j0} = T_{0j}$ components since the corresponding components of these are zero in the fluid's rest frame. The temperature conductivity term does not necessarily have zero contributions for time components, so the projection tensor is used to project the temperature conductivity term into the 3-space orthogonal to the fluid's 4-velocity, which in this case is just the spatial components of the fluid's rest frame. The addition of all of these terms gives the complete stress-energy tensor for a viscous fluid with diffusive heat conduction.

6. TB Exercise 24.14 *Proper Reference Frame*

(a) *Show that the coordinate transformation:*

$$x^i = x^{\hat{i}} + \frac{1}{2}a^{\hat{i}}\left(x^{\hat{0}}\right)^2 + \epsilon^{\hat{i}}{}_{\hat{j}\hat{k}}\Omega^{\hat{j}}x^{\hat{k}}x^{\hat{0}} \tag{51a}$$

$$x^0 = x^{\hat{0}}\left(1 + a_{\hat{j}}x^{\hat{j}}\right) \tag{51b}$$

brings the metric $ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta$ into the form:

$$ds^2 = -(1 + 2\mathbf{a}\cdot\mathbf{x})(dx^{\hat{0}})^2 + 2(\mathbf{\Omega}\times\mathbf{x})\cdot d\mathbf{x}dx^{\hat{0}} + \delta_{\hat{j}\hat{k}}dx^{\hat{j}}dx^{\hat{k}} \tag{52}$$

accurate to linear order in separation $x^{\hat{j}}$ from the origin of coordinates.

Solution

Computing the time component of the metric:

$$\begin{aligned}\eta_{00}dx^0dx^0 &= -(dx^{\hat{0}})^2[1 + 2a_{\hat{j}}x^{\hat{j}} + (a_{\hat{j}}x^{\hat{j}})^2] \\ &= -\left[1 + 2\mathbf{a}\cdot\mathbf{x} + (\mathbf{a}\cdot\mathbf{x})^2\right]\left(dx^{\hat{0}}\right)^2\end{aligned} \tag{53}$$

and the spatial components:

$$\begin{aligned}\eta_{jk}dx^jdx^k = \delta_{\hat{j}\hat{k}}dx^{\hat{j}}dx^{\hat{k}} + \frac{1}{4}a_{\hat{j}}a^{\hat{j}}\left(x^{\hat{0}}\right)^4 + \left(\epsilon^{\hat{j}}{}_{\hat{a}\hat{b}}\Omega^{\hat{a}}x^{\hat{b}}\right)\left(\epsilon_{\hat{j}\hat{a}\hat{b}}\Omega^{\hat{a}}x^{\hat{b}}\right)\left(dx^{\hat{0}}\right)^2 \\ + a_{\hat{j}}dx^{\hat{j}}\left(x^{\hat{0}}\right)^2 + 2\epsilon_{\hat{j}\hat{a}\hat{b}}\Omega^{\hat{a}}x^{\hat{b}}dx^{\hat{j}}dx^{\hat{0}} + \epsilon_{\hat{j}\hat{a}\hat{b}}\Omega^{\hat{a}}x^{\hat{b}}a^{\hat{j}}dx^{\hat{0}}\left(x^{\hat{0}}\right)^2\end{aligned}$$

$$
\begin{aligned}
&= \delta_{\hat{j}\hat{k}} dx^{\hat{j}} dx^{\hat{k}} + \frac{1}{4}\mathbf{a}\cdot\mathbf{a}\left(x^{\hat{0}}\right)^4 + (\mathbf{\Omega}\times\mathbf{x})\cdot(\mathbf{\Omega}\times\mathbf{x})\left(dx^{\hat{0}}\right)^2 \\
&\quad + \mathbf{a}\cdot d\mathbf{x}\left(x^{\hat{0}}\right)^2 + 2(\mathbf{\Omega}\times\mathbf{x})\cdot d\mathbf{x}dx^{\hat{0}} + (\mathbf{\Omega}\times\mathbf{x})\cdot\mathbf{a}dx^{\hat{0}}\left(x^{\hat{0}}\right)^2 \\
&= \delta_{\hat{j}\hat{k}} dx^{\hat{j}} dx^{\hat{k}} + \left[\frac{1}{2}\mathbf{a}\left(x^{\hat{0}}\right)^2\right]^2 + (\mathbf{\Omega}\times\mathbf{x})^2\left(dx^{\hat{0}}\right)^2 + 2\left[\frac{1}{2}\mathbf{a}\left(x^{\hat{0}}\right)^2\right]\cdot d\mathbf{x} \\
&\quad + 2(\mathbf{\Omega}\times\mathbf{x})\cdot d\mathbf{x}dx^{\hat{0}} + 2(\mathbf{\Omega}\times\mathbf{x})\cdot\left[\frac{1}{2}\mathbf{a}\left(x^{\hat{0}}\right)^2\right]dx^{\hat{0}} \\
&= \delta_{\hat{j}\hat{k}} dx^{\hat{j}} dx^{\hat{k}} + \mathbf{x}\cdot\mathbf{x} + (\mathbf{\Omega}\times\mathbf{x})^2\left(dx^{\hat{0}}\right)^2 + 2\mathbf{x}\cdot d\mathbf{x} \\
&\quad + 2(\mathbf{\Omega}\times\mathbf{x})\cdot d\mathbf{x}dx^{\hat{0}} + 2(\mathbf{\Omega}\times\mathbf{x})\cdot\mathbf{x}dx^{\hat{0}}
\end{aligned} \tag{54}
$$

Keeping only terms linear in the x-coordinates, and eliminating the final term since $(\mathbf{\Omega}\times\mathbf{x})\cdot\mathbf{x} = 0$, combining Eq. 53 and Eq. 54 gives the metric in the form:

$$
\boxed{ds^2 = -(1 + 2\mathbf{a}\cdot\mathbf{x})(dx^{\hat{0}})^2 + 2(\mathbf{\Omega}\times\mathbf{x})\cdot d\mathbf{x}dx^{\hat{0}} + \delta_{\hat{j}\hat{k}} dx^{\hat{j}} dx^{\hat{k}}} \tag{55}
$$

(b) *Compute the connection coefficients for this coordinate basis at an arbitrary even on the observer's worldline.*

Solution

At an arbitrary event on the observer's worldline, the rest frame metric on the worldline is just $\eta_{\hat{\alpha}\hat{\beta}}$, but in the local space near the worldline, the metric takes the form of Eq. 55, which can also be written in components as:

$$
g_{\hat{0}\hat{0}} = -(1 + 2\mathbf{a}\cdot\mathbf{x}) \quad , \quad g_{\hat{j}\hat{0}} = g_{\hat{0}\hat{j}} = \mathbf{\Omega}\times\mathbf{x} \quad , \quad g_{\hat{j}\hat{k}} = \delta_{\hat{j}\hat{k}} \tag{56}
$$

Note, for the inverse metric, I also am only keeping terms linear in the x-coordinates, so in this case $g^{\hat{\alpha}\hat{\beta}} \equiv \text{diag}(-1, 1, 1, 1)$.

From these, in this frame, the time derivatives of the metric components are zero, and the only other non-zero derivatives are:

$$
\begin{aligned}
g_{\hat{0}\hat{0},\hat{i}} &= -\left(1 + 2a_{\hat{k}}x^{\hat{k}}\right)_{,\hat{i}} \\
&= -2\left(a_{\hat{k},\hat{i}}x^{\hat{k}} + a_{\hat{k}}x^{\hat{k}}{}_{,\hat{i}}\right) \\
&= -2a_{\hat{k}}\delta^{\hat{k}}{}_{\hat{i}} = -2a_{\hat{i}}
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
g_{\hat{j}\hat{0},\hat{i}} = g_{\hat{0}\hat{j},\hat{i}} &= \left(\epsilon_{\hat{j}\hat{a}\hat{b}}\Omega^{\hat{a}}x^{\hat{b}}\right)_{,\hat{i}} \\
&= \epsilon_{\hat{j}\hat{a}\hat{b}}\Omega^{\hat{a}}x^{\hat{b}}{}_{,\hat{i}} \\
&= \epsilon_{\hat{j}\hat{a}\hat{i}}\Omega^{\hat{a}}
\end{aligned} \tag{58}
$$

From these, the only non-zero connection coefficients are then:

$$\Gamma^{\hat{i}}{}_{\hat{0}\hat{0}} = \frac{1}{2} g^{\hat{i}\hat{i}} \left(g_{\hat{i}\hat{0},\hat{0}} + g_{\hat{i}\hat{0},\hat{0}} - g_{\hat{0}\hat{0},\hat{i}} \right) = a^{\hat{i}} \tag{59}$$

$$\begin{aligned}
\Gamma^{\hat{i}}{}_{\hat{0}\hat{j}} = \Gamma^{\hat{i}}{}_{\hat{j}\hat{0}} &= \frac{1}{2} g^{\hat{i}\hat{i}} \left(g_{\hat{i}\hat{j},\hat{0}} + g_{\hat{i}\hat{0},\hat{j}} - g_{\hat{j}\hat{0},\hat{i}} \right) \\
&= \frac{1}{2} \left[\epsilon_{\hat{i}\hat{a}\hat{j}} \Omega^{\hat{a}} - \epsilon_{\hat{j}\hat{a}\hat{i}} \Omega^{\hat{a}} \right] \\
&= \frac{1}{2} \left[\epsilon_{\hat{i}\hat{a}\hat{j}} \Omega^{\hat{a}} - \left(-\epsilon_{\hat{i}\hat{a}\hat{j}} \Omega^{\hat{a}} \right) \right] \\
&= \epsilon_{\hat{i}\hat{a}\hat{j}} \Omega^{\hat{a}}
\end{aligned} \tag{60}$$

See attached *Mathematica* notebook `conn_coeff_calc.nb` for computer-calculated connection coefficients.

(c) *Using the connection coefficients from part (b), show that the rate of change of the basis vectors* $\mathbf{e}_{\hat{\alpha}}$ *along the observer's worldline is given by:*

$$\vec{\nabla}_{\vec{U}} \vec{e}_{\hat{0}} = \vec{a} \tag{61a}$$

$$\vec{\nabla}_{\vec{U}} \vec{e}_{\hat{j}} = \left(\vec{a} \cdot \vec{e}_{\hat{j}} \right) \vec{U} + \boldsymbol{\epsilon}\left(\vec{U}, \vec{\Omega}, \vec{e}_{\hat{j}}, \_ \right) \tag{61b}$$

Solution

Starting with $\vec{\nabla}_{\vec{U}} \vec{e}_{\hat{\alpha}} = \vec{\nabla}_{\hat{0}} \vec{e}_{\hat{\alpha}} = \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{0}} \vec{e}_{\hat{\mu}}$, using the above connection coefficients then gives the time component as:

$$\begin{aligned}
\vec{\nabla}_{\vec{U}} \vec{e}_{\hat{0}} &= \Gamma^{\hat{\mu}}{}_{\hat{0}\hat{0}} \vec{e}_{\hat{\mu}} \\
&= \Gamma^{\hat{i}}{}_{\hat{0}\hat{0}} \vec{e}_{\hat{i}} \\
&= a^{\hat{i}} \vec{e}_{\hat{i}} \\
\rightarrow \quad & \boxed{\vec{\nabla}_{\vec{U}} \vec{e}_{\hat{0}} = \vec{a}}
\end{aligned} \tag{62}$$

and the spatial components as (using $\vec{e}_{\hat{0}} = \vec{U}$ on the observer's worldline):

$$\begin{aligned}
\vec{\nabla}_{\vec{U}} \vec{e}_{\hat{j}} &= \Gamma^{\hat{\mu}}{}_{\hat{j}\hat{0}} \vec{e}_{\hat{\mu}} \\
&= \Gamma^{\hat{0}}{}_{\hat{j}\hat{0}} \vec{e}_{\hat{0}} + \Gamma^{\hat{i}}{}_{\hat{j}\hat{0}} \vec{e}_{\hat{i}} \\
&= a^{\hat{j}} \vec{U} + \epsilon_{\hat{0}\hat{i}\hat{a}\hat{j}} U^{\hat{0}} \Omega^{\hat{a}} \vec{e}_{\hat{i}} \\
\rightarrow \quad & \boxed{\vec{\nabla}_{\vec{U}} \vec{e}_{\hat{j}} = \left(\vec{a} \cdot \vec{e}_{\hat{j}} \right) \vec{U} + \boldsymbol{\epsilon}\left(\vec{U}, \vec{\Omega}, \vec{e}_{\hat{j}}, _ \right)}
\end{aligned} \tag{63}$$

where $\vec{a} = a^{\hat{j}} \vec{e}_{\hat{j}}$ was used and some contracted indices were rearragned in the last term: $\hat{i}$ swapped with $\hat{a}$, incurring a factor of -1, and then $\hat{a}$ and $\hat{j}$ for another factor of -1.

(d) *Using the connection coefficients from part (b), show that the low-velocity limit of the geodesic equation is given by:*

$$\frac{\mathrm{d}^2 x^{\hat{j}}}{(\mathrm{d}x^{\hat{0}})^2} = -a^{\hat{i}} - 2\epsilon^{\hat{i}}{}_{\hat{j}\hat{k}}\Omega^{\hat{j}} v^{\hat{k}} \tag{64}$$

Solution

Starting with the low-velocity form of the geodesic equation, and using the connection coefficients computed above:

$$\begin{aligned}\frac{\mathrm{d}^2 x^{\hat{j}}}{(\mathrm{d}x^{\hat{0}})^2} &= -\Gamma^{\hat{i}}{}_{\hat{0}\hat{0}} - \left(\Gamma^{\hat{i}}{}_{\hat{j}\hat{0}} + \Gamma^{\hat{i}}{}_{\hat{0}\hat{j}}\right) v^{\hat{j}} \\ &= -a^{\hat{i}} - 2\epsilon_{\hat{i}\hat{k}\hat{j}}\Omega^{\hat{k}} v^{\hat{j}} \\ &= -a^{\hat{i}} - 2\epsilon_{\hat{i}\hat{j}\hat{k}}\Omega^{\hat{j}} v^{\hat{k}}\end{aligned}$$

where the two sets of fully contracted indices in the last term were renamed; this incurs no factor of -1 in this case. This is equivalent to Eq. 64 and can be written in the form:

$$\boxed{\frac{\mathrm{d}^2 \mathbf{x}}{(\mathrm{d}x^{\hat{0}})^2} = -\mathbf{a} - 2\boldsymbol{\Omega} \times \mathbf{v}}$$