Lecture17

Wednesday, March 20, 2019 2:03 PM

Relativistic Boltzmann Equation

- for a system of sufficiently large number N of equal-mass particles, phase space distribution function:

$$f(x^a, p^a) = \frac{dN}{d^3\hat{x}\, d^3 p}$$

$d^3\hat{x}$: physical space volume element

$d^3\hat{p}$: momentum space volume element

$\rightarrow$ 6D

- f and $d^3\hat{x}\, d^3\hat{p}$ are Lorentz invariant
- f satisfies relativistic Boltzmann eqn.:

$$\boxed{\frac{\Delta f}{d\lambda} \equiv \left(\frac{dx^a}{d\lambda}\right)\frac{\partial f}{\partial x^a} + \left(\frac{dp^a}{d\lambda}\right)\frac{\partial f}{\partial p^a} = \left(\frac{\delta f}{\delta \lambda}\right)_{coll}}$$

- derivative D taken along trajectory λ in phase space

- affine parameter λ defined by,

$$p^a \equiv \frac{dx^a}{d\lambda}$$

$\rightarrow \quad \lambda = \tau / m$ for finite mass particles

- in absence of forces other than gravity, and collisions,

$$\frac{dp^a}{d\lambda} = -\Gamma^a{}_{bc}\, p^b p^c \quad \text{.... geodesic eqn.}$$

$\rightarrow$ particles move along geodesics

- define operator,

$$\frac{\mathcal{D}}{dx^a} \equiv \frac{\partial}{\partial x^a} - \Gamma^b{}_{ac}\, p^c \frac{\partial}{\partial p^b}$$

then using geodesic eqn.

$\hookrightarrow$
$$\frac{\mathcal{D}f}{d\lambda} = \left(\frac{dx^a}{d\lambda}\right)\frac{\partial f}{\partial x^a} - \Gamma^b{}_{ac}\, p^a p^c \frac{\partial f}{\partial p^b} = \left(\frac{\delta f}{\delta \lambda}\right)_{coll}$$

by definition $p^a = dx^a / d\lambda$

$\hookrightarrow$
$$\frac{\mathcal{D}f}{d\lambda} = p^a\left[\frac{\partial f}{\partial x^a} - \Gamma^b{}_{ac}\, p^c \frac{\partial f}{\partial p^b}\right] = \left(\frac{\delta f}{\delta \lambda}\right)_{coll}$$

$$= p^a \frac{\mathcal{D}f}{dx^a} = \left(\frac{\delta f}{\delta \lambda}\right)_{coll}$$

Radiation transport eqn.

- along the beam λ, photons may be created & destroyed by interactions & emission from matter
- rate of destruction must be proportional to number of photons in ...

∝ number of photons in $dx\,dp$ along λ

- for photons $f \propto I_\nu/\nu^3$

$$\rightarrow p^a \frac{D(I_\nu/\nu^3)}{dx^a} = \left(\frac{\delta(I_\nu/\nu^3)}{\delta\lambda}\right)_{coll} = \underset{\text{emission}}{e} - \underset{\text{absorption}}{a}(I_\nu/\nu^3)$$

- recognizing e & a must be emissivity & opacity, $e = \eta_\nu/\nu^2$, $a = \nu\chi_\nu$

$$\hookrightarrow p^a \frac{D(I_\nu/\nu^3)}{dx^a} = (\eta_\nu - \chi_\nu I_\nu)/\nu^2$$

- in spherically-symmetric medium & co-moving coords., the metric is,

$$ds^2 = -e^{2\Psi}dt^2 + e^{2\Lambda}dr^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

r: Langrangian radial coord.

Ψ, Λ, R: functions of r, t only

- now introduce operators,

$$D_t = e^{-\Psi}(\partial/\partial t), \quad D_r = e^{-\Lambda}(\partial/\partial r)$$

& variables,

$$U = D_t R, \quad \Gamma = D_r R$$

↳ then transfer eqn. becomes (Lindquist 1966),

$$D_t(I_\nu/\nu^3) + \mu D_r(I_\nu/\nu^3) - \nu\left[\mu D_r\Psi + \mu^2 D_t\Lambda + (1-\mu^2)(U/R)\right](\partial(I_\nu/\nu^3)/\partial\nu)$$

$$+ (1-\mu^2)\{(\Gamma/R) - D_r \Psi + \mu[(U/R) - D_t \Lambda]\}$$
$$\times (\partial(I_\nu/\nu^3)/\partial\mu) = (\eta_\nu - I_\nu \chi_\nu)/\nu^3$$

- solving in co-moving, or fluid rest frames is nice since there interaction terms (RHS) are easier to compute. Radiation in equilibrium is also there closest to Planck function at local matter temperature.
  - not always feasible, however

<u>Moment equations</u>

- similar to constructing moments of the distribution function f, can take angular moments of the relativistic Boltzmann eqn.
- see: Thorne et al (1981): moment hierarchy
  Shibata et al (2011): truncated evolution eqns.
  Cardall et al (2013): rederivation starting from conservative Boltzmann eqn.
- in 3+1: $n^a = (\alpha^{-1}, -\beta^k \alpha^{-1})$ .... normal vector
  $\gamma_{ab} = g_{ab} + n_a n_b$ .... spatial metric

momentum of massless particle:

$$\frac{dx^a}{d\tau} = p^a = \nu(u^a + l^a)$$

l^a: unit normal 4-vector orthogonal to u^a

$$l_a l^a = 1\,, \quad u_a l^a = 0$$

→ contains angular dependence,

$$\int d\Omega\, l^a = 0 = \int d\Omega\, l^a l^b l^c$$

$$\frac{1}{4\pi}\int d\Omega\, l^a l^b = \frac{1}{3} h^{ab}$$

$$\frac{1}{4\pi}\int d\Omega\, l^a l^b l^c l^d = \frac{1}{15}\left(h^{ab}h^{cd} + h^{ac}h^{bd} + h^{ad}h^{bc}\right)$$

where

$$h_{ab} \equiv g_{ab} + u_a u_b \quad \text{... projection operator}$$

- Hierarchy of moments is then:

$$M_{(\nu)}^{a_1 a_2 \ldots a_k} = \nu^3 \int f(\nu, \Omega, x^\mu)(u^{a_1} + l^{a_1})(u^{a_2} + l^{a_2}) \ldots\ldots (u^{a_k} + l^{a_k})\, d\Omega$$

- Comoving frame moments:

$$J_{(\nu)} = \nu^3 \int f(\nu, \Omega, x^\mu)\, d\Omega$$

$$H_{(\nu)}^{a} = \nu^3 \int l^a f(\nu, \Omega, x^\mu)\, d\Omega$$

$$L_{(\nu)}^{ab} = \nu^3 \int l^a l^b f(\nu, \Omega, x^\mu)\, d\Omega$$

$$N_{(\nu)}^{abc} = \nu^3 \int l^a l^b l^c f(\nu, \Omega, x^\mu)\, d\Omega$$

then 2^{nd} & 3^{rd} rank moments are

$$M^{ab} = J u^a u^b + H^a u^b + u^b H^a + L^{ab}$$

$$M^{ab}_{(\nu)} = J_{(\nu)} u^a u^b + H^a_{(\nu)} u^b + H^b_{(\nu)} u^a + L^{ab}_{(\nu)}$$

$$M^{abc}_{(\nu)} = J_{(\nu)} u^a u^b u^c + H^a_{(\nu)} u^b u^c + H^b_{(\nu)} u^a u^c + H^c_{(\nu)} u^a u^b + L^{ab}_{(\nu)} u^c + L^{ac}_{(\nu)} u^b + L^{ac}_{(\nu)} u^b + L^{bc}_{(\nu)} u^a + N^{abc}_{(\nu)}$$

- total radiation stress-energy tensor

$$T^{ab}_{rad} = \int_0^\infty d\nu \, M^{ab}_{(\nu)}$$

- in lab frame, moments are projections of T^{ab}_{rad}:

$$E_{(\nu)} = M^{ab}_{(\nu)} n_a n_b , \quad F^i_{(\nu)} = - M^{ab}_{(\nu)} n_a \gamma_b^{\ i}$$

$$P^{ij}_{(\nu)} = M^{ab}_{(\nu)} \gamma_a^{\ i} \gamma_b^{\ j}$$

↳ second moment:

$$M^{ab}_{(\nu)} = E_{(\nu)} n^a n^b + F^a_{(\nu)} n^b + F^b_{(\nu)} n^a + P^{ab}_{(\nu)}$$

- in 3+1, for zeroth moment $E_{(\nu)}$ & first moment $F_{(\nu)i}$ in lab frame, evolution eqns. are:

$$\partial_t \left(\sqrt{\gamma} E_{(\nu)} \right) + \partial_j \left[\sqrt{\gamma} \left(\alpha F^j_{(\nu)} - \beta^j E_{(\nu)} \right) \right] + \frac{\partial}{\partial \nu} \left(\nu \alpha \sqrt{\gamma} \, n_a M^{abc}_{(\nu)} \nabla_c u_b \right) = \alpha \sqrt{\gamma} \left[P^{ij}_{(\nu)} K_{ij} - F^j_{(\nu)} \partial_j \ln \alpha - S^a_{(\nu)} n_a \right]$$

$$\partial_t \left(\sqrt{\gamma} F_{(\nu)i} \right) + \partial_j \left[\sqrt{\gamma} \left(\alpha P^j_{(\nu)i} - \beta^j F_{(\nu)i} \right) \right]$$

$$- \frac{\partial}{\partial \nu}\left(\nu \alpha \sqrt{\gamma}\, \gamma_{ia}\, M^{abc}_{(\nu)} \nabla_c u_b\right)$$
$$= \sqrt{\gamma}\left[-E_{(\nu)} \partial_i \alpha + F_{(\nu)k} \partial_i \beta^k + \frac{\alpha}{2} P^{jk}_{(\nu)} \partial_i \gamma_{jk} + \alpha S^a_{(\nu)} \gamma_{ia}\right]$$

- hyperbolic system of PDEs!
- higher-order moments (ie, $P^{ab}_{(\nu)}$) found via closure relation
- many closures; common is analytic "M1"
- source terms on RHS determine via interactions w/ matter

Magnetohydrodynamics

- "ideal" MHD: infinite conductivity
  - good approx. for most astro. plasmas.

EM field eqns.

- Faraday, or EM field, tensor:

$$F^{ab} = n^a E^b - n^b E^a + n_d \epsilon^{dabc} B_c$$

Levi-Civita: $\epsilon_{abcd} = \sqrt{-g}\,[abcd]$

E^a + B^a: electric + magnetic fields as observed by normal observer n^a.

(purely spatial, $E^a n_a = 0 = B^a n_a$)

• fields from Faraday:

$$E^a = F^{ab} n_b \quad , \quad B^a = \frac{1}{2} \epsilon^{abcd} n_b F_{dc}$$

• EM S-E:

$$4\pi T_{em}^{ab} = F^{ac} F^{b}{}_{c} - \frac{1}{4} g^{ab} F_{cd} F^{cd}$$

$$\hookrightarrow \quad 4\pi T_{em}^{ab} = \frac{1}{2}(n^a n^b + \gamma^{ab})(E_i E^i + B_i B^i) + 2 n^{(a} \epsilon^{b)cd} E_c B_d - (E^a E^b + B^a B^b)$$

where $\epsilon^{abc} = n_d \epsilon^{dabc}$ ... spatial Levi-Civita

• in presence of a "perfect" fluid:

$$T^{ab} = \rho_0 h u^a u^b + P g^{ab} + T_{em}^{ab}$$

... total S-E

• EM contribution to 3+1 source terms:

$$\rho_{em} = n_a n_b T_{em}^{ab} = \frac{1}{4\pi} \frac{1}{2} (E_i E^i + B_i B^i)$$

$$S_i^{em} = -\gamma_{ia} n_b T_{em}^{ab} = \frac{1}{4\pi} \epsilon_{ijk} E^j B^k \quad \text{... Poynting flux}$$

$$S_{ij}^{em} = \gamma_{ia} \gamma_{jb} T_{em}^{ab} = \frac{1}{4\pi} \left[-E_i E_j - B_i B_j + \frac{1}{2} \gamma_{ij} (E_i E^i + B_i B^i) \right]$$

$$S_{em} = \gamma^{ij} S_{ij}^{em} = \frac{1}{4\pi} \frac{1}{2} (E_i E^i + B_i B^i)$$

• for a perfect conductor (ie, MHD approx), electric field vanishes in fluid rest frame,

$$E^a_{(u)} = F^{ab} u_b = 0$$

$\therefore$ F^{ab} completely determined by $B^a_{(u)}$:

$$F^{ab} = \epsilon^{abcd} u_c B_d^{(u)}$$

$$B^a_{(u)} = \frac{1}{2} \epsilon^{abcd} u_b F_{dc}$$

- Conversion between fields of normal & rest-frame observers:

$$E^a = - \epsilon^{abc} u_b B_c^{(u)}$$

$$B^a = -n_b u^b B^a_{(u)} + n_b B^b_{(u)} u^a$$

MHD evolution eqns.

- define dual to Faraday:

$$F^{*ab} = \frac{1}{2} \epsilon^{abcd} F_{cd}$$

using definition of F^{ab} from $B_d^{(u)}$

$\hookrightarrow$
$$F^{*ab} = u^b B^a_{(u)} - u^a B^b_{(u)}$$

- "magnetic" Maxwell's eqns.:

$$\nabla_a F^{*ac} = 0$$

$$\rightarrow \partial_a \left[\sqrt{-g} \left(u^c B^a_{(u)} - u^a B^c_{(u)} \right) \right] = 0$$

- define $b^a \equiv B^a_{(u)} / (4\pi)^{1/2}$ and $v^a = u^a/u^t$

$\hookrightarrow$

$$\partial_a[\gamma^{1/2} W(v^c b^a - v^a b^c)] = 0$$

- introduce magnetic field variable:

$$\mathcal{B}^i = (4\pi)^{1/2}\gamma^{1/2} W (b^i - v^i b^t)$$

→ spatial component of B^i as measured by normal observer:

$$\mathcal{B}^i = \sqrt{\gamma}\, B^i$$

- split above equation in 2:

set index $c=t$:

$$\partial_a[\gamma^{1/2} W(\cancel{v^t}^{\,1} b^a - v^a b^t)] = 0$$

$$\therefore\ \partial_a \mathcal{B}^a = 0 \quad \text{.... solenoidal constraint}$$

set $c=i$ to give induction eqn.

$$\partial_t \mathcal{B}^i - \partial_j(v^i \mathcal{B}^j - v^j \mathcal{B}^i) = 0$$