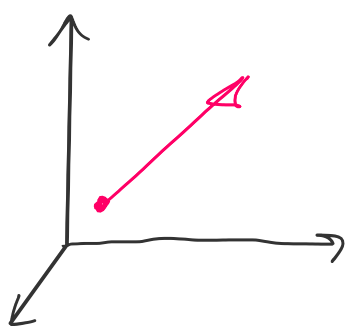
Lecture 1

Friday, January 4, 2019 12:16 PM

Newtonian Physics: Geometric Viewpoint

- Physical laws can be expressed in "geometric" form, independent of any coordinate system or basis vectors

Example: a vector



arrows in 3D Euclidean space.
No coordinates needed!

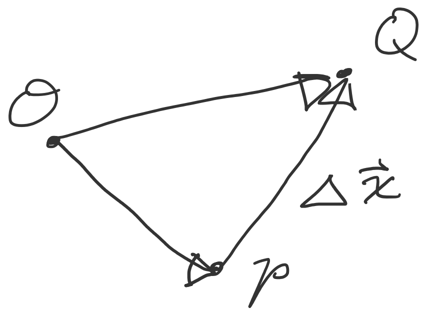
(Principle of Relativity)

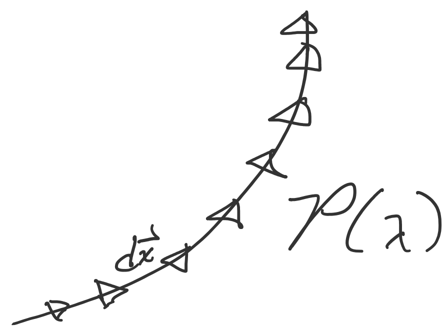
$\boxed{\text{Geometric Principle}}$: the Newtonian laws of physics are all geometric relationships between geometric objects.

- Components of geometric objects (vectors, tensors) only exist after choosing a set of basis vectors.

Foundational Concepts

- Euclidean distance $\Delta\sigma$ between two points $\mathcal{P}$ & Q can be measured w/o a coordinate system.
- $\Delta\sigma$ is the length $|\Delta\vec{x}|$ of the vector from $\mathcal{P}$ to Q





$$|\Delta\vec{x}|^2 \equiv (\Delta\vec{x})^2 \equiv (\Delta\sigma)^2$$

λ: affine parameter; $d\vec{x} = \left(\dfrac{d\mathcal{P}}{d\lambda}\right) d\lambda$

$d\vec{x}$: differential distance

- if a particle or fluid element goes through $d\vec{x}$ in some unit (universal) time dt, then

$$\vec{v} = d\vec{x}/dt \quad \text{... a vector}$$

if we do this for all $\mathcal{P}(\lambda)$:

$$\vec{v}(\mathcal{P}) \quad \text{--- a vector field}$$

- if $\vec{v}(\mathcal{P}(\lambda)) \neq \vec{v}(\mathcal{P}(\lambda + d\lambda))$

$$\hookrightarrow \quad \vec{a} = d\vec{v}/dt \quad \text{... a vector}$$

where dt is the (universal) time to

where $d\tau$ is the (infinitesimal) time to go from $\mathcal{P}(\lambda)$ to $\mathcal{P}(\lambda + d\lambda)$

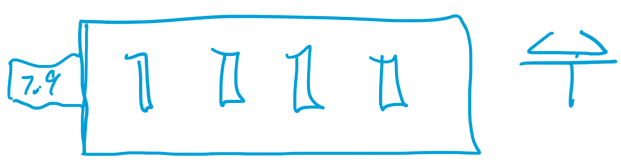
- Multiplying $\vec{a}$ by a scalar is still a vector, the force: $\vec{F} = m\vec{a}$

Tensor algebra coordinate-free

- in flat space, vector is defined entirely by length & direction. Origin does NOT matter.
  - ↳ vectors are totally unchanged by parallel transport.

Eats bunch of vectors, spits out a number

- rank-n tensor: real-valued, linear function of n vectors.



$$\underbrace{\overleftrightarrow{T}(\_,\_,\_,\_)}_{n \text{ slots for vectors}}$$

- only takes on a value when vectors are input into slots.
- e.g. value of rank-3 $\overleftrightarrow{T}$ for vectors $\vec{A}, \vec{B}, \vec{C}$ is $\overleftrightarrow{T}(\vec{A}, \vec{B}, \vec{C})$.
- "Linearity" of this function implies

$$\overleftrightarrow{T}(e\vec{E}+f\vec{F},\vec{B},\vec{C}) = e\overleftrightarrow{T}(\vec{E},\vec{B},\vec{C}) + f\overleftrightarrow{T}(\vec{F},\vec{B},\vec{C})$$

where e, f are real numbers

- inner product

$$\vec{A}\cdot\vec{B} = \frac{1}{4}\Big[\underbrace{(\vec{A}+\vec{B})}_{\text{vector}}{}^2 - \underbrace{(\vec{A}-\vec{B})^2}_{\text{scalar}}\Big]$$

↳ this is a linear function of 2 vectors which returns a real number. ∴ it's a rank-2 tensor!

→ inner product := metric tensor $\overleftrightarrow{g}(\_,\_)$

$$\overleftrightarrow{g}(\vec{A},\vec{B}) \equiv \vec{A}\cdot\vec{B}$$

- since $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$, $\overleftrightarrow{g}(\_,\_)$ is symmetric in its slots.

- using the metric, we can say any vector is a rank-1 tensor $\overleftrightarrow{A}(\_)$ such that

$$\overleftrightarrow{A}(\vec{C}) \equiv \vec{A}\cdot\vec{C}$$

- can construct a tensor from any number of vectors w/ "tensor" or "outer" product

$$\vec{A}\otimes\vec{B}\otimes\vec{C} = \overleftrightarrow{T}(\_,\_,\_)$$

$$\rightarrow \vec{A}\otimes\vec{B}\otimes\vec{C}(\vec{E},\vec{F},\vec{G}) \equiv \underbrace{\vec{A}(\vec{E})\,\vec{B}(\vec{F})\,\vec{C}(\vec{G})}_{\text{scalar}}$$

$$= (\vec{A}\cdot\vec{E})(\vec{B}\cdot\vec{F})(\vec{C}\cdot\vec{G})$$

- since vectors are just rank-1 tensors,

$$\overleftrightarrow{T}\otimes\overleftrightarrow{S}(\vec{E},\vec{F},\vec{G},\vec{H},\vec{I}) \equiv \overleftrightarrow{T}(\vec{E},\vec{F})\,\overleftrightarrow{S}(\vec{G},\vec{H},\vec{I})$$

$T \otimes S(E, F, G, H, J) = T(E, F)\, S(G, H, J)$

- Tensor "contraction": any tensor can be defined as sum of tensor products:

$$\overleftrightarrow{T} = \vec{A} \otimes \vec{B} + \vec{C} \otimes \vec{D} + \ldots$$

$\rightarrow$ contraction $(\overleftrightarrow{T}) = \vec{A} \cdot \vec{B} + \vec{C} \cdot \vec{D} + \ldots$

$\therefore$ we construct inner product from outer

& contraction lowers rank by 2

for rank-3, (must specify slots to contract)

$$1\,\&\,3 \text{ contraction}\left(\vec{A} \otimes \vec{B} \otimes \vec{C} + \vec{E} \otimes \vec{F} \otimes \vec{G} + \ldots\right)$$

$$\equiv (\vec{A} \cdot \vec{C})\,\vec{B} + (\vec{E} \cdot \vec{G})\,\vec{F} + \ldots$$

.... a vector

Note! We have not specified any basis in the above. $\therefore$ this all carries over to any vector space over real numbers. (even 4D spacetime)

Particle Kinetics and Lorentz Force

In geometry-free language,

particle trajectory $\vec{x}(t)$

velocity $\vec{v}(t)$

momentum $\vec{p}(t)$

acceleration $\vec{a}(t)$

acceleration $\vec{a}(t)$

$$\vec{v}(t) = \frac{d\vec{x}}{dt} ; \quad \vec{p}(t) = m\vec{v}(t)$$

$$\vec{a}(t) \quad \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt}$$

$$E(t) = \frac{1}{2} m \vec{v}^2$$

- since points in Euclidean 3-space are geometric objects, so are all of the above.
- Newton's Second Law

$$\frac{d\vec{p}}{dt} = m\vec{a} = \vec{F}$$

for Lorentz force,

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

→ all are geometric relations between geometric objects

Exercise 1.1 – Without coordinates, show energy change of particle interacting w/ $\vec{E}$ & $\vec{B}$ is

$$\frac{dE}{dt} = q\vec{v} \cdot \vec{E} .$$

the energy $E = \frac{1}{2} m \vec{v}^2$

$$\hookrightarrow \quad \frac{dE}{dt} = \frac{1}{2} m \frac{d}{dt}\left(\vec{v}^{\,2}\right) = m\,\vec{v}\cdot\frac{d\vec{v}}{dt}$$

now $$m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$\therefore \frac{dE}{dt} = \vec{v}\cdot q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$= q\,\vec{v}\cdot\vec{E} + q\,\vec{v}\cdot\vec{v}\times\vec{B}$$

$$= q\,\vec{v}\cdot\vec{E} + q\,\vec{B}\cdot\cancelto{0}{\vec{v}\times\vec{v}}$$

$$= q\,\vec{v}\cdot\vec{E}$$ q.e.d.