AST 900 Homework 2

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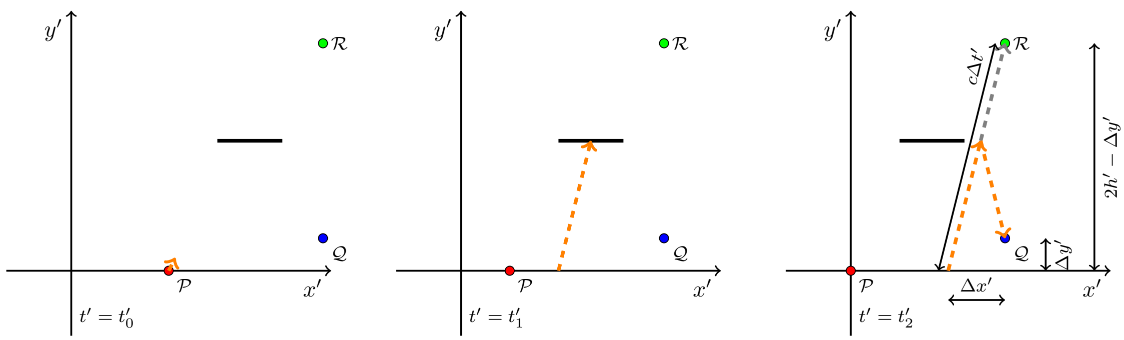
1. TB Exercise 2.2 *Invariance of the Interval*

Complete the derivation of the invariance of the interval given in **TB Box 2.4***, using the Principle of Relativity in the form that the laws of physics must be the same in the primed and unprimed frames.*

Solution



The image above shows the primed frame moving to the right at a speed β relative to the unprimed frame, such that the mirror (thick black horizontal line) and points $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ move to the left at a speed β. In the primed frame, the emitted photon travels in a straight line at speed c, and the emitting point $\mathcal{P}$ moves left after emitting the photon. The photon reflects off the mirror, and adhering to the principle of relativity, that the physics observed must be the same in every frame, the photon reflects at an angle of reflection equal to its angle of incidence. In the primed frame, this angle appears smaller than in the unprimed frame.

After this the photon arrives at point $\mathcal{Q}$ (or $\mathcal{R}$ if the mirror were not present) after a time $c\Delta t' = t'_2 - t'_0$. In comparison to the unprimed frame, the distance x' distance between when the photon is emitted and received is $\Delta x' < \Delta x$.

If the y-distances were not the same in both frames, then the mirror would also be at a lower or higher height, causing the photon to hit it sooner or later, respectively. For both frames to agree on the photon reflecting back to point $\mathcal{Q}$, this would require angle of reflection to be smaller than the angle of incidence if $h' < h$, and a larger angle of reflection if $h' > h$.

This would violate the principle of relativity, so the vertical distances must be equal, both $h' = h$ and $\Delta y' = \Delta y$. In terms of the intervals in the unprimed frame:

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 = -(2h - \Delta y)^2 + (\Delta y)^2$$

and the primed frame:

$$(\Delta s')^2 = -(\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 = -(2h' - \Delta y')^2 + (\Delta y')^2$$

where as shown in the diagram, by the Pythagorean theorem, $(\Delta x')^2 + (2h' - \Delta y')^2 = (\Delta t')^2$ ($(\Delta x)^2 + (2h - \Delta y)^2 = (\Delta t)^2$ in the unprimed frame). Since $2h' - \Delta y' = 2h - \Delta y$ though, this means that:

$$\begin{aligned} -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 &= -(\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 \\ \rightarrow \quad & \boxed{(\Delta s')^2 = (\Delta s)^2} \end{aligned}$$

2. TB Exercise 2.5 *Component Manipulation Rules*

Derive the relativistic component manipulation rules:

$$[\text{Contravariant components of } \boldsymbol{T}(\_,\_,\_) \otimes \boldsymbol{S}(\_,\_)] = T^{\alpha\beta\gamma} S^{\delta\epsilon} \tag{1a}$$

$$\vec{A} \cdot \vec{B} = A^\alpha B_\alpha = A_\alpha B^\alpha, \quad \boldsymbol{T}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = T_{\alpha\beta\gamma} A^\alpha B^\beta C^\gamma = T^{\alpha\beta\gamma} A_\alpha B_\beta C_\gamma \tag{1b}$$

$$\text{Covariant components of } [\text{1\&3contraction of } \boldsymbol{R}] = R^\mu{}_{\alpha\mu\beta} \tag{1c}$$

$$\text{Contravariant components of } [\text{1\&3contraction of } \boldsymbol{R}] = R^{\mu\alpha}{}_\mu{}^\beta \tag{1d}$$

Solution

For the contravariant components of $\boldsymbol{T}(\_,\_,\_) \otimes \boldsymbol{S}(\_,\_)$, first obtain the covariant components by inserting the proper basis vectors:

$$\boldsymbol{T}(\vec{e}_\alpha, \vec{e}_\beta, \vec{e}_\gamma) \otimes \boldsymbol{S}(\vec{e}_\delta, \vec{e}_\epsilon) = T_{\alpha\beta\gamma} S_{\delta\epsilon}$$

In a simlar fashing, the contravariant components are then just obtained by applying the inverse of the metric to each applied basis vector:

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{g}(\_,\vec{e}_\mu), \boldsymbol{g}(\_,\vec{e}_\nu), \boldsymbol{g}(\_,\vec{e}_\sigma)) \otimes \boldsymbol{S}(\boldsymbol{g}(\_,\vec{e}_\lambda), \boldsymbol{g}(\_,\vec{e}_\varphi)) &= T_{\alpha\beta\gamma} S_{\delta\epsilon} g^{\mu\alpha} g^{\nu\beta} g^{\sigma\gamma} g^{\lambda\delta} g^{\varphi\epsilon} \\ &= T^{\alpha\beta\gamma} S^{\delta\epsilon} \end{aligned}$$

The scalar product of two 4-vectors is obtained by use of the metric:

$$\vec{A} \cdot \vec{B} = \boldsymbol{g}\left(\vec{A}, \vec{B}\right) = g_{\alpha\beta} A^\alpha B^\beta = A^\alpha B_\alpha$$

Contracting on β instead and renaming indices afterwards:

$$\vec{A}\cdot\vec{B} = \boldsymbol{g}\left(\vec{A},\vec{B}\right) = g_{\alpha\beta}A^{\alpha}B^{\beta} = A_{\beta}B^{\beta} = A_{\alpha}B^{\alpha}$$

Likewise for the scalar obtained from inserting three vectors into $\boldsymbol{T}$:

$$\begin{aligned}\boldsymbol{T}(\mathbf{A},\mathbf{B},\mathbf{C}) &= T^{\alpha\beta\gamma}\vec{e}_{\alpha}A^{\mu}\vec{e}_{\mu}\vec{e}_{\beta}B^{\nu}\vec{e}_{\nu}\vec{e}_{\gamma}A^{\sigma}\vec{e}_{\sigma}\\ &= T^{\alpha\beta\gamma}g_{\alpha\mu}g_{\beta\nu}g_{\gamma\sigma}A^{\mu}B^{\nu}C^{\sigma}\\ &= T^{\alpha\beta\gamma}A_{\alpha}B_{\beta}C_{\gamma}\end{aligned}$$

or contracting on $\alpha\beta\gamma$ instead and renaming the indices after then gives:

$$\begin{aligned}\boldsymbol{T}(\mathbf{A},\mathbf{B},\mathbf{C}) &= T^{\alpha\beta\gamma}g_{\alpha\mu}g_{\beta\nu}g_{\gamma\sigma}A^{\mu}B^{\nu}C^{\sigma}\\ &= T_{\mu\nu\sigma}A^{\mu}B^{\nu}C^{\sigma}\\ &= T_{\alpha\beta\gamma}A^{\alpha}B^{\beta}C^{\gamma}\end{aligned}$$

The covariant components of a 1+3 contraction can be obtained from inserting the same basis vector into the 1 and 3 slots of a tensor:

$$\boldsymbol{R}(\vec{e}_{\mu},\vec{e}_{\alpha},\vec{e}_{\mu},\vec{e}_{\beta}) = g^{\mu\sigma}R_{\sigma\alpha\mu\beta} = R^{\mu}{}_{\alpha\mu\beta}$$

The contravariant components of this same contraction can then easily be obtained by repeated application of the metric on each free index:

$$g^{\delta\alpha}g^{\epsilon\beta}R^{\mu}{}_{\alpha\mu\beta} = R^{\mu\delta}{}_{\mu}{}^{\epsilon} \quad\rightarrow\quad R^{\mu\alpha}{}_{\mu}{}^{\beta}$$

3. TB Exercise 2.8 *Index Gymnastics*

(a) *Simplify the following expression so the metric does not appear in it:*

$$A^{\alpha\beta\gamma}g_{\beta\rho}S_{\gamma\lambda}g^{\rho\delta}g^{\lambda}{}_{\alpha}$$

Solution

Applying one metric at a time:

$$\begin{aligned}A^{\alpha\beta\gamma}g_{\beta\rho}S_{\gamma\lambda}g^{\rho\delta}g^{\lambda}{}_{\alpha} &= A^{\alpha}{}_{\rho}{}^{\gamma}S_{\gamma\lambda}g^{\rho\delta}g^{\lambda}{}_{\alpha}\\ &= A^{\alpha\delta\gamma}S_{\gamma\lambda}g^{\lambda}{}_{\alpha}\\ &= A^{\alpha\delta\gamma}S_{\gamma\alpha}\end{aligned}$$

(b) *The quantity $g_{\alpha\beta}g^{\alpha\beta}$ is a scalar since it has no free indices. What is its numerical value?*

Solution

$$g_{\alpha\beta}g^{\alpha\beta} = g_{\beta}{}^{\beta} = \delta_{\beta\beta} = 1$$

(c) *What is wrong with the following expression and equation?*

$$A_{\alpha}{}^{\beta\gamma}S_{\alpha\gamma}; \quad A_{\alpha}{}^{\beta\gamma}S_{\beta}T_{\gamma} = R_{\alpha\beta\delta}S^{\beta}$$

Solution

For $A_{\alpha}{}^{\beta\gamma}S_{\alpha\gamma}$, there is a repeated dummy index, α that is a covariant component in both appearances. Valid summation over indices requires a summed pair to have one contravariant and one covariant index.

For $A_{\alpha}{}^{\beta\gamma}S_{\beta}T_{\gamma} = R_{\alpha\beta\delta}S^{\beta}$, there is a summation over β and γ on the left, and over β on the right, and the free index α appears as a covariant index on each side. However, the free index δ on the right-hand side does not appear at all on the left; it would need to appear as a covariant index on the left in this case to make this a valid equation.

4. TB Exercise 2.11 *Doppler Shift Derived without Lorentz Transformation*

(a) *An observer at rest in some inertial frame receives a photon that was emitted in direction* $\mathbf{n}$ *by an atom moving with ordinary velocity* $\mathbf{v}$*. The photon frequency and energy as measured by the emitting atom are* ν_{em} *and* $\mathcal{E}_{em}$*; those measured by the receiving observer are* ν_{rec} *and* $\mathcal{E}_{rec}$*. By a calculation carried out solely in the receiver's inertial frame, and without the aid of any Lorentz transformation, derive the standard formula for the photon's Doppler shift:*

$$\frac{\nu_{\text{rev}}}{\nu_{\text{em}}} = \frac{\sqrt{1-v^2}}{1-\mathbf{v}\cdot\mathbf{n}} \tag{2}$$

Solution

In the receivers frame, the emitting atom's 4-velocity components are:

$$U^0 = \gamma \quad , \quad U^i = \gamma v^i$$

and the photon's 4-momentum components in the receiver's frame are:

$$p^0 = \mathcal{E}_{\text{rec}} \quad , \quad p^i = \mathcal{E}_{\text{rec}} n^i$$

where v^i and n^i are the components of the atom's ordinary velocity **b** and th emitted photon's direction **n**.

The emitted photon's energy in the emitter frame can then be calculated as:

$$\begin{aligned}\mathcal{E}_{\text{em}} &= -\vec{p}\cdot\vec{U}\\ &= -\left(-\gamma\mathcal{E}_{\text{rec}} + \gamma\mathcal{E}_{\text{rec}} n_i v^i\right)\\ &= \gamma\mathcal{E}_{\text{rec}}(1-\mathbf{v}\cdot\mathbf{n})\end{aligned}$$

Solving for the ratio of the energies then gives:

$$\frac{\mathcal{E}_{\text{rec}}}{\mathcal{E}_{\text{em}}} = \frac{1}{\gamma(1-\mathbf{v}\cdot\mathbf{n})}$$

Substituting in $\gamma = (1-v^2)^{-1}$ and the relationship between photon energy and freqeuncy $\mathcal{E} = 2\pi\hbar\nu$ then gives the standard formula for the photon's doppler shift:

$$\boxed{\frac{\nu_{\text{rec}}}{\nu_{\text{em}}} = \frac{\sqrt{1-v^2}}{1-\mathbf{v}\cdot\mathbf{n}}}$$

(b) *Suppose that instead of emitting a photon, the emitter produces a particle with finite rest mass m. Using the same method as in part (a), derive an expression for the ratio of the received energy to emitted energy $\mathcal{E}_{rec}/\mathcal{E}_{em}$, expressed in terms of the emitter's ordinary velocity* **v** *and the particles ordinary velocity* **V** *(both measured in the receiver's frame).*

Solution

If a particle of mass m instead of a photon, with an ordinary velocity **V**, the ratio of the receiver and emitter frame energies can be obtained in a similar fashion. In this case, the emitters 4-velocity is still the same, but the emitted particle now has a 4-momentum with components (in the receiver frame):

$$p^0 = \mathcal{E}_{\text{rec}} \quad , \quad p^i = \gamma m V^i$$

Calculating the energy in the emitter frame then gives:

$$\begin{aligned}\mathcal{E}_{\text{em}} &= -\vec{p}\cdot\vec{U}\\ &= -\left(-\gamma\mathcal{E}_{\text{rec}} + \gamma^2 m V_i v^i\right)\\ &= -\left(-\gamma\mathcal{E}_{\text{rec}} + \gamma\mathcal{E}_{\text{rec}} V_i v^i\right)\\ &= \gamma\mathcal{E}_{\text{rec}}(1-\mathbf{v}\cdot\mathbf{V})\end{aligned}$$

where the relationship $\mathcal{E}_{\text{rec}} = \gamma m$ was used. Solving for the ratio of the energies then yields (and substituting in $\gamma = (1-v^2)^{-1}$):

$$\boxed{\frac{\mathcal{E}_{\text{rec}}}{\mathcal{E}_{\text{em}}} = \frac{\sqrt{1-v^2}}{1-\mathbf{v}\cdot\mathbf{V}}}$$

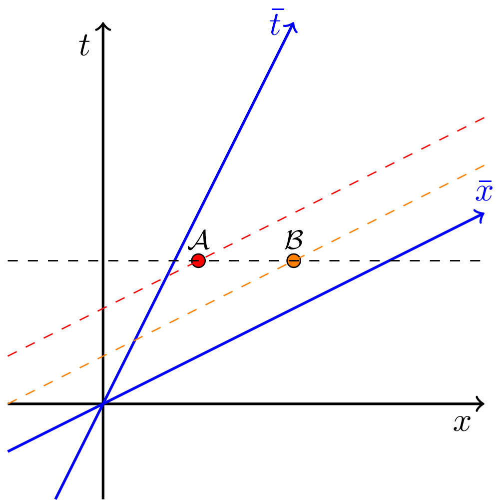
For the case of $\mathbf{V} = \mathbf{V}$, this reduces to $\mathcal{E}_{\text{rec}} = \gamma\mathcal{E}_{\text{em}}$.

5. TB Exercise 2.14 *Spacetime Diagrams*

Use spacetime diagrams to prove the following:

(a) *Two events that are simultaneous in one interial frame are not necessarily simultaneous in another.*

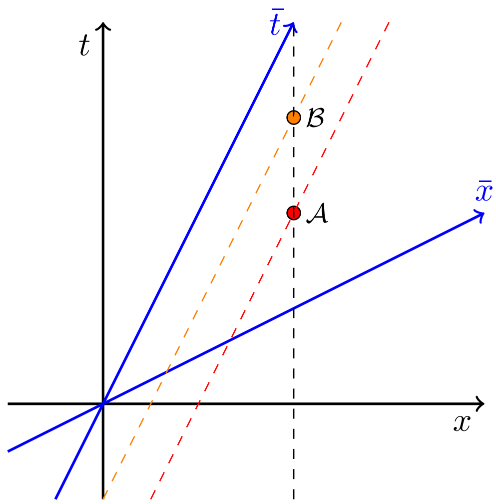
Solution



Here the events $\mathcal{A}$ and $\mathcal{B}$ occur at the same time t in the unbarred frame (depicted by the dashed black line) but in the barred frame, $\mathcal{B}$ before $\mathcal{A}$, as seen by the orange and red dashed lines intersecting the $\bar{t}$ axis.

(b) *Two events that occur at the same spatial location in one interial frame do not necessarily occur at the same spatial location in another.*

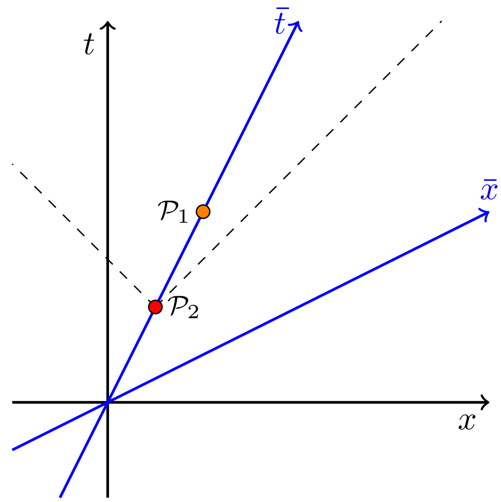
Solution



Here the events $\mathcal{A}$ and $\mathcal{B}$ occur at the same location x in the unbarred frame (depicted by the dashed black line) but in the barred frame, $\mathcal{A}$ and $\mathcal{B}$ occur at different locations as seen by the orange and red dashed lines intersecting the $\bar{x}$ axis.

(c) *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two events with a timelike separation, then there exists an inertial reference frame in which they occur at the same spatial location, and in that frame the time lapse between them is equal to $\Delta t = \Delta\tau = \sqrt{-(\Delta s)^2}$.*

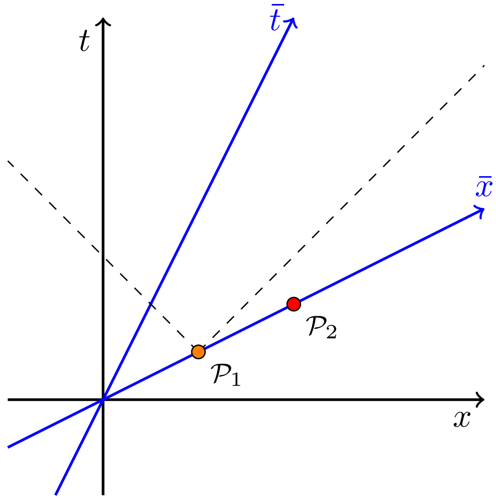
Solution



In the unbarred frame, $\mathcal{P}_2$ is in the lightcone (dashed black line) of $\mathcal{P}_1$, so these events are timelike separated. In the barred frame, both events occur at the same location, so the time measured between them is $\Delta\tau$, but this is just the interval $\Delta\bar{s}$ for $\Delta\bar{x} = 0$.

(d) *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two events with a spacelike separation, then there exists an inertial reference frame in which they are simultaneous, and in that frame the spatial distance between them is equal to $\sqrt{g_{ij}\Delta x^i \Delta x^j} = \Delta s = \sqrt{(\Delta s)^2}$.*

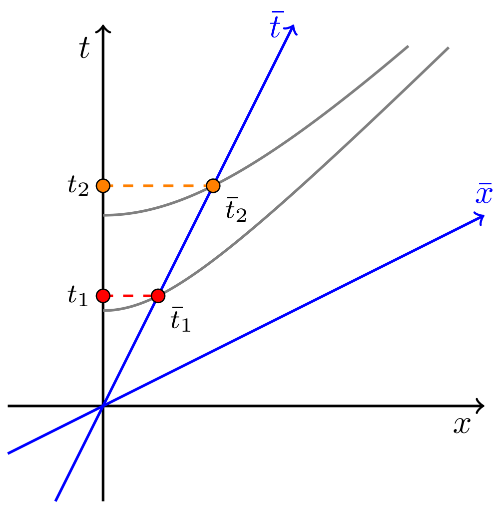
Solution



In the unbarred frame, $\mathcal{P}_2$ is outside of the lightcone (dashed black line) of $\mathcal{P}_1$, so these events are spacelike separated. In the uarred frame, both events occur at the same time, so the distance measured between them is $\Delta\bar{x} = \sqrt{(\Delta\bar{x})^2} = \sqrt{g_{ij}\Delta x^i \Delta x^j}$, but this is just the interval $\Delta\bar{s}$ for $\Delta\bar{t} = 0$.

(e) *If the inertial frame $\bar{\mathcal{F}}$ moves with speed β relative the the frame $\mathcal{F}$, then a clock at rest in $\bar{\mathcal{F}}$ ticks more slowly as view from $\mathcal{F}$ then as viewed from $\bar{\mathcal{F}}$ by a factor of γ.*

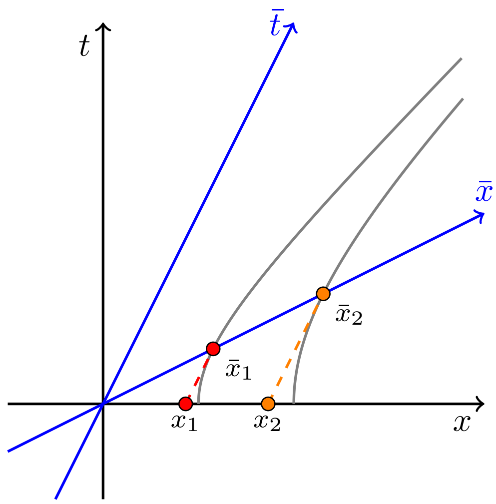
Solution



Here the hyperbola describing the invariant intervals $\Delta s^2 = -\Delta t^2 = -\Delta\bar{t}^2 = -\Delta\tau^2$ are shown to calibrate the unbarred and barred axes. The points at which each hyperbola intersect each axis correspond to equal time coordinates in the respective frames. As seen by the interval $\Delta\tau = \bar{t}_2 = \bar{t}_1$, this unit of time difference when projected onto the t axis in the unbarred frame corresponds to a longer time interval Δt, so the unbarred clock runs "faster" than the barred frame clock.

(f) *If the inertial frame $\bar{\mathcal{F}}$ moves with the velocity $\vec{v} = \beta \vec{e}_x$ relative to the frame $\mathcal{F}$, then an object at rest in $\bar{\mathcal{F}}$ as studied in $\mathcal{F}$ appears shortened by a factor of γ along the x-direction.*

Solution



Here the hyperbola describing the invariant intervals $\Delta s^2 = \Delta x^2 = \Delta \bar{x}^2$ are shown to calibrate the unbarred and barred axes. The points at which each hyperbola intersect each axis correspond to equal spatial coordinates in the respective frames. In the unbarred frame, the length is measured at two locations at the same time, which occur where the worldlines of $\bar{x}_1$ and $\bar{x}_2$ intersect the x-axis. As seen in the diagram, the locations these intersect at x_1 and x_2 form a shorter interval: $x_2 - x_1 < \bar{x}_2 - \bar{x}_1$, so the overall length of the moving object appears shorter in the unbarred frame.

6. TB Exercise 2.20 *Frame-Dependent Version of Maxwell's Equations*

By performing a 3+1 split on the geometric version of Maxwell's equations:

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha \tag{3a}$$

$$\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta;\beta} = 0 \quad \text{i.e.} \quad F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \tag{3b}$$

derive the elementary, frame-dependent version:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi \rho_e \tag{4a}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{j} \tag{4b}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{4c}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{4d}$$

Solution

In an arbitrary inertial reference frame, the components of the antisymmetric electromagnetic field tensor $\boldsymbol{F}$ are:

$$F^{0j} = -F^{j0} = +F_{j0} = -F_{0j} = E_j \quad , \quad F^{ij} = F_{ij} = \epsilon_{ijk} B_k$$

and the components of the charge-current 4-vector $\vec{J}$ are:

$$J^0 = \rho_e \quad , \quad J^i = J_i = j^i$$

where j^i are the components of the current density vectory $\mathbf{j}$.

Taking the $\alpha = 0$ component of the first equation in Eq. 3 gives:

$$\begin{aligned} F^{0\beta}{}_{;\beta} &= 4\pi J^0 \\ \rightarrow \quad \frac{\partial E^j}{\partial x^j} &= 4\pi \rho_e \\ \rightarrow \quad &\boxed{\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi \rho_e} \end{aligned}$$

Taking the spatial components $\alpha = i$ of the first equation in Eq. 3 gives:

$$\begin{aligned} F^{i\beta}{}_{;\beta} &= 4\pi J^i \\ \rightarrow \quad F^{i0}{}_{;0} + F^{ij}{}_{;j} &= 4\pi j^i \\ \rightarrow \quad -\frac{\partial E^j}{\partial t} + \epsilon_{ijk} \frac{\partial B_k}{\partial x^j} &= 4\pi j^i \\ \rightarrow \quad &\boxed{\boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{j}} \end{aligned}$$

Taking the spatial components $\alpha = i$, $\beta = j$, and $\gamma = k$ of the second equation in Eq. 3 gives:

$$\begin{aligned} 0 &= F_{ij;k} + F_{jk;i} + F_{ki;j} \\ &= \epsilon_{ijk} \frac{\partial B^k}{\partial x^k} + \epsilon_{jki} \frac{\partial B^i}{\partial x^i} + \epsilon_{kij} \frac{\partial B^j}{\partial x^j} \\ &= 3\epsilon_{ijk} \frac{\partial B^k}{\partial x^k} \\ \rightarrow \quad &\boxed{0 = \boldsymbol{\nabla} \cdot \mathbf{B}} \end{aligned}$$

Taking $\alpha = 0$ the spatial components $\beta = i$, $\gamma = j$ of the second equation in Eq. 3 gives:

$$0 = F_{0i;j} + F_{ij;0} + F_{kj0;i}$$

$$
\begin{aligned}
&= -\frac{\partial E^i}{\partial x^j} + \epsilon_{ijk}\frac{\partial B^k}{\partial t} + \frac{\partial E^j}{\partial x^i} \\
&= \epsilon_{ijk}\frac{\partial E^j}{\partial x^i} + \epsilon_{ijk}\frac{\partial B^k}{\partial t} \\
\rightarrow \quad & \boxed{0 = \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}}
\end{aligned}
$$

7. TB Exercise 2.26 *Stress-Energy Tensor and Energy-Momentum Conservation for a Perfect Fluid*

(a) *Derive the frame-independent expression of:*

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + Pg^{\alpha\beta} \quad \text{i.e.} \quad \boldsymbol{T} = (\rho + P)\vec{u} \otimes \vec{u} + P\boldsymbol{g}$$

for a perfect fluid stress-energy tensor from its rest-frame components:

$$T^{00} = \rho \quad , \quad T^{jk} = P\delta^{jk}$$

Solution

Since any tensor can be created from the linear combination of other tensors, the perfect-fluid stree energy tensory depends on the 4-velocity of the fluid $\vec{u}$ and the metric $\boldsymbol{g}$. The most general linear combination of these is then:

$$T^{\alpha\beta} = au^\alpha u^\beta + bg^{\alpha\beta}$$

where a and b are constants to be determined. In the fluid's rest frame, its 4-velocity has components $u^0 = 1$ and $u^i = 0$. Using this along with the values of T^{00} and T^{jk}, a system of equations can be derived to obtain a and b from:

$$
\begin{aligned}
T^{00} &= au^0u^0 + bg^{00} \\
\rightarrow \quad \rho &= a - b
\end{aligned}
$$

and

$$
\begin{aligned}
T^{jk} &= au^ju^k + bg^{jk} \\
\rightarrow \quad P\delta^{jk} &= b\delta^{jk} \\
\rightarrow \quad P &= b
\end{aligned}
$$

giving $a = \rho + P$ and $b = P$, which give the frame independent form of the stress energy tensor as:

$$\boxed{T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + Pg^{\alpha\beta}}$$

(b) *Explain why the projection of $\vec{\nabla} \cdot \boldsymbol{T} = 0$ along the 4-velocity, $\vec{u} \cdot \left(\vec{\nabla} \cdot \boldsymbol{T}\right) = 0$, should represent energy conservation as viewed by the fluid itself. Show that this equation reduces to:*

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = -(\rho + P)\vec{\nabla} \cdot \vec{u}$$

with the aid of:

$$\vec{\nabla} \cdot \vec{u} = \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau}$$

bring this into the form:

$$\frac{\mathrm{d}(\rho V)}{\mathrm{d}\tau} = -P\frac{\mathrm{d}V}{\mathrm{d}\tau}$$

What are the physical interpretations of the left- and right-hand sides of this equation, and how is it related to the first law of thermodynamics?

Solution

Projecting $\vec{\nabla} \cdot \boldsymbol{T} = 0$ along the 4-velocity, $\vec{u} \cdot \left(\vec{\nabla} \cdot \boldsymbol{T}\right) = 0$, picks out the $\alpha = 0$ piece of the divergence of the stress-energy tensor since the only non-zero component of $\vec{u}$ in the fluid's rest frame is $u^0 = 1$. These components are T^{00} = energy density and T^{0j} = energy flux, so the divergence of these gives that the time rate of change of the energy density must equal the flux of energy out the system (assuming positve rate of change for energy density).

Calculating this:

$$\begin{aligned}
0 &= \vec{u} \cdot \left(\vec{\nabla} \cdot \boldsymbol{T}\right) \\
&= g_{\alpha\mu} u^\mu T^{\alpha\beta}{}_{;\beta} \\
&= u_\alpha T^{\alpha\beta}{}_{;\beta} \\
&= u_\alpha \left[(\rho + P) u^\alpha u^\beta\right]_{;\beta} + u_\alpha \left[P g^{\alpha\beta}\right]_{;\beta} \\
&= u_\alpha \Big[(\rho_{;\beta} + P_{;\beta}) u^\alpha u^\beta + (\rho + P)\Big(u^\alpha{}_{;\beta} u^\beta + u^\alpha u^\beta{}_{;\beta}\Big)\Big] + u_\alpha \Big[P_{;\beta} g^{\alpha\beta} + P g^{\alpha\beta}{}_{;\beta}\Big] \\
&= -\rho_{;\beta} u^\beta + (\rho + P)\big(u_\alpha u^\alpha{}_{;\beta} u^\beta - u^\sigma{}_{;\sigma}\big)
\end{aligned}$$

where β is relabeled as σ in the contraction $u^\beta{}_{;\beta}$. Taking this in the rest frame of the fluid where the only non-zero component of the stress-energy tensor is $u^0 = 1$, so $\alpha = 0$ and $\beta = 0$:

$$\begin{aligned}
0 &= -\rho_{;0} u^0 + (\rho + P)\big(u_0 u^0{}_{;0} u^0 - u^\sigma{}_{;\sigma}\big) \\
&= -\rho_{;0} - (\rho + P) u^\sigma{}_{;\sigma}
\end{aligned}$$

Since this is in the fluid's rest frame, the time component is τ, so re-writting this in vector notation gives:

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = -(\rho + P)\vec{\nabla} \cdot \vec{u}}$$

This can further be simplified:

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}\tau} &= -(\rho + P)\left(\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau}\right) \\
\frac{\mathrm{d}\rho}{\mathrm{d}\tau} + \rho\left(\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau}\right) &= -P\left(\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau}\right) \\
V\frac{\mathrm{d}\rho}{\mathrm{d}\tau} + \rho\frac{\mathrm{d}V}{\mathrm{d}\tau} &= -P\frac{\mathrm{d}V}{\mathrm{d}\tau} \\
\rightarrow \quad \boxed{\frac{\mathrm{d}(\rho V)}{\mathrm{d}\tau} = -P\frac{\mathrm{d}V}{\mathrm{d}\tau}}
\end{aligned}$$

The left-hand side of this equation is the time rate-of-change of the total energy, and the right-hand side of this equation is the time rate-of-change of the work that the fluid is performing on its surrounding volume. This is directly related to the first law of thermodynamics in that the total energy must be conserved, and in this case the increase in the system's energy must equal the amount of energy it expends, per unit time.

(c) *Explain why* $P_{\mu\alpha}T^{\alpha\beta}{}_{;\beta} = 0$ *should represent the law of momentum conservation as seen by the fluid. Show that this equation reduces to:*

$$(\rho + P)\vec{a} = -\boldsymbol{P} \cdot \vec{\nabla}P$$

where $\boldsymbol{P}$ is the projection tensor:

$$\boldsymbol{P} = \boldsymbol{g} + \vec{u} \otimes \vec{u}$$

Explain the physical meanings of the left- and right-hand sides.

Solution

The projection of $T^{\alpha\beta}{}_{;\beta} = 0$ orthogonal to the fluid's rest frame 4-velocity, i.e. into the fluid's free space, picks out the $\alpha = i$ spatial components, which gives the $T^{i0} =$ momentum density and $T^{ij} =$ stress components. The divergence of these are the time rate of change of the momentum density (which then becomes the force density), and the gradient of the pressure, which is also a force density. The change in momentum density is caused by external forces, and the gradient of the pressure is the force density of the fluid on its surrounding volume. To conserve momentum, these are equal in magnitude but opposite in sign.

Calculating this:

$$
\begin{aligned}
0 &= P_{\mu\alpha} T^{\alpha\beta}{}_{;\beta} \\
&= g_{\mu\alpha} T^{\alpha\beta}{}_{;\beta} + u_\mu u_\alpha T^{\alpha\beta}{}_{;\beta} \\
&= g_{\mu\alpha} \big[(\rho + P) u^\alpha u^\beta + P g^{\alpha\beta}\big]_{;\beta} + u_\mu u_\alpha \big[(\rho + P) u^\alpha u^\beta + P g^{\alpha\beta}\big]_{;\beta} \\
&= g_{\mu\alpha} \Big[(\rho_{;\beta} + P_{;\beta}) u^\alpha u^\beta + (\rho + P) u^\alpha{}_{;\beta} u^\beta + (\rho + P) u^\alpha u^\beta{}_{;\beta}\Big] + g_{\mu\alpha} \Big[P_{;\beta} g^{\alpha\beta} + P g^{\alpha\beta}{}_{;\beta}\Big] \\
&\quad + u_\mu u_\alpha \Big[(\rho_{;\beta} + P_{;\beta}) u^\alpha u^\beta + (\rho + P) u^\alpha{}_{;\beta} u^\beta + (\rho + P) u^\alpha u^\beta{}_{;\beta}\Big] + u_\mu u_\alpha \Big[P_{;\beta} g^{\alpha\beta} + P g^{\alpha\beta}{}_{;\beta}\Big] \\
&= (\rho_{;\beta} + P_{;\beta}) u_\mu u^\beta + (\rho + P) a_\mu + (\rho + P) u_\mu u^\beta{}_{;\beta} + g_\mu{}^\beta P_{;\beta} + P g_\mu{}^\beta{}_{;\beta} \\
&\quad - (\rho_{;\beta} + P_{;\beta}) u_\mu u^\beta + (\rho + P) u_\mu u_\alpha a^\alpha u^\beta - (\rho + P) u_\mu u^\beta{}_{;\beta} + u_\mu u^\beta P_{;\beta} + u_\mu u_\alpha P g^{\alpha\beta}{}_{;\beta} \\
&= (\rho + P) a_\mu + g_\mu{}^\beta P_{;\beta} + u_\mu u^\beta P_{;\beta}
\end{aligned}
$$

where u^0 is the only non-zero component of $\vec{u}$ and is used to calculate $a^\alpha = \mathrm{d}u^\alpha/\mathrm{d}\tau$. Additionally, $\vec{u}$ and $\vec{a}$ are orthogonal so $u_\alpha a^\alpha = 0$. Multiplying both sides of the equation by the metric $g^{\mu\alpha}$ then gives:

$$
\begin{aligned}
0 &= g^{\mu\alpha} (\rho + P) a_\mu + g^{\mu\alpha} g_\mu{}^\beta P_{;\beta} + g^{\mu\alpha} u_\mu u^\beta P_{;\beta} \\
&= (\rho + P) a^\alpha + g^{\alpha\beta} P_{;\beta} + u^\alpha u^\beta P_{;\beta}
\end{aligned}
$$

which is equivalent to:

$$
\boxed{(\rho + P)\vec{a} = -\boldsymbol{P} \cdot \vec{\nabla} \vec{P}}
$$

In the fluid's rest frame, the total energy is the total rest mass, so the left hand side of this equation says that the total inertial mass per unit volume $\rho + P$ times the 4-acceleration $\vec{a}$ is a force per unit volume, and the right-hand side is the spatial component of the pressure gradient. If both sides are integrated over the same volume, this equation then says that the force exerted on the fluid is equal to the work done by the fluid.