Lecture23

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$\boxed{\text{BSSN Evolution Equations}}$

- ADM 3+1 equations are not in a form suitable for longterm 3D evolution!

  → ADM is only weakly hyperbolic,

  ∴ evolution cannot be expected stable, necessarily

- Many approaches for solving this issue ..
- Most general, & generally successful, is BSSN approach of Shibata & Nakamura (1995) / Baumgarte & Shapiro (1998)
- evolve a conformal factor & K separately
- define conformal factor as:

$$\psi = e^{\phi}$$

so the conformally-related 3-metric is

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$$

- require determinant of $\tilde{\gamma}_{ij}$ to be η_{ij}

$\hookrightarrow$ $$\phi = \frac{1}{12}\ln\left(\frac{\gamma}{\eta}\right)$$

or $$e^{4\phi} = \gamma^{1/3} \equiv \det(\gamma_{ij})^{1/3}$$

- now choose Cartesian $(\tilde{\gamma} = \eta = 1)$
- split extrinsic curvature into its trace and a traceless part:

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K$$

where $$K = \gamma^{ij}K_{ij}$$

and conformally rescale traceless part:

$$\tilde{A}_{ij} = e^{-4\phi}A_{ij}$$

- indices of $\tilde{A}_{ij}$ raised & lowered with conformal metric so that

$$\tilde{A}^{ij} = \tilde{\gamma}^{ik}\tilde{\gamma}^{jl}\tilde{A}_{kl} = e^{4\phi}A^{ij}$$

- recall trace of γ_{ij} evolution eqn.,

$$\partial_t \ln \gamma^{1/2} = -\alpha K + D_i \beta^i$$

and extrinsic curvature,

$$\partial_t K = -D^2\alpha + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho + S)\right) + \beta^i D_i K$$

- recasting the trace of the γ & K evolution equations using the above split & rescaling gives,

$$\boxed{\partial_t \phi = -\frac{1}{6}\alpha K + \beta^i \partial_i \phi + \frac{1}{6}\partial_i \beta^i} \qquad (1)$$

$$\boxed{\begin{aligned}\partial_t K = -\gamma^{ij} D_j D_i \alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) \\ + 4\pi\alpha(\rho + S) + \beta^i \partial_i K\end{aligned}} \qquad (2)$$

where the Hamiltonian constraint,

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

is used to eliminate R.

- now subtract these eqns. from full evolution equation:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

and

$$\partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k_j + K K_{ij}) - D_i D_j \alpha$$

$$\partial_t K_{ij} = \alpha(K_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) - D_i D_j \alpha$$
$$- 8\pi\alpha\left(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S-\rho)\right)$$
$$+ \beta^k \partial_k K_{ij} + K_{ik}\partial_j \beta^k + K_{kj}\partial_i \beta^k$$

to yield evolution eqns for the traceless parts:

$$\boxed{\begin{aligned}\partial_t \tilde{\gamma}_{ij} = &-2\alpha \tilde{A}_{ij} + \beta^k \partial_k \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta^k \\ &+ \tilde{\gamma}_{kj}\partial_i \beta^k - \tfrac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k\end{aligned}} \tag{3}$$

and

$$\boxed{\begin{aligned}\partial_t \tilde{A}_{ij} = \; & e^{-4\phi}\left(-(D_i D_j \alpha)^{TF} + \alpha(R_{ij}^{TF} - 8\pi S_{ij}^{TF})\right) \\ &+ \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j\right) \\ &+ \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k \\ &- \tfrac{2}{3}\tilde{A}_{ij}\partial_k \beta^k\end{aligned}} \tag{4}$$

where "TF" means "trace-free", e.g.,

$$R_{ij}^{TF} = R_{ij} - \tfrac{1}{3}\gamma_{ij} R$$

- Ricci tensor can then be written,

$$R_{ij} = \tilde{R}_{ij} + R^{\phi}_{ij}$$

where

$$R^{\phi}_{ij} = -2\tilde{D}_i \tilde{D}_j \phi - 2\tilde{\gamma}_{ij} \tilde{D}^l \tilde{D}_l \phi + 4(\tilde{D}_i \phi)(\tilde{D}_j \phi) - 4\tilde{\gamma}_{ij}(\tilde{D}^l \phi)(\tilde{D}_l \phi)$$

$\tilde{D}_i$ is associated w/ $\tilde{\gamma}_{ij}$,

$$\tilde{D}^i = \tilde{\gamma}^{ij}\tilde{D}_j$$

- $\tilde{R}_{ij}$ can be found by inserting $\tilde{\gamma}_{ij}$ into Ricci tensor evolution eqn.

- Better yet, can make Ricci tensor truly elliptic by introducing conformal connection functions:

$$\tilde{\Gamma}^i \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} = -\tilde{\gamma}^{ij}{}_{,j}$$

(last equality true because $\tilde{\gamma} = 1$)

$\rightarrow$

$$\tilde{R}_{ij} = -\frac{1}{2}\tilde{\gamma}^{lm}\tilde{\gamma}_{ij,lm} + \tilde{\gamma}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k \tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm}\left(2\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}\right)$$

where $\tilde{\gamma}^{lm}\tilde{\gamma}_{ij,lm}$ is a Laplace operator acting on components of $\tilde{\gamma}_{ij}$

- expression for $\tilde{R}_{ij}$ is then elliptic

& diagonally-dominated

- second derivatives in original expression for R_{ij} are absorbed in derivatives of $\tilde{\Gamma}^i$
- for the right coord. system (e.g. $\beta^i = 0$), evolution equations for $\tilde{\gamma}_{ij}$ & $\tilde{A}_{ij}$ reduce to a coupled set of non-linear inhomogeneous wave eqns. for $\tilde{\gamma}_{ij}$ w/ K, $\tilde{\Gamma}^i$, e^ϕ & matter appearing as sources.

→ hyperbolic system

- Now, $\tilde{\Gamma}^i$ are pure gauge functions
  → could be freely chosen
  - in BSSN, choose β^i instead and evolve $\tilde{\Gamma}^i$
- start by permuting time derivative w/ space derivative to give

$$\partial_t \tilde{\Gamma}^i = -\partial_j \left(2\alpha \tilde{A}^{ij} - 2\tilde{\gamma}^{m(j}\beta^{i)}{}_{,m} + \frac{2}{3}\tilde{\gamma}^{ij}\beta^l{}_{,l} + \beta^l \tilde{\gamma}^{ij}{}_{,l} \right)$$

- divergence of $\tilde{A}^{ij}$ is problematic

→ use momentum constraint to eliminate it:

$$\boxed{\begin{aligned}\partial_t \tilde{\Gamma}^i = & -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\Big(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K \\ & - 8\pi\tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\Big) \\ & + \beta^j\partial_j\tilde{\Gamma}^i - \tilde{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_j\beta^j \\ & + \frac{1}{3}\tilde{\gamma}^{li}\partial_l\partial_j\beta^j + \tilde{\gamma}^{lj}\partial_j\partial_l\beta^i\end{aligned}} \qquad (5)$$

- now fundamental variables of system are:

$$\phi,\ K,\ \tilde{\gamma}_{ij},\ \tilde{A}_{ij},\ \tilde{\Gamma}^i$$

and are evolved using eqn. (1) → (5)

- note that required by BSSN, but not directly constrained, are that:

$$\det(\tilde{\gamma}_{ij}) = 1$$

&

$$\tilde{\gamma}^{ij}\tilde{A}_{ij} = 0$$

→ these can be used as numerical checks on solution accuracy

- this system is strongly hyperbolic

- $\angle$'s unknowns + eqns. to solve