Lecture13

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Constructing Initial Data

- To solve evolution eqns., must first specify γ_{ij} & K_{ij} (and T^{ab} if there is matter) on Σ. Must satisfy constraints:

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i$$

- γ_{ij} & K_{ij} have 12 independent components
  - constraints give 4, coordinate choice gives 4
  - remaining 4 related to 2 polarizations of GWs (these are dynamical $\rightarrow$ can't get them from constraints)

Conformal Transformations

- rewrite spatial metric: $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$

  where ψ is the conformal factor

  $\bar{\gamma}_{ij}$ is the conformally-related metric

- usual choice: $\bar{\gamma}_{ij} = \gamma^{-1/3}\gamma_{ij}$

$\hookrightarrow$ $\gamma = \psi^{12}$ .... conformal factor characterizes the metric scale

- connection coefficients:

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} + 2(\delta^i_j \bar{D}_k \ln\psi + \delta^i_k \bar{D}_j \ln\psi - \bar{\gamma}_{jk}\bar{\gamma}^{il}\bar{D}_l \ln\psi)$$

where $\gamma^{ij} = \psi^{-4}\bar{\gamma}^{ij}$

→ $\bar{D}_i \bar{\gamma}_{jk} = 0$

- Ricci tensor:

$$R_{ij} = \bar{R}_{ij} - 2(\bar{D}_i\bar{D}_j \ln\psi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m \ln\psi) + 4\left[(\bar{D}_i \ln\psi)(\bar{D}_j \ln\psi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l \ln\psi)(\bar{D}_m \ln\psi)\right]$$

- Curvature scalar:

$$R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi$$

where $\bar{D}^2 = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j$ is covariant Laplacian

→ Hamiltonian constraint:

$$8\bar{D}^2\psi - \psi\bar{R} - \psi^5 K^2 + \psi^5 K_{ij}K^{ij} = -16\pi\psi^5\rho$$

Conformal transform of extrinsic curvature

- split K_{ij} into trace K & traceless part A_{ij}:

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K$$

& conformal transform separately

- significant freedom in how to do this

Consider, ...

$$A^{ij} = \psi^{\alpha} \bar{A}^{ij} \quad , \quad K = \psi^{\beta} \bar{K}$$

where α, β are undetermined exponents

- now, any symmetric, traceless tensor A^{ij} satisfies, $\quad D_j A^{ij} = \psi^{-10} \bar{D}_j \left(\psi^{10+\alpha} \bar{A}^{ij} \right)$

$\rightarrow$ adopt $\alpha = -10$

$\hookrightarrow \quad A^{ij} = \psi^{-10} \bar{A}^{ij} \; ; \; A_{ij} = \psi^{-2} \bar{A}_{ij}$

$\therefore \; D_j A^{ij} = 0 \quad$ if $\quad \bar{D}_j \bar{A}^{ij} = 0$

- apply this decomposition to momentum constraint,

$$\psi^{-10} \bar{D}_j \bar{A}^{ij} - \frac{2}{3} \psi^{\beta-4} \gamma^{ij} \bar{D}_j \bar{K} - \frac{2}{3} \beta \psi^{\beta-5} \bar{K} \bar{D}_j \psi = 8\pi S^i$$

- if we take $\beta = 0$, $K = \bar{K}$ .... "conformally invariant"

$\hookrightarrow \quad 8 \bar{D}^2 \psi - \psi \bar{R} - \frac{2}{3} \psi^5 K^2 + \psi^{-7} \bar{A}_{ij} \bar{A}^{ij} = -16\pi \psi^5 \rho$

.... Hamiltonian constraint

and $\quad \bar{D}_j \bar{A}^{ij} - \frac{2}{3} \psi^6 \bar{\gamma}^{ij} \bar{D}_j K = 8\pi \psi^{10} S^i$

.... momentum constraint

Conformal transverse-traceless

- for any traceless, symmetric tensor:

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + \bar{A}^{ij}_{L}$$

where "transverse traceless" part,

$$\bar{D}_j \bar{A}^{ij}_{TT} = 0$$

& the "longitudinal" part

and the longitudinal part,

$$\bar{A}_L^{ij} = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k \equiv (\bar{L}W)^{ij}$$

W^i is a vector potential

longitudinal operator $\bar{L}$ produces a symmetric, traceless tensor.

- now,

$$\bar{D}_j A^{ij} = \bar{D}_j \bar{A}_L^{ij} = \bar{D}_j (\bar{L}W)^{ij}$$

$$= \bar{D}^2 W^i + \frac{1}{3}\bar{D}^i(\bar{D}_j W^j) + \bar{R}^i{}_j W^j$$

$$\equiv (\bar{\Delta}_L W)^i$$

where $\bar{\Delta}_L$ is vector Laplacian

- using this decomposition in momentum constraint,

$$(\bar{\Delta}_L W)^i - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_j K = 8\pi\,\psi^{10} S^i$$

- CTT decomposition summary:

1. $\bar{\gamma}_{ij}$, K, $\bar{A}_{TT}^{ij}$ are freely specifiable
2. solve above momentum constraint for W^i
3. Hamiltonian constraint,

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5\rho$$

where $\bar{A}^{ij} = \bar{A}_{TT}^{ij} + \bar{A}_L^{ij} = \bar{A}_{TT}^{ij} + (\bar{L}W)^{ij}$

is solved for ψ

4. physical (i.e. not "conformal") spatial metric,

$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij}$$

physical extrinsic curvature,

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K = \psi^{-2}\bar{A}_{ij} + \frac{1}{3}\gamma_{ij}K$$

- choice of background data ($\bar{\gamma}_{ij}$, K, $\bar{A}^{ij}_{TT}$, and matter sources) must be made carefully for the physical system of interest
  - common choice: $\bar{A}^{ij}_{TT} = 0$ [minimizes GW content of background]
- maximal slicing: $K = 0$, maximizes volume of Σ in M
  - assuming $\rho = 0 = S^i$ (no matter), Hamiltonian constraint,

    $$(\bar{\Delta}_L W)^i = 0$$ ... can be solved w/o first specifying background
  - assume conformal flatness: $\bar{\gamma}_{ij} = \eta_{ij}$

    ↳ vector Laplacian in Cartesian:

    $$\partial^j \partial_j W^i + \frac{1}{3}\partial^i \partial_j W^j = 0$$

    $\Rightarrow$ Bowen-York solutions
- CTT can be prone to initial transients since derivative info cannot be specified

Conformal thin-sandwich (CTS)

- CTT does not tell us anything about time evolution of γ_{ij}, K_{ij}
  $\rightarrow$ fine for equilibrium solutions & steady-state
- seek new decomposition that gives info on time derivative of γ_{ij}
- traceless part of spatial metric time derivative:

$$u_{ij} \equiv \gamma^{1/3} \partial_t \left(\gamma^{-1/3} \gamma_{ij} \right)$$

- recall evolution equation,

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

w/ above definition

$$\rightarrow \quad u^{ij} = -2\alpha A^{ij} + (L\beta)^{ij}$$

where L is "physical" vector gradient (based on γ_{ij} not $\bar{\gamma}_{ij}$)

- define $\bar{u}_{ij} \equiv \partial_t \bar{\gamma}_{ij}$ and $\bar{\gamma}^{ij} \bar{u}_{ij} = 0$

$$\left.\begin{aligned} \bar{u}_{ij} &\equiv \partial_t \bar{\gamma}_{ij} \\ \bar{\gamma}^{ij} \bar{u}_{ij} &= 0 \end{aligned}\right\} \; 0 = \bar{\gamma}^{ij} \bar{u}_{ij} = \bar{\gamma}^{ij} \partial_t \bar{\gamma}_{ij} = \partial_t \ln \bar{\gamma}$$

- with above, can show $u_{ij} = \psi^4 \bar{u}_{ij}$
  and $(L\beta)^{ij} = \psi^{-4} (\bar{L}\beta)^{ij}$
- w/ $A^{ij} = \psi^{-10} \bar{A}^{ij}$, can conformally transform evolution equation to find

$$\bar{A}^{ij} = \frac{\psi^6}{2\alpha} \left((\bar{L}\beta)^{ij} - \bar{u}^{ij} \right)$$

- define densitized lapse: $\alpha = \psi^6 \bar{\alpha}$

$\hookrightarrow$ $\bar{A}^{ij} = \frac{1}{2\bar{\alpha}}\left((\bar{L}\beta)^{ij} - \bar{u}^{ij}\right)$

.... relates $\bar{A}^{ij}$ to shift β^i

- insert this into momentum constraint to solve for shift,

$$(\bar{\Delta}_L \beta)^i - (\bar{L}\beta)^{ij}\bar{D}_j \ln(\bar{\alpha}) = \bar{\alpha}\bar{D}_j\left(\bar{\alpha}^{-1}u^{ij}\right) + \frac{4}{3}\bar{\alpha}\psi^6\bar{D}^i K + 16\pi\bar{\alpha}\psi^{10}S^i$$

- Summary of CTS:

1. specify $\bar{\gamma}_{ij}$, $\bar{u}_{ij}$, K, & $\bar{\alpha}$
2. input into reduced momentum constraint & solve for β^i
3. Solve for ψ from Hamiltonian constraint:
$$\bar{D}^2\psi - \frac{1}{8}\psi\bar{R} - \frac{1}{12}\psi^5K^2 + \frac{1}{8}\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -2\pi\psi^5\rho$$
where $\bar{A}^{ij} = \frac{1}{2\bar{\alpha}}\left((\bar{L}\beta)^{ij} - \bar{u}^{ij}\right)$
4. Construct physical solution:
$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij}$$
$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K = \psi^{-2}\bar{A}_{ij} + \frac{1}{3}\gamma_{ij}K$$
$$\alpha = \psi^6\bar{\alpha}$$

- CTS has 16 unknowns vs. 12 for CTT.
  - four extra are α & β^i which account for inclusion of time derivatives, i.e. info

extrinsic to Σ_i.

- instead of specifying lapse $\tilde{\alpha}$ can instead specify $\partial_t K$ & solve:

$$\Delta^2 \alpha = -\partial_t K + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho + S)\right) + \beta^i D_i K$$

for α.

$\rightarrow$ "extended" CTS

Mass, momentum, angular momentum

- continuity eqn. for conservation of rest-mass:

$$\nabla_a(\rho_0 u^a) = 0$$

where ρ_0 is rest-mass density.

- M_0 will be total conserved rest-mass or "baryon" mass.
- integrating continuity eqn. over 4D Ω:

$$\int_\Omega d^4x \sqrt{-g}\, \nabla_a(\rho_0 u^a) = 0$$

- Gauss' theorem $\rightarrow$

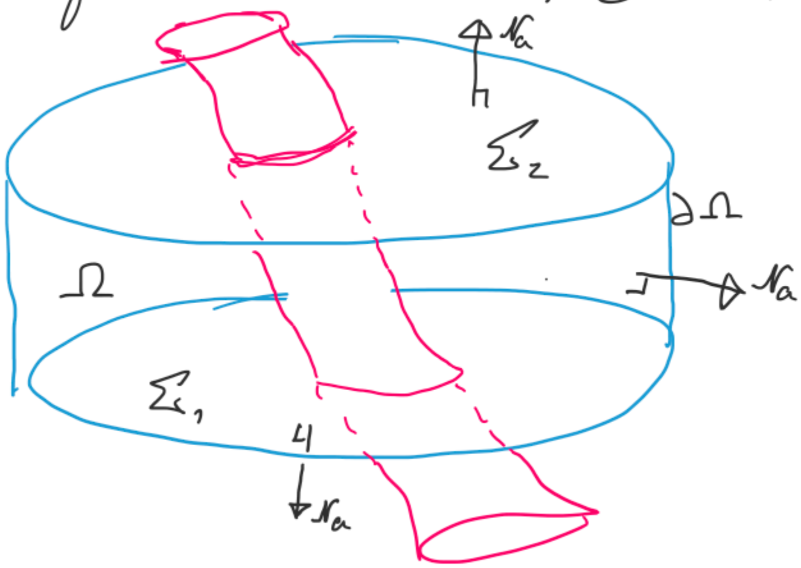
$$\int_\Omega d^4x \sqrt{-g}\, \nabla_a(\rho_0 u^a) = \int_{\partial\Omega} d^3\Sigma_a\, \rho_0 u^a$$

where

$$d^3\Sigma_a = \epsilon\, N_a \sqrt{\gamma}\, d^3x$$

N_a : outward-pointing unit normal on $\partial\Omega$.

- for spacelike $\partial\Omega$, $\epsilon = -1$
- for timelike $\partial\Omega$, $\epsilon = +1$



- normal vector: $N_a u^a = n_a u^a = -\alpha u^t$ .... on Σ_2
- on Σ_1, $N^a = -n^a$

$$\hookrightarrow \int_{\Sigma_1} d^3x \sqrt{\gamma}\, \alpha u^t \rho_0 - \int_{\Sigma_2} d^3x \sqrt{\gamma}\, \alpha u^t \rho_0 = 0$$

$\therefore$ rest mass $M_0 = \int_{\Sigma} d^3x \sqrt{\gamma}\, \alpha u^t \rho_0$

is conserved

- Defining total mass-energy of a system in GR is tricky since it is non-local.

→ define "ADM" mass as total mass-energy of isolated system at an instant of time measured in a spatial surface out to infinity:

$$M_{ADM} = \frac{1}{16\pi} \int_{\partial\Sigma_\infty} \sqrt{\gamma}\, \gamma^{jn} \gamma^{im} \left(\partial_j \gamma_{mn} - \partial_m \gamma_{jn}\right) dS_i$$

where $dS_i = \sigma_i \sqrt{\gamma^{\partial\Sigma_\infty}}\, d^2 z$

is outward-oriented surface element, z^i are coords on $\partial\Sigma_\infty$, $\gamma_{ij}^{\partial\Sigma_\infty}$ is the induced metric on $\partial\Sigma_\infty$, + σ^i is the unit normal ($\sigma^i \sigma_i = 1$) to $\partial\Sigma_\infty$

- formally, integral carried out to infinity but in practice must at least be taken to asymptotically flat space:

$$g_{ab} - \eta_{ab} = \mathcal{O}(r^{-1})$$

- ADM angular momentum:

$$J_i^{ADM} = \frac{1}{8\pi} \int_{\partial\Sigma_\infty} dS_m \left(K^m{}_n - \delta^m{}_n K\right) \xi^n_{(i)}$$

where $\xi^n_{(i)}$ is some global Killing vector field

- adopting $\xi^{(\phi)} = \xi^{(i)}_k = \epsilon_{kml}\, e^m_{(i)}\, x^l$

in Cartesian, where $e^m_{(i)} = \delta_i$ is basis vector along x^i

$$\hookrightarrow \quad J_i^{ADM} = \frac{1}{8\pi} \epsilon_{ijn} \int_{\partial\Sigma_\infty} dS_m \, x^j \left(K^{mn} - \delta^{mn} K \right)$$

- can obtain linear ADM momentum by adopting a translational Killing vector along x^i: $\xi^n_{(i)} = e^n_{(i)}$

$$\hookrightarrow \quad P_i^{ADM} = \frac{1}{8\pi} \int_{\partial\Sigma_\infty} dS_m \left(K^m{}_i - \delta^m{}_i K \right)$$

- M^{ADM}, J_i^{ADM}, P_i^{ADM} are globally conserved and this should be monitored during numerical simulations.