Lecture22

Monday, April 15, 2019 8:17 PM

Gravitational Waves

- Luminosity of GWs from some source:

$$L_{GW} = -\frac{dE}{dt} = \frac{1}{5}\langle \dddot{\mathcal{I}}_{jk} \dddot{\mathcal{I}}^{jk} \rangle$$

where the reduced quadrupole moment:

$$\mathcal{I}_{ij} \equiv I_{ij} - \frac{1}{3}\eta_{ij} I$$

& quadrupole moment is:

$$I^{ij}(t) \equiv \int d^3x' \rho_0(t, x'^k) x'^i x'^j$$

- gravitational radiation results in a loss of angular momentum from system:

$$\frac{dJ_i}{dt} = \lim_{r\to\infty} \frac{r^2}{16\pi}\int Re\langle \dot{H}^* \hat{J}_i H \rangle d\Omega$$

$$H \equiv h_+ - i h_\times$$

$$\hat{J}_x = -\sin\phi\, \partial_\theta - \cos\phi\left(\cot\theta\, \partial_\phi + 2i \csc\theta\right)$$

$$\hat{J}_y = \cos\phi\, \partial_\theta - \sin\phi\left(\cot\theta\, \partial_\phi + 2i \csc\theta\right)$$

$$\hat{J}_z = \partial_\phi$$

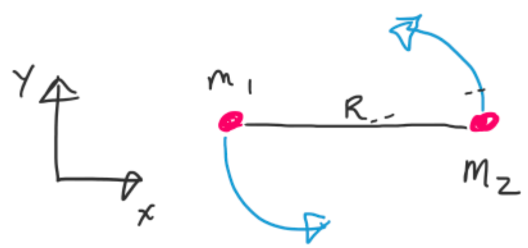
- for angular momentum vector aligned w/ z-axis:

$$\frac{dJ_z}{dt} = \lim_{r\to\infty} \frac{r^2}{16\pi} \int \langle \partial_t h_+ \partial_\phi h_+ + \partial_t h_\times \partial_\phi h_\times \rangle d\Omega$$

- quadruplole approximation:

$$\frac{dJ_i}{dt} = -\frac{2}{5} \epsilon_{ijk} \langle \ddot{I}^{jm} \dddot{I}^{k}{}_{m} \rangle$$

<u>Example</u>: Binary point masses



- in the center-of-mass frame, reduced quadrupole moment is

$$I_{ij} = \frac{1}{2}\mu R^2 \begin{pmatrix} \cos 2\Omega t + \frac{1}{3} & \sin 2\Omega t & 0 \\ \sin 2\Omega t & -\cos 2\Omega t + \frac{1}{3} & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ 0 & 0 & -2/3 \end{pmatrix}$$

Ω : orbital angular velocity

$\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ ... reduced mass

R : binary separation

- for observer at distance r & angle relative to orbital plane θ, waveforms are

$$h_+ = -\frac{2}{r}\Omega^2 \mu R^2 (1+\cos^2\theta)\cos 2\Omega(t-r)$$

$$h_\times = -\frac{4}{r}\Omega^2 \mu R^2 \cos\theta \sin 2\Omega(t-r)$$

taking $\theta = 0$, peak magnitude of strain is then

$$h \simeq \frac{4\mu\Omega^2 R^2}{r}$$

using Kepler's law, $\Omega^2 = \dfrac{M}{R^3}$ & $R^2 = \dfrac{M^{2/3}}{\Omega^{4/3}}$

$$h \approx \frac{4}{r}\frac{\mu M}{R} \quad \text{where } M = m_1 + m_2$$

- Note! strain falls off like $1/r$, unlike flux ($\propto 1/r^2$)

- to compute energy loss of system, need time derivatives of quad. mom.

$$\ddot{\underset{\sim}{I}}_{ij} = \frac{1}{2}\mu R^2 \begin{pmatrix} -4\Omega^2 \cos 2\Omega t & -4\Omega^2 \sin 2\Omega t & 0 \\ -4\Omega^2 \sin 2\Omega t & 4\Omega^2 \cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\dddot{\underset{\sim}{I}}_{ij} = \frac{1}{2}\mu R^2 \begin{pmatrix} -8\Omega^3 \sin 2\Omega t & -8\Omega^3 \cos 2\Omega t & 0 \\ -8\Omega^3 \cos 2\Omega t & 8\Omega^3 \sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then energy loss is:

$$\frac{dE}{dt} = -\frac{1}{5}\langle \dddot{\underset{\sim}{I}}_{jk} \dddot{\underset{\sim}{I}}^{jk} \rangle$$

$$= -\frac{1}{5}\frac{1}{4}\mu^2 R^4 \langle 64\Omega^6 \sin^2 2\Omega t + 64\Omega^6 \cos^2 2\Omega t + 64\Omega^6 \cos^2 2\Omega t + 64\Omega^6 \sin^2 2\Omega t \rangle$$

$$= -\frac{1}{20}\mu^2 R^4 \langle 128\,\Omega^6 \rangle = -\frac{32}{5}\mu^2 R^4 \Omega^6$$

similar for angular momentum loss:

$$\frac{dJ_z}{dt} = -\frac{2}{5}\epsilon_{zjk} \langle \ddot{\underset{\sim}{I}}^{jm} \dddot{\underset{\sim}{I}}^{k}{}_{m} \rangle$$

$$= -\frac{32}{5}\mu^2 R^4 \Omega^5$$

$\therefore$ the size of the binary orbit shrinks

$\rightarrow$ inspiral

- the frequency of the GWs is 2x the orbital frequency

- the above shows $dE = \Omega \, dJ$

  $\hookrightarrow$ circular orbits will remain circular during inspiral

$\boxed{\text{Sources of GWs}}$

- typical GW amplitudes are small!
- for equal-mass binary, use example:

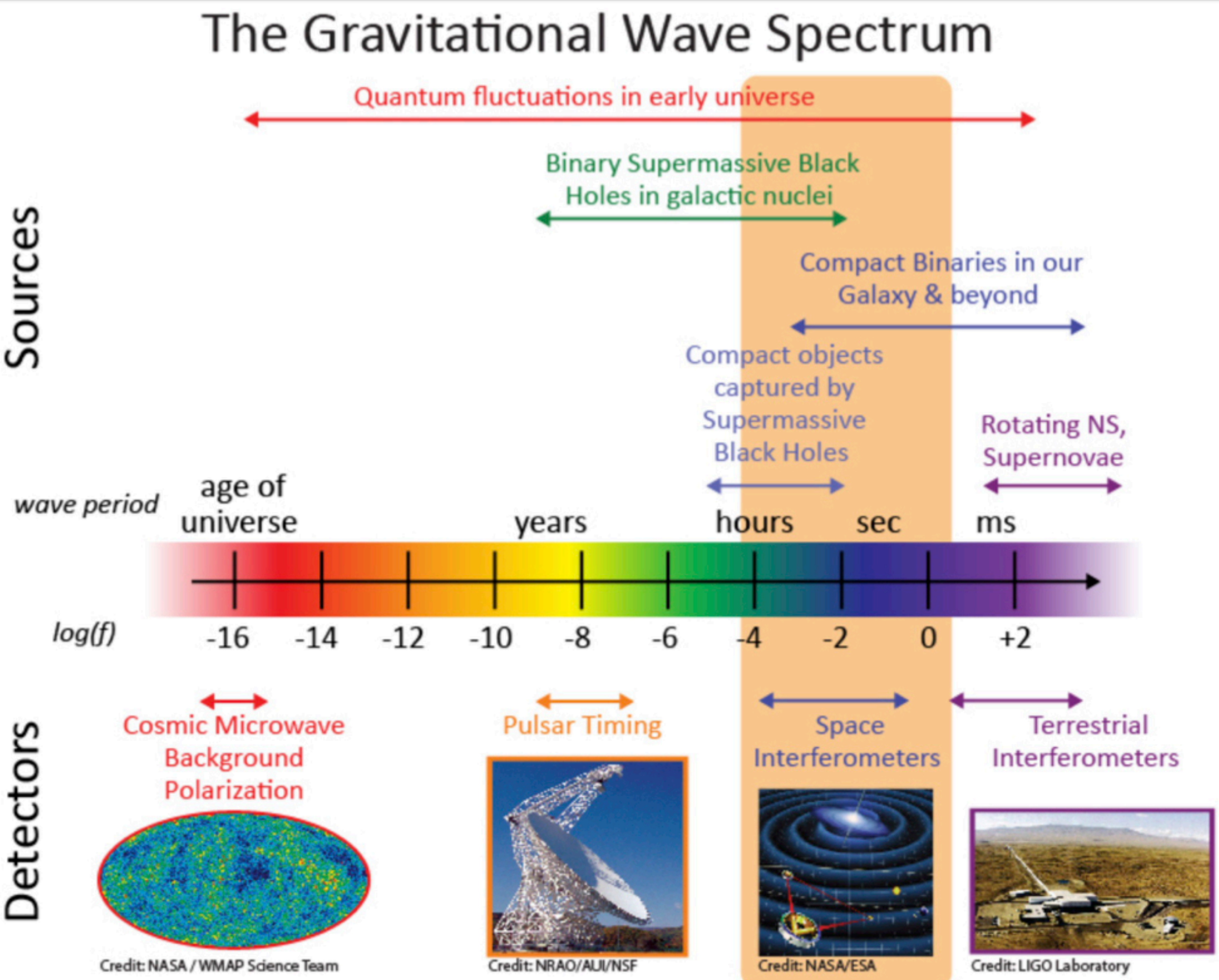
$$h \simeq \frac{4}{r}\frac{\mu M}{R} \simeq 5\times10^{-20}\left(\frac{1\,\mathrm{Mpc}}{r}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{M}{R}\right)$$

where $\mu = \dfrac{m_1 m_2}{m_1+m_2} = \dfrac{M}{4}$ for $m_1 = m_2$

- using either Kepler's law for binaries or the inverse dynamical time for compact objects: $f \sim \tau_{dyn}^{-1} \sim \sqrt{\rho}$, we find

$$f_{GW} \simeq \frac{1}{M}\left(\frac{M}{R}\right)^{3/2} \simeq 2\times10^{5}\,\text{Hz}\left(\frac{M_\odot}{M}\right)\left(\frac{M}{R}\right)^{3/2}$$

↳ wide frequency range for different astrophysical sources



<u>High frequency</u>: $1\text{ Hz} \leq f \leq 10^4\text{ Hz}$

- compact binaries

  NS-NS: Hulse-Taylor pulsar, GW170817

BH-BH : several LIGO detections

BH-NS : soon? O3 maybe?

- waveform predictions + models are key to both detection & extraction of physical parameters

- often characterized by chirp mass:

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

- characterizes the orbital evolution of binary

- frequency evolution of binary emitting GWs:

$$\frac{df}{dt} = \frac{96}{5}\pi^{8/3}\mathcal{M}^{5/3} f^{11/3}$$ ... to leading order

- thus observing the frequency and its first time derivative, can measure chirp mass:

$$\mathcal{M} = \left(\frac{5}{96}\pi^{-8/3} f^{-11/3}\dot{f}\right)^{3/5}$$

- stellar collapse
  - rotating stars
  - convection & other instabilities in post-collapse environment
- "mountains" on neutron stars