Lecture15

Wednesday, March 13, 2019 8:32 PM

$\boxed{\text{Matter Sources}}$

- "matter sources" will refer to anything that contributes to stress-energy tensor T^{ab}
- 4D evolution eqn. for "matter":

$$\nabla_b T^{ab} = 0 \quad \text{... conservation of 4-momentum}$$

- source terms in 3+1 eqns. are projections of T^{ab} into n^a & Σ:

$$\rho = n_a n_b T^{ab} \quad \text{... mass-energy density as measured by normal observer.}$$

$$S_i = -\gamma_{ia} n_b T^{ab} \quad \text{... momentum density}$$

$$S_{ij} = \gamma_{ia}\gamma_{jb} T^{ab} \quad \text{.... stress}$$

$$S = \gamma^{ij} S_{ij} \quad \text{... trace of stress}$$

$\boxed{\text{Hydrodynamics}}$

- stress-energy for perfect gas:

$$T^{ab} = \rho_0 h u^a u^b + P g^{ab}$$

where ρ_0 is rest-mass density, P is pressure,

$$h = 1 + \epsilon + P/\rho_0 \quad \text{---- specific enthalpy}$$

& ϵ is specific internal energy density

- source terms for a perfect fluid,

$$\rho^{\text{fluid}} = \rho_0 h W^2 - P$$

$$S_i^{\text{fluid}} = \rho_0 h W u_i$$

$$S_{ij}^{\text{fluid}} = P \gamma_{ij} + \frac{S_i^{\text{fluid}} S_j^{\text{fluid}}}{\rho_0 h W^2}$$

$$S^{\text{fluid}} = 3P + \rho_0 h (W^2 - 1)$$

where

$$W = \alpha u^t = (1 + \gamma^{ij} u_i u_j)^{1/2} = -n_a u^a$$

... Lorentz factor

- total mass-energy density as measured by comoving observer,

$$\rho_* = \rho_0 (1 + \epsilon)$$

- conservation of 4-momentum supplemented by conservation of rest-mass,

$$\nabla_a (\rho_0 u^a) = 0$$

- recall that the covariant divergence can be written,

$$\nabla_a A^a = \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g}\, A^a\right)$$

where g is determinant of 4-metric and in ADM, $\sqrt{-g} = \alpha \gamma^{1/2}$

Wilson scheme

- many possible 3+1 schemes
- that due to Jim Wilson is straightforward & similar to Newtonian hydro.
- define new variables:

$$D \equiv \rho_0 W \quad \text{... rest-mass density}$$

$$E \equiv \rho_0 \epsilon W \quad \text{.... internal energy density}$$

$$S_a \equiv \rho_0 h W u_a$$

$$= (D + E + PW) u_a \quad \text{... momentum density}$$

where $W \equiv -n_a u^a = \alpha u^t$ .... Lorentz factor

- now, $\nabla_a(\rho_0 u^a) = 0$ becomes,

$$\frac{1}{g^{1/2}} \partial_a \left(g^{1/2} \rho_0 u^a\right) = 0$$

$$\partial_t\left(g^{1/2} \rho_0 u^t\right) + \partial_j\left(g^{1/2} \rho_0 u^j\right) = 0$$

$$\partial_t\left(g^{1/2} \frac{D}{W} u^t\right) + \partial_j\left(g^{1/2} \frac{D}{W} u^j\right) = 0$$

$$\partial_t\left(\frac{g^{1/2}}{\alpha} D\right) + \partial_j\left(g^{1/2} \frac{D}{\alpha} \frac{u^j}{u^t}\right) = 0$$

$$\partial_t\left(\gamma^{1/2} D\right) + \partial_j\left(\gamma^{1/2} D\, v^j\right) = 0$$

where spatial 3-velocity,

$$v^j = u^j / u^t$$

- contracting 4-mom. cons. w/ u_a:

$$u_a \nabla_b T^{ab} = 0 \,, \quad \text{gives energy equation}$$

$$\hookrightarrow \partial_t(\gamma^{1/2} E) + \partial_j(\gamma^{1/2} E v^j) = -P\Big(\partial_t(\gamma^{1/2} W) + \partial_t(\gamma^{1/2} W v^j)\Big)$$

- spatial components of $\nabla_b T^{ab} = 0$ give

$$\partial_t(\gamma^{1/2} S_i) + \partial_j(\gamma^{1/2} S_i v^j) = -\alpha \gamma^{1/2}\left(\partial_i P + \frac{S_a S_b}{2\alpha S^t}\partial_i g^{ab}\right)$$

... relativistic Euler eqns.

where $(-g)^{1/2} = \alpha \gamma^{1/2}$

- can relate pressure to density + energy via simple polytrope,

$$P = (\Gamma - 1)\rho_0 \epsilon$$

+ simplification of energy eqn.:

$$\partial_t(\gamma^{1/2} E^*) + \partial_j(\gamma^{1/2} E^* v^j) = 0$$

where $E^* \equiv (\rho_0 \epsilon)^{1/\Gamma} W$

- Lorentz factor,

$$W = \alpha u^t = \left(1 + \gamma^{ij} u_i u_j\right)^{1/2}$$

since $u_a u^a = -1$

and spatial 3-velocity:

$$v^i = \alpha \gamma^{ij} u_j / W - \beta^i$$

- original Wilson scheme used finite difference & required artificial viscosity to capture shocks.

High resolution shock capturing

- finite difference scheme of Wilson not automatically conservative.
  → solve "flux-conservative" form & evolve cell "averages"
- general form,

$$\underbrace{\partial_t \mathcal{U}}_{\text{Conserved variables}} + \underbrace{\partial_i \mathcal{F}^i}_{\text{Flux vectors}} = \underbrace{\mathcal{S}}_{\text{Source terms}}$$

- conserved variable built from

primitive variables,

$$\mathcal{P} = (\rho_0, v^i, P)^T$$

- $\mathcal{S}$ source vector contains no derivatives of primitives
- some flexibility in construction of conserved variables
- on useful choice,

$$\mathcal{U} = \begin{pmatrix} \tilde{D} \\ \tilde{S}_j \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} \gamma^{1/2} W \rho_0 \\ \gamma^{1/2} \alpha T^0{}_j \\ \alpha^2 \gamma^{1/2} T^{00} - \tilde{D} \end{pmatrix}$$

then fluxes are,

$$\mathcal{F}^i = \begin{pmatrix} \tilde{D} v^i \\ \alpha \gamma^{1/2} T^i{}_j \\ \alpha^2 \gamma^{1/2} T^{0i} - \tilde{D} v^i \end{pmatrix}$$

with source vector,

$$\mathcal{S} = \begin{pmatrix} 0 \\ \frac{1}{2} \alpha \gamma^{1/2} T^{ab} g_{ab,j} \\ \alpha \gamma^{1/2} \left(T^{a0} \partial_a \alpha - {}^{(4)}\Gamma^0_{ab} T^{ab} \alpha \right) \end{pmatrix}$$

- first eqn. is continuity, $\nabla_a (\rho_0 u^a) = 0$
- second eqns. are from energy-momentum conserv.

$\nabla_a T^a \ldots = 0 \ldots$

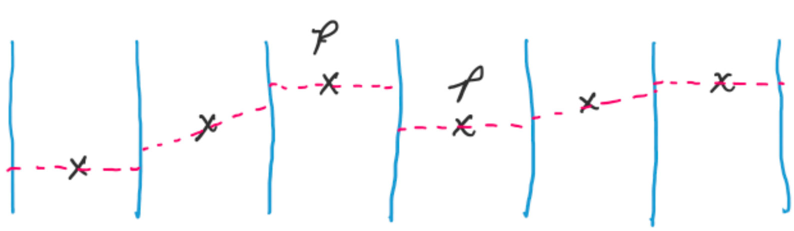
$\nabla_a T^a{}_b = 0$ or

$$\partial_t\left(\sqrt{-g}\; T^0{}_a\right) + \partial_i\left(\sqrt{-g}\; T^i{}_a\right) = \frac{1}{2}\sqrt{-g}\; T^{bc}\partial_a g_{bc}$$

($a = j$ component of this eqn.)

- third eqn. from projecting eqns. of motion along normal: $n_a \nabla_b T^{ab} = 0$ & subtracting off continuity
- procedure is to 1. integrate coupled eqns., 2. combine new conserveds to algebraically solve for new primitives, 3. simultaneously solve 3+1 eqns. for new spacetime metric.

- HRSC schemes begin with reconstruction of primitives to cell interfaces:



- reconstruction means to go from cell averges to point-wise values on interface
- at 2^{nd} order ("linear" reconstruction) cell-center value and cell-average

value are the same. Not true (always) for higher-order schemes.
- "slope limiters" are critical to avoiding over/under-shooting (maintain monotonicity)

- interface values at same spatial point are used as input to a Riemann solver that computes instantaneous flux on the interface via some (approximate) solution to shock tube problem.
- numerical fluxes $\mathcal{F}^i$ are then used to update $\mathcal{U}$ via some form of quadrature rule to integrate the 1st order ODEs
- recovery step then concludes by solving for $\mathcal{P}$ from new $\mathcal{U}$.
  - non-trivial & must be done numerically

Rankine - Hugoniot conditions

- advantage of finite volume schemes is ability to handle shocks.

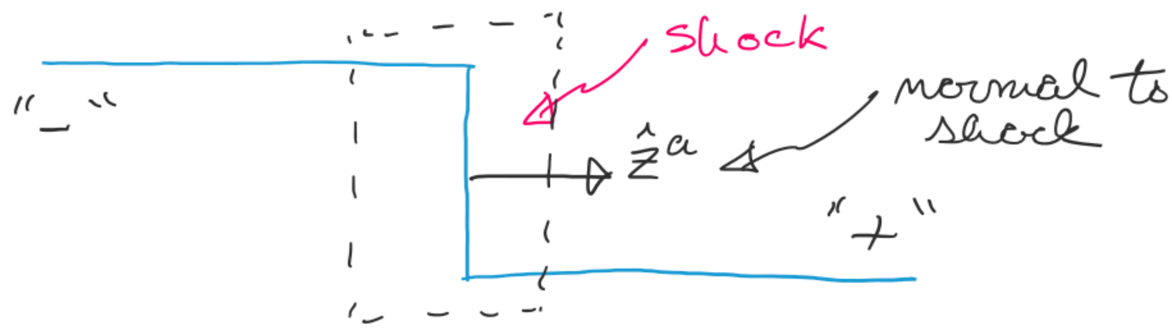
- physical width of shock waves set by microphysical interactions
  → unresolved in simulations

∴ numerically, shocks are infinite discontinuities. This is bad ....

- must handle this somehow in code
- can get relativistic shock jump conditions by integrating hydro eqns.:

$$\nabla_b T^{ab} = 0 \quad , \quad \nabla_a (\rho_0 u^a) = 0$$

over the shock transition.



- even at discontinuity, these conservation equations must hold.
- bracket notation:

$$[V] \equiv V_+ - V_-$$

upstream downstream

- shock jump conditions:

$$[\rho_0 u^a \hat{z}_a] = 0$$

$$[T^{ab} \hat{z}_b] = 0$$

$\hat{z}^a$ a spacelike normal vector

- re-expressing first condition,

$$F \equiv \rho_0^+ u^a_+ \hat{z}_a = \rho_0^- u^a_- \hat{z}_a$$

.... conserved rest-mass flux

- assuming perfect gas stress-energy, second condition is,

$$F(h_+ u^a_+ - h_- u^a_-) = \hat{z}^a (P_- - P_+)$$

- contract w/ u^+_a:

$$F(h_+ \overbrace{u^+_a u^a_+}^{-1} - h_- u^a_- u^+_a) = u^+_a \hat{z}^a (P_- - P_+)$$

- contract w/ u^-_a:

$$F(h_+ u^+_a u^a_- - h_- \overbrace{u^a_- u^-_a}^{-1}) = u^-_a \hat{z}^a (P_- - P_+)$$

- difference two eqns:

$$F(h_+ u^+_a u^a_- + h_-) - F(-h_+ - h_- u^a_- u^+_a)$$

$$= u^-_a \hat{z}^a (P_- - P_+) - u^+_a \hat{z}^a (P_- - P_+)$$

$$h_+ u^+_a u^a_- + h_- + h_+ + h_- u^a_- u^+_a = \frac{(P_- - P_+)}{\rho_0^-} - \frac{(P_- - P_+)}{\rho_0^+}$$

$$= \left(\frac{1}{\rho_0^+} - \frac{1}{\rho_0^-}\right)(P_+ - P_-)$$

$$\left(\frac{h_+}{\rho_+}+\frac{h_-}{\rho_-}\right) F\left(h_+u_+^a - h_-u_-^a\right) = \left(\frac{h_+}{\rho_+}+\frac{h_-}{\rho_-}\right)\hat{z}^a\left(P_- - P_+\right)$$

$$\left(\frac{h_+^2}{\rho_+}u_+^a - \frac{h_-h_+}{\rho_+}u_-^a + \frac{h_-h_+}{\rho_-}u_+^a - \frac{h_-^2}{\rho_-}u_-^a\right)F = \left(\frac{h_+}{\rho_+}+\frac{h_-}{\rho_-}\right)\hat{z}^a\left(P_- - P_+\right)$$

$$h_+^2 \overbrace{u_+^{a2}}^{-1} - \cancel{h_-h_+u_-^a u_+^a} + \cancel{h_-h_+u_+^a} u_-^a - h_-^2 \underbrace{u_-^{a2}}_{-1} = \left(\frac{h_+}{\rho_+}+\frac{h_-}{\rho_-}\right)\left(P_- - P_+\right)$$

$$h_+^2 - h_-^2 = \left(\frac{h_+}{\rho_+}+\frac{h_-}{\rho_-}\right)\left(P_+ - P_-\right)$$

.... Rankine-Hugoniot or Taub adiabat

- non-dimensionalizing:

$$H^2 - 1 = \delta^2 (y-1)\left(\frac{H}{\eta}+1\right)$$

where $H = \frac{h_+}{h_-}$, $y = \frac{P_+}{P_-}$, $\eta = \frac{\rho_0^+}{\rho_0^-}$

$$\delta^2 = \frac{P_-}{\rho_0^- h_-}$$

- using $P = (\Gamma - 1)\rho_0 \epsilon$ and $q \equiv \frac{1}{h_-}$

↳ $\left[y(\Gamma-1)+(\Gamma q+1)\right]\eta^2 - q\left[y(\Gamma+1)+(\Gamma-1)\right]\eta$

$$- (1-q)\, y\, (y + \Gamma - 1) = 0$$

- q measures degree of relativity:

$q \to 1 \implies$ non-relativistic (NR)

$q \to 0 \implies$ ultra-relativistic (UR)

- NR limit:

$$\eta = \frac{y(\Gamma+1) + (\Gamma-1)}{y(\Gamma-1) + (\Gamma+1)}$$

- y is shock strength: $y \to \infty$ for strong

$y \to 1$ for weak (acoustic)

$\hookrightarrow \quad \eta \to \dfrac{\Gamma+1}{\Gamma-1}$ ... strong NR shock

- UR limit:

$$\eta = \left[\frac{y(y+\Gamma-1)}{y(\Gamma-1)+1}\right]^{1/2}$$

$\hookrightarrow \quad \eta \to \left[\dfrac{y}{\Gamma-1}\right]^{1/2}$ ... strong UR shock

- increases w/o bound!