Lecture9

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Numerical Relativity by Baumgarte & Shapiro

Notational notes:

- geometrized units, $G = c = 1$, used throughout
- Latin indices, "ab", go from 0 to 3, unlike Thorne & Blandford!
- BS will use "abstract index notation" which is equivalent to "slot-naming index notation" from TB.

Basis 1-forms

- introduce basis 1-form $\tilde{\omega}^a$ dual to basis vectors $\vec{e}_a$
- arbitrary 1-form $\tilde{\vec{B}} = B_a \tilde{\omega}^a$
- scalar product of 1-forms:

$$\tilde{A} \cdot \tilde{B} = (A_a \tilde{\omega}^a) \cdot (B_b \tilde{\omega}^b)$$

$$= g^{ab} A_a B_b$$

where $g^{ab} = \tilde{\omega}^a \cdot \tilde{\omega}^b$ is the inverse of g_{ab}

- duality requires,

$$\tilde{\omega}^a \cdot \vec{e}_b = \delta^a_b$$

→ scalar product of 1-form + vector does not involve metric,

$$\vec{A} \cdot \tilde{B} = (A^a \vec{e}_a) \cdot (B_b \tilde{\omega}^b)$$
$$= A^a \delta_a{}^b B_b = A^a B_a$$

- vector $\vec{A}$ + 1-form $\tilde{A}$ carry same information and are related via

$$A_a = g_{ab} A^b \quad , \quad A^a = g^{ab} A_b$$

- coordinate basis of 1-forms: $\tilde{\omega}^a = \tilde{d}x^a$ where $\tilde{d}x^a$ are surfaces of constant coordinate x^a
- orthonormal basis: $\tilde{\omega}^{\hat{a}} \cdot \tilde{\omega}^{\hat{b}} = \eta^{\hat{a}\hat{b}}$
- gradient of arbitrary scalar field is 1-form:

$$\tilde{d}f = \partial_a f \, \tilde{d}x^a$$

→ components are ordinary partial derivs.

- directional derivative of f along $\vec{v}$:

$$\vec{v} \cdot \tilde{d}f = (v^a \vec{e}_a) \cdot (\partial_b f \, \tilde{d}x^b)$$
$$= v^a \partial_a f$$

- basis transformation: $\vec{e}_{a'} = \vec{e}_b M^b{}_{a'}$

$$\tilde{\omega}^{a'} = M^{a'}{}_b \tilde{\omega}^b$$

where $\| M^b{}_{a'} \|$ is transformation matrix and $\| M^{a'}{}_{b} \| = \| M^{b'}{}_{a} \|^{-1}$

$\hookrightarrow$ $A^{a'} = M^{a'}{}_{b} A^{b}$, $B_{a'} = B_b M^{b}{}_{a'}$

- if both bases are coordinate, then

$$M^{b}{}_{a'} = \partial_{a} x^{b}$$

- Extension to higher rank tensors straightforward

$$\overleftrightarrow{T} = T^{a}{}_{b} \vec{e}_{a} \tilde{\omega}^{b}$$

where $\vec{e}_a \tilde{\omega}^b$ is a tensor product

- likewise for transformations,

$$T^{a'}{}_{b'} = M^{a'}{}_{c} T^{c}{}_{d} M^{d}{}_{b'}$$

- covariant derivative : measures change of tensor w.r.t. parallel transport,

$$\nabla_c T^{a}{}_{b} = \partial_c T^{a}{}_{b} + \Gamma^{a}{}_{dc} T^{d}{}_{b} - \Gamma^{d}{}_{bc} T^{a}{}_{d}$$

$\rightarrow$ rank increased by one

Black Holes

- area of spacetime that cannot communicate w/ outside, bounded by 3-surface called event horizon
- a causaly disconnected singularity forms

inside BH.

- general solution for BH spacetime metric is "Kerr-Newman". Depends only on mass M, angular momentum J, & charge Q
  - special cases:

    $Q=0$ .... Kerr metric

    $J=0$ .... Reissner-Nordstrom metric

    $J=Q=0$ .... Schwarzschild metric

Schwarzschild BHs

metric in spherical coordinates:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

- "r" is the "areal" radius since it is defined by Euclidean expression:

$$r = (A/4\pi)^{1/2} \quad \text{for area } A$$

- applies to vacuum regions, $r > 2M$
- $r = 2M$ is radius of EH, Schwarzschild radius
- because of symmetry in t & ϕ, there are two Killing vectors, $\vec{e}_t = \partial_t$ & $\vec{e}_\phi = \partial_\phi$
  - associated conserved quantities are

energy $E = -p_t$ & orbital angular momentum $l = p_\phi$

- circular orbits exist down to $r = 3M$

- for particle of mass μ in circular orbit,

$$(E/\mu)^2 = \frac{(r-2M)^2}{r(r-3M)}$$

$$(l/\mu)^2 = \frac{Mr^2}{r-3M}$$

- at $r=3M$, $(E/\mu) \to \infty$ ... photon orbit

- circular orbits stable for $r > 6M$

↳ innermost stable circular orbit (ISCO)

- singularity in metric at $r = 2M$ is a coordinate singularity; can be removed w/ transformation

- physical singularity at $r = 0$. Consider curvature invariant:

$$I \equiv R_{abcd} R^{abcd} = 48 M^2 / r^6$$

→ blows up at origin ∴ spacetime curvature there is infinite

[May come back to Kruskal-Szekeres]

Kerr BHs

- in Boyer-Lindquist coordinates,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dt\,d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2 + \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\,d\phi^2$$

where,

$$a \equiv J/M\,, \quad \Delta \equiv r^2 - 2Mr + a^2$$

$$\Sigma \equiv r^2 + a^2\cos^2\theta$$

- reduces to Schwarzschild w/ $a=0$
- spin must be $0 \leq a/M \leq 1$
- due to stationarity & axisymmetry, two Killing vectors: ∂_t & ∂_ϕ
- BH horizon found from largest root of $\Delta = 0$
  ↳ $r_+ = M + (M^2 - a^2)^{1/2}$
- static limit within which no static observers, solve $g_{tt} = 0$ → $r_0 = M + (M^2 - a^2\cos^2\theta)^{1/2}$
- between r_+ & r_0: ergosphere
  ↳ all timelike observers are dragged w/ $\Omega > 0$

Global theorems for BHs

- "Four laws of BH mechanics", analogous to laws of thermodynamics

<u>2nd law</u>: in an isolated system, sum of surface areas of all BHs can never decrease (Hawking)

- for Kerr BH, area A is area of EH at some instant time t. The 2-metric is then

$$^{(2)}ds^2 = \left(r_+^2 + a^2\cos^2\theta\right) d\theta^2 + \frac{(2Mr_+)^2}{r_+^2 + a^2\cos^2\theta} \sin^2\theta \, d\phi^2$$

$\hookrightarrow$ $A = \iint \sqrt{^{(2)}g} \, d\theta \, d\phi$, $^{(2)}g$ determinant

$$= 8\pi M\left[M + (M^2 - a^2)^{1/2}\right]$$

- define irreducible mass,

$$A \equiv 16\pi M_{irr}^2$$

$\hookrightarrow$ $M^2 = M_{irr}^2 + \dfrac{J^2}{4M_{irr}^2}$ $\qquad \left[> M_{irr}^2 \text{ if } J^2 > 0\right]$

- classically, M_{irr} can never decrease

- including quantum (Hawking), BHs have temperature & entropy,

$$S = \frac{k c^3}{G \hbar} \frac{A}{4}$$, k is Boltzmann

- accounting for Hawking radiation, 2^{nd} law states that total BH entropy cannot decrease

Oppenheimer - Volkov stars

- metric for a spherical star:

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\lambda} dr^2 + r^2 d\Omega^2$$

where Φ & λ functions of r & t and

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

- assume perfect fluid:
  - mass-energy density $\rho = \rho(n_b, s)$ where n_b is baryon number density & s is specific entropy
  - pressure from 1^{st} law of thermo,

$$P = n_b^2 \frac{\partial(\rho/n_b)}{\partial n_b} = P(n_b, s)$$

→ reduces to just function of ρ in certain cases (cold $s=0$ or supermassive star $s=$ const.)

- can derive GR eqns. of stellar structure from this metric $\to$ 1st order, coupled, ODEs

- define metric function $m(r)$ by

$$e^{\lambda} \equiv \left(1 - \frac{2m}{r}\right)^{-1}$$

then from Einstein eqns.,

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{\rho m}{r^2}\left(1 + \frac{P}{\rho}\right)\left(1 + \frac{4\pi P r^3}{m}\right)\left(1 - \frac{2m}{r}\right)^{-1}$$

$$\frac{d\Phi}{dr} = -\frac{1}{\rho}\frac{dP}{dr}\left(1 + \frac{P}{\rho}\right)^{-1}$$

$\to$ "Tolman-Oppenheimer-Volkov" eqns.

- total ("gravitational") mass of star,

$$M = \int_0^R 4\pi r^2 \rho \, dr \quad , \quad R = r(P = \rho = 0)$$

$\hookrightarrow$ this matches smoothly onto vacuum Schwarzschild metric.

- total rest-mass is

$$M_0 = \int_0^R 4\pi r^2 \rho_0 \left(1 - 2m/r\right)^{-1/2} dr$$

where ρ_0 is rest-mass density & $\rho = \rho_0(1+\epsilon)$ w/ ϵ the internal energy density per mass

$\to$ bound stars have $M < M_0$

Oppenheimer - Snyder (OS) collapse

- collapse of uniform density, pressureless cloud or star ("dust")
- each mass shell follows geodesic ("free-fall")
- interior metric is closed Friedman metric

$$ds^2 = -d\tau^2 + a^2\left(d\chi^2 + \sin^2\chi \, d\Omega^2\right)$$

τ : time from onset of collapse

χ : Lagrangian radial coordinate

$$a = \frac{1}{2} a_m (1 + \cos\eta)$$

$$\tau = \frac{1}{2} a_m (\eta + \sin\eta)$$

η : conformal time parameter $0 \le \eta \le \pi$

- "synchronous", "Gaussian normal", or "geodesic" coordinates
- surface of star at $\chi = \chi_0$

- exterior metric is Schwarzschild,

$$ds^2 = -\left(1 - \frac{2M}{r_s}\right)dt^2 + \left(1 - \frac{2M}{r_s}\right)^{-1} dr_s^2 + r_s^2 \, d\Omega^2$$

where $r_s = R(\tau)$ is surface of star

$\rightarrow \quad K = \frac{1}{2} K_0 (1 + \cos\eta)$

$$\tau = \left(\frac{R_0^3}{8M}\right)^{1/2} (\eta + \sin\eta)$$

- matching interior & exterior metrics,

$$a_m = \left(\frac{R_0^3}{2M}\right)^{1/2}$$

$$\sin\chi_0 = \left(\frac{2M}{R_0}\right)^{1/2} \rightarrow 0 \leq \chi_0 \leq \pi/2$$

- 4-velocity $u^a = \partial_t$ from geodesic eqn.,

$$0 = u^b \nabla_b u^a = \frac{d^2 x^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda}$$

- rest-mass energy density is function of proper time,

$$\frac{\rho_0(\tau)}{\rho_0(0)} = Q^{-3}(\tau)$$

where $Q(\tau) = \frac{a}{a_m} = \frac{1}{2}(1 + \cos\eta)$

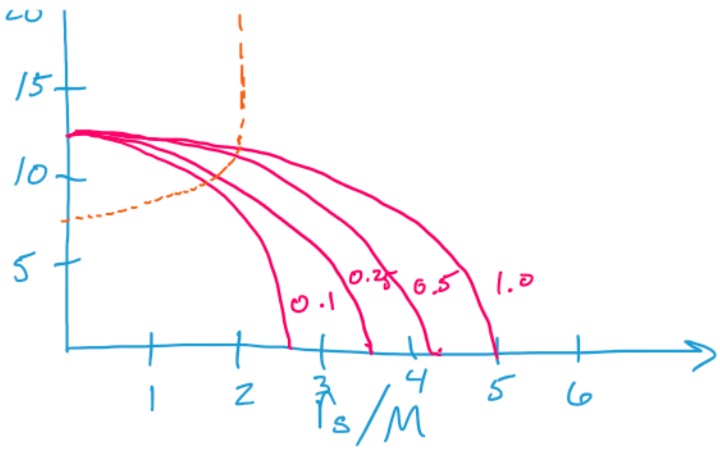
$\rightarrow$ star remains homogeneous ($\rho_0 \neq \rho_0(\chi)$)

- collapse complete at $\eta = \pi$,

$$\tau = \pi (R_0^3 / 8M)^{1/2}$$

$\rightarrow$ singularity formation ($\rho_0 \rightarrow \infty$)

Spacetime diagram:



- location of EH requires knowing global metric
  - for OS collapse, can solve analytically
  - can get from null rays:

$$ds^2 = 0 \quad \text{or} \quad d\tau = a(\tau)\, d\chi$$

w/ defitions of a & τ above,

$$d\chi / d\eta = 1$$

↳ for ray emitted at η_e & χ_e, trajectory

$$\chi = \chi_e + (\eta - \eta_e)$$

- trajectory traces world line of last ray that manages to escape
- originates at center, crosses surface when $R = 2M$

- from $R = \frac{1}{2} R_0 (1 + \cos\eta) = 2M$ Check this

↳ $\eta \equiv \eta_{AH} = 2\cos^{-1}(2M/R_0)^{1/2}$