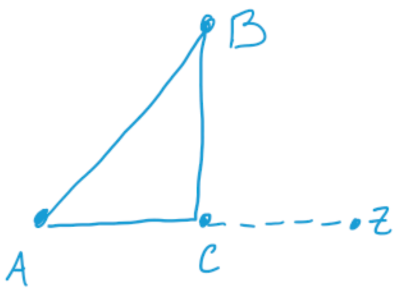
Lecture 2

Monday, January 7, 2019   2:58 PM

"Euclidean" geometry

- See MTW Box 1.3



if geometry is "Euclidean"
then $S_{AB}^2 = S_{AC}^2 + S_{CB}^2$
(Pythagorean)

- can have a "local" Euclidean
- Not true in curved space

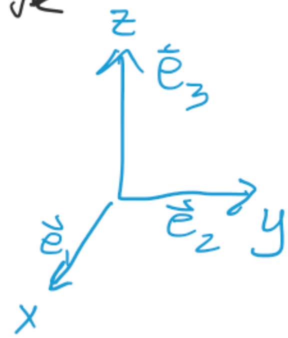
Component Representation of Tensor Algebra

- there are "orthonormal" basis vectors

$$\{\vec{e}_x, \vec{e}_y, \vec{e}_z\} \equiv \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$$

orthogonality $\rightarrow$ $\vec{e}_j \cdot \vec{e}_k = \delta_{jk}$

where $j, k = [1, 2, 3]$



- so a 3-space vector is

$$\vec{A} = A_j \vec{e}_j$$

orthonormality $\rightarrow$ $A_j = \vec{A} \cdot \vec{e}_j$

- similar for any tensor $\overleftrightarrow{T}(-, -, -)$

$$\overleftrightarrow{T} = T_{ijk}\ \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$$

$$T_{ijk} = \overleftrightarrow{T}(\vec{e}_i, \vec{e}_j, \vec{e}_k)$$

- recall, the metric

$$g_{jk} = \overleftrightarrow{g}(\vec{e}_j, \vec{e}_k) = \vec{e}_j \cdot \vec{e}_k = \delta_{jk}$$

- tensor product $\overleftrightarrow{T}(-, -, -) \otimes \overleftrightarrow{S}(-, -)$

$$\overleftrightarrow{T}(\vec{e}_i, \vec{e}_j, \vec{e}_k) \otimes \overleftrightarrow{S}(\vec{e}_l, \vec{e}_m) = T_{ijk} S_{lm}$$

- inner product

$$\vec{A} \cdot \vec{B} = A_j B_j \ ; \ \overleftrightarrow{T}(\vec{A}, \vec{B}, \vec{C}) = T_{ijk} A_i B_j C_k$$

... Einstein summation

- tensor contraction

$C = \text{const} [1\&3 \text{ contraction of } \overleftrightarrow{R}] = R_{\dots}$

components [1-3 contraction of R] = R_{ijik}

∴ rank-4 to rank-2

Slot-naming Index Notation

Define two tensors:

$$\overleftrightarrow{G}(\vec{A}, \vec{B}) = \overleftrightarrow{F}(\vec{B}, \vec{A})$$

need a more compact way to represent this relation.

could say $\overleftrightarrow{F}(\_\_{}_a, \_\_{}_b)$

$\overleftrightarrow{G}(\_\_{}_b, \_\_{}_a)$

→ NAME the slots.

Cool, but .... eww

- how about just $\overleftrightarrow{F}(-_a, -_b) = F_{ab}$?

AHHH. But its fine. Just breath.

∴ $G_{ab} = F_{ba}$

This looks like we have picked a basis, but we shall pretend we have not. since if $G_{ab} = F_{ba}$ is ONLY true if $\overleftrightarrow{G}(-_a, -_b) = \overleftrightarrow{F}(\_\_{}_b, \_\_{}_a)$.

- Consider contraction. The components of a contraction in a particular basis are

$$\left[1 \,\&\, 3 \text{ contraction} \left(\mathbf{T} \right) \right] = T_{aba}$$

in slot-naming index notation, we represent the contraction itself, i.e. not the components, as T_{aba}

<u>Particle Kinetics in index notation</u>

$$v_i = \frac{dx_i}{dt}, \quad p_i = m v_i, \quad a_i = \frac{dv_i}{dt} = \frac{d^2 x_i}{dt}$$

$$E = \frac{1}{2} m v_j v_j, \quad \frac{dp_i}{dt} = q\left(E_i + \epsilon_{ijk} v_j B_k \right)$$

where ϵ_{ijk} is Levi-Civita and produces the cross product

- we can choose to interpret the above as either basis-independent or components in a Cartesian system.

$$\boxed{\text{Orthogonal Transformation of Bases}}$$

- it is possible to expand the basis vectors of a Cartesian coord. system in terms of those of another:

$$\vec{e}_i = \vec{e}_{\bar{p}} R_{\bar{p}i} \quad ; \quad \vec{e}_{\bar{p}} = \vec{e}_i R_{i\bar{p}}$$

Here, $R_{\bar{p}i}$ & $R_{i\bar{p}}$ are the elements of transformation matrices (NOT tensors).
They are also inverse of each other:

$$R_{\bar{p}i} R_{i\bar{q}} = \delta_{\bar{p}\bar{q}} \,, \quad R_{i\bar{p}} R_{\bar{p}j} = \delta_{ij}$$

- Orthonormality $\Rightarrow$ $\delta_{ij} = \vec{e}_i \cdot \vec{e}_j = (\vec{e}_{\bar{p}} R_{\bar{p}i})$

$$(\vec{e}_{\bar{q}} R_{\bar{q}j}) = R_{\bar{p}i} R_{\bar{q}j} (\vec{e}_{\bar{p}} \cdot \vec{e}_{\bar{q}})$$

$$= R_{\bar{p}i} R_{\bar{q}j} \delta_{\bar{p}\bar{q}} = R_{\bar{p}i} R_{\bar{p}j}$$

$$\therefore \; [R_{i\bar{p}}] \equiv \text{Inverse}([R_{\bar{p}i}]) = \text{Transpose}([R_{\bar{p}i}])$$

$\hookrightarrow$ matrix is orthogonal
$\Rightarrow$ reflection or rotation

- in other words, the bases of two Euclidean systems can be related by a reflection or rotation

- since geometric objects (i.e. vector $\vec{A}$) are basis independent:

$$\vec{A} = A_i \vec{e}_i = A_i (\vec{e}_{\bar{p}} R_{\bar{p}i})$$

$$\vec{A} = A_i \vec{e}_i = A_i (\vec{e}_{\bar{p}} R_{\bar{p}i})$$

$$= (R_{\bar{p}i} A_i) \vec{e}_{\bar{p}} = A_{\bar{p}} \vec{e}_{\bar{p}}$$

since $A_{\bar{p}} = R_{\bar{p}i} A_i$

- for rank-3 tensor:

$$T_{\bar{p}\bar{q}\bar{r}} = R_{\bar{p}i} R_{\bar{q}j} R_{\bar{r}k} T_{ijk}$$

- if two coord. systems share common origin, then the vector from origin to some point P, $\vec{x}$, has components which are themselves the coordinates in the respective bases. ∴ since the vectors obey the usual transformation laws, so do the coordinates.

$$x_{\bar{p}} = R_{\bar{p}i} x_i \qquad x_i = R_{i\bar{p}} x_{\bar{p}}$$

- product rule for transformation:

$$[R_{i\bar{p}} R_{\bar{p}\bar{\bar{s}}}] = [R_{i\bar{\bar{s}}}]$$

which transforms $\vec{e}_{\bar{\bar{s}}}$ to $\vec{e}_i$

$$\boxed{\text{Directional derivatives, gradients, Levi-Civita tensor, cross product, \& curl}}$$

• directional derivative in Euclidean 3-space of some tensor $\overleftrightarrow{T}$ along some vector $\vec{A}$,

$$\nabla_{\vec{A}} \overleftrightarrow{T} \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\overleftrightarrow{T}(\vec{x}_p + \epsilon \vec{A}) - \overleftrightarrow{T}(\vec{x}_p)\right]$$

where $\vec{x}_p$ is the vector from origin to point $\mathcal{P}$ where derivative is evaluated.

→ tensor rank is unchanged

→ this is a linear function of $\vec{A}$

• another way to express directional derivative:

$\overleftrightarrow{T}$: rank n tensor field

let $\nabla \overleftrightarrow{T}$ be some other $n+1$ rank tensor field

$$\rightarrow \nabla_{\vec{A}} \overleftrightarrow{T} = \underbrace{\nabla \overleftrightarrow{T}}_{\text{"gradient of } \overleftrightarrow{T}\text{"}}(\_, \_, \_, \underbrace{\vec{A}}_{\text{"differentiation slot"}})$$

• Using slot-naming index notation,

$$\nabla \overleftrightarrow{T} \Longleftrightarrow T_{abc;d}$$

$$\hookrightarrow \nabla_{\vec{A}} \overleftrightarrow{T} \Longleftrightarrow T_{abc;j} A_j$$

• in Cartesian systems, components of gradient are partial derivatives of original tensor,

$$T \quad = \quad \frac{\partial T_{abc}}{} \quad - \quad T$$

$$T_{abc;j} \qquad \overline{\partial x_j} \quad = T_{abc,j}$$

- The gradient and directional derive obey usual rules:

$$\nabla_{\vec{A}}(\overleftrightarrow{S} \otimes \overleftrightarrow{T}) = (\nabla_{\vec{A}}\overleftrightarrow{S}) \otimes \overleftrightarrow{T} + \overleftrightarrow{S} \otimes \nabla_{\vec{A}}\overleftrightarrow{T}$$

$$\left[\ (S_{ab}T_{cde})_{;j}A_j = (S_{ab;j}A_j)T_{cde} + S_{ab}(T_{cde;j}A_j)\right]$$

and

$$\nabla_{\vec{A}}(f\overleftrightarrow{T}) = (\nabla_{\vec{A}}f)\overleftrightarrow{T} + f\nabla_{\vec{A}}\overleftrightarrow{T}$$

$$\left[\ (fT_{abc})_{;j}A_j = (f_{;j}A_j)T_{abc} + fT_{abc;j}A_j\right]$$

- in flat space, Cartesian:

$$\nabla\overleftrightarrow{g} = 0 \quad ; \quad g_{ab;j} = 0$$

- Contraction of the gradient of a vector gives the divergence (a scalar),

$$\nabla\cdot\vec{A} \equiv (\text{contraction of } \nabla\vec{A}) = A_{a;a}$$

- for tensors, divergence can be on certain slots:

$$T_{abc;b} \quad \text{or} \quad T_{abc;c}$$

- taking the double gradient of tensor then contracting on the 2 new slots gives the Laplacian,

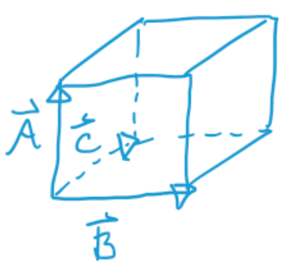
$$\nabla^2\overleftrightarrow{T} \equiv (\nabla\cdot\nabla)\overleftrightarrow{T} \quad ; \quad T_{abc;jj}$$

$\rightarrow$ all indices after semicolon or comma differentiation, $T_{abc,jk} \equiv \dfrac{\partial^2 T_{abc}}{\partial x_j \partial x_k}$

- recall, metric tensor gives the inner product in some space, hence gives notion of distance
- geometric object that embodies a space's volume is Levi-Civita tensor $\overleftrightarrow{\epsilon}$
- parallelopiped w/ vector edges $\vec{A}, \vec{B}, \dots, \vec{F}$:

$$\text{Volume} = \overleftrightarrow{\epsilon}(\vec{A}, \vec{B}, \dots, \vec{F})$$ .... general n-dimension

for 3-space: $\vec{A}, \vec{B}, \vec{C}$



$$\text{Volume} = \overleftrightarrow{\epsilon}(\vec{A}, \vec{B}, \vec{C})$$

- Levi-Civita is anti-symmetric:

$$\overleftrightarrow{\epsilon}(\vec{B}, \vec{A}, \dots \vec{F}) = -\overleftrightarrow{\epsilon}(\vec{A}, \vec{B}, \dots, \vec{F})$$

- for a right-handed orthonormal basis:

$\epsilon_{123} = +1$

$\epsilon_{abc} = +1$ if a, b, c an even permutation of $1, 2, 3$

$= -1$ if a, b, c an odd permutation

of 1, 2, 3

$= 0$ if a, b, c not all different

- Levi-Civita generates cross product & curl

$\vec{A} \times \vec{B} \equiv \overleftrightarrow{\epsilon}(\_, \vec{A}, \vec{B})$

$\epsilon_{ijk} A_j B_k$ ... slot-naming notation

and

$\nabla \times \vec{A} \equiv \epsilon_{ijk} A_{k;j}$

→ this is the double contraction of rank-5 tensor $\overleftrightarrow{\epsilon} \otimes \nabla \vec{A}$ on the second and fifth slots and then on 3^{rd} & 4^{th} slots

- in Euclidean 3-space

$\epsilon_{ijm} \epsilon_{klm} = \delta^{ij}_{kl} \equiv \delta^i_k \delta^j_l - \delta^i_l \delta^j_k$

... property of Levi-Civita

here δ^i_k is Kronecker delta

→ for this to be non-zero, either

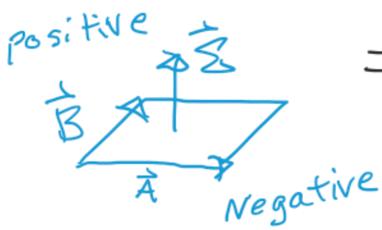
$i = k$ and $j = l$ w/ + sign

or, $i = l$ and $j = k$ w/ − sign

Volumes, Integration, and Conservation Laws

- in 2D, volume is just the area

$$\text{2-volume} = \overleftrightarrow{\epsilon}(\vec{A}, \vec{B}) = \epsilon_{ab} A_a B_b$$



$$= A_1 B_2 - A_2 B_1 = \det \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$$

- in 3D,

$$\text{3-volume} = \overleftrightarrow{\epsilon}(\vec{A}, \vec{B}, \vec{C}) = \epsilon_{ijk} A_i B_j C_k$$
$$= \vec{A} \cdot (\vec{B} \times \vec{C})$$
$$= \det \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

- vectorial surface area in 3-space:

$$\vec{\Sigma} = \vec{A} \times \vec{B} = \overleftrightarrow{\epsilon}(\_, \vec{A}, \vec{B})$$

+ add a third "leg" to 1st slot, get a volume.

- Can make "negative" volume:

$\downarrow \vec{C}$ $\quad \vec{\Sigma}(\vec{C}) = \overleftrightarrow{\epsilon}(\vec{C}, \vec{A}, \vec{B})$ = negative

$\uparrow \vec{C}$ $\quad \vec{\Sigma}(\vec{C})$ = positive

- can use this notation to write Gauss & Stokes theorems for a 3-volume $\mathcal{V}_3$ bounded by 2D surface $\partial \mathcal{V}_3$:

$$\int_{\mathcal{V}_3} (\nabla\cdot\vec{A})\, dV = \int_{\partial\mathcal{V}_3} \vec{A}\cdot d\vec{\Sigma} \quad \ldots \text{ Gauss}$$

and

$$\int_{\mathcal{V}_2} \nabla\times\vec{A}\cdot d\vec{\Sigma} = \int_{\partial\mathcal{V}_2} \vec{A}\cdot d\vec{\ell}$$

where $\partial\mathcal{V}_2$ is the closed curve bounding $\mathcal{V}_2$

- particle number & charge conservation:

ρ_e : charge density ; n : number density

then conservation requires, "current density"

$$\underbrace{\frac{d}{dt}\int_{\mathcal{V}_3} \rho_e\, dV}_{\substack{\text{time rate-of-change of}\\ \text{charge in volume}}} + \underbrace{\int_{\partial\mathcal{V}_3} \vec{j}\cdot d\vec{\Sigma}}_{\substack{\text{flux of charge through}\\ \text{surface of volume}}} = 0$$

particle flux

and

$$\frac{d}{dt}\int_{\mathcal{V}_3} n\, dV + \int_{\partial\mathcal{V}_3} \vec{S}\cdot d\vec{\Sigma} = 0$$

- pulling time derivative inside & using Gauss' law,

$$\int_{\mathcal{V}} \left(\frac{\partial\rho_e}{\partial t} + \nabla\cdot\vec{j}\right) dV = 0$$

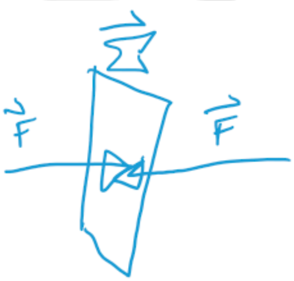
$\longrightarrow$ must be true for ALL integrands

$$\therefore \quad \frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{j} = 0$$

similarly, $$\frac{\partial n}{\partial t} + \nabla \cdot \vec{S} = 0$$

- Note! This derivation required NO coordinate basis. $\to$ Geometric Principle

$\boxed{\text{The Stress tensor and Momentum Consrv.}}$



- the stress tensor is that object that returns a force when an area vector is inserted in its last slot:

$$\vec{F}(\_) = \overleftrightarrow{T}(\_, \vec{\Sigma}) \; ; \quad F_i = T_{ij} \Sigma_j$$

- sign of $\vec{\Sigma}$ determines direction of force
- can define the components of stress tensor as

T_{jk} = j-component of force per unit area across a surface perpendicular to $\vec{e}_k$

= j-component of momentum that crosses a unit area which is

perpendicular to $\vec{e}_k$, per unit
time, crossing from $-x_k$ to $+x_k$

- stress tensor is always symmetric in its two slots:

$$T_{ij} = T_{ji}$$

Stress Tensor of a Perfect Fluid

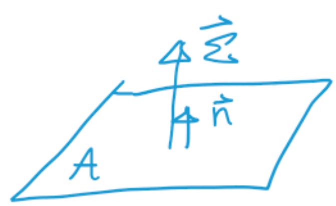
"perfect" fluid is one in which pressure is isotropic and there is no shear stresses.

↳ stress tensor is diagonal in Cartesian,

$$T_{xx} = T_{yy} = T_{zz} = P$$

or $\quad T_{ij} = P\delta_{ij} = Pg_{ij} \quad$ (Euclidean)

.... This is true coordinate free!



- for figure, the vector force exert by fluid across $\vec{\Sigma}$ is $\quad F_j = T_{jk}\Sigma_k = Pg_{jk}An_k = PAn_j$

→ exactly what we expect! pressure is force times area, normal to surface

- from wordy definition above, stress tensor is the flux of momentum.

We can arrive at law of momentum conservation from this:

$$\vec{G} := \text{momentum density}$$

$\to \int_{\mathcal{V}_3} \vec{G}\, dV$ ... total momentum in $\mathcal{V}_3$

since stress tensor is momentum flux,

$$\frac{d}{dt}\int_{\mathcal{V}_3} \vec{G}\, dV + \int_{\partial\mathcal{V}_3} \overleftrightarrow{T} \cdot d\vec{\Sigma} = 0$$

Use same trick as before w/ Gauss' law

$$\to \int_{\mathcal{V}_3} \left(\frac{\partial \vec{G}}{\partial t} + \nabla \cdot \overleftrightarrow{T} \right) dV = 0$$

$$\therefore \frac{\partial \vec{G}}{\partial t} + \nabla \cdot \overleftrightarrow{T} = 0 \; ; \quad \frac{\partial G_j}{\partial t} + T_{jk;k} = 0$$