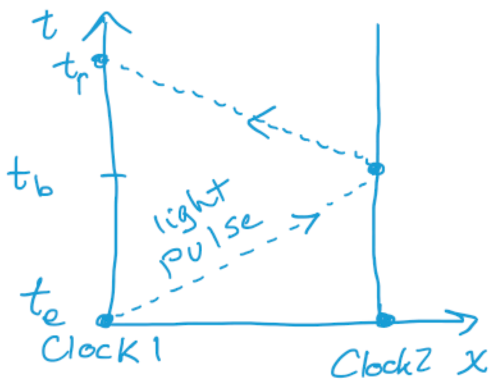
Lecture3

Friday, January 11, 2019 3:29 PM

Special Relativity: Geometric Viewpoint

- introduce 4D Minkowski spacetime
  - "special" because there is no gravity
  - acceleration is permitted
- inertial reference frame or Lorentz frame
  1. purely conceptual latticework; does not gravitate
  2. moves freely in spacetime (no forces act) & non-rotating
  3. forms orthogonal lattice and length marks are uniform (ie, Cartesian)
  4. clocks measuring time at every point
  5. clocks are "ideal": tick uniformly compared to physical quantities
  6. clocks are "Einstein" synchronized:



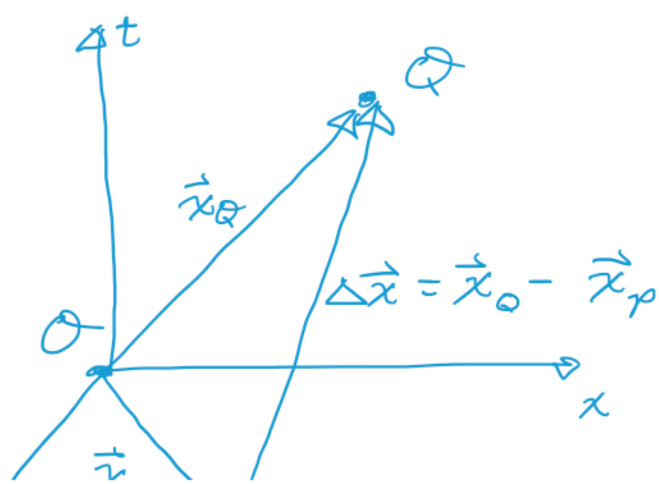
$$t_b = \frac{1}{2}(t_e + t_r)$$

- "event": a precise location at a precise time, i.e., a point in 4D spacetime.
- spacetime vector: 4D vector, book uses arrow notation, my notes will just be confusing as they will use arrows for both ...
- in inertial frame, coordinates of an event:

  $$(x^0, x^1, x^2, x^3) = (t, x, y, z)$$

  ↳ time t is measured by clock that resides at the event's location.

- spacetime diagram:



$$\Delta x^0 = \Delta t = t_Q - t_P \qquad \Delta x^1 = \Delta x = x_Q - x_P$$

$$\Delta x^2 = \Delta y = y_Q - y_P \qquad \Delta x^3 = \Delta z = z_Q - z_P$$

or Δx^α where Greek indices : $(0, 1, 2, 3)$

Principle of Relativity & Light speed

- "All special relativistic laws are the same in every inertial reference frame everywhere in spacetime."
  .... Einstein
- "All special relativistic laws must be expressible as geometric, frame-independent relationships between geometric objects."
  .... "Modern" version
- in practice, any experiment testing fundamental physics in two different inertial reference frames must find the same results.

- Constancy of speed of light in all frames
  follows from Maxwell's equations ...

follows from Maxwell's equations. Also, an experemental fact.

The interval & invariance

- the interval $(\Delta s)^2$ is distance between events in spacetime.

$$(\Delta s)^2 \equiv -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$
$$= -(\Delta t)^2 + \sum_{i,j} \delta_{ij} \Delta x^i \Delta x^j$$

- if $(\Delta s)^2 > 0 \Rightarrow$ "spacelike"
- if $(\Delta s)^2 < 0 \Rightarrow$ "timelike"
- if $(\Delta s)^2 = 0 \Rightarrow$ "null"
- timelike intervals: $(\Delta \tau)^2 \equiv -(\Delta s)^2$

  .... avoids imaginary

- Principle of Relativity requires that the interval be the same in all frames. It is just a geometric object of the separation vector $\Rightarrow (\Delta \vec{x})^2 \equiv (\Delta s)^2$

- Tensor algebra in spacetime behaves the same as for Euclidean 3-space, using the interval in place of the invariant distance.

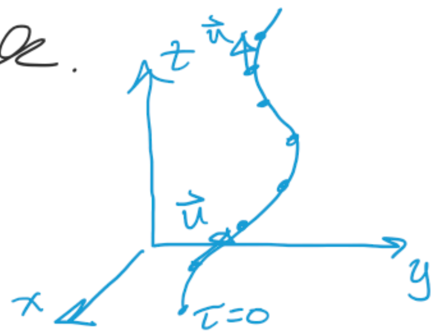
• spacetime metric still generates the inner product: $\overleftrightarrow{g}(\vec{A}, \vec{B}) \equiv \vec{A} \cdot \vec{B}$

... where $\vec{A}$ + $\vec{B}$ are now 4-vectors

Particle Kinetics and Lorentz Force

• "ideal" clock: unaffected by accelerations
ticks same as unaccelerated clocks
momentarily at rest with them

• "proper time τ: time as measured
by a particle's ideal clock.



• particle 4-velocity:

$$\vec{u} = \frac{d\mathcal{P}}{d\tau} = \frac{d\vec{x}}{d\tau}$$

where $\mathcal{P}(\tau)$ is particle world line.

$$\frac{d\mathcal{P}}{d\tau} = \frac{d\vec{x}}{d\tau} \equiv \lim_{\Delta\tau \to 0} \frac{\mathcal{P}(\tau+\Delta\tau) - \mathcal{P}(\tau)}{\Delta\tau}$$

• magnitude of 4-velocity:

$$\vec{u}^2 \equiv \overleftrightarrow{g}(\vec{u}, \vec{u}) = \frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} = \frac{(\Delta s)^2}{(d\tau)^2} = -1$$

... since $\tau = \sqrt{-s^2}$ along particle's world line

• 4-momentum: $\vec{p} \equiv m\vec{u} = m\frac{d\vec{x}}{d\tau} \equiv \frac{d\vec{x}}{d\zeta}$

where $\zeta \equiv \tau/m$ is a renormalized proper time — affine parameter

- taking the square of the 4-momentum and using $\vec{u}^2 = -1$, we have
$$\vec{p}^2 = -m^2$$
- for zero-rest mass particles (photons, gravitons), must take limit as $m \to 0$ & $d\tau \to 0$ but keep $d\zeta = d\tau/m$ finite.

  ↳ $\vec{p} = \frac{d\vec{x}}{d\zeta}$ .... same

  but $\vec{u} = \vec{p}/m$ is undefined.

  also $\tau = m\zeta = 0$ & since $\tau = \sqrt{-s^2}$, the interval $\Delta s = 0$ .... "null"

  → world line is ∴ null
- for massive particle, $d\tau^2 > 0$, $ds^2 < 0$

  → "timelike"
- if no external forces, $\vec{p}$ is conserved along world line:
$$\frac{d\vec{p}}{d\zeta} = 0$$

  like $\vec{u}$, $\vec{p}$ is tangent to world line,

  ∴ free particles move in straight lines through spacetime.
- must apply a force to change direction:

$$\frac{d\vec{p}}{d\tau} = \vec{F}$$

if rest mass is conserved (ie. for fundamental particles)

$$0 = \frac{dm^2}{d\tau} = -\frac{d\vec{p}^{\,2}}{d\tau} = -2\vec{p}\cdot\frac{d\vec{p}}{d\tau} = -2\vec{p}\cdot\vec{F}$$

$\hookrightarrow$ 4-force must be orthogonal to 4-momentum.

Lorentz force law

Consider particle of charge q interacting w/ $\vec{E}$ & $\vec{B}$.

- Newtonian force is

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \quad \ldots \text{ a vector}$$

$\to$ proportional to scalar q & <u>linear</u> in the vector 3-velocity $\vec{v}$

$\therefore$ there must be a tensor that gives this force when 4-velocity is inserted:

$$\frac{d\vec{p}}{d\tau} = \vec{F}(\_) = q\,\overleftrightarrow{F}(\_,\vec{u})$$

... electromagnetic field tensor

- from above, we know forces must be orthogonal to momenta: $\vec{F}\cdot\vec{u} = \vec{F}(\vec{u}) = 0$

$$\overleftrightarrow{F}(\vec{u},\vec{u}) = 0$$

or $\quad F(u,u) = 0$

$\hookrightarrow$ $\vec{F}$ is antisymmetric: $\overleftrightarrow{F}(\vec{A},\vec{B}) = -\overleftrightarrow{F}(\vec{B},\vec{A})$

Component Representation of Tensor Algebra

- in inertial reference frame, there are Lorentz coords. $\{t, x, y, z\} = \{x^0, x^1, x^2, x^3\}$ and associated basis vectors:

$$\{\vec{e}_t, \vec{e}_x, \vec{e}_y, \vec{e}_z\} = \{\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$$

$$\vec{e}_0 \cdot \vec{e}_0 = -1 \; ; \; \vec{e}_a \cdot \vec{e}_a = +1 \, , \; a = 1,2,3$$

$$\vec{e}_a \cdot \vec{e}_b = 0 \quad , \; a \neq b$$

$$\rightarrow \quad \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} \quad \text{... spacetime Kronecker}$$

- if this relation is true for given basis then it is orthonormal in 4D
- in order to deal with non-orthonormal bases and/or non-flat spacetime must introduce different types of components for tensors:

  Contravariant ; upstairs, $T^{\alpha\beta\gamma}$

  Covariant : downstairs, $T_{\alpha\beta\gamma}$
- contravariant $\Rightarrow$ expansion coefficients in chosen basis:

$$\vec{A} \equiv A^{\alpha}\vec{e}_{\alpha} \quad ; \quad \overleftrightarrow{T} \equiv T^{\alpha\beta\gamma}\vec{e}_{\alpha}\otimes\vec{e}_{\beta}\otimes\vec{e}_{\gamma}$$

[Greek indices are summed over when repeated w/ one up & one down]

- Covariant $\Rightarrow$ numbers produced by evaluating tensor on basis vectors:

$$A_{\alpha} \equiv \vec{A}(\vec{e}_{\alpha}) = \vec{A}\cdot\vec{e}_{\alpha}$$

$$T_{\alpha\beta\gamma} \equiv \overleftrightarrow{T}(\vec{e}_{\alpha}, \vec{e}_{\beta}, \vec{e}_{\gamma})$$

- Consequences of these definitions:

(i) $\quad g_{\alpha\beta} = \overleftrightarrow{g}(\vec{e}_{\alpha}, \vec{e}_{\beta}) = \vec{e}_{\alpha}\cdot\vec{e}_{\beta} = \eta_{\alpha\beta}$

(ii) covariant from contravariant:

$$\begin{aligned} T_{\lambda\mu\nu} &= \overleftrightarrow{T}(\vec{e}_{\lambda}, \vec{e}_{\mu}, \vec{e}_{\nu}) \\ &= T^{\alpha\beta\gamma}\vec{e}_{\alpha}\otimes\vec{e}_{\beta}\otimes\vec{e}_{\gamma}(\vec{e}_{\lambda}, \vec{e}_{\mu}, \vec{e}_{\nu}) \\ &= T^{\alpha\beta\gamma}(\vec{e}_{\alpha}\cdot\vec{e}_{\lambda})(\vec{e}_{\beta}\cdot\vec{e}_{\mu})(\vec{e}_{\gamma}\cdot\vec{e}_{\nu}) \end{aligned}$$

$$T_{\lambda\mu\nu} = T^{\alpha\beta\gamma} g_{\alpha\lambda}\, g_{\beta\mu}\, g_{\gamma\nu}$$

.... metric is used to raise (and lower) indices.

(iii) since $g_{00} = -1$, when lowering an index this way there is a "sign-flip-if-temporal"

$$T^{ijk} = T_{ijk} \; ; \; T_{0jk} = -T^{0jk}$$

$$T_{0j0} = +T^{0j0} \; ; \; T_{000} = -T^{000}$$

→ note that the metrics components are numerically identical, up or down

(iv) same rule applys when raising indices

$$T^{\alpha\beta\gamma} = T_{\lambda\mu\nu}\, g^{\lambda\alpha}\, g^{\mu\beta}\, g^{\nu\gamma}$$

(v) mixed component tensors:

$$T^{\alpha}{}_{\mu\nu} = T^{\alpha\beta\gamma} g_{\beta\mu}\, g_{\gamma\nu} = T_{\lambda\mu\nu}\, g^{\lambda\alpha}$$

↳ $g^{\alpha}{}_{\beta} = \delta_{\alpha\beta}$ .... +1 if $\alpha = \beta$

- Algebra rules:

$$[\text{contravariant } \overleftrightarrow{T}(\_,\_,\_) \otimes \overleftrightarrow{S}(\_,\_)] = T^{\alpha\beta\gamma} S^{\delta\epsilon}$$

$$\vec{A} \cdot \vec{B} = A^{\alpha} B_{\alpha} = A_{\alpha} B^{\alpha}$$

$$\overleftrightarrow{T}(\vec{A}, \vec{B}, \vec{C}) = T_{\alpha\beta\gamma} A^{\alpha} B^{\beta} C^{\gamma} = T^{\alpha\beta\gamma} A_{\alpha} B_{\beta} C_{\gamma}$$

Covariant comps. of [1 & 3 contraction $\overleftrightarrow{R}$]

$$= R^{\mu}{}_{\alpha\mu\beta}$$

Contravariant Comps. o [1 & 3 contraction of $\overleftrightarrow{R}$]

$$= R^{\mu\alpha}{}_{\mu}{}^{\beta}$$

- Can still use slot-naming index notation:

$$q\, \overleftrightarrow{F}(\_, \vec{u}) \iff dp_{\mu}/d\tau = q\, F_{\mu\nu} u^{\nu}$$

... frame-independent Lorentz force

- invariant interval from x^{α} to $x^{\alpha} + dx^{\alpha}$:

$$ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} = -dt^2 + dx^2 + dy^2 + dz^2$$

.... special relativistic line element

Particle Kinetics in Lorentz Frame

- components of 4-velocity:

$$u^\alpha = dx^\alpha / d\tau$$

... where dx^α are contravariant components of two neighboring events along particle world line

$$\rightarrow v^j \equiv \frac{dx^j}{dt} = \frac{dx^j/d\tau}{dt/d\tau} = \frac{u^j}{u^0}$$

where v^j is 3-velocity w.r.t. reference frame

- recall, $-1 = \vec{u}^{\,2} = g_{\alpha\beta} u^\alpha u^\beta = -(u^0)^2 + \delta_{ij} u^i u^j$
$$= -(u^0)^2 (1 - \delta_{ij} v^i v^j)$$

$\rightarrow$ $u^0 = \gamma$; $u^j = \gamma v^j$; $\gamma = (1 - \delta_{ij} v^i v^j)^{-1/2}$

- $\vec{v}$ is the "ordinary" velocity at some constant Lorentz ("coordinate") time t
this 3-space is called the "slice of simultaneity": all events are simultaneous at t, according to the frames observers.
$\therefore$ $\vec{v}$ relies on the choice of reference frame for its existence.
- time component of 4-momentum $\vec{p}$ is particles relativistic energy $\mathcal{E}$ as

measured in frame

$$\mathcal{E} \equiv p^0 = m u^0 = m\gamma = \frac{m}{\sqrt{1-\vec{v}^2}}$$

3-velocity

$$\simeq \underset{\text{rest mass}}{m} + \underset{\text{Kinetic energy } E}{\tfrac{1}{2} m \vec{v}^2} \ldots |\vec{v}| \ll 1$$

- spatial part of 4-momentum is just the momentum vector in Lorentz frame's 3-space:

$$p^j = m u^j = m\gamma v^j = \frac{m v^j}{\sqrt{1-\vec{v}^2}} = \mathcal{E} v^j$$

- for zero rest mass particles, same rules apply:

$$\mathcal{E} \equiv p^0 = \hbar\omega \quad \ldots \text{particle energy}$$

where ω is angular frequency of corresponding quantum wave

- recall for lightlike world lines:

$$\vec{p}^{\,2} = -(p^0)^2 + p^j p_j = -m^2 = 0$$

↳ 3-momentum in chosen frame:

$$\vec{p} = \mathcal{E}\vec{n} = \hbar\omega\vec{n}$$

where $\vec{n}$ is unit 3-vector in direction of particle motion

since particle moves at $c\,(=1)$, $\vec{n}$ is its ordinary velocity.

- above illustrates a "3+1 split" of

the 4-momentum into a 3-vector and a scalar, $\vec{p}$ and $p^0 = \mathcal{E}$

→ seperate laws of energy and momentum conservation.

- concepts of energy & 3-momentum require choice of frame for definition

  ∴ less fundamental than conservation of frame-independent 4-momentum

- Consider an observer at rest in their frame measuring 4-momentum of some particle. Observer's 4-velocity (in their rest frame) is $U^0 = 1$, $U^i = 0$

  or $\vec{U} = U^\alpha \vec{e}_\alpha = \vec{e}_0$

→ energy of particle as measured by observer is

$$\mathcal{E} = p^0 = -p_0 = -\vec{p} \cdot \vec{e}_0$$
$$= -\vec{p} \cdot \vec{U}$$

.... using the above rules of index gymnastics

→ this is a <u>frame-independent</u> geometric relation! same in all frames!

- this is just the time part for a 3+1 split of particle's 4-momentum

Lorentz Transformations

- transformation between 2 different inertial reference frames in Minkowski space:

$$\vec{e}_\alpha = \vec{e}_{\bar\mu} L^{\bar\mu}{}_\alpha \ , \quad \vec{e}_{\bar\mu} = \vec{e}_\alpha L^\alpha{}_{\bar\mu}$$

where $\{x^\alpha\}$ & $\{x^{\bar\mu}\}$ are their Lorentz coordinates w/ basis vectors $\{\vec{e}_\alpha\}$ & $\{\vec{e}_{\bar\mu}\}$.

- As in 3-space, L are the components of transformation <u>matrices</u> that must be inverse of one another:

$$L^{\bar\mu}{}_\alpha L^\alpha{}_{\bar\nu} = \delta^{\bar\mu}{}_{\bar\nu} \ ; \quad L^\alpha{}_{\bar\mu} L^{\bar\mu}{}_\beta = \delta^\alpha{}_\beta$$

[by convention first index on L is always up]

- in Minkowski, orthonormality implies

$$g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta = (\vec{e}_{\bar\mu} L^{\bar\mu}{}_\alpha) \cdot (\vec{e}_{\bar\nu} L^{\bar\nu}{}_\beta)$$

$$= L^{\bar\mu}{}_\alpha L^{\bar\nu}{}_\beta \, g_{\bar\mu\bar\nu}$$

$$\rightarrow g_{\bar\mu\bar\nu} L^{\bar\mu}{}_\alpha L^{\bar\nu}{}_\beta = g_{\alpha\beta}$$

& $g_{\alpha\beta} L^\alpha{}_{\bar\mu} L^\beta{}_{\bar\nu} = g_{\bar\mu\bar\nu}$ ... Lorentz transformation

$\eta_{\alpha\beta} \ldots {}_{\bar\mu} \ldots {}_{\bar\nu} = \eta_{\bar\mu\bar\nu}$ ... Lorentz transformation

- transformation of components in spacetime:

$$A^{\bar\mu} = L^{\bar\mu}{}_{\alpha} A^{\alpha}, \quad T^{\bar\mu\bar\nu\bar\rho} = L^{\bar\mu}{}_{\alpha} L^{\bar\nu}{}_{\beta} L^{\bar\rho}{}_{\gamma} T^{\alpha\beta\gamma}$$

→ same for coordinates of two frames sharing an origin

- Pure boost along x-direction:

$$\| L^{\alpha}{}_{\bar\mu} \| = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \| L^{\bar\mu}{}_{\alpha} \| = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $|\beta| < 1$, $\gamma \equiv (1-\beta^2)^{-1/2}$

- these matrices are inverse of each other

→ applying these to coordinates,

$$x^{\alpha} = L^{\alpha}{}_{\bar\mu} x^{\bar\mu}, \quad x^{\bar\mu} = L^{\bar\mu}{}_{\alpha} x^{\alpha}$$

↳ $t = \gamma(\bar t + \beta \bar x)$, $x = \gamma(\bar x + \beta \bar t)$, $y = \bar y$, $z = \bar z$

$\bar t = \gamma(t - \beta x)$, $x = \gamma(x - \beta t)$, $\bar y = y$, $\bar z = z$

∴ for observer's at rest in boosted frame, unboosted frame moves with uniform velocity $\vec v = -\beta \vec e_{\bar x}$.

+ the opposite for observers in unboosted frame. ie, <u>a boost</u>