Lecture12

Wednesday, February 20, 2019 1:30 PM

The ADM Equations

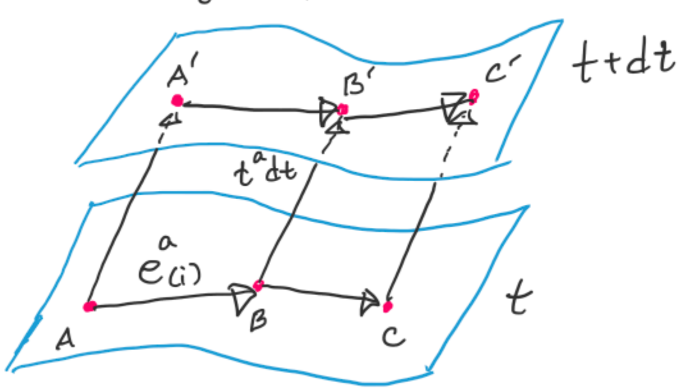
- Evolution eqns. above simplify if we choose our basis wisely.
- introduce spatial basis vectors $e^a_{(i)}$, where $i = 1, 2, 3$ distinguish the 3 vectors, not their components (represented by "a") such that,

$$\Omega_a e^a_{(i)} = 0$$

↳ vectors reside entirely in time slice Σ

- vectors are extended to other slices by "Lie dragging" them along t_a

$$\mathcal{L}_{\vec{t}}\, e^a_{(i)} = 0$$



- for fourth basis vector, choose $e^a = t^a$

for fourth basis vector, choose $e_{(0)} = t$

- want this to connect events on neighbouring slices w/ same spatial coords.

• recall duality, $t^a \Omega_a = 1$

↳ $t^a = e^a_{(0)} = (1, 0, 0, 0)$

∴ Lie derivative reduces to partial derivative along t: $\mathcal{L}_{\vec{t}} = \partial_t$

• now, $\Omega_a e^a_{(i)} = -\frac{1}{\alpha} n_a e^a_{(i)} = 0$

• this choice of coordinates implies

$$n_i = 0$$

↳ all tensors with contravariant index $= 0$ vanish. e.g.,

$$n_a \beta^a = n_0 \beta^0 = 0$$

$$\rightarrow \quad \beta^a = (0, \beta^i)$$

• so, $t^a = \alpha n^a + \beta^a = (1, 0, 0, 0)$

$$\rightarrow \quad n^a = (\alpha^{-1}, -\alpha^{-1}\beta^i)$$

and

$$n_a n^a = -1 \quad \text{as it must}$$

↳ $n_a = (-\alpha, 0, 0, 0)$

• from $\gamma_{ab} = g_{ab} + n_a n_b$

$$\rightarrow \quad \gamma_{ij} = g_{ij}$$

- spatial metric is just the spatial components of the full metric

- since all $\gamma^{a0} = 0$,

$$g^{ab} = \gamma^{ab} - n^a n^b = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ \alpha^{-2}\beta^j & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}$$

- γ^{ij} and γ_{ij} are 3D inverses,

$$\gamma^{ik}\gamma_{kj} = \delta^i_j$$

→ can be used for raising & lowering indices of spatial tensors,

$$\beta_i = \gamma_{ij}\beta^j$$

- inverting the inverse 4-metric,

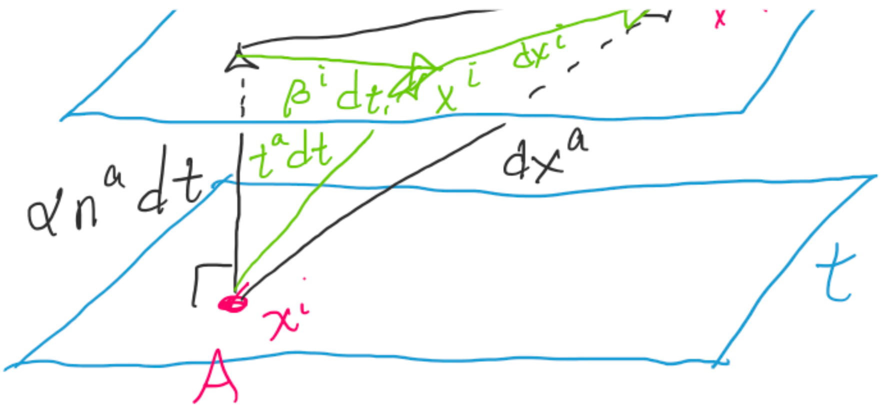
$$g_{ab} = \begin{pmatrix} -\alpha^2 + \beta_l\beta^l & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}$$

or,

$$ds^2 = -\alpha^2 dt + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

.... 3+1 metric

- like Pythagorean theorem in 4D, determines invariant interval between events



- with this choice of 3+1 decomposition and basis vectors, entire content of Einstein eqns. is in their spatial components alone

∴ Hamiltonian constraint,

$$R + K^2 - K_{ij} K^{ij} = 16\pi \rho$$

- Momentum constraint,

$$D_j \left(K^{ij} - \gamma^{ij} K \right) = 8\pi S^i$$

- evolution equation for extrinsic curvature,

$$\begin{aligned} \partial_t K_{ij} = & -D_i D_j \alpha + \alpha \left(R_{ij} - 2K_{ik} K^k{}_j + K K_{ij} \right) \\ & - 8\pi\alpha \left(S_{ij} - \tfrac{1}{2} \gamma_{ij} (S - \rho) \right) \\ & + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k \end{aligned}$$

- evolution for spatial metric,

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

---- Arnowitt, Deser, Misner (ADM) eqns.

- determinant $\gamma = \det(\gamma_{ij})$ satisfies

$\partial_t \ln \delta = -\alpha K + D_i \beta$

& trace $K = K^i{}_i$ satisfies,

$$\partial_t K = -D^2 \alpha + \alpha\left(K_{ij} K^{ij} + 4\pi(\rho + S)\right) + \beta^i D_i K$$

where $D^2 = \gamma^{ij} D_i D_j$ is Laplacian

- matter source terms:

$$\rho = n_a n_b T^{ab} \qquad S^i = -\gamma^{ij} n^a T_{aj}$$

$$S_{ij} = \gamma_{ia} \gamma_{jb} T^{ab} \qquad S = \gamma^{ij} S_{ij}$$

- Evaluating RHS of ADM eqns, can disregard all "0" components, eg. $D_a D_b \alpha = \partial_a \partial_b \alpha - \partial_c \alpha \Gamma^c{}_{ba}$
- & from $\gamma^{a0} = 0$, $\Gamma^0{}_{ba} = 0$

$\therefore$ connection coeffs can be computed from spatial components alone:

$$\Gamma^i{}_{jk} = \frac{1}{2} \gamma^{il}\left(\partial_k \gamma_{lj} + \partial_j \gamma_{lk} - \partial_l \gamma_{jk}\right)$$

- Ricci tensor,

$$R_{ij} = \partial_k \Gamma^k{}_{ij} - \partial_j \Gamma^k{}_{ik} + \Gamma^k{}_{ij} \Gamma^l{}_{kl} - \Gamma^k{}_{il} \Gamma^l{}_{jk}$$

or

$$R_{ij} = \frac{1}{2} \gamma^{kl}\left(\partial_i \partial_l \gamma_{kj} + \partial_k \partial_j \gamma_{il} - \partial_i \partial_j \gamma_{kl} - \partial_k \partial_l \gamma_{ij}\right) + \gamma^{kl}\left(\Gamma^m{}_{il} \Gamma_{mkj} - \Gamma^m{}_{ij} \Gamma_{mkl}\right)$$

- for Schwarzschild:

$$ds^2 = -\left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right)$$

$\rightarrow$ lapse:

$$\alpha = \frac{1 - M/2r}{\ldots}$$

$\alpha = 1 + M/2r$

shift: $\beta^i = 0$

spatial metric:

$$\gamma_{ij} = \left(1 + \frac{M}{2r}\right)^4 \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$$

- takes the form,

$$\gamma_{ij} = \psi^4 \eta_{ij}$$

where $\psi = 1 + M/2r$ is the conformal factor

- since γ_{ij} is independent of β^i,

$$K_{ij} = 0$$

- Ricci tensor,

$$R_{rr} = -\frac{8rM}{(2r^2+Mr)^2} \qquad R_{\theta\theta} = \frac{4r^3M}{(2r^2+Mr)^2}$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

[in vacuum, ${}^{(4)}R_{ij} = 0$, but $R_{ij} \neq 0$]

- above relations satisfy constraints
  - Hamiltonian: $R + K^2 - K_{ij}K^{ij} = 16\pi\rho$

$\to K_{ij} = \rho = 0 \to R = 0$

so, $\gamma^{ij} R_{ij} = 0$

- momentum constraint,

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i$$

$K^{ij} = K = S^i = 0$ .... so easy

- spatial metric evolution eqn. gives $K_{ij} = 0$
- only non-trivial eqn. then is extrinsic curvature evolution equation

$\hookrightarrow$ $\partial_t K_{ij} = 0$ ... as expected