Lecture21

Wednesday, April 10, 2019 3:07 PM

Gravitational Waves

- take the weak field limit of GR:

$$g_{ab} = \eta_{ab} + h_{ab} \qquad |h_{ab}| \ll 1$$

h_{ab} : "metric perturbation"

- introduce trace-reversed perturbation:

$$\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab} h_c{}^c$$

- choose Lorentz gauge such that,

$$\nabla_a \bar{h}_{ab} = 0$$

which reduce the field eqns. to wave eqn. in vacuum:

$$\Box \bar{h}_{ab} \equiv \nabla^c \nabla_c \bar{h}_{ab} = 0$$

$\Box$: 4D Laplacian

- $\bar{h}_{ab}$ is not yet uniquely identified

→ need more gauge! (choices)

- transverse - traceless gauge:

$$\bar{h}^{TT}_{a0} = 0 \quad , \quad \bar{h}^{TT a}{}_{a} = 0$$

→ the only non-zero components of $\bar{h}^{TT}_{ab}$ are spatial

→ $h_c{}^c = 0 \quad \therefore \quad \bar{h}_{ab} = h_{ab}$

- TT components of Riemann tensor,

$${}^{(4)}R_{i0j0} = -\frac{1}{2}\ddot{h}^{TT}_{ij} \quad \left(\begin{array}{c}\text{second time}\\ \text{derivative}\end{array}\right)$$

- Lorentz & TT gauges specify 8 of 10 free components of h_{ab}

→ last two represent polarization states of GWs

- introduce polarization tensors: e^{+}_{ab}, $e^{\times}_{ab}$ where, (in Cartesian)

$$e^{+}_{xx} = -e^{+}_{yy} = 1 \;,\; e^{\times}_{xy} = e^{\times}_{yx} = 1$$

and all other components are zero

- then GW specified by:

$$h_{jk}^{TT} = h_+ e_{ij}^+ + h_\times e_{ij}^\times$$

h_+ & $h_\times$ dimensionless amplitudes

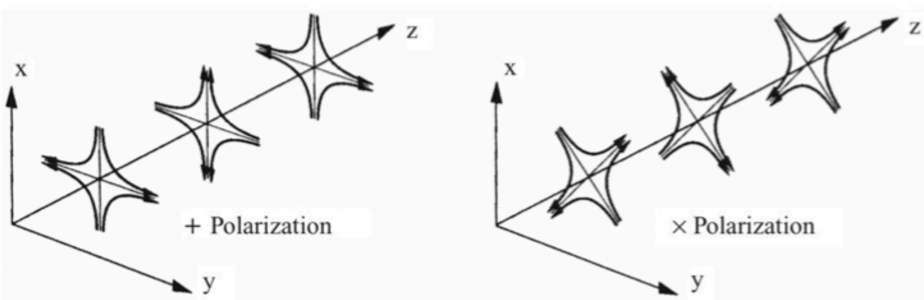


Figure 9.1 Lines of force associated with the two polarization states h_+ and $h_\times$ of a linear plane gravitational wave traveling in vacuum in the z-direction. [From Abramovici *et al.* (1992).]

- GW passing two test particles in free-fall w/ initial separation ξ^i:

$$\ddot{\xi}_i = \frac{1}{2} \ddot{h}_{ij}^{TT} \xi^j$$

↳ induces acceleration & change in separation

- relative strain between 2 particles:

$$\delta\xi / \xi \propto h_{ij}^{TT}$$

↳ call h_{ij}^{TT} "strain" very often

- weak-field, slow-velocity source:

$$\Box \bar{h}_{ij} = \nabla^c \nabla_c \bar{h}_{ij} = -16\pi T_{ij}$$

$$\Box \bar{h}_{ab} = \nabla^c \nabla_c \bar{h}_{ab} = -16\pi T_{ab}$$

- w/ appropriate Green's function,

$$\bar{h}_{ab}(t, x^i) = 4\int d^3x' \frac{T_{ab}(t - |x^i - x'^i|, x'^i)}{|x^i - x'^i|}$$

- taking limit of distant source, can expand integral in powers of x'^i/r:

$$\bar{h}_{ij}(t, x^k) = \frac{4}{r}\int d^3x' \, T_{ij}(t-r, x'^k)$$

- use virial theorem:

(1) $$\int d^3x' \, T^{ij} = \frac{1}{2}\frac{d^2}{dt^2}\int d^3x' \, T^{tt} x'^i x'^j$$

(2) and $T^{tt} \approx \rho_0$ in Newtonian limit

and definition of second moment of mass:

(3) $$I^{ij}(t) \equiv \int d^3x' \rho_0(t, x'^k) x'^i x'^j$$

- (1) + (2) + (3) give

$$\bar{h}_{ij}(t, x^k) = \frac{2}{r}\ddot{I}_{ij}(t-r)$$

- define reduced quadrupole moment:

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\eta_{ij} I \qquad (I = I^a{}_a)$$

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I \qquad (I = I^a{}_a)$$

.... traceless part of I_{ij}

- use projection operator,

$$P_i{}^j \equiv \eta_i{}^j - n_i n^j$$

$$n^i = x^i/r \quad \text{direction of propagation}$$

to remove TT part:

$$\mathcal{I}_{ij}^{TT} \equiv \left(P_i{}^k P_j{}^l - \frac{1}{2} P_{ij} P^{kl} \right) \mathcal{I}_{kl}$$

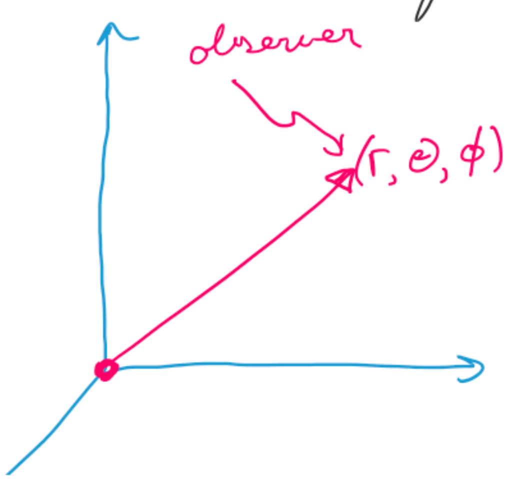
- then

$$h_{ij}^{TT}(t, x^k) = \frac{2}{r} \ddot{\mathcal{I}}_{ij}^{TT}(t-r)$$

... weak-field, slow

$\longrightarrow$ "quadrupole" approximation

- consider GWs from source at origin:



- 3-metric:

$$d\ell^2 = dr^2 + (1+h_+)r^2 d\theta^2 + (1-h_+)\sin^2\theta\, d\phi^2 + 2h_+ \sin\theta\, d\theta\, d\phi$$

- in an orthonormal basis $(\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}})$, polarization modes are

$$h_+ = \frac{1}{r}\left(\ddot{\mathcal{I}}_{\hat{\theta}\hat{\theta}} - \ddot{\mathcal{I}}_{\hat{\phi}\hat{\phi}}\right)$$

$$h_\times = \frac{1}{r}\ddot{\mathcal{I}}_{\hat{\phi}\hat{\phi}}$$

- and in Cartesian basis,

$$h_+ = \frac{1}{r}\Bigg[\frac{\ddot{\mathcal{I}}_{xx} - \ddot{\mathcal{I}}_{yy}}{2}(1+\cos^2\theta)\cos(2\phi) + \ddot{\mathcal{I}}_{xy}(1+\cos^2\theta)\sin(2\phi) + \left(\ddot{\mathcal{I}}_{zz} - \frac{\ddot{\mathcal{I}}_{xx} + \ddot{\mathcal{I}}_{yy}}{2}\right)\sin^2\theta\Bigg]$$

$$h_\times = \frac{1}{r}\Bigg[-\frac{\ddot{\mathcal{I}}_{xx} - \ddot{\mathcal{I}}_{yy}}{2}\cos\theta\sin(2\phi) + \ddot{\mathcal{I}}_{xy}\cos\theta\cos(2\phi)\Bigg]$$

- GWs carry energy, momentum, & angular momentum

- in a nearly Minkowski frame, effective S-E of GWs is

$$T_{ab}^{GW} = \frac{1}{32\pi} \langle \partial_a h_{ij}^{TT} \, \partial_b h^{TT\,ij} \rangle$$

where averaging is done over several wavelengths of GWs

  - outgoing radial energy flux in GWs is then T_{GW}^{tr}

- total GW luminosity is then,

$$L_{GW} \equiv - \frac{dE}{dt} = - \lim_{r \to \infty} \int T_{tr}^{GW} \, r^2 \, d\Omega$$

.... integrated energy flux passing through large sphere centered on source

  - in terms of h_+ & $h_\times$:

$$L_{GW} = - \frac{dE}{dt} = \lim_{r \to \infty} \frac{r^2}{16\pi} \int \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \, d\Omega$$

where $\partial_r h_{ij} = -\partial_t h_{ij}$ for radiation moving out radially

- this is equal to the energy <u>lost</u> by the source to GWs

- expressing h^{TT}_{ij} in terms of quadrupole,

$$L_{GW} = -\frac{dE}{dt} = \frac{1}{5}\langle \dddot{I}_{jk} \dddot{I}^{jk} \rangle$$