Lecture6

Wednesday, January 23, 2019 12:01 PM

Differentiation in non-orthonormal bases

- must distinguish between covariant and contravariant components
- define general spacetime connection coefficients via

$$\nabla_\beta \vec{e}_\alpha \equiv \nabla_{\vec{e}_\beta} \vec{e}_\alpha = \Gamma^\mu{}_{\alpha\beta} \vec{e}_\mu$$

- duality $\vec{e}^\nu \cdot \vec{e}_\alpha = \delta^\nu{}_\alpha$ implies

$$\nabla_\beta \vec{e}^\mu \equiv \nabla_{\vec{e}_\beta} \vec{e}^\mu = -\Gamma^\mu{}_{\alpha\beta} \vec{e}^\alpha$$

- sign flip because $\nabla_\beta (\vec{e}^\mu \cdot \vec{e}_\alpha) = 0$
- Duality $\rightarrow$

$$\Gamma^\mu{}_{\alpha\beta} = \vec{e}^\mu \cdot \nabla_\beta \vec{e}_\alpha = -\vec{e}_\alpha \nabla_\beta \vec{e}^\mu$$

- the gradient of a vector in any basis:

$$A^\mu{}_{;\beta} \vec{e}_\mu = \nabla_\beta \vec{A} = \nabla_\beta (A^\mu \vec{e}_\mu)$$

$$= (\nabla_\beta A^\mu) \vec{e}_\mu + A^\mu \nabla_\beta \vec{e}_\mu$$

$$= A^\mu{}_{,\beta} \vec{e}_\mu + A^\mu \Gamma^\alpha{}_{\mu\beta} \vec{e}_\alpha$$

$$= \left(A^{\mu}{}_{,\beta} + A^{\alpha}\Gamma^{\mu}{}_{\alpha\beta}\right)\vec{e}_{\mu}$$

where, again,

$$A^{\mu}{}_{,\beta} \equiv \partial_{\vec{e}_\beta} A^{\mu}$$

→ comma means the result of letting a basis vector act as a differential operator on the component of the vector.

$$\therefore A^{\mu}{}_{;\beta} = A^{\mu}{}_{,\beta} + A^{\alpha}\Gamma^{\mu}{}_{\alpha\beta}$$

- second term "corrects" the gradient for changes in $\vec{e}_\mu$
- and the covariant components of gradient,

$$A_{\alpha;\beta} = A_{\alpha,\beta} - \Gamma^{\mu}{}_{\alpha\beta} A_{\mu}$$

- extending this to tensors,

$$F^{\alpha\beta}{}_{;\gamma} = F^{\alpha\beta}{}_{,\gamma} + \Gamma^{\alpha}{}_{\mu\gamma} F^{\mu\beta} + \Gamma^{\beta}{}_{\mu\gamma} F^{\alpha\mu}$$

&

$$F_{\alpha\beta;\gamma} = F_{\alpha\beta,\gamma} - \Gamma^{\mu}{}_{\alpha\gamma} F_{\mu\beta} - \Gamma^{\mu}{}_{\beta\gamma} F_{\alpha\mu}$$

→ must "correct" both α & β indices

<u>Computing connection coefficients</u>

1. evaluate commutation coefficients:

$$[\vec{e}_\alpha, \vec{e}_\beta] \equiv c_{\alpha\beta}{}^{\rho}\, \vec{e}_\rho \; ; \; c_{\alpha\beta}{}^{\rho} \equiv \vec{e}^{\,\rho} \cdot [\vec{e}_\alpha, \vec{e}_\beta]$$

2. lower the last index of $c_{\alpha\beta}{}^{\rho}$:

$$c_{\alpha\beta\gamma} = c_{\alpha\beta}{}^{\rho}\, g_{\rho\gamma}$$

3. compute:

$$\Gamma_{\alpha\beta\gamma} \equiv \frac{1}{2}\left[g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha} \right]$$

- in a coordinate basis, "c" terms are zero
- in orthonormal basis, $g_{\mu\nu}$ terms are constant and so "g" terms are zero making $\Gamma_{\alpha\beta\gamma}$ antisymmetric in α, β
- in Cartesian or Lorentz basis (both coordinate AND orthonormal) $\Gamma_{\alpha\beta\gamma}$ will vanish

4. raise first index:

$$\Gamma^{\mu}{}_{\beta\gamma} = g^{\mu\alpha}\, \Gamma_{\alpha\beta\gamma}$$

- in a coordinate basis these are the Christoffel symbols

• One could do the above....

Or, use software!

Integration

- integration requires Levi-Civita tensor in a general basis
- components of L-C then differ from those in orthonormal basis by factors of $\sqrt{|g|}$ where

$$g \equiv \det \| g_{\alpha\beta} \|$$

- define the orthonormal values of L-C as

$$[12\ldots n] = +1$$

$$[\alpha\beta\ldots\nu] = +1 \quad \text{even permutation}$$
$$= -1 \quad \text{odd permutation}$$
$$= 0 \quad \text{not all different}$$

then general components are

$$\epsilon_{\alpha\beta\ldots\nu} = \sqrt{|g|}\,[\alpha\beta\ldots\nu] \quad \text{.... covariant}$$

$$\epsilon^{\alpha\beta\ldots\nu} = \pm \frac{1}{\sqrt{|g|}}\,[\alpha\beta\ldots\nu] \quad \text{.... contravariant}$$

(plus for Euclidean, minus for spacetime)

- Example: Euclidean spherical coords. (r, θ, ϕ) metric tensor components:

$g_{rr} = 1, \; g_{\theta\theta} = r^2 \;, \; g_{\phi\phi} = r^2 \sin^2\theta$

$\hookrightarrow \; g = \det \| g_{\alpha\beta} \| = r^4 \sin^2\theta$

volume element:

$$d\Sigma = \overleftrightarrow{\epsilon}\left(dr \frac{\partial}{\partial r}, \; d\theta \frac{\partial}{\partial \theta}, \; d\phi \frac{\partial}{\partial \phi}\right)$$

$$= \epsilon_{r\theta\phi} \, dr \, d\theta \, d\phi$$

$$= \sqrt{g} \, dr \, d\theta \, d\phi \;\; = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

q.e.d.

- gets more complex in curved spacetime but procedure is same

- in curved manifolds, life gets complicated:

$$\int_{\partial V} T^{\alpha\beta} d\Sigma_\beta$$

$\rightarrow$ the vector $T^{\alpha\beta} d\Sigma_\beta$ lies in a multitude of different tangent spaces.

in order to compute integral, must transport all elements to a common tangent space

$\longrightarrow$ requires transport over large

(non-infinitessimal) distances,
unlike the gradient.

- such transport depends on the route taken, no preferred route

∴ integrals such as the above are ill-defined on curved manifolds!

→ the only well-defined integrals are those of scalar integrands, ie.

$$\int_{\partial V} S^\alpha \, d\Sigma_\alpha$$

- this becomes very important for conservation laws in curved spacetime...

$\boxed{\text{Stress - Energy Tensor}}$

- Recall,

$\overleftrightarrow{T}(_, \vec{\Sigma}) = $ (total 4-momentum $\vec{P}$ that flows through $\vec{\Sigma}$)

or, $T^{\alpha\beta}\Sigma_\beta = P^\alpha$

- in flat spacetime, local conservation:

$$\nabla \cdot \overleftrightarrow{T} = 0$$

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$\to$ $$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i} = 0$$ ... energy conservation

& $$\frac{\partial T^{j0}}{\partial t} + \frac{\partial T^{jk}}{\partial x^k} = 0$$ ... momentum cons.

- Much more complicated in curved spacetime since these are derived from global conservation:

$$\int_{\partial V} T^{\alpha\beta} d\Sigma_\beta$$ ... ill-defined in curved

- stress-energy tensor for perfect fluid:

$$\overleftrightarrow{T} = (\rho + P)\vec{u} \otimes \vec{u} + P\overleftrightarrow{g}$$

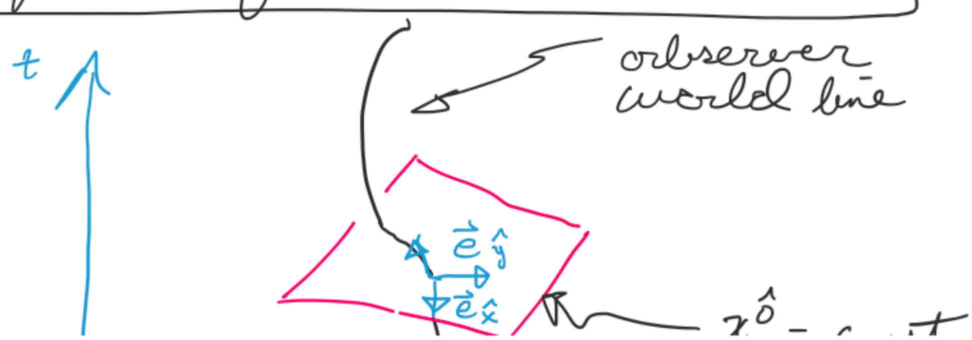
or

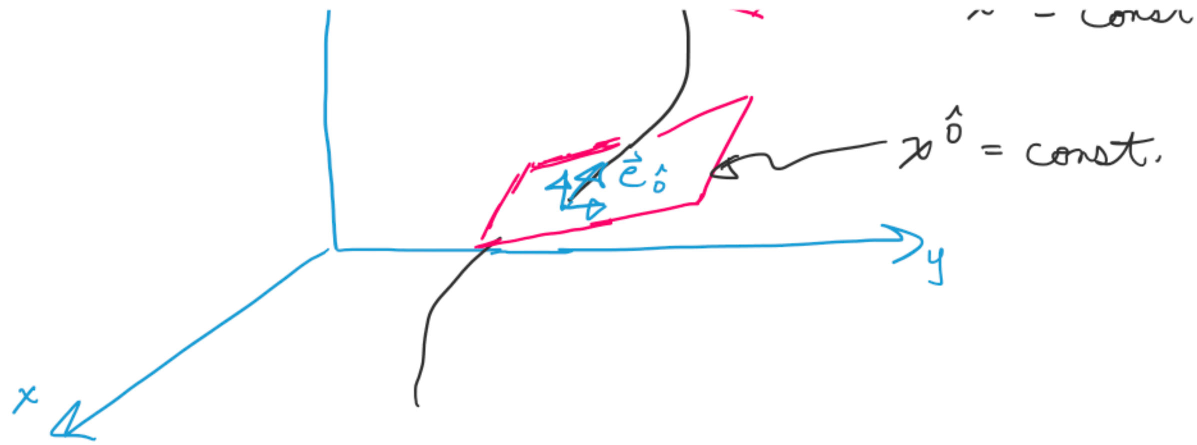
$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P g^{\alpha\beta}$$

where, P is isotropic pressure and
ρ is total energy density

$\to$ frame independent geometric object

$$\boxed{\text{Proper rest frame of accelerated observer}}$$





- accelerated observer w/ 4-velocity $\vec{U}$:

$$\vec{a} = \nabla_{\vec{U}} \vec{U}$$

- Construct a proper reference frame:
  1. spatial origin always at world line $x^{\hat{j}} = 0$
  2. along world line, time coordinate $x^{\hat{0}}$ is their proper time
  3. near world line, $x^{\hat{j}}$ measure physical distance on a Cartesian lattice

↳ the metric is $ds^2 = \eta_{\hat{\alpha}\hat{\beta}}\, dx^{\hat{\alpha}} dx^{\hat{\beta}}$

or

$$g_{\hat{\alpha}\hat{\beta}} = \frac{\partial}{\partial x^{\hat{\alpha}}} \cdot \frac{\partial}{\partial x^{\hat{\beta}}} = \eta_{\hat{\alpha}\hat{\beta}} \quad \text{at } x^{j} = 0$$

- (1) & (2) imply $\vec{e}_{\hat{0}} \equiv \frac{\partial}{\partial x^{\hat{0}}} = \vec{U}$ .... 4-vel.
- for general rotating frame, angular velocity is 3D, spatial vector $\vec{\Omega}$ orthogonal to world line, or

$\Omega^{\hat{j}} \neq 0, \quad \Omega^{\hat{0}} = 0$

- the acceleration of frame (ie. observer)

$$a^{\hat{0}} = 0, \quad a^{\hat{j}} = (\hat{j} \text{ component of measured } \vec{a})$$

.... 4-acceleration

Relation to inertial reference frame

- transform between inertial & proper frames at an event $x^{\hat{0}} = 0$, $x^{\hat{j}} = 0$ ... $x^0 = x^{\hat{0}}$

$$x^i = x^{\hat{i}} + \frac{1}{2} a^{\hat{i}} (x^{\hat{0}})^2 + \epsilon^{\hat{i}}{}_{\hat{j}\hat{k}} \Omega^{\hat{j}} x^{\hat{k}} x^{\hat{0}}$$

vectorial displacement   rotational displacement

- relativistically must also include Lorentz boost between frames:

$$x^0 = x^{\hat{0}} (1 + a_{\hat{j}} x^{\hat{j}})$$

- the metric in inertial frame:

$$ds^2 = -(dx^0)^2 + \delta_{ij}\, dx^i dx^j$$

along with transformation components gives form of metric in proper frame:

$$ds^2 = -(1 + 2\vec{a}\cdot\vec{x})(dx^{\hat{0}})^2 + 2(\vec{\Omega} \times \vec{x}) \cdot d\vec{x}\, dx^{\hat{0}} + \delta_{jk}\, dx^{\hat{j}} dx^{\hat{k}}$$

(3D space vectors)

→ on observer world line,

$$g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$$

Geodesic for a freely falling particle

- consider particle w/ 4-velocity $\vec{u}$ in free fall near an accelerated observer

→ in inertial reference frame, particle moves in a straight line, or geodesic

- geodesic law of motion:

$$\nabla_{\vec{u}} \vec{u} = 0$$

→ it parallel transports its 4-velocity along itself.

- in proper reference frame, $u^{\alpha} = \dfrac{dx^{\alpha}}{d\tau}$

↳
$$u^{\hat{\alpha}}{}_{;\hat{\mu}}\, u^{\hat{\mu}} = 0$$

$$u^{\hat{\alpha}}{}_{,\hat{\mu}}\, u^{\hat{\mu}} + \Gamma^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}}\, u^{\hat{\mu}} u^{\hat{\nu}} = 0$$

$$\left(\frac{\partial}{\partial x^{\hat{\mu}}} \frac{dx^{\hat{\alpha}}}{dt}\right) \frac{dx^{\hat{\mu}}}{d\tau} + \Gamma^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}}\, u^{\hat{\mu}} u^{\hat{\nu}} = 0$$

$$\Longrightarrow \quad \frac{d^2 x^{\hat{\alpha}}}{d\tau^2} + \Gamma^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}} \frac{dx^{\hat{\mu}}}{d\tau} \frac{dx^{\hat{\nu}}}{d\tau} = 0$$

$$d\tau \qquad {}_{\mu\nu} \quad d\tau \quad d\tau = 0$$

- Now, assume velocity is small so that $$v^{\hat{j}} = \frac{dx^{\hat{j}}}{dx^{\hat{0}}} \approx \frac{dx^{\hat{j}}}{d\tau} = u^{\hat{j}} \ll 1$$

$$\rightarrow u^{\hat{0}} = \frac{dx^{\hat{0}}}{d\tau} \approx 1$$

$\hookrightarrow$ to first order in $v^{\hat{j}}$,

$$\frac{d^2 x^{\hat{i}}}{(dx^{\hat{0}})^2} = -\Gamma^{\hat{i}}_{\hat{0}\hat{0}} - \left(\Gamma^{\hat{i}}_{\hat{j}\hat{0}} + \Gamma^{\hat{i}}_{\hat{0}\hat{j}}\right) v^{\hat{j}}$$

- from the components of Γ (see homework) we have,

$$\frac{d^2 x^{\hat{i}}}{(dx^{\hat{0}})^2} = -a^{\hat{i}} - 2\epsilon^{\hat{i}}{}_{\hat{j}\hat{k}}\,\Omega^{\hat{j}} v^{\hat{k}}$$

or,

coriolis acceleration

$$\frac{d^2 \vec{x}}{(dx^{\hat{0}})^2} = -\vec{a} - 2\vec{\Omega} \times \vec{v}$$

.... nonrelativistic E.O.M. for particle in free fall