Lecture5

Sunday, January 20, 2019 2:06 PM

$\boxed{\text{From Special to General Relativity}}$

Review of basic definitions

- tensor: linear, real-valued function of vectors:

$$\overleftrightarrow{F}(\vec{A}, b\vec{B} + c\vec{C}) = b\overleftrightarrow{F}(\vec{A}, \vec{B}) + c\overleftrightarrow{F}(\vec{A}, \vec{C})$$

$\overleftrightarrow{F}(\_, \vec{B}) \to$ vector

$\vec{D}(\vec{B}) = \vec{D} \cdot \vec{B} = \overleftrightarrow{g}(\vec{D}, \vec{B})$ ... metric tensor

- for events $\mathcal{P}(\tau)$ & $\mathcal{P}(\tau + \Delta\tau)$ along particle world line (τ is proper time):

$$\overleftrightarrow{g}(\Delta\vec{x}, \Delta\vec{x}) \equiv \Delta\vec{x} \cdot \Delta\vec{x} \equiv -(\Delta\tau)^2$$

- $\overleftrightarrow{g}(\_, \_)$ is symmetric
- 4-velocity $\vec{u} = \dfrac{d\mathcal{P}}{d\tau}$, $\vec{u} \cdot \vec{u} = \overleftrightarrow{g}(\vec{u}, \vec{u}) = -1$

  $\hookrightarrow$ a "timelike" vector

- since inner product of two timelike vectors is negative, $\vec{e}_0 \cdot \vec{e}_0 = -1$, while for spacelike vectors it is positive:

  $\vec{e}_1 \cdot \vec{e}_1 = +1$, we must distinguish

  between covariant & contravariant

between covariant - contravariant

components.

Covariant : $F_{\alpha\beta} = \overleftrightarrow{F}(\vec{e}_\alpha, \vec{e}_\beta)$

contravariant : $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$

and sign-flip-if-temporal

- components of 4-velocity in an inertial reference frame:

$$u^0 = \gamma\ , \ u^j = \gamma v^j\ , \ v^j = \frac{dx^j}{dt}$$

$$\gamma \equiv (1 - \delta_{ij} v^i v^j)^{-1/2}$$

and 4-momentum:

$$E \equiv p^0 = m\gamma\ , \ p^j = m\gamma v^j$$

.... a "3+1" split

- Interval between two events:

$$(\Delta s)^2 \equiv \Delta\vec{x} \cdot \Delta\vec{x} = \overleftrightarrow{g}(\Delta\vec{x}, \Delta\vec{x}) = g_{\alpha\beta} \Delta x^\alpha \Delta x^\beta$$

$$= -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

... invariant under Lorentz transformation

General Bases and Curved Manifolds

To treat GR, need the following new concepts:

1. non-orthonormal bases:

$$\vec{e}_\alpha \cdot \vec{e}_\beta \neq \eta_{\alpha\beta}$$

2. generalized definition of tensors using tangent spaces.

3. generalized gradients & integration that work in curved spaces.

Non-orthonormal Bases

- manifold: a space that is "flat" on small scales, though the metric may be non-Euclidean. Same smoothness and topology as Euclidean on small scales.

two torii ....

- want to maintain all the old rules of tensor index gymnastics:

$$F_{\alpha\beta} = \overleftrightarrow{F}(\vec{e}_\alpha, \vec{e}_\beta), \quad \overleftrightarrow{F} = F^{\mu\nu}\vec{e}_\mu \otimes \vec{e}_\nu$$

$$F_{\alpha\beta} = g_{\alpha\mu}\, g_{\beta\nu}\, F^{\mu\nu}$$

<u>except</u> the orthonormality of basis.

or $\vec{e}_\alpha \cdot \vec{e}_\beta = g_{\alpha\beta} \neq \eta_{\alpha\beta}$ ... general metric

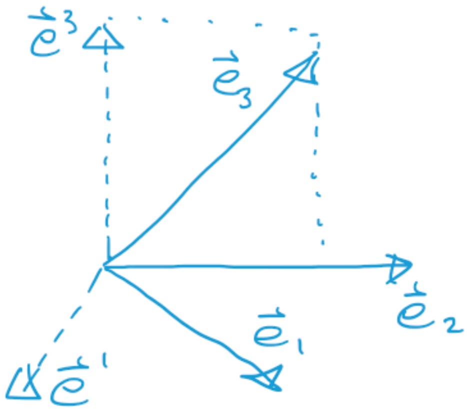
- to do this, must define a basis set that is local to original basis:

that is dual to original basis.

$$\vec{e}^{\mu} \cdot \vec{e}_{\beta} \equiv \overleftrightarrow{g}(\vec{e}^{\mu}, \vec{e}_{\beta}) = \delta^{\mu}{}_{\beta}$$

↳ $\vec{e}^{1}$ always perpendicular to all $\vec{e}_{\alpha}$ but $\vec{e}_{1}$ and $\vec{e}^{1} \cdot \vec{e}_{1} = 1$

- in Minkowski spacetime, $\vec{e}^{0} = -\vec{e}_{0}$



- index gymnastics is the same if

$$F^{\mu\nu} = \overleftrightarrow{F}(\vec{e}^{\mu}, \vec{e}^{\nu}), \quad F_{\alpha\beta} = \overleftrightarrow{F}(\vec{e}_{\alpha}, \vec{e}_{\beta})$$

$$F^{\mu}{}_{\beta} = \overleftrightarrow{F}(\vec{e}^{\mu}, \vec{e}_{\beta})$$

- requiring duality of the bases implies:

(i) $g^{\mu\beta} g_{\beta\nu} = \delta^{\mu}{}_{\nu}$, or $\| g^{\mu\nu} \| = \| g_{\alpha\beta} \|^{-1}$

(ii) $\overleftrightarrow{F} = F^{\mu\nu} \vec{e}_{\mu} \otimes \vec{e}_{\nu} = F_{\alpha\beta} \vec{e}^{\alpha} \otimes \vec{e}^{\beta} = F^{\mu}{}_{\beta} \vec{e}_{\mu} \otimes \vec{e}^{\beta}$

(iii) $\overleftrightarrow{F}(\vec{p}, \vec{q}) = F^{\alpha\beta} p_{\alpha} q_{\beta}$ Wrong in book!

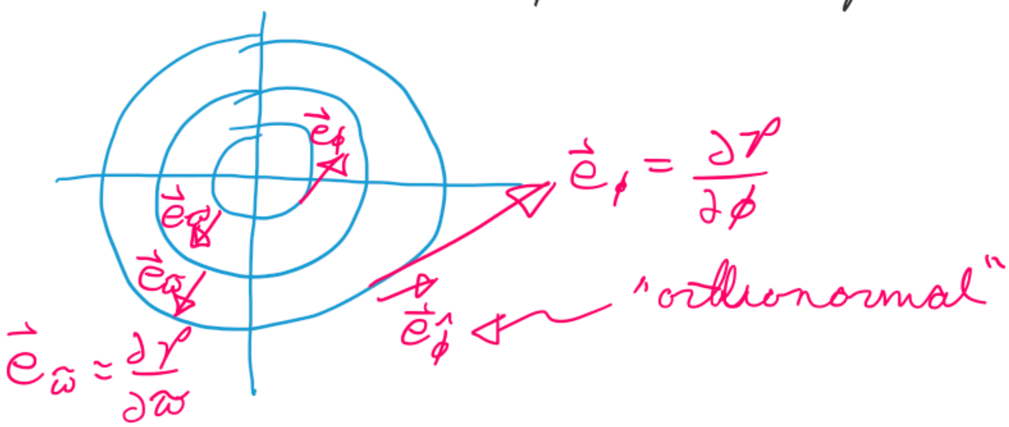
↳ component form still looks the exact same as slot-naming index notation

- can define a coordinate basis:

$$\vec{e}_{\alpha} \equiv \frac{\partial \mathcal{P}}{\partial x^{\alpha}}$$

that defines coordinate system $x^\alpha(\mathcal{P})$
for a scalar field of points $\mathcal{P}$

- consider 2D spherical-polar (ϖ, ϕ):



here, orthonormal angular basis is:

$$\vec{e}_{\hat\phi} = \frac{1}{\varpi}\vec{e}_\phi = \frac{1}{\varpi}\frac{\partial}{\partial\phi}$$

for radial direction,

$$\vec{e}_{\hat\varpi} = \vec{e}_\varpi = \left(\frac{\partial \mathcal{P}}{\partial \varpi}\right)_\phi$$ is already unit length

$\therefore$ we can deduce the metric components:

$$g_{\phi\phi} = \varpi^2, \quad g_{\varpi\varpi} = 1, \quad g_{\varpi\phi} = g_{\phi\varpi} = 0$$

$\hookrightarrow ds^2 = g_{ij}\,dx^i dx^j = d\varpi^2 + \varpi^2 d\phi^2$

- in the orthonormal basis,

$$g_{\hat i \hat j} = \delta_{ij}$$

- can construct dual basis to coordinate basis $\{\vec{e}_\alpha\} = \{\partial \mathcal{P}/\partial x^\alpha\}$,

$$\vec{e}^{\,\mu} = \nabla x^\mu$$

$\vec{e}^{\mu} = \nabla x^{\mu}$

$\rightarrow \vec{e}^{\mu} \cdot \vec{e}_{\alpha} = \vec{e}_{\alpha} \cdot \nabla x^{\mu} = \nabla_{\vec{e}_{\alpha}} x^{\mu}$

$= \nabla_{\partial \mathcal{P}/\partial x^{\alpha}} x^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\alpha}} = \delta^{\mu}{}_{\alpha}$

..... duality of bases

- line element & the metric:

by definition $g_{\alpha\beta} \vec{e}^{\alpha} \otimes \vec{e}^{\beta} = g_{\alpha\beta} \nabla x^{\alpha} \otimes \nabla x^{\beta}$

now consider vector displacement $d\vec{x} = dx^{\alpha} \frac{\partial}{\partial x^{\alpha}}$

$\rightarrow$ can construct interval by putting displacement in both metric slots:

$$ds^2 = \overleftrightarrow{g}(d\vec{x}, d\vec{x}) = g_{\alpha\beta} \nabla x^{\alpha} \otimes \nabla x^{\beta} (d\vec{x}, d\vec{x})$$

$$= g_{\alpha\beta} (d\vec{x} \cdot \nabla x^{\alpha})(d\vec{x} \cdot \nabla x^{\beta})$$

$$ds^2 = g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$$

[note for any scalar ψ, $d\vec{x} \cdot \nabla \psi$ is change $d\psi$ along $d\vec{x}$]

Lorentz transformation in general

- can expand any basis $\{\vec{e}_{\alpha}\}$ in terms of another $\{\vec{e}_{\bar{\mu}}\}$:

$$\vec{e}_{\alpha} = \vec{e}_{\bar{\mu}} L^{\bar{\mu}}{}_{\alpha}, \quad \vec{e}_{\bar{\mu}} = \vec{e}_{\alpha} L^{\alpha}{}_{\bar{\mu}}$$

where

$$L^{\bar{\mu}}{}_{\alpha} L^{\alpha}{}_{\bar{\nu}} = \delta^{\bar{\mu}}{}_{\bar{\nu}} , \quad L^{\alpha}{}_{\bar{\mu}} L^{\bar{\mu}}{}_{\beta} = \delta^{\alpha}{}_{\beta}$$

.... inverse relationships

→ for components of tensors

$$A_{\bar{\mu}} \, L^{\alpha}{}_{\bar{\mu}} A_{\alpha} , \quad T^{\bar{\mu}\bar{\nu}}{}_{\bar{\rho}} = L^{\bar{\mu}}{}_{\alpha} L^{\bar{\nu}}{}_{\beta} L^{\gamma}{}_{\bar{\rho}} T^{\alpha\beta}{}_{\gamma}$$

.... note mixed contra- & co-variance

- for coordinate basis,

$$L^{\bar{\mu}}{}_{\alpha} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\alpha}} , \quad L^{\alpha}{}_{\bar{\mu}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\mu}}}$$

since $\vec{e}_{\alpha} = \frac{\partial P}{\partial x^{\alpha}} = \frac{\partial x^{\mu}}{\partial x^{\alpha}} \frac{\partial P}{\partial x^{\mu}} = \vec{e}_{\mu} \frac{\partial x^{\mu}}{\partial x^{\alpha}}$

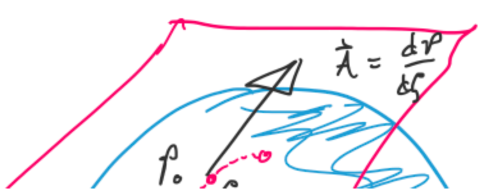
Vectors and Tangent spaces

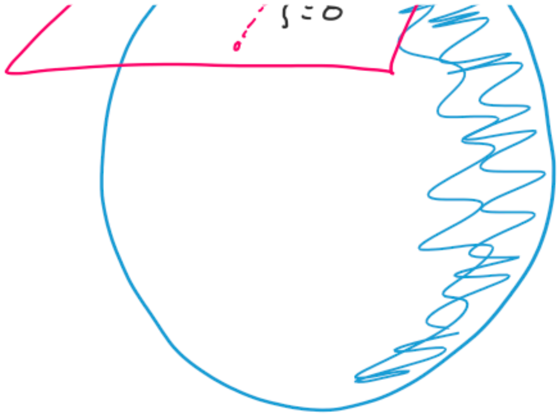
- usual definition of vector:

$$\vec{A} \equiv \frac{dP}{d\zeta} \equiv \lim_{\Delta\zeta \to 0} \frac{P(\Delta\zeta) - P(0)}{\Delta\zeta}$$

as $\Delta\zeta \to 0$, effects of curvature on a manifold become negligible, but what then does a vector mean? what space does it live in?

- Consider embedding curved manifold in a higher-dimension space:





- Can think of the vector "arrow" as living in the flat plane tangent to the curved manifold

→ same dimensionality as manifold

- tensors at P_0 are linear functions of the vectors that are in the tangent space at P_0.

- More generally, vectors at P_0 are the directional derivatives $\partial_{\vec{A}}$ at P_0

$$\rightarrow \frac{\partial P}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}, \quad \vec{A} = \partial_{\vec{A}}$$

$$+ \qquad \vec{A} = A^\alpha \frac{\partial}{\partial x^\alpha} \quad \text{in a coordinate basis}$$

- Commutator: $[\vec{A}, \vec{B}]$ is vector that is given by operator $[\partial_{\vec{A}}, \partial_{\vec{B}}]$.

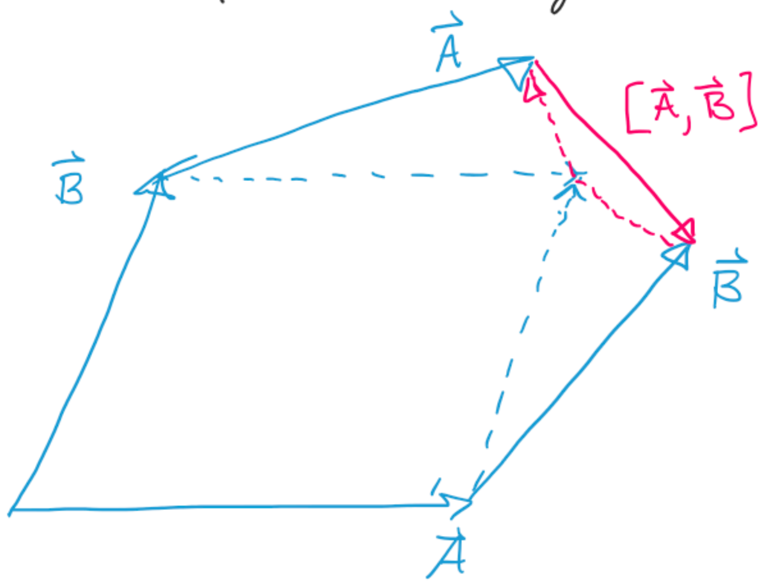
components of commutator:

$$[\vec{A},\vec{B}] \equiv \left[A^\alpha \frac{\partial}{\partial x^\alpha},\ B^\beta \frac{\partial}{\partial x^\beta}\right] = \left(A^\alpha \frac{\partial B^\beta}{\partial x^\alpha} - B^\alpha \frac{\partial A^\beta}{\partial x^\alpha}\right)\frac{\partial}{\partial x^\beta}$$

.... operates on some scalar field

components : $A^\alpha B^\beta{}_{,\alpha} - B^\alpha A^\beta{}_{,\alpha}$

- for coordinate bases : $[\vec{e}_\alpha, \vec{e}_\beta] = 0$
- for non-coordinate bases, some component of $[\vec{e}_\alpha, \vec{e}_\beta]$ is non-zero



→ if $[\vec{A},\vec{B}] = 0$ then $\vec{A}$ + $\vec{B}$ can be used as coordinate vectors.

Differentiation of tensors in general

- On curved manifolds, definition of derivative requires comparing values of vectors/tensors in different tangent spaces. Tricky.

→ need to transport tensors between them

- easy in flat spacetime: keep vector parallel to ~~itself~~: Parallel transport

or, keep all the components fixed in an orthonormal coordinate basis.

- Can define a local orthonormal basis on arbitrarily small scales where things look Euclidean

→ transport $\overleftrightarrow{F}$ from $\mathcal{P}(\Delta\zeta)$ to $\mathcal{P}(0)$ holding its components fixed, take difference in tangent space at $\mathcal{P}_0 = \mathcal{P}(0)$, divide by $\Delta\zeta$ & let $\Delta\zeta \to 0$

$\Longrightarrow$ $\nabla_{\vec{A}} \overleftrightarrow{F}$ ... directional derivative

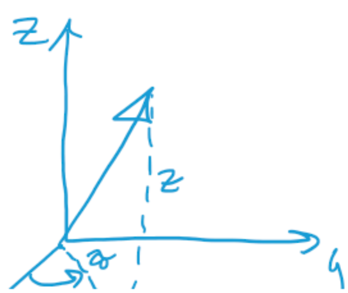
& gradient $\nabla_{\vec{A}} \overleftrightarrow{F} = \nabla \overleftrightarrow{F}(\_,\_,\_,\vec{A})$

Connection Coefficients

- used in treating non-Cartesian coordinates
- consider cylindrical coordinates (ϖ, ϕ, z)

$\varpi = \sqrt{x^2+y^2}$

$\phi = \arctan(y/x)$



Basis vectors:

$$\vec{e}_{\varpi} = \frac{x}{\varpi}\vec{e}_x + \frac{y}{\varpi}\vec{e}_y$$

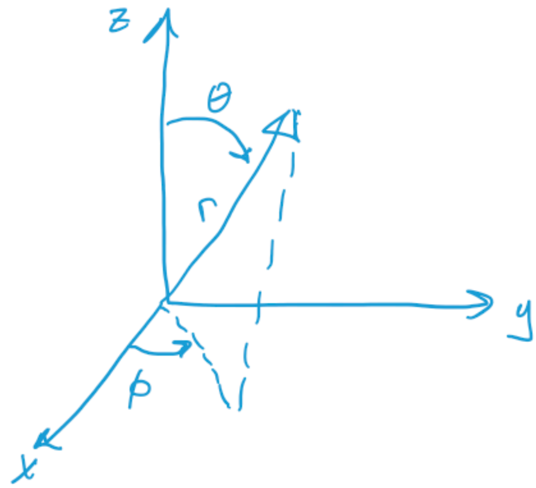
$$\vec{e}_{\phi} = -\frac{y}{\varpi}\vec{e}_x + \frac{x}{\varpi}\vec{e}_y \qquad \vec{e}_z = \vec{e}_z$$

- spherical coordinates (r, ϕ, ϕ)

$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = \arccos(z/r)$$

$$\phi = \arctan(y/x)$$



and

$$\vec{e}_r = \frac{x}{r}\vec{e}_x + \frac{y}{r}\vec{e}_y + \frac{z}{r}\vec{e}_z$$

$$\vec{e}_\theta = \frac{z}{r}\vec{e}_{\varpi} - \frac{\varpi}{r}\vec{e}_z$$

$$\vec{e}_\phi = -\frac{y}{\varpi}\vec{e}_x + \frac{x}{\varpi}\vec{e}_y$$

- Both bases are orthonormal:

$$g_{jk} \equiv \vec{e}_j \cdot \vec{e}_k = \delta_{jk}$$

- Connection coefficients Γ_{ijk}:
  quantify the turning of the orthonormal basis vectors & how are the basis vectors at one point connected to those at some other point.
  Definition:

$$\nabla_k \vec{e}_j = \Gamma_{ijk}\vec{e}_i \,, \quad \Gamma_{ijk} = \vec{e}_i \cdot (\nabla_k \vec{e}_j)$$

antisymmetry: $\nabla_k(\vec{e}_i \cdot \vec{e}_j) = 0$

$$\vec{e}_j \cdot (\nabla_k \vec{e}_i) + \vec{e}_i \cdot (\nabla_k \vec{e}_j) = 0$$

$$\rightarrow \quad \Gamma_{ijk} = -\Gamma_{jik}$$

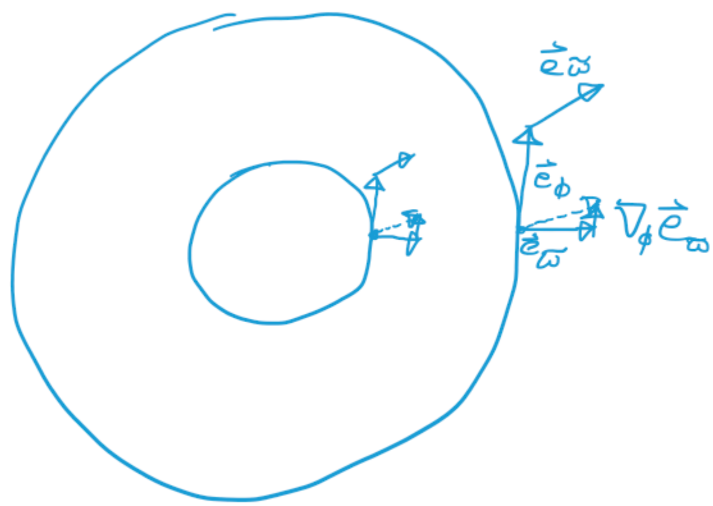
- connection coefficients in cylindrical:

$$\nabla_\phi \vec{e}_\varpi = \frac{\vec{e}_\phi}{\varpi}$$

$$\nabla_\phi \vec{e}_\phi = -\frac{\vec{e}_\varpi}{\varpi}$$

$$\rightarrow \Gamma_{\varpi\phi\phi} = -\frac{1}{\varpi}$$

$$\Gamma_{\phi\varpi\phi} = \frac{1}{\varpi}$$



- Connection coefficients are key to differentiating vectors & tensors

consider gradients of some displacement

$$\overleftrightarrow{W} = \nabla \vec{\xi}$$

$$\rightarrow \nabla_k(\xi_i \vec{e}_j) = (\nabla_k \xi)\vec{e}_j + \xi_j(\nabla_k \vec{e}_j)$$

$$= \xi_{j,k}\vec{e}_j + \xi_j \Gamma_{ljk}\vec{e}_l$$

in components:

$$W_{ik} = \xi_{i;k} = \xi_{i,k} + \Gamma_{ijk}\xi_j$$