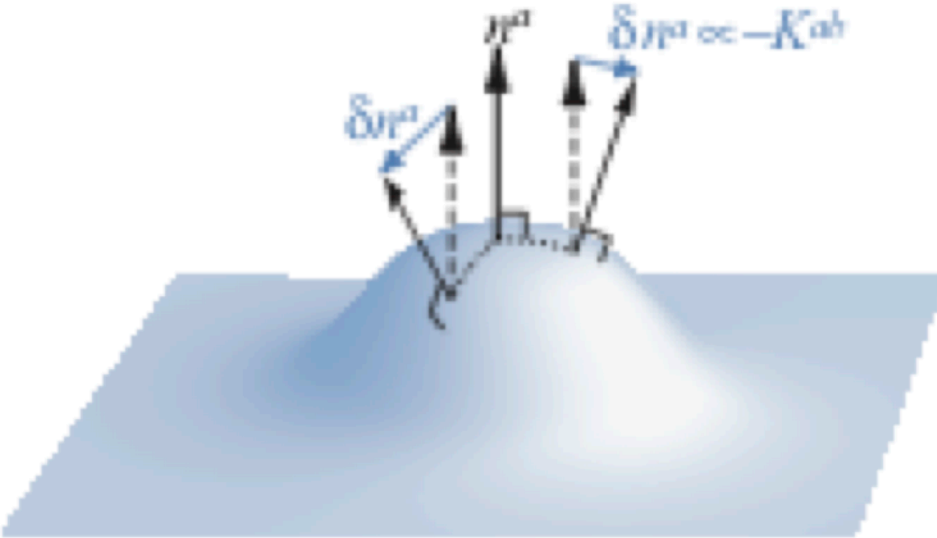
Lecture11

Saturday, February 16, 2019 8:25 AM



$\boxed{\text{Extrinsic Curvature}}$

- K_{ab} : from projection of gradients of normal vector onto Σ; also, first time derivative of γ_{ab}
- together (γ_{ab}, K_{ab}) are analogous to position & velocity in classical mechanics but for gravitational field
- projection of gradient of normal vector :

  $$\gamma_a{}^c \gamma_b{}^d \nabla_c n_d$$

- split into symmetric part, expansion tensor,

$$\theta_{ab} = \gamma_a{}^c \gamma_b{}^d \nabla_{(c} n_{d)}$$

and antisymmetric twist,

$$\omega_{ab} = \gamma_a{}^c \gamma_b{}^d \nabla_{[c} n_{d]}$$

- the twist must vanish since the normal vector is rotation free:

$\therefore$ extrinsic curvature is

$$K_{ab} \equiv -\gamma_a{}^c \gamma_b{}^d \nabla_{(c} n_{d)} = -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d$$

.... since $\nabla_c n_d$ is symmetric

$\rightarrow$ $K_{ab} = K_{ba}$ & is purely spatial

- n^a are normalized & vary only in direction so K_{ab} says how much the direction changes between different points in Σ

$\rightarrow$ how much Σ deforms as it is carried forward along n^a

- Can also express K_{ab} in terms of acceleration of the normal vector:

$$a_a \equiv n^b \nabla_b n_a$$

$\hookrightarrow$

$$K_{ab} = -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d$$

$$= -(\delta_a{}^c + n_a n^c)(\delta_b{}^d + n_b n^d)\nabla_c n_d$$
$$= -(\delta_a{}^c + n_a n^c)\delta_b{}^d \nabla_c n_d$$
$$= -\nabla_a n_b - n_a a_b$$

- can also write in terms of the Lie derivative $\mathcal{L}_{\vec{n}}$ along n^a

$$K_{ab} = -\frac{1}{2}\mathcal{L}_{\vec{n}}\gamma_{ab}$$

- for scalar f,

$$\mathcal{L}_{\vec{X}} f = X^b D_b f = X^b \partial_b f \quad \text{.... partial}$$

- for vector v^a:

$$\mathcal{L}_{\vec{X}} v^a = X^b \partial_b v^a - v^b \partial_b X^a = [X, v]^a$$

... commutator

- 1-form:

$$\mathcal{L}_{\vec{X}} \omega_a = X^b \partial_b \omega_a + \omega_b \partial_a X^b$$

- tensor:

$$\mathcal{L}_{\vec{X}} T^a{}_b = X^c \partial_c T^a{}_b - T^c{}_b \partial_c X^a + T^a{}_c \partial_b X^c$$

→ measures change of tensor along a vector relative to a simple coordinate transform

- if two hypersurfaces differed by only

a coordinate transform (steady-state),

$$\mathcal{L}_{\vec{n}} \gamma_{ab} = 0$$

- thus, this definition of K_{ab} makes apparent its connection to a time derivative of γ_{ab} (since n^a is timelike)

↳ time derivative of γ_{ab} is proportional to K_{ab}

- derive it: write γ_{ab} in terms of g_{ab} & n_a:

$$\begin{aligned}
\mathcal{L}_{\vec{n}} \gamma_{ab} &= \mathcal{L}_{\vec{n}} (g_{ab} + n_a n_b) \\
&= 2 \nabla_{(a} n_{b)} + n_a \mathcal{L}_{\vec{n}} n_b + n_b \mathcal{L}_{\vec{n}} n_a \\
&= 2 (\nabla_{(a} n_{b)} + n_{(a} a_{b)}) = -2 K_{ab}
\end{aligned}$$

where $\mathcal{L}_{\vec{X}} g_{ab} = \nabla_a X_b + \nabla_b X_a$

.... symmetries of metric

- trace of extrinsic curvature is mean curvature,

$$\begin{aligned}
K &= g^{ab} K_{ab} = \gamma^{ab} K_{ab} \quad \ldots \text{(purely spatial)} \\
&= -\tfrac{1}{2} \gamma^{ab} \mathcal{L}_{\vec{n}} \gamma_{ab} \\
&= -\frac{1}{2\gamma} \mathcal{L}_{\vec{n}} \gamma = -\frac{1}{\gamma^{1/2}} \mathcal{L}_{\vec{n}} \gamma^{1/2} \quad \text{(chain rule)} \\
&= -\mathcal{L}_{\vec{n}} \ln \gamma^{1/2}
\end{aligned}$$

→ $\gamma^{1/2} d^3x$ is the proper volume element in Σ_t, so $-K$ measures change in 3-volume element along n^a

$\boxed{\text{Equations of Gauss, Codazzi + Ricci}}$

- γ_{ab} + K_{ab} cannot be arbitrary; must satisfy constraints so that they "fit" in M
- to determine constraints, must relate $R^a{}_{bcd}$ to ${}^{(4)}R^a{}_{bcd}$. This involves:
  - take completely spatial projection of ${}^{(4)}R^a{}_{bcd}$
  - then a projection w/ one index in the normal direction
  - then a projection w/ two indices in the normal direction

  → 3 different types of projections
- $R^a{}_{bcd}$ contains only spatial derivatives, so relating it to ${}^{(4)}R^a{}_{bcd}$ will require the

relating [illegible] to [illegible] 3D which requires [illegible]
extrinsic curvature (and derivatives)

- for purely spatial V^q, $n_q V^q = 0$ by definition

$$\mapsto \nabla_p (n_q V^q) = V^q \nabla_p n_q + n_q \nabla_p V^q = 0$$

$$\rightarrow \quad n_q \nabla_p V^q = - V^q \nabla_p n_q$$

- now expand spatial gradient of vector,

$$\begin{aligned} D_a V^b &= \gamma_a{}^p \gamma_q{}^b \nabla_p V^q = \gamma_a{}^p (g_q{}^b + n_q n^b) \nabla_p V^q \\ &= \gamma_a{}^p \nabla_p V^b - \gamma_a{}^p n^b V^q \nabla_p n_q \\ &= \gamma_a{}^p \nabla_p V^b - n^b V^e \gamma_a{}^p \gamma_e{}^q \nabla_p n_q \\ &= \gamma_a{}^p \nabla_p V^b + n^b V^e K_{ae} \end{aligned}$$

where by definition $K_{ae} = - \gamma_a^p \gamma_e^q \nabla_p n_q$

- second spatial derivative:

$$D_a D_b V^c = \gamma_a^p \gamma^q_b \gamma^c_r \nabla_p \nabla_q V^r - K_{ab} \gamma^c_r n^p \nabla_p V^r - K_a{}^c K_{bp} V^p$$

- re-writing definition of 3D Riemann tensor,

$$R^{dc}{}_{ba} V_d = 2 D_{[a} D_{b]} V^c$$

using above expression for second derivative,

$$R^{dc}{}_{ba} V_d = 2 \gamma^p_a \gamma^q_b \gamma^c_r \nabla_{[p} \nabla_{q]} V^r - 2 K_{[ab]} \gamma^c_r n^p \nabla_p V^r - 2 K_{[a}{}^c K_{b]p} V^p$$

- since $R_{[ab]} = 0$ second term on R.H.S. disappears
- recall, $V_d \, {}^{(4)}R^d{}_{cab} = \nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c$

$$= 2 \nabla_{[a} \nabla_{b]} V_c$$

so we can re-express above as,

$$R_{dcba} V^d = \gamma_a{}^p \gamma^q_b \gamma^r_c \, {}^{(4)}R_{drqp} V^d - 2 K_{c[a} K_{b]d} V^d$$

(after lowering c)

- V^d is an arbitrary vector, so we can take it out,

$$R_{abcd} + K_{ac} K_{bd} - K_{ad} K_{cb} = \gamma_a{}^p \gamma^q_b \gamma^r_c \gamma^s_d \, {}^{(4)}R_{pqrs}$$

.... Gauss' equation

- quadratic in extrinsic curvature

- next need projection of ${}^{(4)}R^a{}_{bcd}$ w/ one index in normal direction. start w/ spatial derivative of extrinsic curvature.

$$D_a K_{bc} = \gamma_a^p \gamma_b^q \gamma_c^r \nabla_p K_{qr}$$
$$= - \gamma_a^p \gamma_b^q \gamma_c^r \left(\nabla_p \nabla_q n_r + \nabla_p (n_q a_r) \right)$$

[recall, $K_{ab} = - \nabla_a n_b - n_a a_b$]

a_b is acceleration of normal vector field,

$$a_a \equiv n^b \nabla_b n_a$$

- since $\gamma_b^q n_q = 0$ by definition,

$$\gamma_a^p \gamma_b^q \gamma_c^r a_r \nabla_p n_q = - a_c K_{ab}$$

↳ $$D_a K_{bc} = - \gamma_a^p \gamma_b^q \gamma_c^r \nabla_p \nabla_q n_r + a_c K_{ab}$$

- from symmetry of K_{ab},

$$D_{[a} K_{b]c} = \tfrac{1}{2}(D_a K_{bc} - D_b K_{ac}) = -\gamma^p \gamma^q \gamma^r \nabla \nabla n$$

$$D_{[a}K_{b]c} = 2(D_a K_{bc} - D_b K_{ac}) = \gamma_a \gamma_b \gamma_c \nabla_{[p} \nabla_{q]} n_r$$

- from the definition of Riemann tensor then,

$$D_b K_{ac} - D_a K_{bc} = \gamma_a^{\,p} \gamma_b^{\,q} \gamma_c^{\,r} n^{s}\, {}^{(4)}R_{pqrs}$$

.... Codazzi equation

- now computing projection of two indices of Riemann tensor into normal direction will require "time" derivatives of K_{ab}

$$\mathcal{L}_{\vec{n}} K_{ab} = n^c \nabla_c K_{ab} + 2 K_{c(a} \nabla_{b)} n^c$$

$$= -n^c \nabla_c \nabla_a n_b - n^c \nabla_c (n_a a_b)$$

$$- 2 K_{c(a} K_{b)}^{\;\;c} - 2 K_{c(a} n_{b)} a^c$$

using ${}^{(4)}R_{dbac} n^d = 2 \nabla_{[c} \nabla_{a]} n_b$,

$$\mathcal{L}_{\vec{n}} K_{ab} = -n^d n^c\, {}^{(4)}R_{dbac} - n^c \nabla_a \nabla_c n_b$$

$$- n^c a_b \nabla_c n_a - n^c n_a \nabla_c a_b$$

$$- 2 K^c_{\;(a} K_{b)c} - 2 K_{c(a} n_{b)} a^c$$

- w/ definition of acceleration and,

$$n^c \nabla_a \nabla_c n_b = \nabla_a a_b - (\nabla_a n^c)(\nabla_c n_b)$$

$$= \nabla_a a_b - K_a^{\;c} K_{cb} - n_a a^c K_{cb}$$

↳

$$\mathcal{L}_{\vec{n}} K_{ab} = -n^d n^c\, {}^{(4)}R_{dbac} - \nabla_a a_b - n^c n_a \nabla_c a_b$$

$$- a_a a_b - K^c_{\;b} K_{ac} - K_{ca} n_b a^c$$

$\boxed{Ex. 2.18}$

$$n^a \mathcal{L}_n K_{..} = - n^a n^d n^c\, {}^{(4)}R_{..} - n^a \nabla_{.} a_{.}$$

$$n^a \mathcal{L}_{\vec{n}} K_{ab} = n^a n^d n^c \, {}^{(4)}R_{dbac} - n^a \nabla_a a_b$$
$$- n^a n^c n_a \nabla_c a_b - n^a a_a a_b$$
$$- n^a K^c{}_b K_{ac} - n^a K_{ca} n_b a^c$$

INSERT SPACE

$\therefore$ $n^a \mathcal{L}_{\vec{n}} K_{ab} = 0 \rightarrow \mathcal{L}_{\vec{n}} K_{ab}$ is purely spatial

- Projecting free indices ab,

$$\gamma_a^q \gamma_b^r \mathcal{L}_{\vec{n}} K_{qr} = \mathcal{L}_{\vec{n}} K_{ab} \quad \text{[purely spatial]}$$

$$\hookrightarrow \mathcal{L}_{\vec{n}} K_{ab} = -n^d n^c \gamma_a^q \gamma_b^r \, {}^{(4)}R_{drqc} - \gamma_a^q \gamma_b^r \nabla_q a_r$$
$$- \gamma_a^q \gamma_b^r n^c n_q \nabla_c a_r - \gamma_a^q \gamma_b^r a_q a_r$$
$$- \gamma_a^q \gamma_b^r K^c{}_r K_{qc} - \gamma_a^q \gamma_b^r K_{cq} n_r a^c$$
$$= -n^d n^c \gamma_a^q \gamma_b^r \, {}^{(4)}R_{drqc} - \gamma_a^q \gamma_b^r \nabla_q a_r$$
$$- \gamma_a^q \gamma_b^r \cancel{n^c n_q} \nabla_c a_r - a_a a_b$$
$$- K^c{}_b K_{ac} - K_{ca} \cancel{n_b} a^c \quad \text{[projections onto normal]}$$

$$\rightarrow \mathcal{L}_{\vec{n}} K_{ab} = -n^d n^c \gamma_a^q \gamma_b^r \, {}^{(4)}R_{drqc} - \gamma_a^q \gamma_b^r \nabla_q a_r$$
$$- a_a a_b - K^c{}_b K_{ac}$$

- Can be simplified using spatial derivative of normal vector acceleration,

$$D_a a_b = \gamma_a^p \gamma_b^q \nabla_p a_q = \gamma_a^r \nabla_p a_b + n_b a_e K_a{}^e$$

$$a_a = n^b \nabla_b n_a \qquad K_{ab} = -\nabla_a n_b - n_a a_b$$
$$\omega_a \equiv \alpha \Omega_a \qquad a_a = D_a \ln \alpha$$

$$\nabla_a V^a = \frac{1}{\alpha} D_a(\alpha V^a)$$ for any spatial vector

$$\rightarrow D_a a_b = -a_a a_b + \frac{1}{\alpha} D_a D_b \alpha$$

$$\hookrightarrow \mathcal{L}_{\vec{n}} K_{ab} = n^d n^c \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} - \frac{1}{\alpha} D_a D_b \alpha - K^c{}_b K_{ac}$$

.... Ricci equation

- relates "time" derivative of K_{ab} to projection of 4D Riemann tensor

Constraint & Evolution equations

- to form 3+1 set of equations, take equations of Gauss, Codazzi, + Ricci & eliminate 4D Riemann tensor using Einstein eqns.,
  $$G_{ab} \equiv {}^{(4)}R_{ab} - \frac{1}{2} {}^{(4)}R \, g_{ab} = 8\pi T_{ab}$$
- this links purely geometric objects from above to physical properties
- contracting Gauss' equation once,
  $$\gamma^{pr} \gamma^q_b \gamma^s_d \, {}^{(4)}R_{pqrs} = R_{bd} + K K_{bd} - K^c{}_d K_{cb}$$
- contracting again,
  $$\gamma^{pr} \gamma^{qs} \, {}^{(4)}R_{pqrs} = R + K^2 - K_{ab} K^{ab}$$

• expanding left hand side,

$$\gamma^{pr}\gamma^{qs}\,{}^{(4)}R_{pqrs} = (g^{pr}+n^p n^r)(g^{qs}+n^q n^s)\,{}^{(4)}R_{pqrs}$$
$$= {}^{(4)}R + 2n^p n^r\,{}^{(4)}R_{pr}$$

where $n^p n^r n^q n^s\,{}^{(4)}R_{pqrs} = 0$ from symmetry

• also, $$2n^p n^r G_{pr} = 2n^p n^r\,{}^{(4)}R_{pr} - n^p n^r g_{pr}\,{}^{(4)}R$$
$$= 2n^p n^r\,{}^{(4)}R_{pr} - n^p n^r(\gamma_{pr} - n_p n_r)\,{}^{(4)}R$$
$$= 2n^p n^r\,{}^{(4)}R_{pr} + {}^{(4)}R = \gamma^{pr}\gamma^{qs}\,{}^{(4)}R_{pqrs}$$

• putting this into contracted Gauss eqn,

$$2n^p n^r G_{pr} = R + K^2 - K_{ab}K^{ab}$$

• w/ ρ: total energy density measured by a normal observer n^a

↳ $\rho \equiv n_a n_b T^{ab}$

$$\Rightarrow \quad R + K^2 - K_{ab}K^{ab} = 16\pi\rho \quad \text{.... Hamiltonian Constraint}$$

• contracting Codazzi eqn. once,

$$D_b K_a{}^b - D_a K = \gamma_a{}^p \gamma^{qr} n^s\,{}^{(4)}R_{pqrs}$$

• expanding RHS,

$$\gamma_a{}^p\gamma^{qr} n^s\,{}^{(4)}R_{pqrs} = -\gamma_a{}^p(g^{qr}+n^q n^r)n^s\,{}^{(4)}R_{qprs}$$
$$= -\gamma_a{}^p n^s\,{}^{(4)}R_{ps} - \gamma_a{}^p n^q n^r n^s\,{}^{(4)}R_{qprs}$$

0 due to symmetry of ${}^{(4)}R_{efgd}$

and, $$\gamma_a^q n^s G_{qs} = \gamma_a^q n^s\,{}^{(4)}R_{qs} - \frac{1}{2}\gamma_a^q n^s g_{qs}\,{}^{(4)}R$$

$$= \gamma^q_a n^s \, {}^{(4)}R_{qs} \quad \left[\gamma^q_a n^s g_{qs} = \gamma_{as} n^s = 0\right]$$

- now the contracted Codazzi eqn is

$$D_b K^b_a - D_a K = -\gamma^q_a n^s G_{qs}$$

- define momentum density as measured by a normal observer n^a:

$$S_a \equiv -\gamma^b_a n^c T_{bc}$$

$\hookrightarrow$

$$D_b K^b_a - D_a K = 8\pi S_a$$

.... Momentum constraint

- Constraint equations:
  - directly equivalent to EM constraints
  - involve only spatial metric, extrinsic curvature, & their spatial derivatives
  - allow Σ (with γ_{ab}, K_{ab}) to be embedded in M (w/ g_{ab})
- Now we need evolution equations for γ_{ab} & K_{ab}. Can start w/ "definition" of extrinsic curvature,

$$K_{ab} = -\frac{1}{2} \mathcal{L}_{\vec{n}} \gamma_{ab}$$

and the Ricci eqn.,

$$\mathcal{L}_{\vec{n}} K_{ab} = n^d n^c \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} - \frac{1}{\alpha} D_a D_b \alpha - K^c_b K_{ac}$$

• But Lie derivative is not a natural time derivative since n^a is not dual to Ω_a,

$$n^a \Omega_a = -\alpha g^{ab} \nabla_a t \nabla_b t = \alpha^{-1} \neq 1$$

• instead define vector

$$t^a = \alpha n^a + \beta^a$$

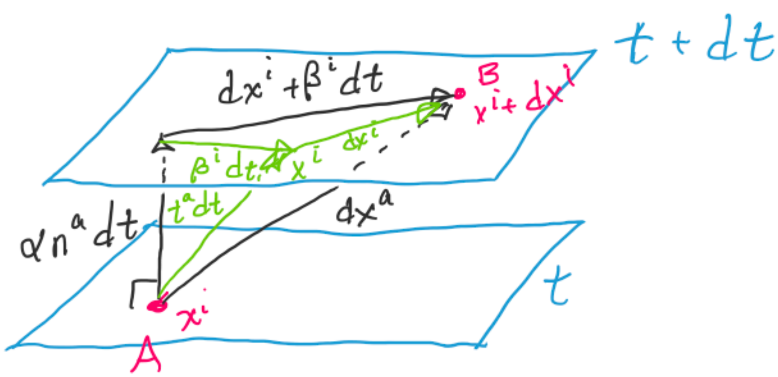
where β^a is a spatial shift vector

↳ t^a is dual to Ω_a:

$$t^a \Omega_a = \alpha n^a \Omega_a + \beta^a \Omega_a = 1$$

• t^a connects points w/ same spatial coordinates on neighboring time slices

→ β^a measures amount that spatial coords are shifted within a slice w.r.t. normal vector



→ α measures elapse of proper time along n^a

∴ α & β^a determine how the coordinates evolve; their choice is arbitrary

→ α & β^a represent the four gauge freedoms

• duality $t^a \nabla_a t = 1$

↳ all vectors $t^a dt$ (and $\alpha n^a dt$) starting one one slice Σ_t will end on same slice Σ_{t+dt} after dt (convenient)

→ also, Lie derivative of any spatial tensor along t^a is also spatial

$$\mathcal{L}_{\alpha\vec{n}} \gamma^a_{\ b} = 0$$

- consider,

$$\mathcal{L}_{\vec{t}} K_{ab} = \mathcal{L}_{\alpha\vec{n}+\vec{\beta}} K_{ab} = \alpha \mathcal{L}_{\vec{n}} K_{ab} + \mathcal{L}_{\vec{\beta}} K_{ab}$$

- use Ricci eqn. to eliminate $\mathcal{L}_{\vec{n}} K_{ab}$, first rewrite projection,

$$n^d n^c \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} = \gamma^{cd} \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} - \gamma^q_a \gamma^r_b \, {}^{(4)}R_{rq}$$

- use Gauss' eqn. to sub 1^{st} term and Einstein eqns. to sub 2^{nd} term:

$$n^d n^c \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} = R_{ab} + K K_{ab} - K_{ac} K^c_b - 8\pi \gamma^q_a \gamma^r_b \left(T_{rq} - \tfrac{1}{2} g_{rq} T\right)$$

where $T \equiv T_{ab} g^{ab}$

- define spatial stress: $S_{ab} \equiv \gamma^c_a \gamma^d_b T_{cd}$; $S \equiv S^a_a$

↳ $$\gamma^q_a \gamma^r_b g_{rq} g^{ef} T_{ef} = \gamma_{ab}\left(\gamma^{ef} - n^e n^f\right) T_{ef} = \gamma_{ab}(S - \rho)$$

- finally,

$$\mathcal{L}_{\vec{t}} K_{ab} = -D_a D_b \alpha + \alpha\left(R_{ab} - 2K_{ac}K^c_b + K K_{ab}\right) - 8\pi\alpha\left(S_{ab} - \tfrac{1}{2}\gamma_{ab}(S-\rho)\right) + \mathcal{L}_{\vec{\beta}} K_{ab}$$

.... extrinsic curvature evolution eqn.

- all differentials & R_{ab} associated w/ γ_{ab}

• from $K_{ab} = -\frac{1}{2}\mathcal{L}_{\vec{n}}\gamma_{ab}$ w/ definition of t^a,

$$\mathcal{L}_{\vec{t}}\,\gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_{\vec{\beta}}\,\gamma_{ab}$$

.... spatial metric evolution eqn.

• 12 evolution eqns., 4 constraints
  - completely determine gravitational field (γ_{ab}, K_{ab})
  - equivalent to Einstein eqns., but are only 1^{st} order in time vs. 2^{nd}
  - evolution eqns. preserve constraints