AST 900 Homework 1

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1. TB Exercise 1.5 *Meaning of Slot-Naming Index Notation*

(a) *The following expressions and equations are written in slot-naming index notation. Convert them to geometric, index-free notation:* A_iB_{jk}, A_iB_{ji}, $S_{ijk} = S_{kji}$, $A_iB_i = A_iB_jg_{ij}$.

Solution

$$\begin{aligned} A_iB_{jk} &\Rightarrow \mathbf{A} \otimes \mathbf{B}(\_,\_) \\ A_iB_{ji} &\Rightarrow \mathbf{B}(\_,\mathbf{A}) \\ S_{ijk} = S_{kji} &\Rightarrow \mathbf{S}\big(\_{i},\_{j},\_{k}\big) = \mathbf{S}\big(\_{k},\_{j},\_{i}\big) \\ A_iB_i = A_iB_jg_{ij} &\Rightarrow \mathbf{A}\cdot\mathbf{B} = \mathbf{A}(\mathbf{B}) = \mathbf{g}(\mathbf{A},\mathbf{B}) \end{aligned}$$

(b) *The following expressions are written in geometric, index-free notation. Convert them to slot-naming index notation:* $\mathbf{T}(\_,\_,\mathbf{A})$, $\mathbf{T}(\_,\mathbf{S}(\mathbf{B},\_),\_)$.

Solution

$$\begin{aligned} \mathbf{T}(\_,\_,\mathbf{A}) &\Rightarrow A_kT_{ijk} \\ \mathbf{T}(\_,\mathbf{S}(\mathbf{B},\_),\_) &\Rightarrow B_\ell S_{\ell j}T_{ijk} \end{aligned}$$

2. TB Exercise 1.6 *Rotation in x-y Plane*

(a) *Show that the rotation matrix that takes the barred basis vectors to the unbarred basis vectors is:*

$$[R_{\bar{p}i}] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

and show that the inverse of this rotation matrix is, indeed, its transpose, as it must be if this is to represent a rotation.

Solution

For two sets of Cartesian basis vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and $(\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{y}}, \mathbf{e}_{\bar{z}})$, the rotation matrix in Eq. 1 takes the barred basis to the unbarred basis:

$$\mathbf{e}_i = \mathbf{e}_{\bar{p}} R_{\bar{p}i} \tag{2}$$

which gives the three equations:

$$\mathbf{e}_x = \mathbf{e}_{\bar{x}} \cos\phi - \mathbf{e}_{\bar{y}} \sin\phi \tag{3}$$
$$\mathbf{e}_y = \mathbf{e}_{\bar{x}} \sin\phi + \mathbf{e}_{\bar{y}} \cos\phi \tag{4}$$
$$\mathbf{e}_z = \mathbf{e}_{\bar{z}} \tag{5}$$

Since the basis vectors are of unit length, these are simply vector addtions and subtractions of the barred basis vectors of lengths $\cos\phi$ and $\sin\phi$. For an angle ϕ, these combinations of the barred basis vectors give the unbarred basis vectors. See Fig. 1 for a depicition of the results of adding/subtracting these vectors.

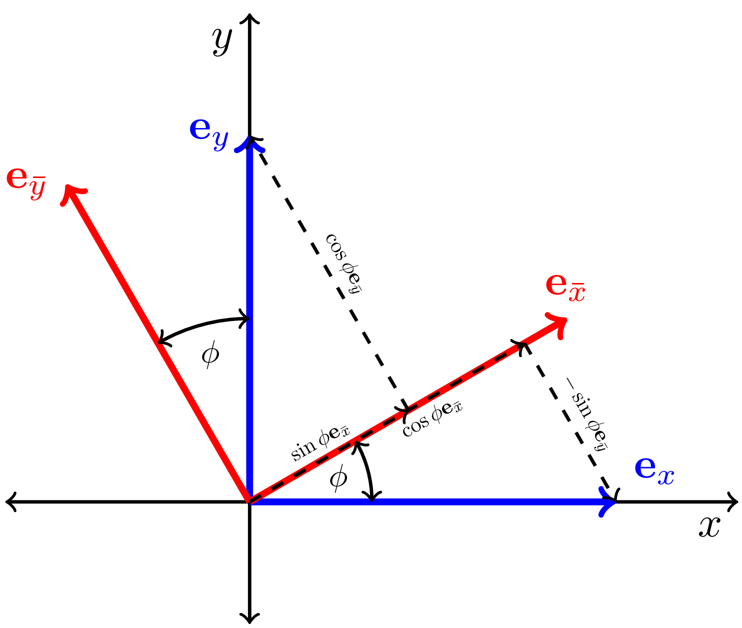

Figure 1: Graphical depiction of rotation matrix performing a linear combination of the barred basis vectors to give the unbarred basis vectors.

The inverse of the rotation matrix is simply:

$$[R_{\bar{p}i}]^{-1} = \frac{1}{\det(R_{\bar{p}i})} [C_{\bar{p}i}]^{\mathrm{T}}$$

where $[C_{\bar{p}i}]$ is the cofactor matrix of rotation matrix, and as easily seen from Eq. 1, $\det(R_{\bar{p}i}) = 1$. With these, the inverse of the rotation matrix becomes:

$$
\begin{aligned}
[R_{\bar{p}i}]^{-1} &= \frac{1}{\det(R_{\bar{p}i})}\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \\
&= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= [R_{\bar{p}i}]^{\mathrm{T}} \\
&= [R_{i\bar{p}}] \qquad (6)
\end{aligned}
$$

(b) *Verify that the two coordinate systems are related by:*

$$x_{\bar{p}} = R_{\bar{p}i}x_i \qquad (7a)$$

$$x_i = R_{i\bar{p}}x_{\bar{p}} \qquad (7b)$$

Solution

To verfiy the coordinate rotation in Eq. 7, consider an arbitrary coordinate in the unbarred frame $(x, y) = (\alpha, \beta)$. Under rotation, this coordinate in the barred frame becomes:

$$
\begin{aligned}
\bar{x} &= R_{\bar{x}x}x + R_{\bar{x}y}y \\
&= \alpha\cos\phi - \beta\sin\phi \qquad (8a) \\
\bar{y} &= R_{\bar{y}x}x + R_{\bar{y}y}y \\
&= \alpha\sin\phi + \beta\cos\phi \qquad (8b)
\end{aligned}
$$

For this transformation to be a rotation through an angle ϕ, the coordinates distance to the shared origin of the coordinate systems must remain the same:

$$
\begin{aligned}
\bar{x}^2 + \bar{y}^2 &= \alpha^2\cos^2\phi + \beta^2\sin^2\phi - 2\alpha\beta\sin\phi\cos\phi + \alpha^2\sin^2\phi + \beta^2\cos^2\phi + 2\alpha\beta\sin\phi\cos\phi \\
&= \alpha^2\big(\cos^2\phi + \sin^2\phi\big) + \beta^2\big(\cos^2\phi + \sin^2\phi\big) \\
&= \alpha^2 + \beta^2 \\
&= x^2 + y^2
\end{aligned}
$$

and the angle ϕ' between the vector to the old coordinate and new coordinate must equal ϕ, which by the law of cosines is:

$$
\begin{aligned}
\cos\phi' &= \frac{1}{2\sqrt{x^2+y^2}\sqrt{\bar{x}^2+\bar{y}^2}}\big[\big(x^2+y^2\big) + \big(\bar{x}^2+\bar{y}^2\big) - (\bar{x}-x)^2 - (\bar{y}-y)^2\big] \\
&= \frac{1}{2(x^2+y^2)}\big[2\big(x^2+y^2\big) - 2\big(x^2+y^2\big) - 2x\bar{x} - 2y\bar{y}\big]
\end{aligned}
$$

$$= \frac{1}{2(\alpha^2+\beta^2)}\left[2\alpha^2\cos\phi - 2\alpha\beta\sin\phi + 2\alpha\beta\sin\phi + 2\beta^2\cos\phi\right]$$
$$= \cos\phi$$

With these confirmed, Eq. 7 can be used to show:

$$\begin{aligned} x_{\bar{p}} &= R_{\bar{p}i}x_i \\ &= R_{\bar{p}i}(R_{i\bar{q}}x_{\bar{q}}) \\ &= (R_{\bar{p}i}R_{i\bar{q}})x_{\bar{q}} \\ &= \delta_{\bar{p}\bar{q}}x_{\bar{q}} \\ &= x_{\bar{p}} \end{aligned}$$

(c) *Let A_j be the components of the electromagnetic vector potential that lies in the xy-plane, so that $A_z = 0$. Show that $A_{\bar{x}} + iA_{\bar{y}} = (A_x + iA_y)e^{-i\phi}$.*

Solution

Vector components follow a similar transformation under rotation as coordinates. For a vector $\mathbf{A}$, its components transform as:

$$A_{\bar{p}} = R_{\bar{p}i}A_i \tag{9a}$$
$$A_i = R_{i\bar{p}}A_{\bar{p}} \tag{9b}$$

With these, the electromagnetic vector potential components transform as:

$$\begin{aligned} A_{\bar{x}} + iA_{\bar{y}} &= R_{\bar{x}i}A_i + iR_{\bar{y}i}A_i \\ &= \cos\phi A_x + \sin\phi A_y - i\sin\phi A_x + i\cos\phi A_y \\ &= A_x\left(\frac{e^{i\phi}+e^{-i\phi}}{2} - \frac{e^{i\phi}-e^{-i\phi}}{2}\right) + A_y\left(\frac{e^{i\phi}-e^{-i\phi}}{2i} - \frac{e^{i\phi}+e^{-i\phi}}{2i}\right) \\ &= A_x e^{-i\phi} + iA_y e^{-i\phi} \\ &= (A_x + iA_y)e^{-i\phi} \end{aligned}$$

(d) *Let h_{jk} be the components of a symmetric tensor that is trace-free (its constraction h_{jj} vanishes) and is confined to the xy-plane (so $h_{zk} = h_{kz} = 0$ for all k). Then the only nonzero components of this tensor are $h_{xx} = -h_{yy}$ and $h_{xy} = h_{yx}$. Show that $h_{\bar{x}\bar{x}} + ih_{\bar{x}\bar{y}} = (h_{xx} + ih_{xy})e^{-2i\phi}$.*

Solution

Tensor components transform with two applications of the rotation matrix:

$$T_{\bar{p}\bar{q}} = R_{\bar{p}i}R_{\bar{q}j}T_{ij} \tag{10a}$$
$$T_{ij} = R_{i\bar{p}}R_{j\bar{q}}T_{\bar{p}\bar{q}} \tag{10b}$$

Applying this (along with the symmetries of the tensor h_{ii}):

$$\begin{aligned}
h_{\bar{x}\bar{x}} + ih_{\bar{x}\bar{y}} &= R_{\bar{x}x}R_{\bar{x}x}h_{xx} + R_{\bar{x}x}R_{\bar{x}y}h_{xy} + R_{\bar{x}y}R_{\bar{x}x}h_{yx} + R_{\bar{x}y}R_{\bar{x}y}h_{yy} \\
&\quad + iR_{\bar{x}x}R_{\bar{y}x}h_{xx} + iR_{\bar{x}x}R_{\bar{y}y}h_{xy} + iR_{\bar{x}y}R_{\bar{y}x}h_{yx} + iR_{\bar{x}y}R_{\bar{y}y}h_{yy} \\
&= h_{xx}\left(\cos^2\phi - \sin^2\phi\right) + 2h_{xy}\cos\phi\sin\phi \\
&\quad - 2ih_{xx}\cos\phi\sin\phi + ih_{xy}\left(\cos^2\phi - \sin^2\phi\right) \\
&= h_{xx}\left(\frac{e^{2i\phi} + e^{-2i\phi}}{2}\right) - ih_{xy}\left(\frac{e^{2i\phi} - e^{-2i\phi}}{2}\right) \\
&\quad - h_{xx}\left(\frac{e^{2i\phi} - e^{-2i\phi}}{2}\right) + ih_{xy}\left(\frac{e^{2i\phi} + e^{-2i\phi}}{2}\right) \\
&= (h_{xx} + ih_{xy})e^{-2i\phi}
\end{aligned}$$

3. TB Exercise 1.7 *Properties of the Levi-Civita Tensor*

From its complete antisymmetry, derive the four properties of the Levi-Civita tensor, in n-dimensional Euclidean space:

i) *The volume vanishes unless all legs are linearly independent.*

Solution

The volume spanned by n-dimensions by the n vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{F}$ is determined by:

$$\mathcal{V}_n = \boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{F}) \equiv \epsilon_{ijk\ldots n}A_iB_jC_k\ldots F_n \tag{11}$$

Since the Levi-Civita tensor is zero if any slots/indices are repeated, e.g. $\epsilon_{iji\ldots n} = 0$, if any leg used to calculate the volume is not linearly independent of all other legs, e.g. $\mathbf{C} = \alpha\mathbf{A} + \beta\mathbf{B}$, the volume vanishes:

$$\begin{aligned}
\boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}, \alpha\mathbf{A} + \beta\mathbf{B}, \ldots, \mathbf{F}) &= \alpha\boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}, \mathbf{A}, \ldots, \mathbf{F}) + \beta\boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}, \mathbf{B}, \ldots, \mathbf{F}) \\
&= \alpha\epsilon_{iji\ldots n}A_iB_jA_i\ldots F_n + \beta\epsilon_{ijj\ldots n}A_iB_jB_j\ldots F_n \\
&= 0
\end{aligned}$$

ii) *Once the volume has been specified for one parallelpiped it is thereby determined for all parallelpipeds.*

Solution

Since tensors are linear functions of vectors, e.g. $\boldsymbol{\epsilon}(\alpha\mathbf{A}, \beta\mathbf{B}, \ldots, \mathbf{F}) = \alpha\beta\boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}, \ldots, \mathbf{F})$, any volume spanned by the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{F}$ is easily determined from the initial volume in Eq. 11:

$$\begin{aligned}
\boldsymbol{\epsilon}(\alpha\mathbf{A}, \beta\mathbf{B}, \gamma\mathbf{C}, \ldots, \omega\mathbf{F}) &= (\alpha\beta\gamma\ldots\omega)\boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{F}) \\
&= (\alpha\beta\gamma\ldots\omega)\mathcal{V}_n
\end{aligned}$$

iii) *Only one number plus antisymmetry required to determine $\boldsymbol{\epsilon}$ fully.*

Solution

Assigning any arbitrary number to any one component of $\boldsymbol{\epsilon}$, e.g.:

$$\epsilon_{123...n} = \alpha$$

allows all other components to be determined by its antisymmetry. Swapping any two component indices results in negating the value:

$$\epsilon_{213...n} = -\epsilon_{123...n} = -\alpha$$

By continued swapping of index pairs, all components with unique index combinations can be obtained and will result in values of $\pm\alpha$. For any components that have repeated indices (swapping repeated 1 and 2 indices with the poisition indicated by bold and non-bold face):

$$\epsilon_{1\mathbf{1}3...n} = -\epsilon_{\mathbf{1}13...n} = 0$$

Only zero is equal to the negative of itself, so the remaining components of $\boldsymbol{\epsilon}$ must be zero for any repeated indices.

iv) *$\boldsymbol{\epsilon}$ is fully determined by its antisymmetry, compatibility with the metric, and a single sign.*

Solution

For the volume spanned by any n basis vectors in n-dimensions, the Levi-Civita tensor gives:

$$\boldsymbol{\epsilon}(\_, \mathbf{e}_j, \mathbf{e}_k, \ldots, \mathbf{e}_n) = \mathbf{e}_i$$

Multiplying both sides by an arbitrary basis vector $\mathbf{e}_\ell$ and applying the metric:

$$g_{\ell i} e_\ell \epsilon_{ijk..n} e_j e_k \ldots e_n = g_{\ell i} e_\ell e_i$$

This reduces to (and choosing a positive sign for the result):

$$g_{\ell i} \epsilon_{ijk..n} e_\ell e_j e_k \ldots e_n = \delta_{\ell i}$$

which gives:

$$\epsilon_{ijk...n} e_i e_j e_k \ldots e_n = +1$$

and as shown in the previous part, this uniquely determines all of the remaining components of $\boldsymbol{\epsilon}$.

4. TB Exercise 1.10 *Volume Elements in Cartesian Coordinates*

Using:

$$\begin{aligned}\mathcal{V}_2 &= \boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}) \\ \mathcal{V}_3 &= \boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}, \mathbf{C})\end{aligned}$$

derive the usual formulas $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ *and* $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *for the 2-dimensional and 3-dimensional integration elements in right-handed Cartesian coordinates.*

Solution

For two-dimensions, the unit volume element is:

$$\begin{aligned}\boldsymbol{\epsilon}(\mathbf{e}_x, \mathbf{e}_y) &= \epsilon_{ab}(e_x)_a(e_y)_b \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 1\end{aligned}$$

With this, the differential area element is:

$$\begin{aligned}\mathrm{d}A &= \boldsymbol{\epsilon}(\mathrm{d}x\,\mathbf{e}_x, \mathrm{d}y\,\mathbf{e}_y) \\ &= \mathrm{d}x\,\mathrm{d}y\,\boldsymbol{\epsilon}(\mathbf{e}_x, \mathbf{e}_y) \\ &= \mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Similarly, for three-dimensions, the unit volume element is:

$$\begin{aligned}\boldsymbol{\epsilon}(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) &= \epsilon_{ijk}(e_x)_i(e_y)_j(e_z)_k \\ &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= 1\end{aligned}$$

and the differential volume element is:

$$\begin{aligned}\mathrm{d}V &= \boldsymbol{\epsilon}(\mathrm{d}x\,\mathbf{e}_x, \mathrm{d}y\,\mathbf{e}_y, \mathrm{d}z\,\mathbf{e}_z) \\ &= \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\boldsymbol{\epsilon}(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \\ &= \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\end{aligned}$$

5. TB Exercise 1.12 *Faraday's Law of Induction*

One of Maxwell's equations says that $\boldsymbol{\nabla} \times \mathbf{E} = -\,\partial\mathbf{B}/\partial t$ *(in SI units), where* $\mathbf{E}$ *and* $\mathbf{B}$ *are the electric and magnetic fields. This is a geometric relationship between geometric objects;*

it requires no coordinates or basis for its statement. By integrating this equation over a 2-dimensional surface $\mathcal{V}_2$ with boundary curve $\partial\mathcal{V}_2$ and applying Stokes' theorem, derive Faraday's law of induction - again, a geometric relationship between geometric objects.

Solution

First, integrating over the surface $\mathcal{V}_2$:

$$\int_{\mathcal{V}_2} (\boldsymbol{\nabla} \times \mathbf{E})\, \mathrm{d}\boldsymbol{\Sigma} = \int_{\mathcal{V}_2} \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \cdot \mathrm{d}\boldsymbol{\Sigma}$$

Applying Stokes' theorem to the left-hand side and assuming a surface that is not changing in time (so that the time derivative can be pulled outside the integral):

$$\int_{\partial\mathcal{V}_2} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}_2} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\Sigma}$$

The left-hand side is now just the change in potential along the boundary curve, also (poorly) referred to as the electromotive-force, and the right-hand side is the negative of the time derivative of the magnetic flux through the surface. This is Faraday's law of induction and can be written as:

$$\mathcal{E}_{\mathrm{emf}} = -\frac{\mathrm{d}\Phi_{\mathbf{B}}}{\mathrm{d}t}$$