Zeeman Effect on The Hydrogen Atom

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Introduction & History

The Zeeman effect comes about from putting an atom in a magnetic field. The magnetic field causes a magnetic moment to be produced and perturbs the energy levels of the atom causing spectral line shifts to appear when modeled or measured with spectroscopy.

Landé g Factor

Consider the Landé g factor for the hydrogen atom Zeeman Effect with a spin value of $s = \frac{1}{2}$,

$$g_L = \frac{12j(j+1) - 4\ell(\ell+1) + 3}{8j(j+1)}. \tag{8}$$

Now, $m_j = \{j + k | -j \leq m_j \leq j \text{ and } k \in \mathbb{Z}\}$ which can be expressed in terms of ℓ as

$$m_j = \{\ell \pm \frac{1}{2} + k | -\ell \leq m_j \pm \frac{1}{2} \leq \ell \pm 1 \text{ and } k \in \mathbb{Z}\}.$$

Using these two forms of m_j and g_L, we can write out energy corrections in terms of a single variable for both $j = \ell + \frac{1}{2}$ and $j = \ell - \frac{1}{2}$ respectively as,

$$E_n^{(1)} = \mu_B B \begin{cases} \frac{2+2l}{1+2\ell}(\ell + \frac{1}{2} - k) & \text{for } 0 \leq k \leq 2\ell + 1 \\ \frac{2l}{1+2\ell}(\ell - \frac{1}{2} - k) & \text{for } 0 \leq k \leq 2\ell - 1 \end{cases}.$$

Now we have a general formula for the corrected energy levels that we can plot for any ℓ value for spin $s = \frac{1}{2}$ hydrogen (demonstrated in figure 1).

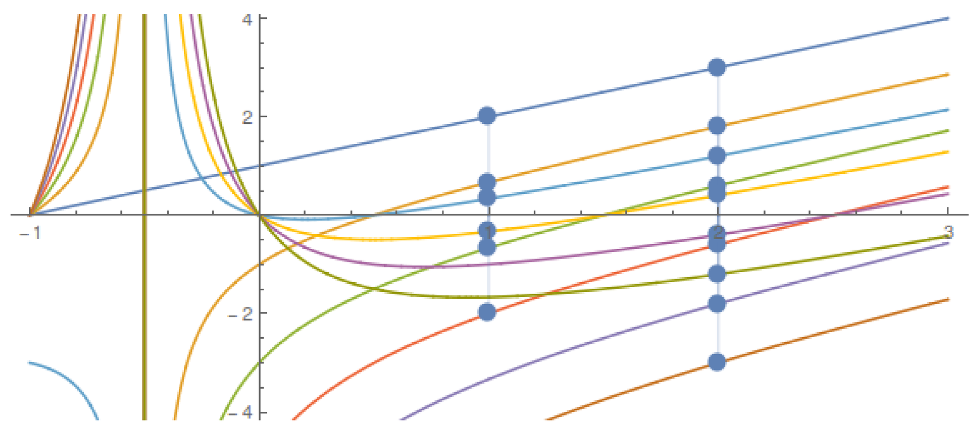

Figure 1. Relative hydrogen Energy corrections for $\ell = 1$ and $\ell = 2$ with units of $\mu_B = B = 1$.

Perturbation Theory & The Zeeman Effect

The unperturbed Hamiltonian of hydrogen is given by

$$H^{(0)} = \underbrace{\frac{\vec{p}^2}{2m}}_{\text{Kinetic term}} - \underbrace{\frac{\hbar c \alpha}{r}}_{\text{Potential term}}. \tag{1}$$

The total Zeeman Effect to the Hamiltonian are given by determining the effect of a constant magnetic field $\vec{B}$,

$$H^{(1)}_{mag} = \frac{e}{2m}(\vec{L} + 2\vec{S}) \cdot \vec{B} = \frac{\mu_B}{\hbar}(\vec{L} + 2\vec{S}) \cdot \vec{B}, \tag{2}$$

where $\mu_B = \frac{e\hbar}{2m}$ is the Bohr magneton. From perturbation theory we can solve for the first order energy corrections that come from $H^{(1)}_{mag}$ by using

$$E_n^{(1)} = \langle n j m_j \ell s | H^{(1)}_{mag} | n j m_j \ell s \rangle. \tag{3}$$

This gives a general result

$$E_n^{(1)} = \mu_B B m_j g_L, \tag{4}$$

where g_L is the Landé g factor. This result in (4) can be used for an explicit state of the hydrogen atom. For example, consider the $2P_{3/2}$ state. This has $n = 2$, $\ell = 1$, $s = \frac{1}{2}$, and $j = \frac{3}{2}$. The m_j values are shown by the set $m_j = \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$, which tells us there will be 4 different energy splits for this state. Using these values gives us a $g_L = \frac{4}{3}$ and so our energy splits are

$$E^{(1)}_{2P_{3/2}} = \begin{cases} 2\mu_B B & \text{for } m_j = \frac{3}{2} \\ \frac{2\mu_B B}{3} & \text{for } m_j = \frac{1}{2} \\ -\frac{2\mu_B B}{3} & \text{for } m_j = -\frac{1}{2} \\ -2\mu_B B & \text{for } m_j = -\frac{3}{2} \end{cases} \tag{5}$$

Changing Magnetic Fields & Induced Electric Fields

Consider $\vec{B} = Bt\hat{z}$. By Maxwell's Equations we have $\frac{\partial \vec{E}}{\partial t} = 0$, which implies the electric field is a constant throughout time. Similarly, the electric field must satisfy $\nabla \times \vec{E} = -B\hat{z}$. If we take the curl of both sides of this we get

$$\begin{aligned} \nabla \times (\nabla \times \vec{E}) &= \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \\ &= -\nabla \times B\hat{z} = 0. \end{aligned}$$

Using the assumption of a vacuum, this becomes

$$\nabla^2 \vec{E} = 0$$

From this and our magnetic field, we require that there be an electric field $\vec{E}_1 = By\hat{x}$ in order for Maxwell's equations to be satisfied. By analogy we can also find another solution to $\vec{E}$ as $\vec{E}_2 = -Bx\hat{y}$. And so by superposition we can also have a solution

$$\vec{E}_{1+2} = B(y\hat{x} - x\hat{y}). \tag{6}$$

This shows that with a changing magnetic field, we would see an electric field appear which could itself cause another corrections to the perturbed energies. This is known as the Stark Effect and in our case would introduce the correction

$$H^{(1)}_{elec} = -e\vec{r} \cdot \vec{E} = 0. \tag{7}$$

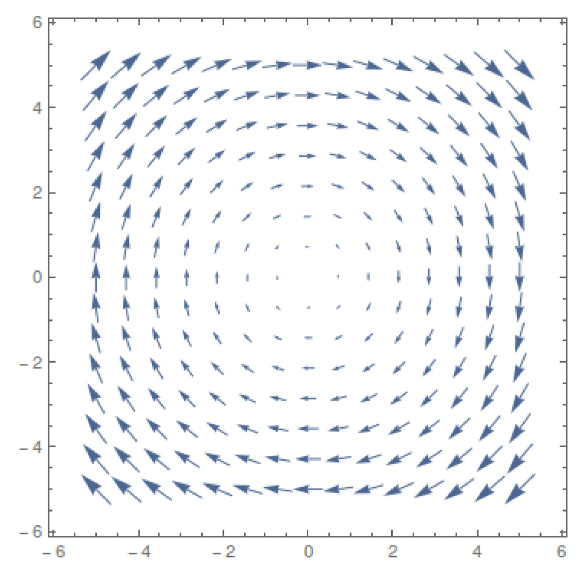

Figure 2. A plot of $\vec{E}_{1+2}$ using B=1 in the x-y plane.

Addendum and Final Remarks

As shown above, the induced electric field of a linearly changing magnetic field in time does not induce the Stark Effect. For any constant magnetic field, we are able to generate a plot of the correction splittings as shown in figure 4 and 5. It is important to note that with a time dependent magnetic field, we would need to use time-dependent perturbation theory to solve for the energy corrections.

Observations

The Zeeman Effect is a real phenomenon that can be observed in spectroscopy when magnetic fields effect the radiation (see figure 3). This can be used to determine where magnetic fields are present and their strength depending on the separation of observed spectral energy levels.

Computational Models

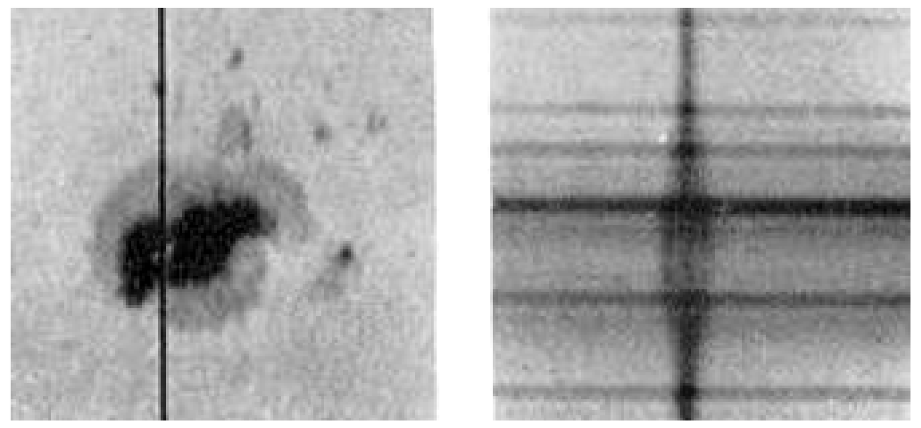

Figure 3. Observations of the Zeeman Effect from spectroscopy of a sunspot [2].

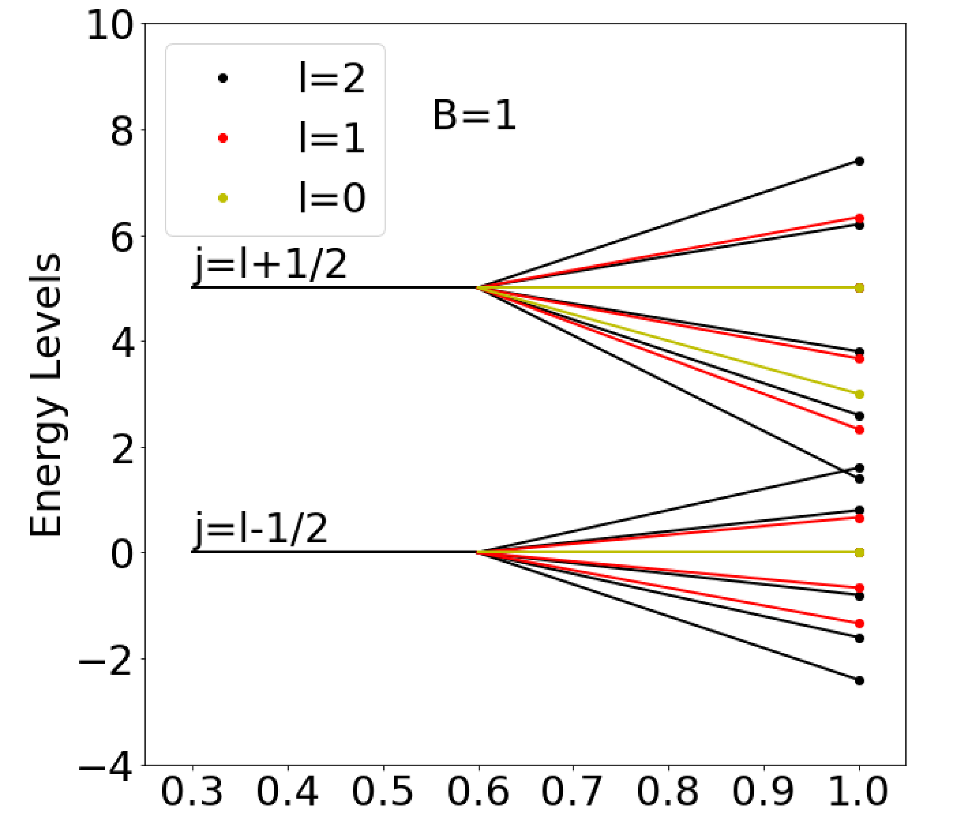

Figure 4. Relative hydrogen Energy corrections.

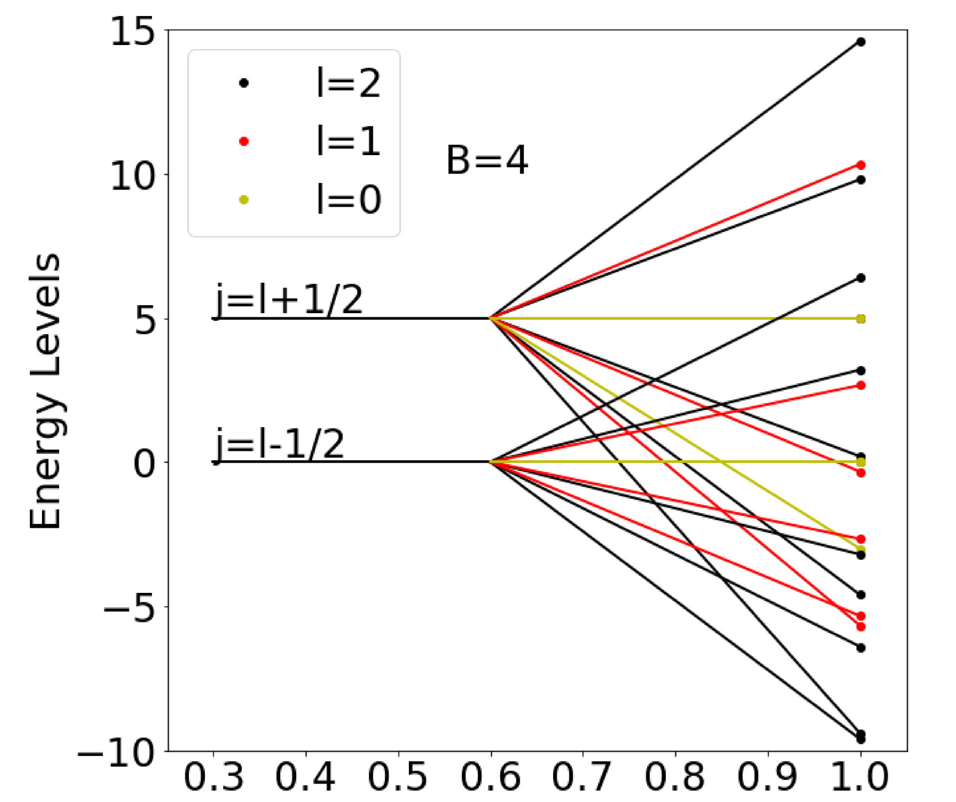

Figure 5. Relative hydrogen Energy corrections with an increased magnetic field.

References

1. Griffiths, David J. Introduction to Quantum Mechanics. 2nd ed. Harlow: Pearson, 2014. Print.
2. Image from https://en.wikipedia.org/wiki/Zeeman_effect