The Quantum Casimir Effect

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Introduction & History

Named after the Dutch physicist Hendrik Casimir, the Casimir effect is a physical force that arises from and is explained by quantized fields in quantum field theory. The typical example of this is an apparent attraction created between two very closely placed parallel plates within a vacuum. Due to the nature of quantized fields dealing with virtual particles in a vacuum, a force becomes present in the system.

Analytic Continuation

In quantum physics, the energy density should be $S(-3) = 1+8+27+64+\cdots$, which is a divergent series and thus does not make much sense as an energy density. When we use the process of analytic continuation, this can be written as

$$\zeta(-3) = 1 + 2^3 + 3^3 + 4^3 + \cdots \to 1/120. \tag{16}$$

The way Ramanujin expresses functions that are divergent such as this [3] (from the Riemann zeta function) is

$$\sum_{k=\alpha}^{x} f(k) \sim \int_{\alpha}^{x} f(t)dt + c + \frac{1}{2}f(x) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x). \tag{17}$$

This is a process of analytically continuing these divergent series and coming up with a finite result without any 'magic'. I say magic because there is a process, which was first shown by Euler around 1735 [1], in which one can ignore the divergent nature of a sum and come up with these results as well. An example is demonstrated in the right panel which is a result that is very important to obtaining the $24+2 = 26$ dimensions in bosonic string theory. It is also a simpler example than that of equation (16) to demonstrate.

References

1. John Baez on the number 24. The Rankin Lectures 2008. Youtube. John Baez. University of California. 16 May. 2012. Web.
2. Gibbs, Philip. "What Is the Casimir Effect?" The Casimir Effect. University of California, 1997. Web. 16 Oct. 2016.
3. Ramanujan Aiyangar, Srinivasa, 1887-l 920. Ramanujans notebooks. Mathematics-Collected works. 1. Berndt, Bruce C., 1939- II. Title. QA3. R33. 1985. 510. 84-20201

Derivation of the Casimir Force

Let k_x and k_y represent the wave numbers in the x and y directions respectively and $k_n = n\pi/a$. If we allow two plates to be parallel in the $x-y$ plane at a distance a apart, the standing waves are

$$\psi_n(x, y, z; t) = e^{ik_x x + ik_y y - i\omega_n t}\sin(k_n z). \tag{1}$$

The frequency of this wave is $\omega_n = c\sqrt{k_x^2 + k_y^2 + k_n^2}$. The vacuum energy is the sum over all possible expectation modes. Taking the expectation value of the energy yields

$$\langle E \rangle = \frac{A\hbar}{4\pi^2} \iint \sum_{n=1}^{\infty} \omega_n dk_x dk_y. \tag{2}$$

This expression is clearly infinite due to the diverging sum. If we use zeta-regulation, we can find a finite energy per unit area which is

$$\frac{\langle E(s) \rangle}{A} = \frac{\hbar}{4\pi^2} \iint \sum_{n=1}^{\infty} \omega_n |\omega|^{-s} dk_x dk_y. \tag{3}$$

Simplifying the above expression gives us

$$\frac{\langle E(s) \rangle}{A} = \frac{\hbar c^{1-s}\pi^{2-s}}{2a^{3-s}(3-s)} \sum_{n=1}^{\infty} |n|^{3-s}. \tag{4}$$

This expression may be analytically continued to $s = 0$ where it becomes finite.

$$\frac{\langle E \rangle}{A} = \lim_{s\to 0} \frac{\langle E(s) \rangle}{A} = -\frac{\hbar c \pi^2}{6a^3}\zeta(-3). \tag{5}$$

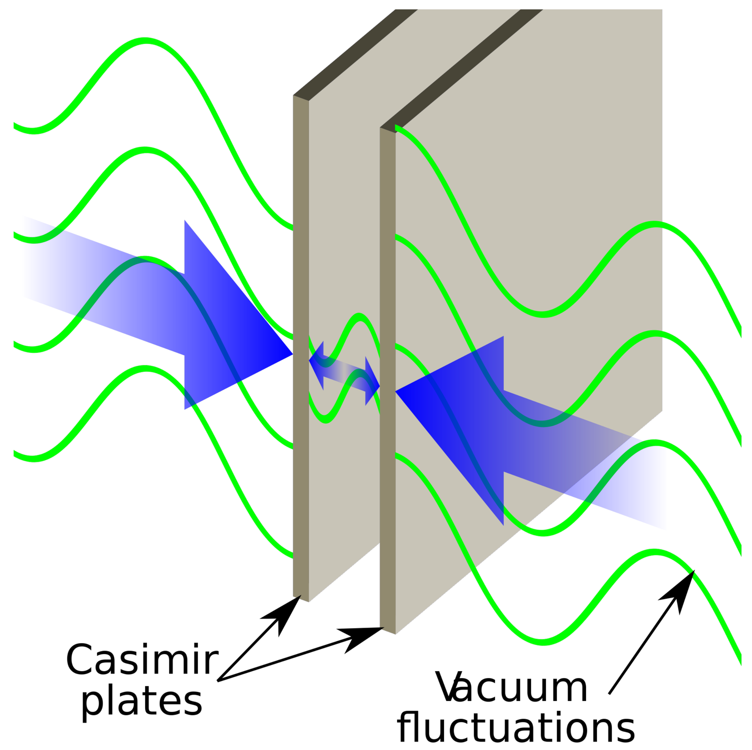
Now, plugging in (16) to the above expression gives us

$$\frac{\langle E \rangle}{A} = \frac{-\hbar c \pi^2}{720 a^3}. \tag{6}$$

The Casimir force per unit area between two parallel plates within a vacuum is therefore given by $F = -\nabla\langle E \rangle$ which is

$$\frac{F_c}{A} = -\frac{d}{da}\frac{\langle E \rangle}{A} = \frac{-\hbar c \pi^2}{240 a^4}. \tag{7}$$

Implications and Applications



Casimir and the Casimir Force between two parallel plates. Images taken from Wikipedia.

The Casimir effect extends quantum field theory to allow for negative energy densities. It has been suggested by numerous physicists such as Stephen Hawking, Kip Thorne, and others that such a thing will allow the possibilities of stabilizing traversable wormholes. Miguel Alcubierre, creator of the Alcubierre Drive has also suggested using the Casimir effect to obtain the negative energy required for his designs. Other possible applications include propulsion drives for space craft and nano-technology.

Addendum and Final Remarks

As we can clearly see, the Casimir force would not have come about without the use of analytical continuation. In a sense, this is due to nature not containing apparent infinities. Rather, the continuation allowed us to arrive at a finite solution which is experimentally confirmed. The facet that our expression came out negative suggests that the force is an attractive force, and due to the presence of $\hbar$, we can see that the force is of a quantum origin. In the original derivation, Casimir computed nonconvergent sums using Euler-Maclaurin summation with a regularizing function [5]

Discovery & Properties

All bosons make a contribution to the Casimir force, but fermions make a repulsive contribution to the force. Even though all of these particles make a contribution to the force, only that from photons is measurable. The theory states that the lowest energy state of a vacuum is infinite when considering all possible photon modes. The Casimir force comes about from a situation in which the differences in infinities cancel out.

Astounding Mathematics

Consider the following well defined sum ($x < 1$)

$$f(x) = 1 + x + x^2 + x^3 + \cdots = (1-x)^{-1} \tag{8}$$

$$\implies f'(x) = 1 + 2x + 3x^2 + \cdots = (1-x)^{-2}. \tag{9}$$

If we evaluate this at $x = -1$ we get

$$f'(-1) = 1 - 2 + 3 - 4 + \cdots = 1/4. \tag{10}$$

Now, if we take $2^{-s}\zeta(s)$ we have

$$2^{-s}\zeta(s) = 2^{-s} + 4^{-s} + 6^{-s} + 8^{-s} \cdots. \tag{11}$$

Now, if we take $g(s) = [1 - 2(2^{-s})]\zeta(s)$ we have

$$g(s) = 1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - \cdots. \tag{12}$$

If we set $s = -1$, we can see that $g(-1) = -3\zeta(-1)$ and so

$$-3\zeta(-1) = 1 - 2 + 3 - 4 + \cdots = 1/4 \tag{13}$$

$$\implies \zeta(-1) = -1/12. \tag{14}$$

Now, notice that plugging in $s = -1$ into the Riemann zeta function gives

$$\zeta(-1) = -\frac{1}{12} \implies \sum_{n=1}^{\infty} n \to -\frac{1}{12}. \tag{15}$$

References

4. Casimir Effect & Black Holes - Sixty Symbols. Prod. Brady Haran. Perf. Mike Merrifield Ph.D. Youtube. University of Nottingham, 20 Mar. 2014. Web.
5. H.B.G. Casimir, Proc. Kon. Ned. Akad. Wetensch. B51, 793 (1948)
6. Sum of Natural Numbers (second proof and extra footage). Prod. Brady Haran. Perf. Ed Copeland Ph.D and Tony Padilla Ph.D. Youtube. University of Nottingham, 11 Jan. 2015. Wb.