We are interested in static magnetic field $\vec{B}_0 = (0, 0, B_0)$ on the possible plasma waves and also we will neglect pressure forces for simplicity. The equation of motion then takes the form

$$\frac{\partial \vec{v}_1^{(j)}}{\partial t} = \frac{q_j}{m_j}\left(\vec{E}_1 + \vec{v}_1^{(j)} \times \vec{B}_0\right) \qquad j = e, i, \ldots$$

where the index $j = e, i$, distinguishes electrons from ions. We will neglect the second order Lorentz force $v_1^{(j)} \times \vec{B}_1$ arising from the wave magnetic field. Our task is to calculate the first-order particle velocities in the first order wave electric field $\vec{E}_1$ including the deflection by the Lorentz force. From this, follow the alternating currents of electrons and ions and hence the conductivity tensor. Now we drop the index 1 again.
Considering the motion in the plane perpendicular to $\vec{B}_0$, the resulting particle velocities are (in the wave field).

$$\hat{v}_x^{(j)} = i\frac{q_j}{\omega m_j}\left(\hat{E}_x + \hat{v}_y^{(j)} B_0\right)$$

$$\hat{v}_y^{(j)} = i\frac{q_j}{\omega m_j}\left(\hat{E}_y + -\hat{v}_x^{(j)} B_0\right)$$

These two equations are usually decoupled by introducing rotating coordinates $\hat{v}_\pm = \hat{v}_x \pm i\hat{v}_y$ and $\hat{E}^\pm = \hat{E}_x \pm i\hat{E}_y$ to give

$$\hat{v}_\pm = \frac{i\, q_j}{\omega m_j}\left(\hat{E}^\pm \mp i\hat{v}_\pm B_0\right)$$

introducing the cyclotron frequencies $\omega_{cj} = |q_j| B_0 / m_j$ the solution takes the compact form

$$\hat{v}_\pm = i\frac{q_j}{m_j}\hat{E}^\pm \frac{1}{\omega \mp s_j \omega_{cj}}$$

Here S_j is the sign of the charge. The transform to Cartesian Coordinates is accomplished by $\hat{v}_x^{(j)} = (\hat{v}^+ + \hat{v}^-)/2$ and $\hat{v}_y^{(j)} = (\hat{v}^+ - \hat{v}^-)/(2i)$ in matrix notation the relationship between the velocity amplitude and wave electric field reads

$$\begin{pmatrix} \hat{v}_x^{(j)} \\ \hat{v}_y^{(j)} \\ \hat{v}_z^{(j)} \end{pmatrix} = i\frac{q_j}{\omega m_j}\begin{pmatrix} \frac{\omega^2}{\omega^2-\omega_{cj}^2} & i\frac{s_j\omega\omega_{cj}}{\omega^2-\omega_{cj}^2} & 0 \\ -i\frac{s_j\omega\omega_{cj}}{\omega^2-\omega_{cj}^2} & \frac{\omega^2}{\omega^2-\omega_{cj}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix}$$

Using our definition of the total current $\hat{j} = \sum_j n_j q_j \hat{v}^{(j)}$ we obtain the conductivity tensor as

$$\sigma = i\omega\varepsilon_0 \begin{pmatrix} \sum_j \frac{\omega_{pj}^2}{\omega^2-\omega_{cj}^2} & i\sum_j s_j \frac{\omega_{pj}^2}{\omega^2-\omega_{cj}^2}\frac{\omega_{cj}}{\omega} & 0 \\ -i\sum_j s_j \frac{\omega_{pj}^2}{\omega^2-\omega_{cj}^2}\frac{\omega_{cj}}{\omega} & \sum_j \frac{\omega_{pj}}{\omega^2-\omega_{cj}} & 0 \\ 0 & 0 & \sum \frac{\omega_{pj}^2}{\omega^2-\omega_{cj}^2} \end{pmatrix}$$

The dielectric tensor has the structure

$$\epsilon = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

Which contains the so-called Stix-parameters

$$S = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2-\omega_{cj}^2} \quad D = \sum_j s_j \frac{\omega_{pj}^2}{\omega^2-\omega_{cj}^2}\frac{\omega_{cj}}{\omega} \quad P = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2}$$

The parameter $P = \epsilon_{zz}$ is our familiar dielectric constant of unmagnetized plasma. This coincidence is obvious because the motion of the charged particles along the magnetic field does not involve the Lorentz force. At last using the definition of the refractive index $N = kc/\omega$ and introducing the angle ψ between the wave vector and magnetic field direction the wave equation becomes

$$\begin{pmatrix} S - N^2\cos^2\psi & -iD & N^2\cos\psi\sin\psi \\ iD & S-N^2 & 0 \\ N^2\cos\psi\sin\psi & 0 & P - N^2\sin^2\psi \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0$$

Here the wave vector $\vec{k} = (k\sin\psi, 0, k\cos\psi)$ has been used to define the x-z plane.

Propogation along magnetic field.

When the wave propogates along the magnetic field, we have $\psi = 0$ and the wave equation takes the form

$$\begin{pmatrix} S - N^2 & -iD & 0 \\ iD & S-N^2 & 0 \\ 0 & 0 & P \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0$$

We immediately see that $P\hat{E}_z = 0$ describes longitudinal plasma oscillations along magnetic fieldlines. The remaining set of equations for transverse polarization can be written in rotating coordinates as

$$(S-D-N^2)\hat{E}^+ + (S+D-N^2)\hat{E}^- = 0$$

Here $\hat{E}^+$ is a left hand circularly polarized wave and requires $S-D-N^2=0$ while $\hat{E}^-$ is a right handed circularly polarized wave and whose dispersion described as $(S+D-N^2) = 0$.

Using the Stix parameters, we find for the refractive index of the R-wave and L-wave

$$N_R = \left[1 - \frac{\omega_{pe}^2}{\omega(\omega - \omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega + \omega_{ci})}\right]^{1/2}$$

$$N_L = \left[1 - \frac{\omega_{pe}^2}{\omega(\omega + \omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega - \omega_{ci})}\right]^{1/2}$$

Let us now consider wave propogation through a magnetized plasma of frequencies far bellow the ion cyclotron or plasma frequencies, which are in turn well bellow the corresponding electron frequencies, In the low-frequency regime ($\omega \ll \Omega_i \Pi_i$)

$$S \simeq 1 + \frac{\Pi_i^2}{\Omega_i^2}$$

$$D \approx 0,$$

$$P = \frac{-\Pi_e^2}{\omega^2}$$

here Use has been made of $\Pi_e^2/\Omega_e \Omega_i = -\Pi_i^2/\Omega_i^2$ thus the eigenmode equation reduces to

$$\begin{pmatrix} 1 + \Pi_i^2/\Omega_i^2 - n^2\cos^2\theta & 0 & n^2 \cos\theta \sin\theta \\ 0 & 1 + \Pi_i^2/\Omega_i^2 - n^2 & 0 \\ n^2 \cos\theta \sin\theta & 0 & -\Pi_e^2/\omega^2 - n^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

the solubility conditions yields the dispersion relation

$$\begin{vmatrix} 1 + \Pi_i^2/\Omega_i^2 - n^2\cos^2\theta & 0 & n^2 \cos\theta \sin\theta \\ 0 & 1 + \Pi_i^2/\Omega_i^2 - n^2 & 0 \\ n^2 \cos\theta \sin\theta & 0 & -\Pi_e^2/\omega^2 - n^2 \sin^2\theta \end{vmatrix}$$

we obtain two roots

$$n^2 \cos^2\theta = 1 + \frac{\Pi_i^2}{\Omega_i^2}$$

and

$$n^2 = 1 + \frac{\Pi_i^2}{\Omega_i^2}$$

from the plasma cyclotron frequencies

$$\frac{\Pi_i^2}{\Omega_i^2} = \frac{c^2}{B_0^2/\mu_0 \rho} = c^2/V_A^2$$

Here $\rho \equiv n M_i$ is the plasma density and

$$V_A = \sqrt{\frac{B_0^2}{\mu_0 \rho}}$$

is called the alfven velocity. Thus the dispersion relation of the two low frequency waves can be written

$$\omega = \frac{K V_A \cos\theta}{\sqrt{1 + V_A^2/c^2}} \simeq K V_A \cos\theta = k_{\parallel} V_A$$

and

$$\omega = \frac{K V_A}{\sqrt{1 + V_A^2/c^2}} \simeq K V_A.$$

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