CSCI 301 HW 4

Isaac Boaz

May 2, 2023

Problem 1

Define a relation R on $\mathbb{Z}$ as xRy if and only if $3 \mid (2x + y)$

a. Prove that R is an equivalence relation.
To show that R is an equivalence relation, we must show that R is reflexive, symmetric, and transitive.

Reflexive R is reflexive if and only if xRx for all $x \in \mathbb{Z}$.
That is to say, R is reflexive if and only if $3 \mid (2x + x)$ for all $x \in \mathbb{Z}$.

Proof. By contradiction.
Suppose $3 \nmid (2x + x)$
We can rewrite $2x + x$ as $3x$.
By definition of divisibility, $3 \mid 3x$.
Thus $2x + x$ is both divisible by 3 and not divisible by 3. Contradiction!
Therefore $3|2x + x$ □

Symmetric R is symmetric if and only if $xRy \implies yRx$ for all $x, y \in \mathbb{Z}$.
That is to say, R is symmetric if and only if $3 \mid (2x + y) \implies 3 \mid (2y + x)$ for all $x, y \in \mathbb{Z}$.

Proof. By definition of divisibility, we can write $2x + y$ as

$$\begin{aligned} 2x + y &= 3a, a \in \mathbb{Z} \\ y &= 3a - 2x \end{aligned}$$

Thus

$$\begin{aligned} 2y + x &= 2(3a - 2x) + x \\ &= 6a - 4x + x \\ &= 6a - 3x \\ &= 3(2a - x) \\ &= 3b \text{ where } b = 2a - x \end{aligned}$$

Therefore by definition of divisibility, $3 \mid (2y + x)$ □

Transitive R is transitive if and only if $xRy \wedge yRz \implies xRz$ for all $x, y, z \in \mathbb{Z}$.
That is to say, R is transitive if and only if $3 \mid (2x + y) \wedge 3 \mid (2y + z) \implies 3 \mid (2x + z)$

Proof. By definition of divisibility, we can rewrite the first two statements as

$$\begin{aligned} 2x + y &= 3a, a \in \mathbb{Z} \\ 2y + z &= 3b, b \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} y &= 3a - 2x \\ z &= 3b - 2y \end{aligned}$$

which allows us to define

$$\begin{aligned} z &= 3b - 2y \\ &= 3b - 2(3a - 2x) \\ &= 3b - 6a + 4x \end{aligned}$$

Plugging this in to $2x + z$ we get

$$\begin{aligned} 2x + z &= 2x + 3b - 6a + 4x \\ &= 6x + 3b - 6a \\ &= 3(2x + b - 2a) \\ &= 3c \text{ where } c = 2x + b - 2a \end{aligned}$$

Therefore by definition of divisibility, $3 \mid (2x + z)$ $\square$

b. Describe the equivalence classes of R.
To determine the equivalence classes of R, we must observe that there are three possible values that result from $3 \mid 2x + y$: $0, 1, 2$.
We can rewrite this equation as $y \equiv -2x \mod 3$.
Therefore, the set of all integers y that satisfies $y \equiv -2x \mod 3$ is the equivalence class $[x]$:

$$[x] = \{y \in \mathbf{Z} : y \equiv -2x \mod 3\}$$

$[0]_R = \{\dots, -6, -3, 0, 3, 6, \dots\}$
$[1]_R = \{\dots, -5, -2, 1, 4, 7, \dots\}$
$[2]_R = \{\dots, -4, -1, 2, 5, 8, \dots\}$

Problem 2

Prove the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $f(m,n) = (5m+4n, 4m+3n)$ is bijective. Find its inverse f^{-1}.

To prove that f is bijective, we must show that f is both injective and surjective.

Injective f is injective if and only if $f(a) = f(b) \implies a = b$ for all $a, b \in \mathbb{Z} \times \mathbb{Z}$.
That is to say, f is injective if and only if $f(m,n) = f(p,q) \implies (m,n) = (p,q)$ for all $m,n,p,q \in \mathbb{Z}$.

Proof. Suppose $f(a,b) = f(c,d)$.
Then

$$\begin{aligned} 5a + 4b &= 5c + 4d \\ 4a + 3b &= 4c + 3d \end{aligned}$$

$$\begin{aligned} 3b &= 4c + 3d - 4a \\ b &= \frac{4c+3d-4a}{3} = \frac{4}{3}(c-a) + d \\ 5a + 4(\frac{4}{3}(c-a) + d) &= 5c + 4d \\ 5a + \frac{16}{3}(c-a) + 4d &= 5c + 4d \\ 5a + \frac{16c}{3} - \frac{16a}{3} &= 5c \\ -\frac{a}{3} + \frac{16c}{3} &= 5c \\ -\frac{a}{3} &= -\frac{c}{3} \\ a &= c \end{aligned}$$

$$\begin{aligned} 4a &= 4c + 3d - 3b \\ a &= c + \frac{3}{4}(d-b) \\ 5(c + \frac{3}{4}(d-b)) + 4b &= 5c + 4d \\ \frac{15}{4}(d-b) + 4b &= 4d \\ \frac{15}{4}d - \frac{15}{4}b + 4b &= 4d \\ \frac{b}{4} &= \frac{d}{4} \\ b &= d \end{aligned}$$

Thus, $a = c$ and $b = d$.
Therefore $f(a,b) = f(c,d) \implies (a,b) = (c,d)$ □

Surjective f is surjective if and only if $f(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$.
That is to say, f is surjective if and only if $\forall (m,n) \in \mathbb{Z}\times\mathbb{Z}, \exists (a,b) \in \mathbb{Z}\times\mathbb{Z}$ such that $f(a,b) = (m,n)$.