CSCI 301 HW 2

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Proposition. *Let $a, b \in \mathcal{Z}$. $(a+5)b^2 \in E$ if and only if $a \in O \vee b \in E$.*

Proof. Let's first show that $(a+5)b^2 \in E \implies a \in O \vee b \in E$ by contraposition.
Suppose $a \notin O \wedge b \notin E$.
Thus a is even and b is odd.
By definition of even and odd, $a = 2n_1, b = 2n_2 + 1$ where $n_1, n_2 \in \mathcal{Z}$.
Thus $(a+5)b^2 = (2n_1+5)(2n_2+1)^2$
$= 8n_1n_2^2 + 20n_2^2 + 8n_1n_2 + 20n_2 + 2n_1 + 5$
$= 2z + 1$ where $z = 4n_1n_2^2 + 10n_2^2 + 4n_1n_2 + 10n_2 + n_1 + 2 \in \mathcal{Z}$
Therefore by definition of odd $(a+5)b^2$ is not even.

Conversely, Suppose $a \in O \vee b \in E$.
Case 1: $a \in O$
Then by definition of odd $a = 2n+1$ where $n \in \mathcal{Z}$.
Thus $(a+5)b^2 = (2n+6)b^2$
$= 2b^2(n+3) = 2z$ where $z = b^2(n+3) \in \mathcal{Z}$
Therefore by definition of even $(a+5)b^2$ is even.
Case 2: $b \in E$
Then by definition of even $b = 2n$ where $n \in \mathcal{Z}$.
Thus $(a+5)b^2 = (a+5)(2n)^2$
$= 4n^2(a+5) = 2z$ where $z = 2n^2(a+5) \in \mathcal{Z}$
Therefore by definition of even $(a+5)b^2$ is even.
Therefore $(a+5)b^2 \in E$.

Therefore $(a+5)b^2 \in E \iff a \in O \vee b \in E$.

$\square$

Proposition. $\forall n \in \mathcal{Z}, 3 \nmid (n^2 - 5)$

Proof. By contradiction.
Suppose $\exists n \in \mathcal{Z}$ s.t. $3 \mid (n^2 - 5)$
By definition of divisibility, $(n^2 - 5) = 3z$, where $z \in \mathcal{Z}$
Solving for n^2 we get $n^2 = 3z + 5$
When dividing by 3, there are 3 possible remainders (0, 1, 2).
Suppose $m \in \mathcal{Z}$
Case 1: $n = 3m$
Then $n^2 = (3m)^2 = 9m^2 = 3(3m^2)$ which has a remainder of 0.
Case 2: $n = 3m + 1$
Then $n^2 = (3m + 1)^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1$ which has a remainder of 1.
Case 3: $n = 3m + 2$
Then $n^2 = (3m + 2)^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m) + 4$ which has a remainder of 4.
Thus n^2 can only have remainders of 0, 1, or 4 when divided by 3.
However, $n^2 = 3z + 5$ can only have a remainder of 2.
Therefore $n^2 = 3z + 5$ cannot be divisible by 3.

$\square$