CSCI 301 HW 3

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Proposition. *If A, B, and C are sets, then $A - (B \cap C) = (A - B) \cup (A - C)$*

Proof. By definition of set equality, $A = B \iff A \subseteq B \land B \subseteq A$.
Proving $A - (B \cap C) \subseteq (A - B) \cup (A - C)$
Suppose $x \in A - (B \cap C)$
By definition of difference, $x \in A \land x \notin (B \cap C)$
By definition of intersection, x could be in B or C, but not both.
WLOG Suppose $x \in B$.
Then $x \in B \land x \notin C$.
By definition of complement, $x \in (A - C)$.
By definition of union, $x \in (A - B) \cup (A - C)$
Therefore $A - (B \cap C) \subseteq (A - B) \cup (A - C)$

Proving $(A - B) \cup (A - C) \subseteq A - (B \cap C)$
Suppose $y \in (A - B) \cup (A - C)$
By definition of union, $y \in (A - B) \lor y \in (A - C)$
WLOG suppose $y \in (A - B)$
By definition of difference, $y \in A \land y \notin B$
Thus $y \in A \land y \notin (B \cap C)$
Thus by definition of difference $y \in A - (B \cap C)$
Therefore $(A - B) \cup (A - C) \subseteq A - (B \cap C)$

Therefore $A - (B \cap C) = (A - B) \cup (A - C)$ $\square$

Proposition. *If $n \in \mathbb{Z}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$*
Note: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = \sum_{i=1}^{n} \frac{i}{(i+1)!}$

Proof. By mathematical induction.
Basis Step: Observe at $n = 1$, $\frac{1}{2!} = 1 - \frac{1}{2!}$ is true.
Inductive Step: Suppose $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$
Then $\sum_{i=1}^{n} \frac{i}{(i+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!}$
$= 1 - \frac{(n+2)!}{(n+1)!(n+2)!} + \frac{(n+1)(n+1)!}{(n+1)!(n+2)!}$
$= 1 - \frac{(n+1)!(n+2)}{(n+1)!(n+2)!} + \frac{(n+1)(n+1)!}{(n+1)!(n+2)!}$
$= 1 - \frac{(n+1)!((n+2)-(n+1))}{(n+1)!(n+2)!} = 1 - \frac{(n+2)-(n+1)}{(n+2)!}$
$= 1 - \frac{1}{(n+2)!}$
Therefore $\sum_{i=1}^{n} \frac{i}{(i+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$ $\square$

Proposition. *If $n \in \mathbb{Z}$, then $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$.*

Proof. By mathematical induction.
Basis Step: Observe at $n = 1$, $\frac{1}{1} + \frac{1}{2} \geq 1 + \frac{1}{2}$ is true.
Inductive Step: Suppose $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$
That is to say, suppose $\sum_{i=1}^{2^n} \frac{1}{i} \geq 1 + \frac{n}{2}$.
Then $\sum_{i=1}^{2^n} \frac{1}{i} + \sum_{i=2^n}^{2^{n+1}} \geq 1 + \frac{n}{2} + \sum_{i=2^n+1}^{2^{n+1}} \frac{1}{i}$
There are 2^n terms in the sum $\sum_{i=2^n+1}^{2^{n+1}} \frac{1}{i}$, and they are all $\geq \frac{1}{2^{n+1}}$, so their sum is $\geq \frac{2^n}{2^{n+1}} = \frac{1}{2}$
Thus $\sum_{i=1}^{2^{n+1}} \frac{1}{i} \geq 1 + \frac{n}{2} + \sum_{i=2^n+1}^{2^{n+1}} \frac{1}{i} \geq 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2}$
Hence, we have shown that $\sum_{i=1}^{2^n} \frac{1}{i} \geq 1 + \frac{n}{2}$ holds for all $n \in \mathbb{Z}$

$\square$