CSCI 301 HW 3

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Proposition. *If A, B, and C are sets, then $A - (B \cap C) = (A - B) \cup (A - C)$.*

Proof. To prove the set equality $A - (B \cap C) = (A - B) \cup (A - C)$, we need to show that every element in $A - (B \cap C)$ is also in $(A - B) \cup (A - C)$ and vice versa.

First, let x be an element of $A - (B \cap C)$. This means that x is in A, but not in $B \cap C$. In other words, x is in A, but not in both B and C. Therefore, x must be in either $(A - B)$ or $(A - C)$, or both. Hence, x is in $(A - B) \cup (A - C)$.

Conversely, let y be an element of $(A - B) \cup (A - C)$. This means that y is in either $(A - B)$ or $(A - C)$, or both. If y is in $(A - B)$, then y is in A, but not in B. If y is in $(A - C)$, then y is in A, but not in C. In either case, y is not in $B \cap C$. Therefore, y is in $A - (B \cap C)$.

Since we have shown that every element in $A - (B \cap C)$ is also in $(A - B) \cup (A - C)$ and vice versa, we can conclude that $A - (B \cap C) = (A - B) \cup (A - C)$. □

Proposition. *If $n \in \mathbb{Z}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.*
Note: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = \sum_{i=1}^{n} \frac{i}{(i+1)!}$.

Proof. We will prove the proposition by mathematical induction.

Base case: Let $n = 1$. Then the left-hand side of the equation is $\frac{1}{2!} = \frac{1}{2}$, and the right-hand side is $1 - \frac{1}{(n+1)!} = 1 - \frac{1}{2!} = \frac{1}{2}$. Hence, the proposition is true for $n = 1$.

Inductive step: Assume that the proposition is true for some integer $k \geq 1$, i.e.,

$$\sum_{i=1}^{k} \frac{i}{(i+1)!} = 1 - \frac{1}{(k+1)!}.$$

We want to show that the proposition is also true for $n = k + 1$, i.e.,

$$\sum_{i=1}^{k+1} \frac{i}{(i+1)!} = 1 - \frac{1}{((k+1)+1)!}.$$

To prove this, we start with the left-hand side of the equation:

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{i}{(i+1)!} &= \sum_{i=1}^{k} \frac{i}{(i+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} && \text{(by the inductive hypothesis)} \\
&= 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+2)!}.
\end{aligned}$$

Therefore, we have shown that the proposition is true for $n = k + 1$.

By the principle of mathematical induction, the proposition is true for all integers $n \geq 1$. □

Proposition. *If $n \in \mathbb{Z}$, then $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$.*

Proof. We will prove the proposition by mathematical induction.

Base case: Let $n = 1$. Then the left-hand side of the inequality is $\frac{1}{1} + \frac{1}{2} = \frac{3}{2}$, and the right-hand side is $1 + \frac{1}{2} = \frac{3}{2}$. Hence, the proposition is true for $n = 1$.

Inductive step: Assume that the proposition is true for some integer $k \geq 1$, i.e.,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} \geq 1 + \frac{k}{2}.$$

We want to show that the proposition is also true for $n = k + 1$, i.e.,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{k+1}} \geq 1 + \frac{k+1}{2}.$$

To prove this, we start with the left-hand side of the inequality:

$$
\begin{aligned}
&\frac{1}{1} + \frac{1}{2} + \frac{1}{2^k} + \frac{1}{2^{k+1}} \\
&= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k}\right) + \frac{1}{2^k} - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k} - \frac{1}{2^k} + \frac{1}{2^{k+1}} && \text{(by the inductive hypothesis)} \\
&= 1 + \frac{k+1}{2^{k+1}} \\
&= 1 + \frac{k+1}{2} \cdot \frac{1}{2^k} \\
&\geq 1 + \frac{k+1}{2},
\end{aligned}
$$

where the last inequality follows from the fact that $2^k \geq 2$, since $k \geq 1$.

Therefore, we have shown that the proposition is true for $n = k + 1$.

By the principle of mathematical induction, the proposition is true for all integers $n \geq 1$. □