

# CSCI 301 HW 4

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## Problem 1

Define a relation  $R$  on  $\mathbb{Z}$  as  $xRy$  if and only if  $3 \mid (2x + y)$

- a. Prove that  $R$  is an equivalence relation.

To show that  $R$  is an equivalence relation, we must show that  $R$  is reflexive, symmetric, and transitive.

**Reflexive**  $R$  is reflexive if and only if  $xRx$  for all  $x \in \mathbb{Z}$ .

That is to say,  $R$  is reflexive if and only if  $3 \mid (2x + x)$  for all  $x \in \mathbb{Z}$ .

*Proof.* By contradiction.

Suppose  $3 \nmid (2x + x)$

We can rewrite  $2x + x$  as  $3x$ .

By definition of divisibility,  $3 \mid 3x$ .

Thus  $2x + x$  is both divisible by 3 and not divisible by 3. Contradiction!

Therefore  $3 \mid 2x + x$

□

**Symmetric**  $R$  is symmetric if and only if  $xRy \implies yRx$  for all  $x, y \in \mathbb{Z}$ .

That is to say,  $R$  is symmetric if and only if  $3 \mid (2x + y) \implies 3 \mid (2y + x)$  for all  $x, y \in \mathbb{Z}$ .

*Proof.* By definition of divisibility, we can write  $2x + y$  as

$$2x + y = 3a, a \in \mathbb{Z}$$

$$y = 3a - 2x$$

Thus

$$2y + x = 2(3a - 2x) + x$$

$$= 6a - 4x + x$$

$$= 6a - 3x$$

$$= 3(2a - x)$$

$$= 3b \text{ where } b = 2a - x$$

Therefore by definition of divisibility,  $3 \mid (2y + x)$

□

**Transitive**  $R$  is transitive if and only if  $xRy \wedge yRz \implies xRz$  for all  $x, y, z \in \mathbb{Z}$ .

That is to say,  $R$  is transitive if and only if  $3 \mid (2x + y) \wedge 3 \mid (2y + z) \implies 3 \mid (2x + z)$

*Proof.* By definition of divisibility, we can rewrite the first two statements as

$$2x + y = 3a, a \in \mathbb{Z}$$

$$2y + z = 3b, b \in \mathbb{Z}$$

$$y = 3a - 2x$$

$$z = 3b - 2y$$

which allows us to define

$$z = 3b - 2y$$

$$= 3b - 2(3a - 2x)$$

$$= 3b - 6a + 4x$$

Plugging this in to  $2x + z$  we get

$$\begin{aligned}2x + z &= 2x + 3b - 6a + 4x \\&= 6x + 3b - 6a \\&= 3(2x + b - 2a) \\&= 3c \text{ where } c = 2x + b - 2a\end{aligned}$$

Therefore by definition of divisibility,  $3 \mid (2x + z)$

□

- b. Describe the equivalence classes of  $R$ .

To determine the equivalence classes of  $R$ , we must observe that there are three possible values that result from  $3 \mid 2x + y$ : 0, 1, 2.

We can rewrite this equation as  $y \equiv -2x \pmod{3}$ .

Therefore, the set of all integers  $y$  that satisfies  $y \equiv -2x \pmod{3}$  is the equivalence class  $[x]$ :

$$[x] = \{y \in \mathbf{Z} : y \equiv -2x \pmod{3}\}$$

$$[0]_R = \{\dots, -6, -3, 0, 3, 6, \dots\}$$

$$[1]_R = \{\dots, -5, -2, 1, 4, 7, \dots\}$$

$$[2]_R = \{\dots, -4, -1, 2, 5, 8, \dots\}$$

## Problem 2

Prove the function  $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$  defined by the formula  $f(m, n) = (5m + 4n, 4m + 3n)$  is bijective. Find its inverse  $f^{-1}$ .

To prove that  $f$  is bijective, we must show that  $f$  is both injective and surjective.

**Injective**  $f$  is injective if and only if  $f(a) = f(b) \implies a = b$  for all  $a, b \in \mathbb{Z} \times \mathbb{Z}$ .

That is to say,  $f$  is injective if and only if  $f(m, n) = f(p, q) \implies (m, n) = (p, q)$  for all  $m, n, p, q \in \mathbb{Z}$ .

*Proof.* Suppose  $f(a, b) = f(c, d)$ .

Then

$$5a + 4b = 5c + 4d$$

$$4a + 3b = 4c + 3d$$

$$3b = 4c + 3d - 4a$$

$$b = \frac{4c + 3d - 4a}{3} = \frac{4}{3}(c - a) + d$$

$$5a + 4\left(\frac{4}{3}(c - a) + d\right) = 5c + 4d$$

$$5a + \frac{16}{3}(c - a) + 4d = 5c + 4d$$

$$5a + \frac{16c}{3} - \frac{16a}{3} = 5c$$

$$-\frac{a}{3} + \frac{16c}{3} = 5c$$

$$-\frac{a}{3} = -\frac{c}{3}$$

$$a = c$$

$$4a = 4c + 3d - 3b$$

$$a = c + \frac{3}{4}(d - b)$$

$$5\left(c + \frac{3}{4}(d - b)\right) + 4b = 5c + 4d$$

$$\frac{15}{4}(d - b) + 4b = 4d$$

$$\frac{15}{4}d - \frac{15}{4}b + 4b = 4d$$

$$\frac{b}{4} = \frac{d}{4}$$

$$b = d$$

Thus,  $a = c$  and  $b = d$ .

Therefore  $f(a, b) = f(c, d) \implies (a, b) = (c, d)$  □

**Surjective**  $f$  is surjective if and only if  $f(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$ .

That is to say,  $f$  is surjective if and only if  $\forall (m, n) \in \mathbb{Z} \times \mathbb{Z}, \exists (a, b) \in \mathbb{Z} \times \mathbb{Z}$  such that  $f(a, b) = (m, n)$ .

*Proof.* Suppose  $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ .

Then

$$(a, b) = f(m, n)$$

$$= (5m + 4n, 4m + 3n)$$

$$a = 5m + 4n$$

$$b = 4m + 3n$$

$$n = \frac{b - 4m}{3}$$

$$a = \frac{4b - m}{3}$$

$$m = 4b - 3a$$

$$5m = a - 4n$$

$$m = \frac{a - 4n}{5}$$

$$b = 4\left(\frac{a - 4n}{5}\right) + 3n$$

$$n = 4a - 5b$$

$$\begin{aligned} f(m, n) &= (5m + 4n, 4m + 3n) \\ &= 5(4b - 3a) + 4(4a - 5b), 4(4b - 3a) + 3(4a - 5b) \\ &= a, b \end{aligned}$$

Therefore  $\forall (m, n) \in \mathbb{Z} \times \mathbb{Z}, \exists (a, b) \in \mathbb{Z} \times \mathbb{Z}$  such that  $f(a, b) = (m, n)$ .

□