

CSCI 305 HW 4

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Problem 1

Show that

$$\sum_{i=1}^n = \frac{n(n+1)}{2} \text{ for } n \geq 1$$

Proof. By induction.

Base case: Observe at $n = 1$,

$$\sum_{i=1}^1 = \frac{1(1+1)}{2} = 1$$

holds true.

Inductive step: Suppose $\sum_{i=1}^n = \frac{n(n+1)}{2}$.

That is to say, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Then $1 + 2 + 3 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$.

$$\begin{aligned} \frac{n(n+1)}{2} + (n+1) &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

Therefore $\sum_{i=1}^n = \frac{n(n+1)}{2}$ for $n \geq 1$.

□

Problem 2

Show that

$$\sum_{i=0}^k ar^i = a \left(\frac{1 - r^{k+1}}{1 - r} \right)$$

Proof. By induction.

Base case: Observe at $k = 0$,

$$\sum_{i=0}^0 ar^i = a \left(\frac{1 - r^{0+1}}{1 - r} \right) = a \left(\frac{1 - r}{1 - r} \right) = a$$

Inductive step: Suppose $\sum_{i=0}^k ar^i = a \left(\frac{1 - r^{k+1}}{1 - r} \right)$.

That is to say, $ar^0 + ar^1 + ar^2 + \cdots + ar^k = a \left(\frac{1 - r^{k+1}}{1 - r} \right)$.

Then $ar^0 + ar^1 + ar^2 + \cdots + ar^k + ar^{k+1} = a \left(\frac{1 - r^{k+1}}{1 - r} \right) + ar^{k+1}$

$$\begin{aligned} a \left(\frac{1 - r^{k+1}}{1 - r} \right) + ar^{k+1} &= a \left(\frac{1 - r^{k+1}}{1 - r} \right) + ar^{k+1} \cdot \frac{1 - r}{1 - r} \\ &= a \left(\frac{1 - r^{k+1}}{1 - r} \right) + a \left(\frac{r^{k+1} - r^{k+2}}{1 - r} \right) \\ &= a \left(\frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} \right) \\ &= a \left(\frac{1 - r^{k+2}}{1 - r} \right) \end{aligned}$$

Therefore $\sum_{i=0}^k ar^i = a \left(\frac{1 - r^{k+1}}{1 - r} \right)$.

□

Problem 3

Show that the Fibonacci numbers defined recursively by

$$\begin{aligned}f(n) &= f(n-1) + f(n-2) \\ f(0) &= 0, f(1) = 1\end{aligned}$$

grow as $f(n) = O(b^n)$ for some $b > 1$.

Proof. By induction.

Note that $f(n) = O(b^n) \implies f(n) \leq b^n$.

First we will show that $f(n) \leq b^n$.

Base case: Observe at $n = 0$,

$$\begin{aligned}f(0) &= 0 \leq b^0 \\ 0 &\leq 1\end{aligned}$$

holds true.

Inductive step: Suppose $f(n) \leq b^n$.

□