CSCI 305 Assignment 4

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1. 1. How many 5-card outcomes are possible?
      Since one player pulls 4 cards for their hand, we have $\binom{52}{4}$ possible combinations, leaving $\binom{48}{1}$ possible combinations for the face-up card.
      $$\binom{52}{4} \times \binom{48}{1} = 12,994,800$$
   2. How many ways to make hand 1?
      - Turned-up card must be a 5
      - Player must have 5, 5, J J with one of the Jacks being the same suit as the turned-up card.
      $$\binom{4}{3} \times \binom{4}{1} \times \binom{3}{1} = 48$$
   3. How many ways to make hand 2?
      - Total hand must have 4, 5, 5, 5, 6
      $$\binom{4}{1} \times \binom{4}{3} \times \binom{4}{1} = 64$$
   4. What is the probability of scoring 23 points (either hand 1 or hand 2)?
      $$\frac{48+64}{12{,}994{,}800} = \frac{1}{116{,}025} \approx 0.0000086$$

2. Search Query
   1. Explain it cannot be the case that every n searches the algorithm takes $\Theta(n^3)$ steps.
      By definition of Θ, suppose $T(n)$ is the search algorithm; then
      $$\begin{aligned} T(n) = \Theta(n) \implies & \exists c_1, c_2, n_0 \in \mathbb{R}^+ \text{ such that} \\ & \forall n \geq n_0, c_1 n \leq T(n) \leq c_2 n \end{aligned}$$

      If $T(n)$ *did* take $\Theta(n^3)$ steps every x steps, then we could show this by accounting for this rate:
      $$\begin{aligned} T'(n) &= T(n) + \frac{\Theta(n^3)}{x} \\ c_1 n \leq & T(n) + \frac{\Theta(n^3)}{x} \leq c_2 n \end{aligned}$$
      Setting c_1 to 0 satisfies the lower bound, however, there is no c_2 that can be set to satisfy the upper bound. Therefore, it cannot be the case that every n searches the algorithm takes $\Theta(n^3)$ steps.
      $$\begin{aligned} T(n) + \frac{\Theta(n^3)}{x} &\leq T(n) + \frac{O(n^3)}{n} \\ &\leq T(n) + \frac{cn^3}{n} \\ T(n) + \frac{cn^3}{n} &\leq c_2 n \\ T(n) + cn^2 &\leq c_2 n \end{aligned}$$

      We can discard the leading $T(n)$ since this is $\leq$, making it smaller.
      $$cn^2 \leq c_2 n$$

2. If the algorithm occasionally took $\Theta(2^n)$, how rarely must this occur?

$$\begin{aligned} T(n) = \Theta(2^n) \implies & \exists c_1, c_2, n_0 \in \mathbb{R}^+ \text{ such that} \\ & \forall n \geq n_0, c_1 2^n \leq T(n) \leq c_2 2^n \end{aligned}$$

If $T(n)$ *did* take $\Theta(2^n)$ steps every x steps, then we could show this by accounting for this rate:

$$\begin{aligned} T'(n) = T(n) + \frac{\Theta(2^n)}{x} \\ c_1 2^n \leq T(n) + \frac{\Theta(2^n)}{x} \leq c_2 2^n \end{aligned}$$

Setting c_1 to 0 satisfies the lower bound, leaving us with the upper bound.

$$\begin{aligned} T(n) + \frac{\Theta(2^n)}{x} &\leq T(n) + \frac{O(2^n)}{x} \\ &\leq T(n) + \frac{c2^n}{x} \end{aligned}$$

Here we must set x to 2^n to avoid adding an additional leading term that would break the $\Theta(n)$ bound. That is to say the occurence of running $\Theta(2^n)$ must happen every 2^n steps.

3. Along the coastline

   1. What is the expected number of boats that return to their home (original) village?
   Since each the probability of boat X returning to its home village is not impacted by other boats, these trials are independent. Mathematically it makes sense for each boat to have a $\frac{1}{100}$ chance of returning to its home village. Since this is being done 100 times, we simply multiply the probability by the number of trials.

$$\frac{1}{100} \times 100 = 1 \text{ boat}$$

   2. What is the expected number of villages where 2 boats show up?
   To calculate the number of villages where only 2 boats show up, let's observe that the first boat as a $\frac{1}{100}$ chance of showing up to one village, and the second boat has a $\frac{1}{100}$ chance of showing up as well. Since this effectively halves the number of ships, this trial is run 50 times.

$$\frac{1}{100} \times \frac{1}{100} \times 50 = \frac{1}{200} \text{ villages}$$

   3. With n boats and villages, show that in $\lim_{n\to\infty}$ the expected fraction of villages with empty docks approaches $1/e$.
   Observe that at $n = 1$, the expected # of villages with empty docks is 0.
   At $n = 2$, the expected # of villages with empty docks is $\frac{1}{1}$.
   At $n = 3$, the expected # of villages with empty docks is

$$\frac{3}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}$$

   At $n = 10$, the expected # of villages with empty docks is

$$\frac{10}{10} \times \left(\frac{9}{10}\right)^{10} = \frac{9^{10}}{10^{10}}$$

   Thus, the expected fraction of villages with empty docks is

$$\lim_{n\to\infty} \frac{(n-1)^n}{n^n} = \frac{1}{e}$$

4. Nodes

1. With $d = 3$ and $n = 10$, what is the number of self-edges in the graph?
   For any given stub, there are $d - 1$ stubs on the same node that could form a self-edge, and $nd - 1$ total stubs it could connect to. Thus, the expected number of self-edges is

$$30 \times \frac{3-1}{30-1} = 30 \times \frac{2}{29} \approx 2.06896$$

2. For arbitrary d and n, the expected number of self-edges is

$$nd \times \frac{d-1}{nd-1}$$