CSCI 305 Assignment 2

(Solo) Isaac Boaz

October 26, 2023

1. 1. Prove that finite sums and Θ commute.

$$\sum_{i=1}^{n} \Theta(f(i)) = \Theta(\sum_{i=1}^{n} f(i))$$

Proof. To show that finite sums and Θ commute, we must show that each has the same inequality constraints.

Left-Hand Side.

$$\begin{aligned} &\sum_{i=1}^{n} \Theta(f(i)) \\ \rightarrow &\sum_{i=1}^{n} c_1 f(i) \leq \sum_{i=1}^{n} \Theta(f(i)) \leq \sum_{i=1}^{n} c_2 f(i) \\ \rightarrow &\, c_1 \sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} \Theta(f(i)) \leq c_2 \sum_{i=1}^{n} f(i) \end{aligned}$$

Right-Hand Side.

$$\begin{aligned} &\Theta(\sum_{i=1}^{n} f(i)) \\ \rightarrow &\, c_1 \sum_{i=1}^{n} f(i) \leq \Theta(\sum_{i=1}^{n} f(i)) \leq c_2 \sum_{i=1}^{n} f(i) \end{aligned}$$

Since both equations have the same lower and upper bounds, Θ is commutive. $\square$

2. Prove that $\log_a n = \Theta(\lg n)$ for any $a > 1$, where lg is $\log_2$.

$$\begin{aligned} \log_a n = \Theta(\lg n) &\implies \\ \exists c_1, c_2, n_0 \in \mathbb{R}^+ &\text{ such that} \\ c_1 \lg n \leq \log_a n \leq c_2 \lg n &\text{ for all } n \geq n_0 \end{aligned}$$

Note that for any a or n, we can set $c_1 = 0$, Leaving us with the right-hand side of the equation.

$$\begin{aligned} \log_a n &\leq c_2 \lg n \\ \frac{\log_2 n}{log_2 a} &\leq c_2 \log_2 n \\ c_2 &\geq \frac{1}{\log_2 a} \end{aligned}$$

Showing us that this holds true for any $a > 1$.

3. Prove that for k integer, $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1}) \implies$$
$$c_1 n^{k+1} \leq \sum_{i=1}^{n} i^k \leq c_2 n^{k+1}$$

Similarly to the previous question, we can set $c_1 = 0$ to resolve the lower bound. As for the upper bound, notice

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}$$
$$\sum_{i=1}^{n} i^k \leq n^{k+1}$$

hence $\sum_{i=1}^{n} i^k = O(n^{k+1})$.

2. 1. Let $p > 0$. Show that $\log n = o(n^p)$.
   To show that $\log n = o(n^p)$, we must show that $\lim_{n\to\infty} \frac{\log n}{n^p} = 0$.

$$\begin{aligned}\lim_{n\to\infty} \frac{\log n}{n^p} &= \lim_{n\to\infty} \frac{\frac{1}{n}}{pn^{p-1}} \\ &= \lim_{n\to\infty} \frac{1}{pn^p} \\ &= 0\end{aligned}$$

   2. Let $q > 0, p > 0$ Show that $2^{qn} = \omega(n^p)$.
   To show that $2^{qn} = \omega(n^p)$, we must show that $\lim_{n\to\infty} \frac{2^{qn}}{n^p} = \infty$.

$$\begin{aligned}\lim_{n\to\infty} \frac{2^{qn}}{n^p} &= \lim_{n\to\infty} \frac{qn2^{qn}}{pn^{p-1}} \\ &= \lim_{n\to\infty} \frac{qn2^{qn}}{pn^p} \\ &= \lim_{n\to\infty} \frac{qn}{pn^p} \\ &= \lim_{n\to\infty} \frac{q}{pn^{p-1}} \\ &= \infty\end{aligned}$$

   3. In Figure 1 we see the plots of $\ln(n)$ and $n^{0.1}$. We can see that $\ln(n)$ grows slower than $n^{0.1}$ for all $n > 1$. Note, however, that both functions have the same limit as $n \to \infty$, namely ∞.

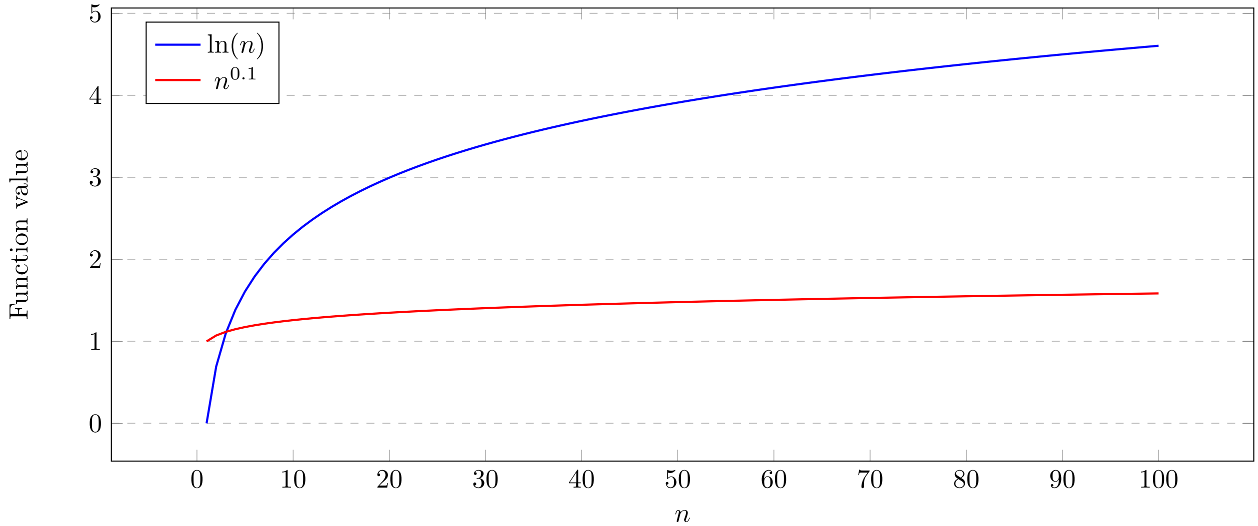

Figure 1: Plot of $\ln(n)$ and $n^{0.1}$

3.

$$2^{\log x} \leq x \log x \leq x \log_{10} x \leq 2^x \leq e^{x \log x} \leq x!$$

4. Let $F(1) = F(2) = 1$ be the 'initial conditions' for the Fibonnaci recurrence given for $n > 2$ by

$$F(n) = F(n-1) + F(n-2)$$

   1. *Proof.* By induction.
   Base Case: $n = 1, n = 2$. Observe at $n = 1$, $F(1) = 1$, and at $n = 2$, $F(2) = 1$
   Lastly, $F(3) = F(2) + F(1) = 1 + 1 = 2 = \frac{a^3 - b^3}{\sqrt{5}}$.
   Inductive Step: Assume $F(n) = \frac{a^n - b^n}{\sqrt{5}}$ where $a = \frac{1+\sqrt{5}}{2}, b = \frac{1-\sqrt{5}}{2}$.
   Then (keeping in mind $a^2 = a + 1$)

$$\begin{aligned} F(n+1) &= F(n) + F(n-1) \\ &= \frac{a^n - b^n}{\sqrt{5}} + \frac{a^{n-1} - b^{n-1}}{\sqrt{5}} \\ &= \frac{a^n - b^n + a^{n-1} - b^{n-1}}{\sqrt{5}} \\ &= \frac{a^{n-1}(a+1) - b^{n-1}(b+1)}{\sqrt{5}} \\ &= \frac{a^{n-1}a^2 - b^{n-1}b^2}{\sqrt{5}} \\ &= \frac{a^{n+1} - b^{n+1}}{\sqrt{5}} \end{aligned}$$

$\square$

   2. Using the formula from the previous question, we can show that $F(n) = \Theta(a^n)$.

$$\begin{aligned} F(n) = \Theta(a^n) \implies \\ \exists c_1, c_2, n_0 \in \mathbb{R}^+ \text{ such that} \\ c_1 x^n \leq F(n) \leq c_2 x^n \text{ for all } n \geq n_0 \\ \to c_1 x^n \leq \frac{a^n - b^n}{\sqrt{5}} \leq c_2 x^n \end{aligned}$$

   Note that for any n or x, we can set $c_1 = 0$ to resolve the lower bound, leaving us with the upper bound. Since we know b is bounded by $-1 \leq b \leq 0$, $-1 \leq b^2 \leq 0$. Thus

$$\frac{a^n - b^n}{\sqrt{5}} \leq \frac{a^n + 1}{\sqrt{5}}$$

since the $+\ 1$ becomes proportionally smaller as $n \to \infty$, we can ignore it.

$$\frac{a^n + 1}{\sqrt{5}} \leq \frac{2a^n}{\sqrt{5}}$$

$$\text{Leaving us with } c_2 = \frac{2}{\sqrt{5}} \text{ for any } n_0.$$

5.

`for i = 1 to n`	n + 1
`    for j = 1 to i`	$\sum_{i=1}^{n+1} i$
`        k = 1`	$\sum_{i=1}^{n} \sum_{j=1}^{i} j$
`            while k <= j`	$\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j+2} k$
`                k = k +1`	$\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j+1} k$

The inner-most loop runs $j + 1$ times, the middle loop runs i times, and the outer loop runs n times. Thus the total runtime is $\Theta(n^3)$.

Ex. Credit Prove

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = C,\ 0 \leq C \leq \infty \implies f(n) = \Theta(g(n))$$

First let's rewrite this using the definition of Θ.

$$\cdots \implies \exists c_1, c_2, n_0 \text{ such that}$$
$$c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0$$

There are three cases to consider for C.
Case 1: $C = 0$.
This implies that $g(n)$ grows much faster than $f(n)$, meaning that $f(n)$ is upper-bounded by $g(n)$ for all $n \geq n_0$.
By this logic $f(n) = O(g(n))$.

Case 2: $0 < C < \infty$.
This implies that $f(n)$ and $g(n)$ grow at the same rate, meaning that $f(n)$ is upper-bounded by $g(n)$ and lower-bounded by $g(n)$ for all $n \geq n_0$.

$$C - \epsilon \leq \frac{f(n)}{g(n)} \leq C + \epsilon$$
$$\to (C - \epsilon)g(n) \leq f(n) \leq (C + \epsilon)g(n)$$

Thus $f(n) = \Theta(g(n))$, where $c_1 = C - \epsilon, c_2 = C + \epsilon$

Case 3: $C = \infty$.
This implies that $g(n)$ grows much slower than $f(n)$, meaning that $f(n)$ is lower-bounded by $g(n)$ for all $n \geq n_0$.
By this logic $f(n) = \Omega(g(n))$.