CSCI 305 HW 4

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Problem 1

Show that

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2} \text{ for } n \geq 1$$

Proof. By induction.
Base case: Observe at $n = 1$,

$$\sum_{i=1}^{1} = \frac{1(1+1)}{2} = 1$$

holds true.
Inductive step: Suppose $\sum_{i=1}^{n} = \frac{n(n+1)}{2}$.
That is to say, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
Then $1 + 2 + 3 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$.

$$\begin{aligned} \frac{n(n+1)}{2} + (n+1) &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

Therefore $\sum_{i=1}^{n} = \frac{n(n+1)}{2}$ for $n \geq 1$. □

Problem 2

Show that

$$\sum_{i=0}^{k} ar^i = a\left(\frac{1-r^{k+1}}{1-r}\right)$$

Proof. By induction.
Base case: Observe at $k = 0$,

$$\sum_{i=0}^{0} ar^i = a\left(\frac{1-r^{0+1}}{1-r}\right) = a\left(\frac{1-r}{1-r}\right) = a$$

Inductive step: Suppose $\sum_{i=0}^{k} ar^i = a\left(\frac{1-r^{k+1}}{1-r}\right)$.
That is to say, $ar^0 + ar^1 + ar^2 + \cdots + ar^k = a\left(\frac{1-r^{k+1}}{1-r}\right)$.
Then $ar^0 + ar^1 + ar^2 + \cdots + ar^k + ar^{k+1} = a\left(\frac{1-r^{k+1}}{1-r}\right) + ar^{k+1}$

$$\begin{aligned}
a\left(\frac{1-r^{k+1}}{1-r}\right) + ar^{k+1} &= a\left(\frac{1-r^{k+1}}{1-r}\right) + ar^{k+1} \cdot \frac{1-r}{1-r} \\
&= a\left(\frac{1-r^{k+1}}{1-r}\right) + a\left(\frac{r^{k+1}-r^{k+2}}{1-r}\right) \\
&= a\left(\frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r}\right) \\
&= a\left(\frac{1-r^{k+2}}{1-r}\right)
\end{aligned}$$

Therefore $\sum_{i=0}^{k} ar^i = a\left(\frac{1-r^{k+1}}{1-r}\right)$. $\square$

Problem 3

Show that the Fibonacci numbers defined recursively by

$$\begin{aligned}
f(n) &= f(n-1) + f(n-2) \\
f(0) &= 0, f(1) = 1
\end{aligned}$$

grow as $f(n) = O(b^n)$ for some $b > 1$.

Proof. By induction.
Note that $f(n) = O(b^n) \implies f(n) \leq b^n$.
First we will show that $f(n) \leq b^n$.
Base case: Observe at $n = 0$,

$$\begin{aligned}
f(0) = 0 &\leq b^0 \\
0 &\leq 1
\end{aligned}$$

holds true.
Inductive step: Suppose $f(n) \leq b^n$.
Then $f(n+1) = f(n) + f(n-1) \leq b^n + b^{n-1}$.

$$\begin{aligned}
b^n + b^{n-1} &= b^{n-1}(b+1) \\
&\leq b^{n-1}(b+b) \\
&= b^{n-1}(2b) \\
&= 2b^n
\end{aligned}$$

Therefore $f(n+1) \leq 2b^n$.

$\square$

Problem 4

Show that the recursion

$$T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + O(n)$$

has a solution which is

$$T(n) = O(n \log n)$$

Note that $T(n) = O(n \log n) \implies T(n) \leq n \log n$

Proof. By substitution.

$\square$