CSCI 305 Assignment 2

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1. 1. Prove that finite sums and Θ commute.

$$\sum_{i=1}^{n} \Theta(f(i)) = \Theta(\sum_{i=1}^{n} f(i))$$

Proof. To show that finite sums and Θ commute, we must show that each has the same inequality constraints.

Left-Hand Side.

$$\sum_{i=1}^{n} \Theta(f(i))$$
$$\rightarrow \sum_{i=1}^{n} c_1 f(i) \leq \sum_{i=1}^{n} \Theta(f(i)) \leq \sum_{i=1}^{n} c_2 f(i)$$
$$\rightarrow c_1 \sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} \Theta(f(i)) \leq c_2 \sum_{i=1}^{n} f(i)$$

Right-Hand Side.

$$\Theta(\sum_{i=1}^{n} f(i))$$
$$\rightarrow c_1 \sum_{i=1}^{n} f(i) \leq \Theta(\sum_{i=1}^{n} f(i)) \leq c_2 \sum_{i=1}^{n} f(i)$$

Since both equations have the same lower and upper bounds, Θ is commutive. $\square$

2. Prove that $\log_a n = \Theta(\lg n)$ for any $a > 1$, where lg is $\log_2$.

$$\log_a n = \Theta(\lg n) \implies$$
$$\exists c_1, c_2, n_0 \in \mathbb{R}^+ \text{ such that}$$
$$c_1 \lg n \leq \log_a n \leq c_2 \lg n \text{ for all } n \geq n_0$$

Note that for any a or n, we can set $c_1 = 0$, Leaving us with the right-hand side of the equation.

$$\log_a n \leq c_2 \lg n$$
$$\frac{\log_2 n}{log_2 a} \leq c_2 \log_2 n$$
$$c_2 \geq \frac{1}{\log_2 a}$$

Showing us that this holds true for any $a > 1$.

3. Prove that for k integer, $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1}) \implies$$

$$c_1 n^{k+1} \leq \sum_{i=1}^{n} i^k \leq c_2 n^{k+1}$$

Similarly to the previous question, we can set $c_1 = 0$ to resolve the lower bound. As for the upper bound, notice

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}$$
$$\sum_{i=1}^{n} i^k \leq n^{k+1}$$

hence $\sum_{i=1}^{n} i^k = O(n^{k+1})$.

2. 1. Let $p > 0$. Show that $\log n = o(n^p)$.
To show that $\log n = o(n^p)$, we must show that $\lim_{n\to\infty} \frac{\log n}{n^p} = 0$.

$$\begin{aligned}\lim_{n\to\infty} \frac{\log n}{n^p} &= \lim_{n\to\infty} \frac{\frac{1}{n}}{pn^{p-1}} \\ &= \lim_{n\to\infty} \frac{1}{pn^p} \\ &= 0\end{aligned}$$