CSCI 305 HW 6

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$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2i\pi}{N}kn}$$

Problem 1

Compute the DFT of ...

A) $[0, 1, 0, -1]$

k=0

$$\begin{aligned}\sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}0n} &= \sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}0n} \\ &= 0 + e^{\frac{-2i\pi}{4}0\cdot 1} + 0 - e^{\frac{-2i\pi}{4}0\cdot 3} \\ &= 0 + 1 + 0 - 1 \\ &= 0\end{aligned}$$

k=1

$$\begin{aligned}\sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}1n} &= \sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}1n} \\ &= 0 + e^{\frac{-2i\pi}{4}1\cdot 1} + 0 - e^{\frac{-2i\pi}{4}1\cdot 3} \\ &= 0 - i + 0 - i \\ &= 0 - 2i\end{aligned}$$

k=2

$$\begin{aligned}\sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}2n} &= \sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}2n} \\ &= 0 + e^{\frac{-2i\pi}{4}2\cdot 1} + 0 - e^{\frac{-2i\pi}{4}2\cdot 3} \\ &= 0 - 1 + 0 + 1 \\ &= 0\end{aligned}$$

k=3

$$\begin{aligned}\sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}3n} &= \sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}3n} \\ &= 0 + e^{\frac{-2i\pi}{4}3\cdot 1} + 0 - e^{\frac{-2i\pi}{4}3\cdot 3} \\ &= 0 + i + 0 + i \\ &= 0 + 2i\end{aligned}$$

$$DFT([0, 1, 0, -1]) = [0, -2i, 0, 2i]$$

Analysis: Since the input does not have a low-frequency presence of data, it makes sense that DFT_0 is 0. As for the complex parts of the DFT, I assume it has something to do with the polarity of the signal / frequency.

B) $[1, 1, 1, 1]$ (In-Class Exercise)

$$DFT([1, 1, 1, 1]) = [4, 0, 0, 0]$$

Analysis: We see that the data has a low frequency of data (all 1-s), so it makes sense that the highest level (lowest frequency) of the DFT is 4.

C) $[0, -1, 0, 1]$

k=0

$$\begin{aligned}\sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}0n} &= \sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}0n} \\ &= 0 - e^{\frac{-2i\pi}{4}0\cdot 1} + 0 + e^{\frac{-2i\pi}{4}0\cdot 3} \\ &= 0 - 1 + 0 + 1 \\ &= 0\end{aligned}$$

k=1

$$\begin{aligned}\sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}1n} &= \sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}1n} \\ &= 0 - e^{\frac{-2i\pi}{4}1\cdot 1} + 0 + e^{\frac{-2i\pi}{4}1\cdot 3} \\ &= 0 + i + 0 + i \\ &= 0 + 2i\end{aligned}$$

k=2

$$\begin{aligned}\sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}2n} &= \sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}2n} \\ &= 0 - e^{\frac{-2i\pi}{4}2\cdot 1} + 0 + e^{\frac{-2i\pi}{4}2\cdot 3} \\ &= 0 + 1 + 0 - 1 \\ &= 0\end{aligned}$$

k=3

$$\begin{aligned}\sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}3n} &= \sum_{n=0}^{3} x_n e^{-\frac{2i\pi}{4}3n} \\ &= 0 - e^{\frac{-2i\pi}{4}3\cdot 1} + 0 + e^{\frac{-2i\pi}{4}3\cdot 3} \\ &= 0 - i + 0 - i \\ &= 0 - 2i\end{aligned}$$

$$DFT([0, -1, 0, 1]) = [0, 2i, 0, -2i]$$

Analysis: Since the input array is actually the same as (1), it makes sense that the output array would be the same with reversed signs.

D) $[0, 1, 0, -1, 0, 1, 0, -1]$

k=0

$$\begin{aligned}\sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{4}0n} &= \sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{8}0n} \\ &= 0 + e^{\frac{-2i\pi}{8}0\cdot 1} + 0 - e^{\frac{-2i\pi}{8}0\cdot 3} + 0 + e^{\frac{-2i\pi}{8}0\cdot 5} + 0 - e^{\frac{-2i\pi}{8}0\cdot 7} \\ &= 0 + 1 + 0 - 1 + 0 + 1 + 0 - 1 \\ &= 0\end{aligned}$$

k=1

$$\begin{aligned}\sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{4}1n} &= \sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{8}1n} \\ &= 0 + e^{\frac{-2i\pi}{8}1\cdot 1} + 0 - e^{\frac{-2i\pi}{8}1\cdot 3} + 0 + e^{\frac{-2i\pi}{8}1\cdot 5} + 0 - e^{\frac{-2i\pi}{8}1\cdot 7} \\ &= 0 + e^{-\frac{i\pi}{4}} + 0 - e^{-\frac{3i\pi}{4}} + 0 + e^{\frac{3i\pi}{4}} + 0 - e^{\frac{i\pi}{4}} \\ &= 0\end{aligned}$$

k=2

$$\begin{aligned}\sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{4}2n} &= \sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{8}2n} \\ &= 0 + e^{\frac{-2i\pi}{8}2\cdot 1} + 0 - e^{\frac{-2i\pi}{8}2\cdot 3} + 0 + e^{\frac{-2i\pi}{8}2\cdot 5} + 0 - e^{\frac{-2i\pi}{8}2\cdot 7} \\ &= 0 - i + 0 - i + 0 - i + 0 - i \\ &= 0 - 4i\end{aligned}$$

k=3

$$\begin{aligned}\sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{4}3n} &= \sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{8}3n} \\ &= 0 + e^{\frac{-2i\pi}{8}3\cdot 1} + 0 - e^{\frac{-2i\pi}{8}3\cdot 3} + 0 + e^{\frac{-2i\pi}{8}3\cdot 5} + 0 - e^{\frac{-2i\pi}{8}3\cdot 7} \\ &= 0 + e^{-\frac{3i\pi}{4}} + 0 - e^{-\frac{i\pi}{4}} + 0 + e^{\frac{i\pi}{4}} + 0 - e^{\frac{3i\pi}{4}} \\ &= 0\end{aligned}$$

k=4

$$\begin{aligned}\sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{4}4n} &= \sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{8}4n} \\ &= 0 + e^{\frac{-2i\pi}{8}4\cdot 1} + 0 - e^{\frac{-2i\pi}{8}4\cdot 3} + 0 + e^{\frac{-2i\pi}{8}4\cdot 5} + 0 - e^{\frac{-2i\pi}{8}4\cdot 7} \\ &= 0 + 1 + 0 - 1 + 0 + 1 + 0 - 1 \\ &= 0\end{aligned}$$

k=5

$$\begin{aligned}\sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{4}5n} &= \sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{8}5n} \\ &= 0 + e^{\frac{-2i\pi}{8}5\cdot 1} + 0 - e^{\frac{-2i\pi}{8}5\cdot 3} + 0 + e^{\frac{-2i\pi}{8}5\cdot 5} + 0 - e^{\frac{-2i\pi}{8}5\cdot 7} \\ &= 0 + e^{-\frac{i\pi}{4}} + 0 - e^{-\frac{3i\pi}{4}} + 0 + e^{\frac{3i\pi}{4}} + 0 - e^{\frac{i\pi}{4}} \\ &= 0\end{aligned}$$

k=6

$$\begin{aligned}
\sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{4}6n} &= \sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{8}6n} \\
&= 0 + e^{\frac{-2i\pi}{8}6\cdot 1} + 0 - e^{\frac{-2i\pi}{8}6\cdot 3} + 0 + e^{\frac{-2i\pi}{8}6\cdot 5} + 0 - e^{\frac{-2i\pi}{8}6\cdot 7} \\
&= 0 + i + 0 + i + 0 + i + 0 + i \\
&= 0 + 4i
\end{aligned}$$

k=7

$$\begin{aligned}
\sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{4}7n} &= \sum_{n=0}^{7} x_n e^{-\frac{2i\pi}{8}7n} \\
&= 0 + e^{\frac{-2i\pi}{8}7\cdot 1} + 0 - e^{\frac{-2i\pi}{8}7\cdot 3} + 0 + e^{\frac{-2i\pi}{8}7\cdot 5} + 0 - e^{\frac{-2i\pi}{8}7\cdot 7} \\
&= 0 + e^{-\frac{3i\pi}{4}} + 0 - e^{-\frac{i\pi}{4}} + 0 + e^{\frac{i\pi}{4}} + 0 - e^{\frac{3i\pi}{4}} \\
&= 0
\end{aligned}$$

$$DFT([0, 1, 0, -1, 0, 1, 0, -1]) = [0, 0, -4i, 0, 0, 0, 4i, 0]$$

Analysis: Using our imagination, we can see that the data follows a frequency of up, down, up, down. Thus, it makes sense for the output array to have minimal values as it can encapsulate a theoretical sin / cos pattern.

Problem 2

Suppose the splits of every level of Quicksort are in $(1 - \alpha, \alpha)$ as a fraction of n, where $0 < \alpha \leq 1/2$ is a constant.

Minimum Depth

Given an arbitrary α, the minimum level is

$$\begin{aligned}
\log_{1/a} n &= \frac{\log_2 n}{\log_2 1/a} \\
&= \frac{\log_2 n}{\log_2 a^{-1}} \\
&= \frac{log_2 n}{-1 \log a} \\
&= -\frac{\log n}{\log a}
\end{aligned}$$

Maximum Depth

Similarly, since α is lower bounded by $1 - \alpha$, the maximum level is

$$\begin{aligned}
\log_{\frac{1}{a-1}} n &= \frac{\lg n}{\lg \frac{1}{1-a}} \\
&= \frac{\lg n}{\lg(1-a)^{-1}} \\
&= \frac{\lg n}{-\lg(1-a)} \\
&= -\frac{\lg n}{\lg(1-a)}
\end{aligned}$$