

# CSCI 305 Assignment 2

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October 26, 2023

1. Prove that finite sums and  $\Theta$  commute.

$$\sum_{i=1}^n \Theta(f(i)) = \Theta\left(\sum_{i=1}^n f(i)\right)$$

*Proof.* To show that finite sums and  $\Theta$  commute, we must show that each has the same inequality constraints.

**Left-Hand Side.**

$$\begin{aligned} & \sum_{i=1}^n \Theta(f(i)) \\ \rightarrow & \sum_{i=1}^n c_1 f(i) \leq \sum_{i=1}^n \Theta(f(i)) \leq \sum_{i=1}^n c_2 f(i) \\ \rightarrow & c_1 \sum_{i=1}^n f(i) \leq \sum_{i=1}^n \Theta(f(i)) \leq c_2 \sum_{i=1}^n f(i) \end{aligned}$$

**Right-Hand Side.**

$$\begin{aligned} & \Theta\left(\sum_{i=1}^n f(i)\right) \\ \rightarrow & c_1 \sum_{i=1}^n f(i) \leq \Theta\left(\sum_{i=1}^n f(i)\right) \leq c_2 \sum_{i=1}^n f(i) \end{aligned}$$

Since both equations have the same lower and upper bounds,  $\Theta$  is commutative.  $\square$

2. Prove that  $\log_a n = \Theta(\lg n)$  for any  $a > 1$ , where  $\lg$  is  $\log_2$ .

$$\begin{aligned} \log_a n = \Theta(\lg n) & \implies \\ \exists c_1, c_2, n_0 \in \mathbb{R}^+ & \text{ such that} \\ c_1 \lg n \leq \log_a n & \leq c_2 \lg n \text{ for all } n \geq n_0 \end{aligned}$$

Note that for any  $a$  or  $n$ , we can set  $c_1 = 0$ , Leaving us with the right-hand side of the equation.

$$\begin{aligned} \log_a n & \leq c_2 \lg n \\ \frac{\log_2 n}{\log_2 a} & \leq c_2 \log_2 n \\ c_2 & \geq \frac{1}{\log_2 a} \end{aligned}$$

Showing us that this holds true for any  $a > 1$ .

3. Prove that for  $k$  integer,  $\sum_{i=1}^n i^k = \Theta(n^{k+1})$

$$\sum_{i=1}^n i^k = \Theta(n^{k+1}) \implies$$

$$c_1 n^{k+1} \leq \sum_{i=1}^n i^k \leq c_2 n^{k+1}$$

Similarly to the previous question, we can set  $c_1 = 0$  to resolve the lower bound. As for the upper bound, notice

$$\sum_{i=1}^n i^k \leq \sum_{i=1}^n n^k = n \cdot n^k = n^{k+1}$$

$$\sum_{i=1}^n i^k \leq n^{k+1}$$

hence  $\sum_{i=1}^n i^k = O(n^{k+1})$ .

2. 1. Let  $p > 0$ . Show that  $\log n = o(n^p)$ .

To show that  $\log n = o(n^p)$ , we must show that  $\lim_{n \rightarrow \infty} \frac{\log n}{n^p} = 0$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log n}{n^p} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{pn^{p-1}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{pn^p} \\ &= 0 \end{aligned}$$

2. Let  $q > 0, p > 0$  Show that  $2^{qn} = \omega(n^p)$ .

To show that  $2^{qn} = \omega(n^p)$ , we must show that  $\lim_{n \rightarrow \infty} \frac{2^{qn}}{n^p} = \infty$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2^{qn}}{n^p} &= \lim_{n \rightarrow \infty} \frac{qn2^{qn}}{pn^{p-1}} \\ &= \lim_{n \rightarrow \infty} \frac{qn2^{qn}}{pn^p} \\ &= \lim_{n \rightarrow \infty} \frac{qn}{pn^p} \\ &= \lim_{n \rightarrow \infty} \frac{q}{pn^{p-1}} \\ &= \infty \end{aligned}$$

3. In Figure 1 we see the plots of  $\ln(n)$  and  $n^{0.1}$ . We can see that  $\ln(n)$  grows slower than  $n^{0.1}$  for all  $n > 1$ . Note, however, that both functions have the same limit as  $n \rightarrow \infty$ , namely  $\infty$ .

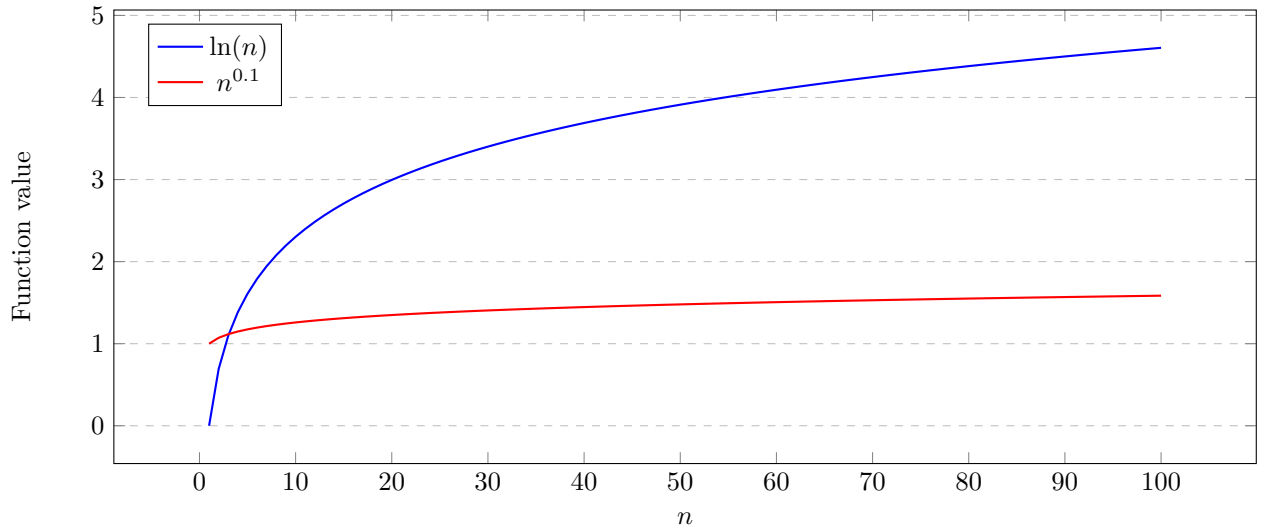


Figure 1: Plot of  $\ln(n)$  and  $n^{0.1}$

3.

$$2^{\log x} \leq x \log x \leq x \log_{10} x \leq 2^x \leq e^{x \log x} \leq x!$$

4. Let  $F(1) = F(2) = 1$  be the ‘initial conditions’ for the Fibonnaci recurrence given for  $n > 2$  by

$$F(n) = F(n-1) + F(n-2)$$

1. *Proof.* By induction.

**Base Case:**  $n = 1, n = 2$ . Observe at  $n = 1$ ,  $F(1) = 1$ , and at  $n = 2$ ,  $F(2) = 1$

Lastly,  $F(3) = F(2) + F(1) = 1 + 1 = 2 = \frac{a^3 - b^3}{\sqrt{5}}$ .

**Inductive Step:** Assume  $F(n) = \frac{a^n - b^n}{\sqrt{5}}$  where  $a = \frac{1+\sqrt{5}}{2}, b = \frac{1-\sqrt{5}}{2}$ .

Then (keeping in mind  $a^2 = a + 1$ )

$$\begin{aligned} F(n+1) &= F(n) + F(n-1) \\ &= \frac{a^n - b^n}{\sqrt{5}} + \frac{a^{n-1} - b^{n-1}}{\sqrt{5}} \\ &= \frac{a^n - b^n + a^{n-1} - b^{n-1}}{\sqrt{5}} \\ &= \frac{a^{n-1}(a+1) - b^{n-1}(b+1)}{\sqrt{5}} \\ &= \frac{a^{n-1}a^2 - b^{n-1}b^2}{\sqrt{5}} \\ &= \frac{a^{n+1} - b^{n+1}}{\sqrt{5}} \end{aligned}$$

□

2. Using the formula from the previous question, we can show that  $F(n) = \Theta(a^n)$ .

$$\begin{aligned} F(n) &= \Theta(a^n) \implies \\ \exists c_1, c_2, n_0 &\in \mathbb{R}^+ \text{ such that} \\ c_1 x^n &\leq F(n) \leq c_2 x^n \text{ for all } n \geq n_0 \\ \rightarrow c_1 x^n &\leq \frac{a^n - b^n}{\sqrt{5}} \leq c_2 x^n \end{aligned}$$

Note that for any  $n$  or  $x$ , we can set  $c_1 = 0$  to resolve the lower bound, leaving us with

the upper bound. Since we know  $b$  is bounded by  $-1 \leq b \leq 0$ ,  $-1 \leq b^2 \leq 0$ . Thus

$$\frac{a^n - b^n}{\sqrt{5}} \leq \frac{a^n + 1}{\sqrt{5}}$$

since the  $+1$  becomes proportionally smaller as  $n \rightarrow \infty$ , we can ignore it.

$$\frac{a^n + 1}{\sqrt{5}} \leq \frac{2a^n}{\sqrt{5}}$$

Leaving us with  $c_2 = \frac{2}{\sqrt{5}}$  for any  $n_0$ .

5.

1	<b>for</b>	$i = 1$	<b>to</b>	$n$	$\sum_{i=1}^{n+1} i$
2	<b>for</b>	$j = 1$	<b>to</b>	$i$	$\sum_{i=1}^n \sum_{j=1}^i j$
3		$k = 1$			$\sum_{i=1}^n \sum_{j=1}^i \sum_{k=1}^{j+2} k$
4		<b>while</b>	$k \leq$	$j$	$\sum_{i=1}^n \sum_{j=1}^i \sum_{k=1}^{j+1} k$
5			$k = k + 1$		

The inner-most loop runs  $j + 1$  times, the middle loop runs  $i$  times, and the outer loop runs  $n$  times. Thus the total runtime is  $\Theta(n^3)$ .

Ex. Credit Prove

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = C, \quad 0 < C < \infty \implies f(n) = \Theta(g(n))$$

First let's rewrite this using the definition of  $\Theta$ .

$$\begin{aligned} \dots &\implies \exists c_1, c_2, n_0 \text{ such that} \\ c_1 g(n) &\leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0 \end{aligned}$$

This implies that  $f(n)$  and  $g(n)$  grow at the same rate, meaning that  $f(n)$  is upper-bounded by  $g(n)$  and lower-bounded by  $g(n)$  for all  $n \geq n_0$ .

$$\begin{aligned} C - \epsilon &\leq \frac{f(n)}{g(n)} \leq C + \epsilon \\ \rightarrow (C - \epsilon)g(n) &\leq f(n) \leq (C + \epsilon)g(n) \end{aligned}$$

Thus  $f(n) = \Theta(g(n))$ , where  $c_1 = C - \epsilon$ ,  $c_2 = C + \epsilon$