Standard vs. Nested Form

$$\text{Standard Form: } a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
$$\text{Nested Form: } a_0 + x(a_1 + a_2 x + \cdots + a_n x^{n-1})$$

Sig Fig Rounding

4-Digit Example

37.21429	37.21
0.002271828	0.002272
3000527.11059	3001000

Bit Preciscion

	sign	exponent	mantissa	machine ϵ
single	1	8	23	2^{-23}
double (default)	1	11	52	2^{-52}
long double	1	15	64	2^{-64}

sign	exponent	mantissa

$\rightarrow 1.b_1 b_2 \cdots b_n \times 2^P$

Error Calculation

$$x_e = \text{Exact},\ x_a = \text{Approximate}$$
$$\text{Abs err} = |x_e - x_a|$$
$$\text{Rel err} = \frac{|x_e - x_a|}{x_e}$$
$$\text{Worst case error} = 0.5 \cdot 10^{-P+1}$$

Loss of Significance

$$a - b = \frac{a^2 - b^2}{a + b}$$

Theorems

Intermediate Value Theorem

If $f(x)$ is continuous on $[a, b]$ and $f(a) \neq f(b)$,
then for any y in $[f(a),\ f(b)]$ $\exists\ c \in [a, b]$ such that $f(c) = y$

Mean Value Theorem

If $f'(x)$ is continuous on $[a, b]$,
then $\exists\ c \in [a, b]$ such that $f'(c) = (f(b) - f(a))/(b - a)$

Rolle's Theorem

If $f'(x)$ is continuous on $[a, b]$ and $f(a) = f(b)$,
then $\exists\ c \in [a, b]$ such that $f'(c) = 0$

Taylor Series

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$E_{k+1} = \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}$$

Methods

Bisection Method

1. Choose a, b such that $f(a) \cdot f(b) < 0$
2. Compute $c = \frac{a+b}{2}$
3. If $f(c)$ and $f(a)$ have opposite signs, then $b = c$
4. If $f(c)$ and $f(b)$ have opposite signs, then $a = c$
5. If $|b - a| < \epsilon$ or $f(c) = 0$ then c is the solution

$$\text{Error} = \frac{b - a}{2^{n+1}}$$
$$\text{Steps}(\epsilon) = \left\lceil \frac{\ln(b - a) - \ln(\epsilon)}{\ln(2)} - 1 \right\rceil$$

Newton's Method

1. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
2. If $|x_n - x_{n-1}| < \epsilon$ or $f(x_n) = 0$ then x_n is the solution

Multiplicity

$$\text{If } f^{(m-1)}(r_0) = 0 \text{ and } f^{(m)}(r_0) \neq 0,$$
$$\text{then } f^{(m)}(r_0) \text{ is a root of multiplicity } m$$

Convergence

$m = 1$

$$M = \frac{1}{2}\left|\frac{f''(r_0)}{f'(r_0)}\right|$$
$$e_{i+1} \approx M e_i^2$$

$m > 1$

$$e_{i+1} \approx \frac{m - 1}{m} e_i$$

Secant Method

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Convergence

$$e_{i+1} \approx M e_i^p$$
$$p = \frac{1 + \sqrt{5}}{2} \text{ (Golden Ratio)}$$

False Position Method

Same as Bisection Method, but

$$c = a - \frac{f(a)(b - a)}{f(b) - f(a)}$$

Convergence - Generally same as Bisection

Fixed Point Iteration

1. Solve $f(x)$ for $x = g(x)$

$$g(x) = 2.8x - x^2 = x$$
$$1.8x - x^2 = 0$$
$$x(1.8 - x) = 0$$
$$x = 0, 1.8$$

$$f(x) = 2x^3 - 6x - 1$$
$$g(x) = \frac{2x^3 - 1}{6}$$

2. If $|g'(r_n)| < 1$, fixed point will converge to r_n

Matrix

Gaussian Elimination $O(\frac{2n^3}{3})$

Back Substitution $O(n^2)$

LU Fact. Gaussian + 2 * Back Sub

Summations

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n+5} (k - 1) = \sum_{i=1}^{n+4} j = \frac{(n+4)(n+5)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Complexity Ratio

$$\frac{s_b^2}{2 \cdot s_e^3/3} = \frac{t_b}{t_e}$$

LU Factorization

Goal: $A = LU \to A\vec{x} = \vec{b}$

Slow Method

1. Row reduce A, recording operations as elementary matrices
2. The row reduced A is now U
3. You now have $E_n E_{n-1} \cdots E_2 E_1 A = LU$
4. Invert the elementary matricies to find L
5. $L = (E_n E_{n-1} \cdots E_2 E_1)^{-1}$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\underset{e_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}} \underset{e_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}} \underset{e_1}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Example

$$\begin{aligned} E_3 E_2 E_1 A &= U \\ A &= E_1^{-1} E_2^{-1} E_3^{-1} U \\ &= (E_3 E_2 E_1)^{-1} U \end{aligned}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Fast Method

1. Apply pivots internally within A
2. Note that you will subtract rather than add with these operations

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} 1 & 2 & -1 \\ \frac{2}{1} & -3 & 0 \\ -\frac{3}{1} & -7 & -2 \end{bmatrix} \quad (2)$$

3. Don't forget to recurse through the inner matrix (row 2 and beyond)

$$\begin{bmatrix} 1 & 2 & -1 \\ \frac{2}{1} & -3 & 0 \\ -\frac{3}{1} & \frac{7}{3} & -2 \end{bmatrix} \quad (3)$$

4. The circled (lower triangular) elements are now L (plus the identity matrix)
5. The remaining upper triangular elements are U

Derivatives

$$\frac{df}{dx} f(g(x)) = f'(g(x))g'(x) \quad \frac{d}{dx} ax^n = nax^{n-1} \quad \frac{d}{dx} ln_a(x) = \frac{1}{xln(a)}$$

$$\frac{d}{dx} a^x = \ln(a)a^x \quad \frac{d}{dx} sin(x) = cos(x) \quad \frac{d}{dx} cos(x) = sin(x)$$

PA = LU Factorization

Goal: $PA = LU \to A\vec{x} = \vec{b}$

1. Same logic as LU, but move the |biggest| element to the top of the column
2. Make sure you do the same for inner matrices
3. You'll have multiple P matrices, such that
$$PA = LU$$
4. Make sure you apply P when solving for $\vec{x}$

Solving

1. Solve $L\vec{y} = \vec{b}$ for $\vec{y}$
2. Solve $U\vec{x} = \vec{y}$ for $\vec{x}$

Example 1: $A = LU$

$$\text{Let } \vec{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

1. $L\vec{y} = \vec{b}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 1 \\ -3 & -\frac{7}{3} & 1 & -1 \end{array}\right] \to \vec{y} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$$

2. $U\vec{x} = \vec{y}$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & -2 & -2 \end{array}\right] \to \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example 2: $PA = LU$

$$\text{Let } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

1. Find P such that A's pivots are the largest in magnitude per column.
In this instance:
$$P = P_2 P_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2. $L\vec{y} = P\vec{b}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 6 \\ \frac{1}{2} & -\frac{1}{2} & 1 & 5 \end{array}\right] \to \vec{y} = \begin{pmatrix} 0 \\ 6 \\ 8 \end{pmatrix}$$

3. $U\vec{x} = \vec{y}$

$$\left[\begin{array}{ccc|c} 4 & 4 & -4 & 0 \\ 0 & 2 & 2 & 6 \\ 0 & 0 & 8 & 8 \end{array}\right] \to \vec{x} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$