M/CS 375 HW 2

Isaac Boaz

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Example 4

Consider $f(x) = \frac{1}{x+1}$

(a) Find the degree 3 Taylor polynomial for f at $x_0 = 0$, $P_3(x)$.

$$\begin{aligned} f(x) &= \frac{1}{x+1} = (x+1)^{-1} \\ f^{(1)}(x) &= -(x+1)^{-2} \\ f^{(2)}(x) &= 2(x+1)^{-3} \\ f^{(3)}(x) &= -6(x+1)^{-4} \\ f^{(4)}(x) &= 24(x+1)^{-5} \end{aligned}$$

$$\begin{aligned} P_3(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 \\ &= f(0) + f'(0)(x) + \frac{f^{(2)}(0)}{2}(x)^2 + \frac{f^{(3)}(0)}{6}(x)^3 \\ &= 1 + (-1)(x) + \frac{2}{2}(x)^2 \; - \frac{6}{6}(x)^3 \\ &= 1 - x + x^2 - x^3 \end{aligned}$$

(b) Find the Taylor remainder, E_4.

$$\begin{aligned} E_{k+1} &= \frac{f^{(k+1)}(C)}{(k+1)!}(x - x_0)^{k+1} \\ E_4 &= \frac{f^{(4)}(C)}{4!}(x - x_0)^4 \\ &= \frac{24(C+1)^{-5}}{24}(x - 0)^4 \\ &= (C+1)^{-5}(x)^4 \\ &= \frac{x^4}{(C+1)^5}, C \in [0, x] \end{aligned}$$

(c) Estimate the worse case error when using $P_3(x)$ to approximate $f(x)$ for $x = 0.02$.

$$\begin{aligned} E_4(0.02) &= \frac{0.02^4}{(C+1)^5} = \frac{1.6 \cdot 10^{-7}}{(C+1)^5} \\ &< \frac{1.6 \cdot 10^{-7}}{(0+1)^5} = \frac{1.6 \cdot 10^{-7}}{1^5} = 1.6 \cdot 10^{-7} \end{aligned}$$

(d) Estimate the worst case error when using $P_3(x)$ to approximate $f(x)$ for $|x| \leq 10^{-4}$

$$E_4(|x| \leq 10^{-4}) \leq E_4(\pm 10^{-4})$$
$$C \in [-10^{-4}, 10^{-4}]$$

$$\begin{aligned} E_4(10^{-4}) &= \frac{(10^{-4})^4}{(C+1)^5} \\ = \frac{10^{-16}}{(C+1)^5} &= \frac{10^{-16}}{(-10^{-4}+1)^5} \\ = \frac{10^{-16}}{0.9998^5} &\approx 10^{-16} \end{aligned}$$