M/CS 375 HW 2

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Example 4

Consider $f(x) = \frac{1}{x+1}$

(a) Find the degree 3 Taylor polynomial for f at $x_0 = 0$, $P_3(x)$.

$$\begin{aligned}
f(x) &= \frac{1}{x+1} = (x+1)^{-1} \\
f^{(1)}(x) &= -(x+1)^{-2} \\
f^{(2)}(x) &= 2(x+1)^{-3} \\
f^{(3)}(x) &= -6(x+1)^{-4} \\
f^{(4)}(x) &= 24(x+1)^{-5}
\end{aligned}$$

$$\begin{aligned}
P_3(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 \\
&= f(0) + f'(0)(x) + \frac{f^{(2)}(0)}{2}(x)^2 + \frac{f^{(3)}(0)}{6}(x)^3 \\
&= 1 + (-1)(x) + \frac{2}{2}(x)^2 \ - \frac{6}{6}(x)^3 \\
&= 1 - x + x^2 - x^3
\end{aligned}$$

(b) Find the Taylor remainder, E_4.

$$\begin{aligned}
E_{k+1} &= \frac{f^{(k+1)}(C)}{(k+1)!}(x-x_0)^{k+1} \\
E_4 &= \frac{f^{(4)}(C)}{4!}(x-x_0)^4 \\
&= \frac{24(C+1)^{-5}}{24}(x-0)^4 \\
&= (C+1)^{-5}(x)^4 \\
&= \frac{x^4}{(C+1)^5}, C \in [0, x]
\end{aligned}$$

(c) Estimate the worse case error when using $P_3(x)$ to approximate $f(x)$ for $x = 0.02$.

$$\begin{aligned}
E_4(0.02) &= \frac{0.02^4}{(C+1)^5} = \frac{1.6 \cdot 10^{-7}}{(C+1)^5} \\
&< \frac{1.6 \cdot 10^{-7}}{(0.02+1)^5} = \frac{1.6 \cdot 10^{-7}}{1.02^5} \approx 1.6 \cdot 10^{-7}
\end{aligned}$$

(d) Estimate the worst case error when using $P_3(x)$ to approximate $f(x)$ for $|x| \leq 10^{-4}$

$$E_4(|x| \leq 10^{-4}) \leq E_4(\pm 10^{-4})$$

$$C \in [-10^{-4}, 10^{-4}]$$

$$\begin{aligned}
E_4(10^{-4}) &= \frac{(10^{-4})^4}{(C+1)^5} \\
&= \frac{10^{-16}}{(C+1)^5} = \frac{10^{-16}}{(-10^{-4}+1)^5} \\
&= \frac{10^{-16}}{0.9998^5} \approx 10^{-16}
\end{aligned}$$