

# M/CS 375 HW 2

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## Example 4

Consider  $f(x) = \frac{1}{x+1}$

(a) Find the degree 3 Taylor polynomial for  $f$  at  $x_0 = 0$ ,  $P_3(x)$ .

$$\begin{aligned}f(x) &= \frac{1}{x+1} = (x+1)^{-1} \\f^{(1)}(x) &= -(x+1)^{-2} \\f^{(2)}(x) &= 2(x+1)^{-3} \\f^{(3)}(x) &= -6(x+1)^{-4} \\f^{(4)}(x) &= 24(x+1)^{-5}\end{aligned}$$

$$\begin{aligned}P_3(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 \\&= f(0) + f'(0)(x) + \frac{f^{(2)}(0)}{2}(x)^2 + \frac{f^{(3)}(0)}{6}(x)^3 \\&= 1 + (-1)(x) + \frac{2}{2}(x)^2 - \frac{6}{6}(x)^3 \\&= 1 - x + x^2 - x^3\end{aligned}$$

(b) Find the Taylor remainder,  $E_4$ .

$$\begin{aligned}E_{k+1} &= \frac{f^{(k+1)}(C)}{(k+1)!}(x - x_0)^{k+1} \\E_4 &= \frac{f^{(4)}(C)}{4!}(x - x_0)^4 \\&= \frac{24(C+1)^{-5}}{24}(x - 0)^4 \\&= (C+1)^{-5}(x)^4 \\&= \frac{x^4}{(C+1)^5}, C \in [0, x]\end{aligned}$$

(c) Estimate the worst case error when using  $P_3(x)$  to approximate  $f(x)$  for  $x = 0.02$ .

$$\begin{aligned}E_4(0.02) &= \frac{0.02^4}{(C+1)^5} = \frac{1.6 \cdot 10^{-7}}{(C+1)^5} \\&< \frac{1.6 \cdot 10^{-7}}{(0.02+1)^5} = \frac{1.6 \cdot 10^{-7}}{1.02^5} \approx 1.6 \cdot 10^{-7}\end{aligned}$$

(d) Estimate the worst case error when using  $P_3(x)$  to approximate  $f(x)$  for  $|x| \leq 10^{-4}$

$$\begin{aligned}E_4(|x| \leq 10^{-4}) &\leq E_4(10^{-4}) \\E_4(10^{-4}) &= \frac{(10^{-4})^4}{(C+1)^5} \\&= \frac{10^{-16}}{(C+1)^5} = \frac{10^{-16}}{(10^{-4}+1)^5} \\&= \frac{10^{-16}}{1.0001^5} \approx 10^{-16}\end{aligned}$$