Final Project: Peano's Axioms and Prolog

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The Problem

In 1889, Peano demonstrated that all natural numbers could be derived from eight axioms which Russell further expanded this approach in his "Principia Mathematica," which established premises for a broader range of mathematical concepts (Hosch, 2023). This underlying simplicity, forming the basis of mathematical complexity recently captivated me, showing that the edifice of mathematics rests on the sturdy base of logic.

Establishing a knowledge base is very potent in mathematics, characterized by its logical, closed, and consistent nature, unlike most real-life domains where this approach has usually failed. Applying knowledge bases and first-order logic through AI techniques has been instrumental in proving and disproving mathematical theorems such as the Vampire algorithm (Kovács & Voronkov, 2013), automating tasks that might have otherwise required years and the collaboration of many mathematicians.

Driven by a fascination with the interplay between logic and mathematics, I am developing a tool using Prolog's first-order logic. Starting with Peano's axioms, I plan to define basic mathematical operations to explore the extent of their applications, working up to fractions, exponentials, etc. While such a tool might have limited practical utility, it offers a unique perspective on mathematics, dissecting it to its fundamental axioms and highlighting the potency of first-order logic.

Solution

While the Peano axioms in first-order logic are incomplete, their soundness and consistency offer a robust foundation. Using first-order logic and tools like Prolog is suitable and practical for modeling and reasoning about natural numbers, given our goals and requirements.

Soundness: The axioms are structured to model the intuitive progression of numbers in nature, starting from zero and advancing sequentially. This natural representation ensures that any valid logical inference made within this system, following its rules, leads to conclusions that are true in the natural number model, thereby confirming the system's soundness.

Consistency: Each axiom introduces a unique aspect of natural numbers, like the existence of a first number (zero) and the principle of succession, without allowing for any contradictory interpretations. This clarity and separation of concepts ensure that no axiom can be used to prove both a statement and its negation, thus maintaining the consistency of the entire system.

Completeness: Peano's axioms in first-order logic are not complete, meaning that they cannot express and prove every true statement about natural numbers. By Gödel's first incompleteness theorem, there are true properties or relations between natural numbers that cannot be derived within this logical framework.

The approach started from the basic definition of Peano's axiom's 0 and its successor. This simple rule and axiom was used to construct an addition and its inverse, subtraction. With each step, we move one level of abstraction up, as multiplication and division were defined as repeated additions or subtractions. The same can be said of exponents, factorials, etc., as long as we define the correct base cases for these definitions. The division allows us to determine the greatest common divisor and check for prime numbers and simplest fractions. This abstraction can be continued as long as we stay within rational numbers. This demonstrates how each natural number computation can be defined as relative to 0.

Relevant tests were then added, and an interface was built to convert natural language queries to prolog, solve them, and return the result.

Analysis

Extension to rational numbers

Peano's axioms inherently deal with natural numbers, i.e., non-negative integers, which limits the scope of this calculator. Despite my initial attempts, since one of the axioms is that 0 cannot be the successor to any number, it was difficult to directly extend this system to negative numbers, which means that subtraction that led to negative results would not produce any results.

However, by building on the initial axioms, I was able to combine two positive integers to create fractions, which would allow us to theoretically deal with all rational numbers, although there would be a practical limit on the degree of accuracy to these as at some point we would reach the maximum allowed recursive depth in Prolog.

Scaling of First Order Logic Operations

Graphs are provided in Appendix.

Addition is inherently a linear operation. A recursive definition where add(s(X), Y, s(Z)) translates to add(X, Y, Z) ensures that each incremental step is a single recursive call. In Prolog, this recursion is a direct query transformation that consistently takes a constant time to compute each step, leading to an overall linear scaling, O(n), concerning the size of the incrementing argument. This mirrors the first-order logic principle where a proposition about n naturally extends to n+1.

Multiplication exhibits quadratic scaling, $O(n^2)$, when both variables increase synchronously (n * n scenario). The recursive definition involves a multiplication operation being reduced to iterative addition operations. As one variable increases, the number of additions required scales linearly. However, when both variables are allowed to grow, the number of operations represents the area of a square with sides of length n, hence the quadratic nature. In first-order logic, this is akin to a nested implication where knowing n * m leads to understanding n * (m+1), which requires a complete additional set of n additions.

Greater Than comparison is implemented as a linear check, decrementing both numbers until the base case is reached (typically when the second number reaches zero). The scaling here is linear, O(n), similar to addition, because it involves a simple progression through the natural numbers.

In conclusion, we explored the role of logic in mathematics and the potential of Prolog's first-order logic as a tool for understanding fundamental mathematical concepts. Pursuing this logical approach promises a unique perspective on the bedrock of mathematics and its enduring connection to logic.

Appendix

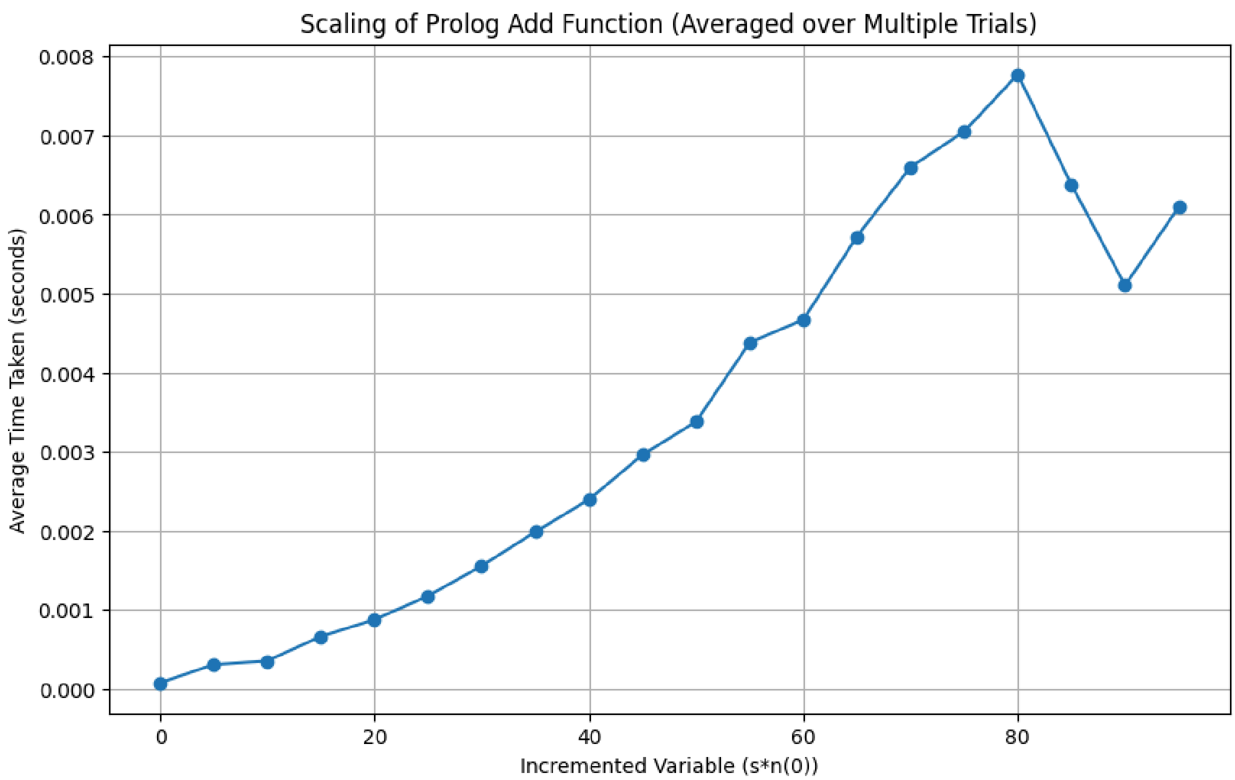

Fig. 1. O(n) scaling of time complexity of the add function as one of the added number increases.

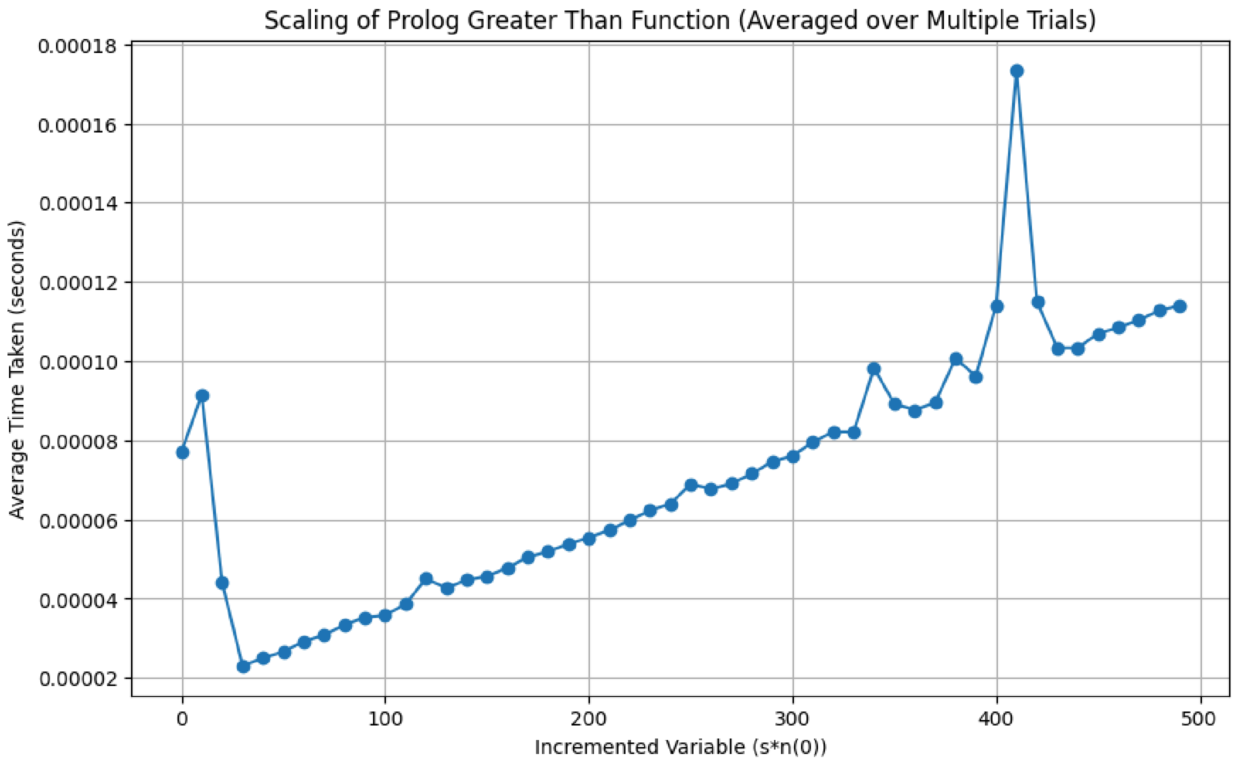

Fig. 2. O(n) scaling of time complexity of the greater than function as one of the mutiplied number increases.

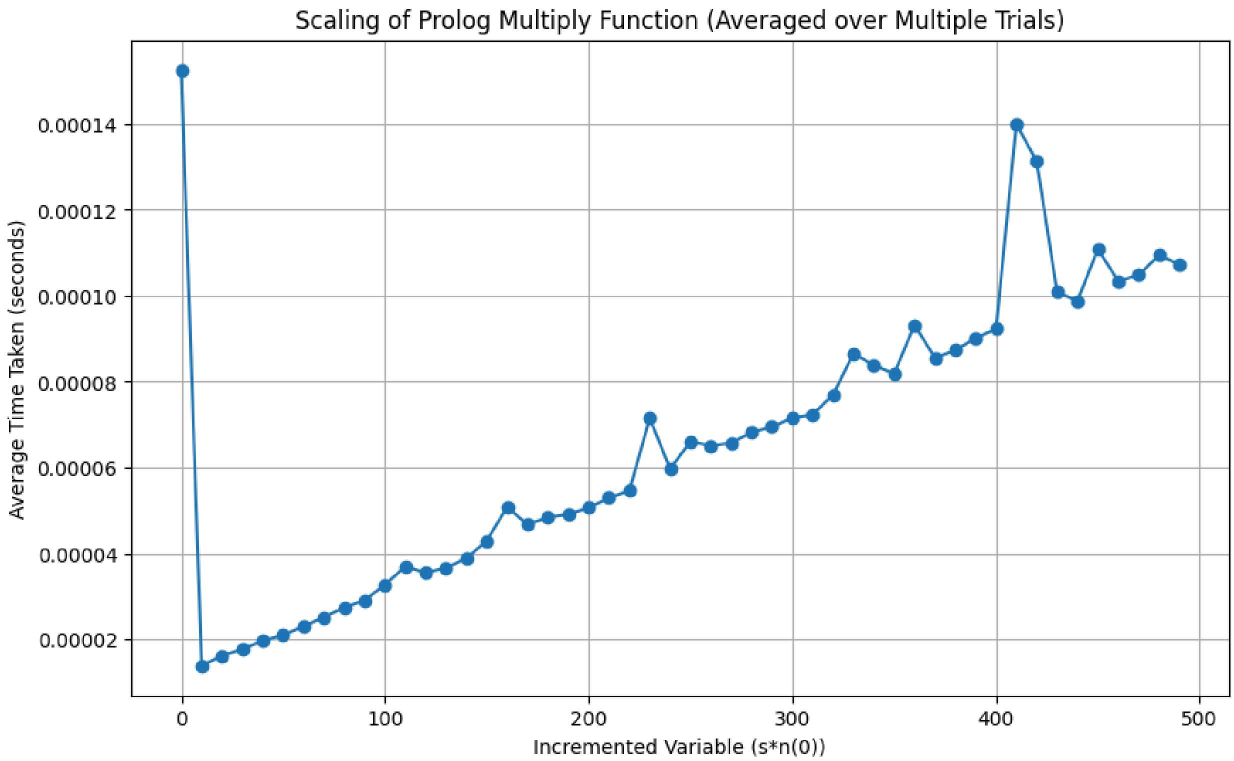

Fig. 3. O(n) scaling of time complexity of the multiply function as one of the mutiplied number increases.

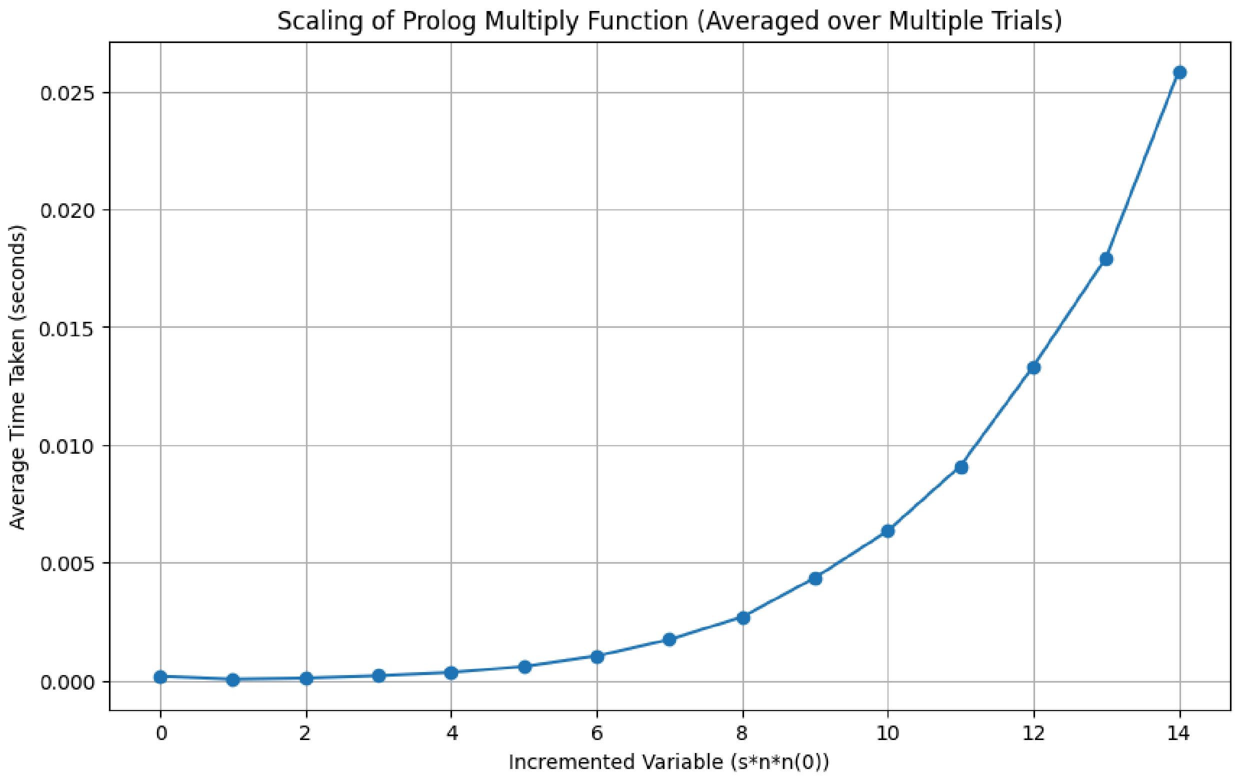

Fig. 4. O(n^2) scaling of time complexity of the multiply function as both of the mutiplied number increases.

References

Hosch, W. L. (2023, December 4). *Peano axioms | Logic, Set Theory, Number Theory*. Britannica. Retrieved December 15, 2023, from https://britannica.com/science/Peano-axioms

Kovács, L., & Voronkov, A. (2013). First-order theorem proving and Vampire. *Computer Aided Verification*, 1-35. 10.1007/978-3-642-39799-8_1