Incentives for the Over-Provision of Public Goods *

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Abstract

A wide range of public goods, for instance open source software and regulatory policy involve both consumption costs and private benefits to contributing. This paper models such an environment. I show that when there are consumption costs but no private benefits to provision, such as learning costs, increasing the number of contributors mitigates the free-rider problem, rather than exacerbating it. When there are both consumption costs and private benefits to contributing, increasing the number of contributors not only mitigates the free-rider problem, but leads to an over-provision problem in which both the number of contributors and the intensity of contributions are inefficiently high. When the population is large, every equilibrium yields over-provision. Lastly, I find that welfare-maximizing policies involve transferring surpluses from consumers to producers, decreasing the utility from consumption and increasing the utility from contributing.

Keywords: congestion, free riding, negative externalities, over-provision, positive externalities, public goods, under-provision.

JEL classifications: C72, D29, D62, D71, H41.

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1 Introduction

The founding developer of Linux, Linus Torvalds, once said, “given enough eyeballs, all bugs are shallow (Raymond, 1999, p. 29).” Dubbed ‘Linus’s law’, this statement claims that increasing the number of contributors to an open source software project increases welfare by increasing the quality of the software. Conversely, there is an English proverb stating, “too many cooks spoil the broth.” The claim is that too many contributors can have a deleterious effect on welfare. This paper shows that these two statements, though seemingly at odds with one-another, can be reconciled within a single framework.

Traditional public goods are characterized by (*i*) nonexcludable and nonrival consumption benefits and (*ii*) contribution costs. These two features lead to free-riding and under-provision. The technology industry is known for the collaborative production of public goods such as open source software (Johnson, 2002; Polanski, 2007; Athey and Ellison, 2014; Sacks, 2015).[1] In addition to consumption benefits and contribution costs, collaboratively-produced public goods often feature two additional traits: (*iii*) consumption costs borne by all consumers and (*iv*) excludable and potentially rivalrous contribution benefits.

When adopting new software, there are often learning (or switching) costs that grow as the software becomes more complex (Sacks, 2015). Moreover, individuals benefit both from consuming and contributing. Lerner and Tirole (2002), Feller et al. (2005), Bitzer et al. (2007), Myatt and Wallace (2008), Fang and Neufeld (2009), and Athey and Ellison (2014) argue that the primary reasons individuals contribute to the development of open source software are not for consumption benefits, but for contribution benefits including altruism, signaling skill, and the joy of coding. Contribution benefits are not unique to open source software. More generally, Andreoni (1990) discusses altruism and Glazer and Konrad (1996) discuss signaling as incentives to contribute to public goods.

In this paper, I develop a static model of the collaborative production of a public good with a finite homogenous population and the above four features. I show that the standard incentive problems are no longer present when collaboratively produced public goods exhibit these four

[1]Similarly the development of regulation and policy, which often entail the provision of public goods, are developed collaboratively by political actors.

features, but new interrelated problems are introduced. Instead of under-provision, over-provision may occur. Over-provision increase consumption costs and lowers the utility from consumption, henceforth referred to as *quality*. Surpluses are transferred from consumers to contributors. Efforts to coordinate contributions to maximize total welfare by limiting either the number of contributors or the magnitude of the contribution benefits amplifies these effects, raising total welfare by further transferring surpluses from consumers to contributors. Implementing quality-control by mandating a lower-bound to the quality limits the degree to which these surpluses are transferred.

Under features $(i) - (iii)$, the free-rider problem persists, though its severity is decreasing in the number of contributors. As the number of contributors increases, the decrease in each individual's contribution is outweighed by the increase in total contributions. Provision becomes socially optimal for large populations and quality is maximized. This result is in stark contrast to existing models, including the well studied congestion phenomenon, which often features consumption costs.[2]

Under all four features, over-provision can occur.[3] When the contribution benefits are large, there is over-provision regardless of the population size. With a large population, over-provision occurs when consumption costs and contribution benefits are present, no matter the size of the benefits. Aggregate contributions in equilibrium exceeds the level that maximizes welfare, which exceeds the level that maximizes quality. Contribution benefits induce a negative externality through the consumption costs. The magnitude of this negative externality is increasing in the number of contributors eventually dominating the positive externality from consumption benefits and contribution costs. Contributors do not internalize the impact of their own contributions on the change in quality experienced by others. This phenomenon is a supply-side corollary to congestion. Supply-side congestion occurs due to the tension between individual and aggregate incentives. Contributors place value not only the quality, but on their own participation in its provision (quantity). Hence there is a quantity-quality tradeoff.

[2]For examples, see Olson (1965), Brown, Jr. (1974), Arnott and Small (1994), and Cornes and Sandler (1996, ch. 8), among others.

[3]Over-provision can also occur under (i), (ii), and (iv), though the implications are substantially different. I present this case in an online appendix.

Stamelos et al. (2002) find such a relationship in open source software after analyzing quality characteristics of one hundred open source projects written for Linux. The authors show that the relationship between the size of a component of an open source application (aggregate contributions) is negatively related to user satisfaction for the application (quality) as measured by a four-point defect severity scale developed by IBM.[4] The model developed here offers a mechanism explaining these results.

Efforts to maximize welfare, e.g., by a project coordinator or social planner yield potentially unintended results. Optimally fixing the number of contributors typically does not maximize quality even though maximizing quality involves maximizing both the net consumption benefits for consumers and the net contribution benefits for contributors. Instead, there is an incentive to increase the number of contributors, which decreases both the consumption and contribution benefits received by each individual but increases the number of individuals receiving the contribution benefits. Surpluses are transferred from consumers to contributors.

Adjusting contributions by manipulating the contribution benefits can increase welfare by worsening the supply-side congestion externality. Contribution benefits can be manipulated in many ways, e.g. by making the signal more visible when there is a signaling benefit (Holmström, 1999). Welfare is maximized by increasing the contribution benefits without bound, sacrificing quality for the benefit of contributors. Again, all surpluses are transferred to the contributors. This outcome can be avoided only if the social planner can commit to mandating a minimal quality. Such a commitment may not be feasible.[5]

Public goods featuring $(i) - (iv)$ are not limited to open source software. Public policies and

[4]In cases where expert consumers reported only superficial errors at worst, the mean length of the program is strictly lower than in cases where (a) consumers reported either all major program functions are working but at least one minor function is disabled or incorrect, (b) at least one major function is disabled or incorrect, and (c) the program is inoperable. Their result is robust to other metrics, including the number of statements per module.

[5]Proprietary software developers often support and coordinate the development of open source software (Mustonen, 2005; Economides and Katsamakas, 2006; Kumar et al., 2011; Llanes and de Elejalde, 2013). A proprietary software developer can inject capital into an open source project thereby increasing the contribution benefits, which can lead to decreases in the quality of its open source competition. The decrease in open source software quality increases consumers' willingness to pay for the proprietary software. Thus the supporters have little incentive to mandate a minimum quality.

regulations are nonexcludable and nonrival: all must comply. Regulators have an incentive to contribute.[6] Therefore, the production of regulations / public policies possesses features $(i)-(iv)$. In response to the Dodd-Frank Wall Street Reform and Consumer Protection Act of 2010, companies have experienced sharp increases in compliance costs (Hogan, 2019), which can be interpreted as a consumption cost. During the COVID-19 pandemic, businesses and universities have heavily invested in personal protective equipment and barriers such as plexiglass in response to social distancing and lockdown policies. Additionally, politicians develop political capital when contributing (Quandt, 1983).

This paper contributes to several strands of literature, including the theory of public goods, industrial organization, and the economics of innovation. The literature on the private provision of public goods began with Bernheim (1986) and Bergstrom et al. (1986). Summaries of the core theories are given by Cornes and Sandler (1996), Sandler and Tschirhart (1997), and Scotchmer (2002). Though some work has been conducted in the context of regulatory complexity, this paper is among the first to formally analyze the effects of consumption costs and contribution benefits on the provision of public goods.[7]

Quandt (1983) and Kearl (1983) show that there exist conditions such that there is an over-investment in regulation. This paper generalizes these results and broadens their applicability. Similarly, Nitzan et al. (2013) argue that the increasing volume and velocity of complexity in regulation over time leads to inefficiencies induced by implementing barriers to entry so only the largest established firms survive. The authors reference accounting regulations, corporate governance and securities regulations, and the market for corporate charters. Combining the results of Nitzan et al. (2013), Quandt (1983), Kearl (1983), and the citations therein with this paper leads to a more complete understanding of the effects of complexity in regulation via the quantity-quality tradeoff.[8] Galasso and Schankerman (2015) find that whether or not patents hinder subsequent innovation depends on the complexity

[6] The regulated have an incentive as well (regulatory capture).

[7] A general model of public goods incorporating non-monotonicity was developed in Mas-Colell (1980), where public projects are elements of a non-linear topological space. However, such a model is unable to capture the characteristics described above.

[8] For examples of regulatory complexity in finance, see Berglund (2014). For examples in IP, see Allison and Lemley (2002). Evidence of a negative impact of regulations is given in Murray and Stern (2007) and Williams (2013).

of the industry and that negative effects are concentrated in more complex industries. If it is relatively more costly to implement IP in more complex industries, then Galasso and Schankerman (2015) can be viewed as evidence in support of the quality-quantity tradeoff. Polborn (2008) also finds over-provision when there is heterogeneous individuals and competition between organizations developing a public good. This paper shows that neither competition nor heterogeneity are necessary for over-provision.

This paper is particularly useful in understanding the structure and outcomes of industries whose interactions are characterized by collaborative production, such as the software industry. I show that collaborative groups will likely over-invest in technological investment, shedding light on the nature of research joint ventures by building upon Kamien et al. (1992) and the theories outlined in Caloghirou et al. (2003). Lastly, this paper offers an alternative mechanism for understanding over-investment in patent races (Gilbert and Newbery, 1982; Fudenberg et al., 1983; Shapiro, 1985; Harris and Vickers, 1987; Aoki, 1991; Baye and Hoppe, 2003).

The paper is structured as follows. Section 2 outlines the model. Section 3 presents the main results. Section 4 presents the social planner's problem. Section 5 concludes.

2 The Model

In this section, I develop a model of the private provision of public goods incorporating the four features from the introduction. In an online appendix, I offer an alternative model illustrating over-provision when there are contribution benefits based on the share of the public good contributed by an individual but no consumption costs. In that specification, all costs of over-provision are borne by the contributors rather than the consumers.

There are $N > 1$ individuals and a single public good. Each individual i has the option to either contribute at level $x_i \geq 0$ and consume or not consume.[9] Let $\mathbf{x} = (x_1, \ldots, x_N)$, $X = \sum_{j=1}^{N} x_j$, and $X_{-i} = \sum_{j \neq i} x_j$.[10] All decisions are made simultaneously and independently.

[9] Allowing individuals to contribute but not consume only strengthens the results of the paper. Contributors will not internalize the utility from consumption, further increasing the intensity of contributing.

[10] It is straightforward to incorporate agents with varying productivity, e.g., $X = \sum_{j=1}^{N} \alpha_j x_j$. In this case, sorting occurs where agents who are more productive contribute while those who are less productive

Each individual's utility consists of four additively-separable components.[11] The first component is the consumption benefit, which depends only on total contributions and is denoted by $B(X)$. I assume that $B(\cdot)$ is twice continuously differentiable, strictly increasing, and weakly concave with $B(0) = 0$. The second component is the consumption cost, which like the consumption benefit, depends only on the aggregate contributions. Denote the consumption cost by $C(X)$, which assumed to be twice continuously differentiable, strictly increasing, and strictly convex with $C(0) = 0$. Define $B(X) - C(X)$ as the public good's *quality* and $\hat{X} \equiv \max_X \{B(X) - C(X)\}$ as the quality-maximizing level of aggregate contributions.

Examples of costs depending only on aggregate contributions include enforcement costs, implementation costs, learning costs, and switching costs. Open source software often entails implementation, learning, and switching costs as new users must learn how to navigate the software and acquire hardware capable of running the software. Larger programs with more features require greater hardware investments.

The third component is the contribution benefit, which depends on individual's contribution, the other individuals' aggregate contributions X_{-i}, and an exogenous parameter $\sigma \geq 0$. Denote the contribution benefit by $\sigma b(x_i, X_{-i})$, which I assume is twice continuously differentiable, strictly increasing in x_i, weakly decreasing in X_{-i}, and weakly concave with $b(0, X_{-i}) = 0$. Furthermore, I assume that $\frac{\partial b(x, X_{-i})}{\partial x_i} = \frac{\partial b(x, X_{-j})}{\partial x_j} > 0$ for every X_{-i} and X_{-j} and that $\frac{\partial^2 b(x_i, X_{-i})}{\partial x_i^2} = \frac{\partial^2 b(x_i, X_{-i})}{\partial x_i \partial X_{-i}}$. These assumptions allow contribution benefits to depend on either the individual's contribution or the share of the individual's contributions to the aggregate.[12] I define σ as the relative strength of the contribution benefits. If $\sigma = 0$, then contributing is solely a cost. If $\sigma > 0$, then individuals benefit not only from the good's quality, but also from contributing to its production. For example, when developing open source software, contributors benefit for reasons such as signaling ability.[13] The fourth component is the contribution cost, which depends only on the individual's contributions, $c(x_i)$.

free-ride, as with technologically savvy individuals and open source software (Sacks, 2015).

[11] Additive separability is assumed for expositional convenience and is not necessary.

[12] For example, $b(x_i, X_{-i}) = x_i$ or $b(x_i, X_{-i}) = \frac{x_i}{x_i + X_{-i}}$.

[13] See, e.g. Lerner and Tirole (2002). Other examples of contribution benefits include signaling wealth (Glazer and Konrad, 1996), forms of impure altruism (Andreoni, 1990), or increases in productivity via learning-by-doing (Irwin and Klenow, 1996; Bramoullé and Kranton, 2007).

I assume that $c(x_i)$ is increasing, twice continuously differentiable and strictly concave. with $c(0) = 0$.

Combining these four components, individual i's utility is

$$u\left(x_i, X_{-i}; \sigma\right) = \underbrace{B\left(X\right) - C\left(X\right)}_{\substack{\textbf{utility from consumption} \\ \textbf{(quality)}}} + \underbrace{\sigma b(x_i, X_{-i}) - c(x_i).}_{\substack{\textbf{utility from contribution} \\ \textbf{(quantity)}}} \tag{1}$$

The utility from neither consuming nor contributing to the public good is normalized to zero. I utilize the Nash Equilibrium solution concept.

3 Analysis

Denote by x_i^* the individual equilibrium contribution to the public good and by X^* the aggregate equilibrium contribution.

Lemma 1. *If $x_i^* > 0$ and $x_j^* > 0$, then $x_i^* = x_j^* = x^*$ for all i and j.*

I order the individuals so if $M < N$ individuals contribute, individuals $1, \ldots, M$ are the contributors and individuals $M + 1, \ldots, N$ are the non-contributors.

The equilibrium contributions are defined by the following system of first-order conditions:

$$\frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} + \sigma \frac{\partial b(x^*, X_{-i}^*)}{\partial x_i} - \frac{dc(x^*)}{dx_i} = 0, \quad \forall\, i = 1, \ldots, M \tag{2}$$

$$\frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} \leq 0, \quad \forall\, i = M+1, \ldots, N. \tag{3}$$

By the implicit function theorem, $x^* = x(M, \sigma)$ and $X^* = X(M, \sigma)$ and by the theorem of the maximum, $\frac{\partial x(M,\sigma)}{\partial \sigma}$ exists and is well defined.[14] Let $x'' > 0$ be the contribution level satisfying

$$\frac{dB\left(X_{-i}^* + x''\right)}{dX} - \frac{dC\left(X_{-i}^* + x''\right)}{dX} + \sigma \frac{\partial b(x'', X_{-i}^*)}{\partial x_i} - \frac{dc(x'')}{dx_i} = 0.$$

There exists an equilibrium in which exactly M individuals contribute if, for all $i = 1, \ldots, M$,

$$B\left(X^*\right) - C\left(X^*\right) + \sigma b(x^*, X_{-i}^*) - c(x_i^*) \geq \max\{B\left(X_{-i}^*\right) - C\left(X_{-i}^*\right), 0\}, \tag{4}$$

[14] For $\sigma = 0$, define the derivative as $\lim_{\Delta \to 0^+} \frac{x(M,\Delta) - x(M,0)}{\Delta}$.

and if $M < N$,

$$\max\{B(X^*) - C(X^*), 0\} \geq B(X^* + x'') - C(X^* + x'') + \sigma b\left(x'', X^*_{-i}\right) - c(x'') \qquad (5)$$

for all $i = M+1, \ldots, N$. Inequality (4) states that there is no profitable deviation in which an individual stops contributing and (5) states that there is no profitable deviation in which a free-rider starts contributing. Denote by $M^*(N)$ the values of M such that $\mathbf{x}^* = \mathbf{x}(M^*(N), \sigma)$ constitutes a Nash equilibrium.

For comparison, it is necessary to characterize the welfare-maximizing contributions. I use $\tilde{x}$ and $\tilde{X}$ to denote the individual and aggregate welfare-maximizing contributions, which are functions of M, N, and σ. Denote by $\tilde{M}(N)$ the values of M that maximize welfare.[15] In what follows, $M^*(N)$ and $\tilde{M}(N)$ denote the number of equilibrium and welfare-maximizing contributors while M represents an arbitrary number of contributors. As a benchmark, suppose that $\sigma = 0$ to isolate the effects of consumption costs.

Proposition 1. *If $\sigma = 0$, then in equilibrium, $M^*(N) = \tilde{M}(N) = N$. For all finite N, $X^* < \tilde{X} < \hat{X}$ and $\lim_{N\to\infty} X^* = \lim_{N\to\infty} \tilde{X} = \hat{X}$.*

The proof, and all subsequent proofs, are contained in the appendix. When contributing is solely a burden, free-riding persists and neither the equilibrium nor the welfare-maximizing contributions maximize quality. However, both the equilibrium and welfare-maximizing contributions approach the quality maximizer as the size of the population grows large. The free-rider problem becomes less severe rather than more severe as the equilibrium contributions converge to maximize both welfare and quality. Proposition 1 provides the economic justification of Linus's law.

In what follows, I show that introducing contribution benefits substantially alters these results. Linus's law is falsified and the economic justification of the "too many cooks spoil the broth" proverb is illustrated. Suppose that $\sigma \geq 0$. In addition to (2)-(5), one of three mutually exclusive and exhaustive properties must be satisfied at the equilibrium contribution levels.

[15]Formal derivations are provided in the proof of Proposition 1.

Property 1. **(P1)** $\frac{dB(X(M,\sigma))}{dX} - \frac{dC(X(M,\sigma))}{dX} > 0 > \sigma \frac{\partial b(x(M,\sigma),X_{-i}(M,\sigma))}{\partial x_i} - \frac{dc(x(M,\sigma))}{dx_i}$.

Property 2. **(P2)** $\frac{dB(X(M,\sigma))}{dX} - \frac{dC(X(M,\sigma))}{dX} < 0 < \sigma \frac{\partial b(x(M,\sigma),X_{-i}(M,\sigma))}{\partial x_i} - \frac{dc(x(M,\sigma))}{dx_i}$.

Property 3. **(P3)** $\frac{dB(X(M,\sigma))}{dX} - \frac{dC(X(M,\sigma))}{dX} = 0 = \sigma \frac{\partial b(x(M,\sigma),X_{-i}(M,\sigma))}{\partial x_i} - \frac{dc(x(M,\sigma))}{dx_i}$.

Property 2 illustrates the quantity-quality tradeoff. Quality is decreasing in aggregate contributions. Yet the individual can benefit from contributing when the contribution benefits outweigh the contribution costs. If σ is large enough, such an incentive is present in equilibrium. The following lemma formalizes the existence of such a σ.

Lemma 2. *There exists a decreasing function $\overline{\sigma}(M) : \mathbb{N} \setminus \{0\} \to \mathbb{R}_+ \cup \{0\}$ such that if $\sigma < \overline{\sigma}(M)$, then $P1$ is satisfied; if $\sigma > \overline{\sigma}(M)$, then $P2$ is satisfied; and if $\sigma = \overline{\sigma}(M)$, then $P3$ is satisfied. As $M \to \infty$, $\overline{\sigma}(M) \to 0$.*

Lemma 2 shows that either $P1$ or $P2$ can be satisfied in equilibrium.[16] When σ is small, the typical under-provision result persists. Over-contributing relative to the quality-maximizing contribution level occurs when the contribution benefits are sufficiently large ($\sigma > \overline{\sigma}(M)$). Contribution benefits impose a negative externality on consumption, which can be severe enough to induce over-provision. As a consequence, some individuals may choose to free-ride or not consume at all, implying that no symmetric equilibrium exists.

Denote by $M^U \in \mathbb{R}_+$ the M that makes (4) hold with equality. The maximum number of contributors is given by $\lfloor M^U \rfloor$. If there are M contributors and $\sigma < \overline{\sigma}(M)$, then (5) cannot be satisfied as both $B(x_i + X_{-i}) - C(x_i + X_{-i})$ and $\sigma b(x_i, X_{-i}) - c(x_i)$ are increasing for small x_i. Thus either $P2$ or $P3$ must be satisfied. If $\sigma \leq \overline{\sigma}(N)$, then all N individuals contribute. If either $P2$ or $P3$ is satisfied, then the value of M that makes (4) hold with equality (M^U) indicates the upper bound on the number of contributors in equilibrium. To see this relationship, note that under $P2$ and $P3$, $\sigma b(x(M,\sigma);\sigma) - c(x(M,\sigma))$ is decreasing in M while $B\left(X^*_{-i}\right) - C\left(X^*_{-i}\right) - \left[B\left(X^*\right) - C\left(X^*\right)\right]$ is increasing in M, so (4) is violated for all $M > M^U$. The value of M, denoted by M^L, that makes (5) hold with equality indicates the lower bound on the number of contributors. It follows that (5) is violated for all $M < M^L$.

[16] $P3$ occurs only if $\sigma = \overline{\sigma}(M)$, so I focus on $P1$ and $P2$.

Consequently, the maximal number of contributors is determined by $\min\left\{\lfloor M^U \rfloor, N\right\}$. If $N > M^U$, then $N - \lfloor M^U \rfloor$ individuals do not contribute in equilibrium. The correspondence

$$M^*(N) : N \rightarrow \left\{Z \in \mathbb{N} \,|\, M^L \leq Z \leq \min\left\{M^U, N\right\}\right\}$$

defines the equilibrium number of contributors as a function of the population size.

Note that $|M^*(N)| \in \{1, 2\}$. That is, there may in fact exist two equilibria due to there being a kink in the best response function at $x_i = 0$, where an individual can choose to either free-ride and consume or not consume. Depending on the relative slopes of the four components of utility, as x_i approaches zero there may exist a situation in which there are two sequential values of M such that there is no unilateral incentive to deviate.

Proposition 2. *Suppose that $\sigma > 0$. For every $M^*(N)$, there exists a Nash equilibrium with $M^*(N)$ contributors, each contributing $x(M^*(N), \sigma) > 0$. Furthermore, $M^*(N) = N$ is necessary for $P1$ to be satisfied in equilibrium.*

While at least one and at most two equilibria may exist, the equilibria share the same characteristics so the analysis can be conducted as if $M^*(N)$ is unique. Although the population is *ex ante* symmetric, the equilibria itself need not be completely symmetric due to the fact that players can choose not only to free-ride, but not consume the good entirely. However, all contributors will act identically (Lemma 1).

Unlike the case with $\sigma = 0$, the equilibrium contribution $X\left(M^*(N), \sigma\right)$ converges neither to the welfare-maximizing nor quality-maximizing contribution level. The relationships between the equilibrium contributions and the quality-maximizing contributions for any $\sigma \geq 0$ are outlined in the following proposition, which is the main result of this section.

Proposition 3. *If $\sigma < \overline{\sigma}(N)$, then $X^* < \tilde{X} < \hat{X}$ and if $\sigma > \overline{\sigma}(N)$, then $X^* > \tilde{X} > \hat{X}$. Optimal provision only occurs if $M^*(N) = 1$. Furthermore, $\lim_{N\rightarrow\infty} X^* > \lim_{N\rightarrow\infty} \tilde{X} = \hat{X}$ for all $\sigma > 0$.*

Proposition 3 formalizes the "too many cooks spoil the broth" proverb. When the contribution benefit is small ($\sigma < \overline{\sigma}(N)$), there is under-provision and the free-rider problem persists and when the private benefits are large ($\sigma > \overline{\sigma}(N)$), supply-side congestion occurs. There is a crowding effect where the contribution benefit induces a large level of contributions. These contributions generate a significant consumption cost that is not fully internalized by the contributors. Hence both Linus's law and the proverb are consistent with the collaborative provision of public goods. Which is applicable is determined by the presence of contribution benefits. Without contribution benefits, Linus's law is supported and with contribution benefits, the proverb is supported.

The following corollary emerges as a consequence of Proposition 3.

Corollary 1. *If $\sigma' > \sigma''$, then $X\left(M^*(N), \sigma'\right) > X\left(M^*(N), \sigma''\right)$.*

The aggregate equilibrium contribution is increasing in σ, which implies that as M becomes increasingly large, the distance between the equilibrium contributions and the quality-maximizing contributions is increasing in the relative strength of the contribution benefit, further increasing the supply-side congestion. Increasing the intensity of contributions further increases aggregate output, which imposes a cost not just on the individual contributor, but on all consumers. Contributors fail to internalize that the increase in their own contributions also imposes a consumption cost on all other consumers, lowering quality.

Regardless of σ, the aggregate welfare-maximizing contribution level converges to the quality-maximizing contribution level as the size of the population grows; however, in contrast to Proposition 1, when $\sigma > 0$ the equilibrium contribution does not converge to the welfare-maximizing contribution level. Instead, contributions converge to a level strictly greater than the welfare-maximizing level. This result follows from Lemma 2 as $\overline{\sigma}(M)$ tends to zero when M becomes large, which implies that the equilibrium always falls under $P2$. Thus there is always over-provision in large populations whenever $\sigma > 0$. The lack of convergence in $X\left(M^*(N), \sigma\right)$ follows from the negative externality imposed by $\sigma > 0$ dominating the positive externality (free-riding). In other words, quality is monotonically increasing in the number of contributors whenever $\sigma = 0$. If $\sigma > 0$, then the relationship between quality and quantity is non-monotonic (the tradeoff).

4 The Social Planner

Depending on the context, there may be many options available to policymakers. I assume that policymakers, henceforth referred to as social planners, cannot directly affect contributions, but can indirectly affect them through two channels: the number of contributors M and the private benefit parameter σ.[17] Call the first channel the contribution channel and the second the benefits channel. Policies that manipulate the number of contributors induce changes on the intensive margin via changes on the extensive margin. Policies that manipulate the private benefit parameter induce changes on the extensive margin via altering decisions on the intensive margin.

4.1 Contribution Channel

Suppose that the social planner is only able to selectively restrict contributions on the extensive margin by fixing the number of contributors, thereby rendering (4) and (5) obsolete. While setting the number of contributors to maximize the quality of the public good will maximize the utility of any individual contributor and any individual free-rider, a social planner interested in maximizing total welfare rarely has the incentive to maximize quality. Instead, a social planner can shift the number of contributors and non-contributors and transfer free-rider surpluses from the quality to contributors in the form of increasing the contribution benefit. This outcome is a generalization of regulatory capture.

The objective of the social planner is given by

$$\max_{M} \Bigg\{ (N - M) \max \{ B(X) - C(X), 0 \} + \sum_{j=1}^{M} \left[B(X) - C(X) + \sigma b(x_j, X_{-j}) - c(x_j) \right] \Bigg\}, \quad (6)$$

subject to

$$\begin{aligned} &x_j = \arg\max \{ B(X) - C(X) + \sigma b(x_j, X_{-j}) - c(x_j) \} \quad \forall\, j \leq M \\ &x_j = 0 \quad \forall\, j > M \end{aligned} \quad (7)$$

[17] For example, the leadership team of an open source software project cannot compel specific contributions, but can influence incentives or reject contributions.

Denote by $\hat{M}$ the solution to (6).

Firstly note that there can be no interior solution $\hat{M} < \overline{\sigma}^{-1}(\sigma)$ such that

$$\overline{\sigma}^{-1}(\sigma) < \frac{\hat{M} + (\hat{M}+1)}{2};$$

otherwise, adding one more contributor increases the net quality free-riders and both the quality and contribution benefit for the original $\hat{M}$ contributors while also increasing the utility of the new contributor by the marginal change in the quality plus the entirety of the contribution benefit, leading to a strict welfare gain.[18] This logic cannot be extended to the other side of $\overline{\sigma}^{-1}(\sigma)$. Suppose that $\hat{M} > \overline{\sigma}^{-1}(\sigma)$ and there exists a value $\hat{M}-1$ such that

$$\overline{\sigma}^{-1}(\sigma) < \frac{(\hat{M}-1) + \hat{M}}{2}.$$

By decreasing $\hat{M}$ to $\hat{M}-1$, the quality increases, as does the contribution benefit for each of the $\hat{M}$ contributors. Yet, this change may actually lead to a decrease in total welfare if the change in the quality, weighted by the number of consumers plus the change in the contribution benefit, weighted by the number of contributors, is less than the loss in the contribution benefit by the $\hat{M}^{th}$ contributor:

$$\begin{aligned}(N-M)\big[&\max\{B(X(\hat{M}-1,\sigma)) - C(X(\hat{M}-1,\sigma)),0\}\\ &-\max\{B(X(\hat{M},\sigma)) - C(X(\hat{M},\sigma)),0\}\big]\\ &+M\big[B(X(\hat{M}-1,\sigma)) - C(X(\hat{M}-1,\sigma))\\ &-[B(X(\hat{M},\sigma)) - C(X(\hat{M},\sigma))]\big]\\ &+(\hat{M}-1)[\sigma b(x(\hat{M}-1,\sigma),X_{-i}(\hat{M}-1,\sigma)) - c(x(\hat{M}-1,\sigma))\\ &-[\sigma b(x(\hat{M},\sigma),X_{-i}(\hat{M},\sigma)) - c(x(\hat{M},\sigma))]]\\ &<\sigma b(x(\hat{M},\sigma),X_{-i}(\hat{M},\sigma)) - c(x(\hat{M},\sigma)).\end{aligned} \tag{8}$$

Increasing quality by decreasing the number of contributors increases the utility for each contributor and each free-rider, but may decrease welfare due to the decrease in the utility of the contributor turned free-rider. Thus depending on how welfare is defined, there are different policy protocols.

[18]If $\sigma = 0$, then the consumption benefit is a net loss on the new contributor, but by (5), the marginal impact of the loss is less than the marginal increase in quality leading to a positive change in utility.

Proposition 4. *If* (8) *is satisfied, then the optimal strategy for a social planner interested in maximizing total welfare need not match the optimal strategy for a social planner interested in maximizing quality.*

Proposition 4 has bite due to the fact that maximizing quality is akin to concurrently maximizing both the quality and the contribution benefit for any given contributor. A tension remains on the extensive margin, whereby adding an additional contributor may decrease both the quality for all consumers and the contribution benefit for the original set of contributors, but increases the new contributor's payoff enough that aggregate welfare of the population at large is increased. Even if there is an integer M such that $M = \overline{\sigma}^{-1}(\sigma)$, it may not be the welfare maximizer. Supply-side congestion exists on both the intensive and extensive margins. It merits mentioning that the differences in outcomes described in Proposition 4 do not violate the Warr/BBV neutrality theorem, as the change is being made along the extensive margin and is thus not net-zero.[19] Thus the theorem does not apply in this type of environment.

Regardless of the type of contributor, the above shows that a social planner is willing to sacrifice the utility of a consumer to increase the utility of a contributor in order to maximize total welfare.

Corollary 2. *If the set of potential contributors is a strict subset of the population and* σ *is small, then for a finite population, a social planner seeking to maximize total welfare will never choose the maximum number of contributors that maximizes quality.*

Corollary 2 provides a general extension of the results in Kearl (1983), Quandt (1983), Nitzan et al. (2013), and the other associated works on regulatory complexity.

4.2 Benefits Channel

Now suppose that the social planner can only influence the individuals' benefit parameter σ. For example, the contribution benefit often occurs through signaling. Signals become

[19] See Warr (1983) and Bergstrom et al. (1986) for details on the neutrality theorem.

more valuable as they become more observable (Holmström, 1999). Thus a social planner can increase σ by making contributions more publicly visible, e.g. by publishing names and posting plaques, or alternatively decrease σ by keeping all contributions anonymous.

Rather than adjusting σ such that the public good is provided as close to $P3$ as is possible (due to the integer nature of contributing on the extensive margin), the social planner can unboundedly increase welfare by increasing the consumption benefit at the expense of decreasing the quality, transferring all surpluses to the contributors and leading all non-contributors to not consume.

The social planner's objective function is given by

$$\max_{\sigma \geq 0} \Bigg\{ \mu \left(N - M\right) \max \left\{ B\left(X(M,\sigma)\right) - C\left(X(M,\sigma)\right), 0 \right\} + (1-\mu) \sum_{j=1}^{M} \big[B\left(X(M,\sigma)\right) \\ - C\left(X(M,\sigma)\right) + \sigma b(x_j(M,\sigma), X_{-j}(M,\sigma)) - c(x_j(M,\sigma)) \big] \Bigg\}, \quad (9)$$

subject to

$$x_j = x(M^*(N), \sigma) \quad \forall j \leq M^*(N)$$
$$x_j = 0 \quad \forall j > M^*(N),$$

where $\mu \in [0,1]$ is the weight placed on the non-contributors. Although σ is continuous, (9) is not continuously differentiable in σ. The objective function has discontinuous jumps in σ because $M^*(N)$ is a discontinuous function of σ (as it must be an integer).[20]

By inspection of (4) and (5), $\lceil M^L \rceil$ and $\lfloor M^U \rfloor$, are decreasing step functions of σ. As σ increases, each contributor responds by contributing a greater amount. Initially, this movement is smooth, first increasing quality and then decreasing net quality. Increases in individual contributions cause quality to decrease at an increasing rate and the contribution benefit to increase at a decreasing rate. There reaches a point where an increase in σ induces a contributor to no longer contribute or consume. This change on the extensive margin decreases total contributions, leading to a sharp increase in quality. Further increases in σ cause this process to repeat itself, inducing another contributor to stop contributing and

[20]The same holds true with respect to maximizing welfare along the intensive margin.

consuming, leading to another positive jump in each remaining contributor's contribution. From (5), there always exists the incentive for at least one contributor. Therefore, for at least one contributor,

$$\frac{dB\left(X(M^*(N),\sigma)\right)}{dX} - \frac{dC\left(X(M^*(N),\sigma)\right)}{dX} + \sigma\frac{\partial b\left(x(M^*(N),\sigma), X_{-i}(M^*(N),\sigma)\right)}{\partial x_i} - \frac{dc(x(M^*(N),\sigma))}{dx_i} = 0$$

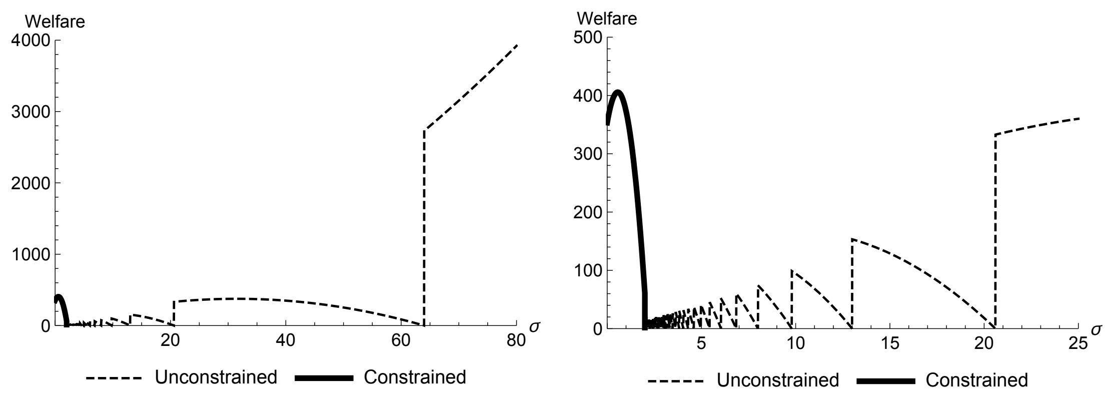
with $x(M^*(N),\sigma) > 0$. Thus the process stabilizes to a fixed number of contributors in the limit (for σ large). Indirect utility is increasing in σ for σ large. Because utility is bounded below by zero, welfare can be made arbitrarily large by increasing σ. Thus the objective function in (9) takes a unique shape. It is initially smooth and hump-shaped, then taking on a sawtooth pattern with jumps representing points at which a contributor changes her decision on the extensive margin followed by smooth decreases due to increases in contributions. This pattern repeats until the process stabilizes with a final set of contributors, at which point the total welfare is strictly increasing in σ. While this pattern is general, a specific example is provided below and is visualized in Figure 1.

There are two options available to the social planner that lead to a solution that does not transfer all surpluses to the contributors: force consumption or directly consider quality in the optimization program.[21] The second solution is both more efficient and practical than the first, so I focus attention on this option.[22] A natural consideration is to ensure that the net quality is no worse than the outside option, which is accomplished by adding the condition that $B\left(X\left(M,\sigma\right)\right) - C\left(X\left(M,\sigma\right)\right) \geq \underline{Q}$ for some $\underline{Q} \in \mathbb{R} \cup \{-\infty\}$ to (9). The most natural value is $\underline{Q} = 0$, though the optimization can be conducted with any feasible $\underline{Q}$. I conclude the section with the formal statement of the result and an illustrative example.

Proposition 5. *For every $\mu \in [0,1)$ and $\underline{Q} = -\infty$, a social planner can make total welfare arbitrarily large by increasing σ. For every finite $\underline{Q} \leq B(\hat{X}) - C(\hat{X})$ and $\mu \in [0,1]$, there exists a finite $\hat{\sigma} \geq 0$ that maximizes total welfare.*

[21] Forcing consumption will lead to a completely symmetric equilibrium with N contributors.

[22] The notion of quality control has been, for the most part, unexplored in the previous literature.



(a) Full figure.

(b) Zoomed region.

Figure 1: Welfare maximization, Example 1.

There are two effects at work: a direct effect and an indirect effect. Increasing σ increases the benefits of all contributors. However, by Corollary 1, increasing the benefits decreases quality through increasing the aggregate contributions. Thus free-riders are impacted negatively by such a policy. Because non-contributors and free-riders will eventually choose the outside option, a large enough increase in the contribution benefit will allow the net effect to grow to be arbitrarily large and positive. The following example illustrates this result.

Example 1. *Consider the utility function*

$$u(X, x_i; \sigma) = \left(\sum_{j=1}^{N} x_j\right) - \frac{1}{40}\left(\sum_{j=1}^{N} x_j\right)^2 + \sigma x_i - \frac{1}{2}x_i^2.$$

Panel (A) of Figure 1 plots the welfare objective function for $\mu = \frac{1}{2}$ and panel (B) zooms in towards the origin to provide a more detailed view. The constrained line corresponds to $\underline{Q} = 0$ and the unconstrained line corresponds to $\underline{Q} = -\infty$. Each jump corresponds to a contributor switching to not contributing. The final jump, which occurs at $\sigma \approx 64$, corresponds to the point at which the number of contributors settles at five.

5 Discussion and Concluding Remarks

This paper has outlined several novel and empirically relevant results. When extending the standard theory of public goods to incorporate consumption costs and contribution benefits, well known incentive problems change and new issues are introduced. While inefficiencies due to free-riding may still persist, over-provision due to supply-side congestion also presents a concern, but is not dealt with in the same manner as free-riding. Over-provision is made more severe by large populations. Moreover, policy prescriptions typically used to internalize the externalities driving these inefficiencies no longer have the same desirable properties. Without further consideration, maximizing welfare by controlling the number of contributors can lead to a slight deterioration of the quality to benefit the contributors, while maximizing welfare by controlling the contribution benefit leads to a severe deterioration in the quality to benefit the contributors unless the policymaker also implements a minimum quality requirement. The notion of quality control has thus far been under-explored.

While a few assumptions were included in the model for tractability, it is worth noting that heterogeneity can be introduced into the model without qualitatively changing the results. Suppose that heterogeneity is introduced such that the ratio $\frac{\partial g(\mathbf{x}^k)}{\partial x_i^k} / \frac{\partial g(\mathbf{x}^k)}{\partial x_j^k} = \frac{\eta_i}{\eta_j}$ for some positive constants η_i and η_j. It is straightforward to see that the only change is in the first-order conditions, where the marginal effect of a change in contributions on the quality function is scaled. The resulting values are scaled accordingly, where individuals with a greater η contribute more, but their relationships remain qualitatively unchanged (i.e., the welfare-maximizing contribution is closer to the quality-maximizing contribution than the equilibrium contribution for each i). Notation would be more cumbersome but no new insights would be gained.

An interesting extension to consider is the dynamic aspects of public good provision. Regulations, statutes, and other legal guidelines such as tax codes tend to evolve over time with contributors (regulators) who have their own private and public incentives. The quality-enhancing properties of early contributions can induce a feedback loop that increases the contributions made in these regulations over time. The early contributions may lead to improvements in the quality of these regulations, but over time, these improvements diminish and eventually have deleterious effects, decaying the quality of these regulations. A similar story can be told with respect to open source software. Such a dynamic analysis focusing on the efficiency of provision over time presents an interesting avenue for future research.

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Appendix

Proof of Lemma 1.

Proof. The (interior) first-order condition with respect to i is

$$\frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} - \frac{dc(x_i^*)}{dx_i}.$$

Hence for an arbitrary i and j,

$$\frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} + \sigma\frac{\partial b(x_i^*, X_{-i}^*)}{\partial x_i} - \frac{dc(x_i^*)}{dx_i} \\ = \frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} + \sigma\frac{\partial b(x_j^*, X_{-j}^*)}{\partial x_j} - \frac{dc(x_j^*)}{dx_i},$$

which simplifies to

$$\sigma\left(\frac{\partial b(x_i^*, X_{-i}^*)}{\partial x_i} - \frac{\partial b(x_j^*, X_{-j}^*)}{\partial x_j}\right) = \frac{dc(x_i^*)}{dx_i} - \frac{dc(x_j^*)}{dx_i}$$

If $x_i^* > x_j^*$, then the RHS is nonpositive given the weak concavity of $b(x, X)$ while the LHS is strictly positive given the strict convexity of $c(x)$. Thus the two sides are equal if and only if $x_i^* = x_j^*$. □

Proof of Proposition 1.

Proof. Set $\sigma = 0$ and suppose that $M < N$ individuals contribute. For $x_i^* > 0$,

$$\frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} - \frac{dc(x_i^*)}{dx_i} \leq 0, \tag{10}$$

with equality for $i = 1, \ldots, M$. As the first two terms are common for all individuals, it follows that $x_i^* = x_j^*$ for all $i, j \leq M$. Moreover,

$$\frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} > 0$$

as $\frac{dc(x_i^*)}{dx_i} > 0$. Inequalities (4) and (5) require that for each i,

$$B\left(X(M,0)\right) - C\left(X(M,0)\right) - c\left(x(M,0)\right) \geq \max\left\{B\left(X_{-i}(M,0)\right) - C\left(X_{-i}(M,0)\right), 0\right\}$$

and

$$\max\left\{B\left(X(M,0)\right) - C\left(X(M,0)\right), 0\right\} \geq B\left(X(M,0) + x''\right) - C\left(X(M,0) + x''\right) - c\left(x''\right).$$

Consider a deviation by individual $i = M + 1$. Inequality (5) can be rewritten as

$$\begin{aligned} &\max\left\{B\left(X(M,0) + x_{M+1}\right) - C\left(\left(X(M,0) + x_{M+1}\right), 0\right\} - c(0) \\ &\qquad \geq \max_{x_{M+1}}\{B\left(X(M,0) + x_{M+1}\right) - C\left(X(M,0) + x_{M+1}\right) - c\left(x_{M+1}\right)\}, \end{aligned}$$

At best, (5) can only be satisfied with equality, so x'' must equal zero. Differentiating the RHS and evaluating at $x_{M+1} = 0$ yields

$$\frac{dB\left(X(M,0)\right)}{dX} - \frac{dC\left(X(M,0)\right)}{dX} > 0,$$

a contradiction. Hence there is no equilibrium with $M < N$ as at least one non-contributing individual has a profitable deviation. It remains to be shown that at $M = N$, (4) is satisfied. Given $M = N$, (4) can be rewritten as

$$\begin{aligned} &\max_x\left\{B\left(X_{-i}(N,0) + x\right) - C\left(X_{-i}(N,0) + x\right) - c\left(x\right)\right\} \\ &\qquad \geq \max\left\{B\left(X_{-i}(N,0)\right) - C\left(X_{-i}(N,0)\right), 0\right\} - c(0), \end{aligned}$$

As

$$\frac{dB\left(X_{-i}(N,0) + 0\right)}{dX} - \frac{dC\left(X_{-i}(N,0) + 0\right)}{dX} > 0,$$

the inequality is necessarily satisfied so no contributing individual has a unilateral incentive to deviate. Thus there exists a symmetric Nash equilibrium with N contributors. The relevant first-order condition is now

$$\frac{dB\left(X(N,0)\right)}{dX} - \frac{dC\left(X(N,0)\right)}{dX} - \frac{dc\left(x_i(N,0)\right)}{dx_i} = 0, \quad \forall\, i. \tag{11}$$

To prove that the equilibrium level of contributions converges to the quality-maximizing level, suppose to the contrary that $x_i^* \to z > 0$ as $N \to \infty$. Convergence to a positive z implies that $X^* \to \infty$ as $N \to \infty$. It follows that

$$\frac{dB(\infty)}{dX} - \frac{dC(\infty)}{dX} < 0.$$

As $\frac{dc(z)}{dx_i} < 0$, (11) cannot be satisfied. Thus $x_i^* \to 0$ as $N \to \infty$. As x_i^* approaches zero and $\lim_{N\to\infty} \frac{dc(x_i^*)}{dx_i} = 0$, (11) holds only if

$$\lim_{N\to\infty} \left(\frac{dB(X^*)}{dX} - \frac{dC(X^*)}{dX} \right) = 0,$$

which is true when $\lim_{N\to\infty} X^* = \hat{X}$.

Now, consider the welfare maximizing case. The welfare maximizing contributions are determined by the solution to

$$\max_{\mathbf{x}} \sum_{j=1}^{N} u(X, x_j; \sigma).$$

The arbitrary first-order condition with respect to contribution $x_i > 0$ is given by

$$\frac{dB(\tilde{X})}{dX} - \frac{dC(\tilde{X})}{dX} - \frac{1}{N}\frac{dc(\tilde{x}_i)}{dx_i} = 0. \tag{12}$$

By inspection, $\tilde{x}_i = \tilde{x}_j$ for all i, j and by an analogous argument to the above, $\tilde{M}(N) = N$. Taking the limit of (12) as $N \to \infty$ yields $\tilde{X} \to \hat{X}$.

Lastly to prove that $X^* < \tilde{X} < \hat{X}$ for all finite $N > 1$, note that $\frac{dc(x_i)}{dx_i} > 0$ for all $x_i > 0$. Evaluating the LHS of (11) at $x_i = x_i(N, 0)$ for all i yields

$$\frac{dB(X(N,0))}{dX} - \frac{dC(X(N,0))}{dX} - \frac{1}{N}\frac{dc(x_i(N,0))}{dx_i}.$$

As $\frac{dc(x_i(N,0))}{dx_i} > 0$,

$$-\frac{dc(x_i(N,0))}{dx_i} < -\frac{1}{N}\frac{dc(x_i(N,0))}{dx_i}.$$

From (11),

$$\frac{dB(X(N,0))}{dX} - \frac{dC(X(N,0))}{dX} - \frac{1}{N}\frac{dc(x_i(N,0))}{dx_i}$$
$$> \frac{dB(X(N,0))}{dX} - \frac{dC(X(N,0))}{dX} - \frac{dc(x_i(N,0))}{dx_i} = 0,$$

so $\tilde{x}_i > x_i^*$ and $\tilde{X} > X^*$ for all i and j and all finite $N > 1$. The presence of $\frac{dc(x_i)}{dx_i} > 0$ for all $x_i > 0$ implies that both X^* and $\tilde{X}$ are less than $\hat{X}$. If $N = 1$, then there is no externality, so $x(1, 0) = \tilde{x}$ and $X^* = \tilde{X}$ by default. $\square$

Proof of Lemma 2.

Proof. By lemma 1, $\frac{\partial x_i(M,\sigma)}{\partial\sigma} = \frac{\partial x_j(M,\sigma)}{\partial\sigma} = \frac{\partial x(M,\sigma)}{\partial\sigma}$. The first step is to show that $\frac{\partial x(M,\sigma)}{\partial\sigma} > 0$. Fix M and recall that, for each contributor $i = 1, \ldots, M$, it must be that

$$\frac{dB\left(Mx^*\right)}{dX} - \frac{dC\left(Mx^*\right)}{dX} + \sigma\frac{\partial b\left(x^*, (M-1)x^*\right)}{\partial x_i} - \frac{dc(x^*)}{dx_i} = 0 \tag{13}$$

holds in equilibrium. Differentiating (13) with respect to σ yields

$$\begin{aligned} &M\underbrace{\left(\frac{d^2B(X(M,\sigma))}{dX^2} - \frac{d^2C(X(M,\sigma))}{dX^2}\right)}_{\Omega_1}\frac{\partial x(M,\sigma)}{\partial\sigma} \\ &\qquad + \sigma M\underbrace{\left(\frac{\partial^2 b(x(M,\sigma),(M-1)x(M,\sigma))}{\partial {x_i}^2}\right)}_{\Omega_2}\frac{\partial x(M,\sigma)}{\partial\sigma} \\ &\qquad\qquad \underbrace{-\frac{d^2c(x^*)}{dx_i^2}}_{\Omega_3}\frac{\partial x(M,\sigma)}{\partial\sigma} = -\frac{\partial b\left(x^*, (M-1)x^*\right)}{\partial x_i} \end{aligned} \tag{14}$$

The RHS of (14) is strictly negative, so $\frac{\partial x(M,\sigma)}{\partial\sigma} \neq 0$ for all $\sigma \geq 0$. Therefore $x(M,\sigma)$ is monotonic in σ. $\Omega_1 < 0$, $\Omega_2 \leq 0$, and $\Omega_3 < 0$. Hence the LHS is negative if and only if $\frac{\partial x(M,\sigma)}{\partial\sigma} > 0$.

As $\frac{\partial x(M,\sigma)}{\partial\sigma} > 0$, $X(M,\sigma)$ is also increasing in σ. At $\sigma = 0$,

$$\frac{dB(X(M,0))}{dX} - \frac{dC(X(M,0))}{dX} > 0,$$

so there must exist a value $\overline{\sigma}(M) > 0$ for every $M \geq 1$ such that for all $\sigma < \overline{\sigma}(M)$, property $P1$ holds and for all $\sigma > \overline{\sigma}(M)$, property $P2$ holds.

Proving that $\overline{\sigma}(M)$ is decreasing in M first requires showing that X^* is increasing in M. Fix σ and suppose, to the contrary, that X^* is decreasing in M on any subset of $\mathbb{N} \setminus \{0\}$. Then,

$$\begin{aligned} &\frac{dB\left(X(M,\sigma)\right)}{dX} - \frac{dC\left(X(M,\sigma)\right)}{dX} + \sigma\frac{\partial b\left(x(M,\sigma),(M-1)x(M,\sigma)\right)}{\partial x_i} \\ &\qquad = \frac{dc(x(M,\sigma)}{dx_i} > \frac{dc(x(M+1,\sigma)}{dx_i} \\ &= \frac{dB\left(X(M+1,\sigma)\right)}{dX} - \frac{dC\left(X(M+1,\sigma)\right)}{dX} + \sigma\frac{\partial b\left(x(M+1), Mx(M+1,\sigma)\right)}{\partial x_i}, \end{aligned}$$

a contradiction. Thus X^* is increasing in M.

Now to show that $\overline{\sigma}(M)$ is decreasing in M, suppose that $X(M,\sigma)$ is such that

$$\frac{dB\left(X(M,\sigma)\right)}{dX}-\frac{dC\left(X(M,\sigma)\right)}{dX}\leq 0$$

and

$$\frac{dB\left(X(M+1,\sigma)\right)}{dX}-\frac{dC\left(X(M+1,\sigma)\right)}{dX}<0.$$

It follows that $\sigma\leq\overline{\sigma}(M)$ but $\sigma>\overline{\sigma}(M+1)$, which implies that $\overline{\sigma}(M)$ is decreasing in M. To show that as $M\to\infty$, $\overline{\sigma}(M)\to 0$ recall that by Proposition 1, $x^*\to 0$ as $M\to\infty$, while $\lim_{M\to\infty}\frac{\partial b(x(M,\sigma),(M-1)x(M,\sigma))}{\partial x_i}-\frac{dc(x(M,\sigma))}{dx_i}>0$ for all $\sigma>0$. The result follows from $P2$. □

Proof of Proposition 2.

Proof. To prove that $M^*(N)=N$ is necessary for the public good to be produced under $P1$, suppose that $M^*(N)<N$. The first-order condition for a non-contributor is

$$\frac{dB\left(X^*\right)}{dX}-\frac{dC\left(X^*\right)}{dX}\leq 0.$$

For $\sigma>0$, $\frac{\partial b(0,X_{-i})}{\partial x}+\frac{dc(0)}{dx_i}>0$. Hence the first-order condition can only be satisfied if

$$\frac{dB\left(X^*\right)}{dX}-\frac{dC\left(X^*\right)}{dX}<0.$$

Thus $P1$ does not hold. □

Proof of Proposition 3.

Proof. The proof proceeds in two cases. Firstly suppose that $\sigma<\overline{\sigma}(N)$. By Lemma 2, $P1$ applies, and the result follows immediately from an identical argument to that of Proposition 1.

Next suppose that $\sigma>\overline{\sigma}(N)$, so by Lemma 2, $P2$ applies. Otherwise, there would exist an equilibrium with $M^*(N)<N$ under $P1$, a contradiction. The first-order conditions for a maximum are given by

$$\frac{dB\left(X^*\right)}{dX}-\frac{dC\left(X^*\right)}{dX}+\frac{\partial b\left(x^*,(M-1)x^*\right)}{\partial x_i}-\frac{dc(x^*)}{dx_i}=0,\quad\forall\, i=1,\ldots,M \tag{15}$$

while the welfare maximizing conditions with M contributors is given by

$$\frac{dB(\tilde{X})}{dX}-\frac{dC(\tilde{X})}{dX}+\frac{1}{N}\left(\frac{\partial b(\tilde{x},(M-1)\tilde{x})}{\partial x_i}-\frac{dc(\tilde{x})}{dx_i}\right)=0,\quad\forall\, i\leq M. \tag{16}$$

Given (15) and Lemma 2,

$$\frac{dB\left(X^{*}\right)}{dX}-\frac{dC\left(X^{*}\right)}{dX}+\frac{1}{N}\left(\frac{\partial b\left(x^{*},(M-1)x^{*}\right)}{\partial x_{i}}-\frac{dc(x^{*})}{dx_{i}}\right)<0.$$

Thus $x^{*}>\tilde{x}$ and by extension, $X^{*}>\tilde{X}$. Both being greater than $\hat{X}$ follows immediately. $\tilde{X}\to\hat{X}$ as $N\to\infty$ follows from taking the limit of (16) as $N\to\infty$. By Proposition 1, $x^{*}\to 0$ as $N\to\infty$. As $\lim_{x\to 0^{+}}\frac{\partial b(x,X_{-i})}{\partial x}-\frac{dc(x)}{dx_{i}}>0$ for $\sigma>0$, $\frac{dB(X(\infty,\sigma))}{dX}-\frac{dC(X(\infty,\sigma))}{dX}<0$, which implies that $X^{*}>\hat{X}$ for all $\sigma>0$. □

Proof of Corollary 1.

Proof. The proof of Corollary 1 is contained in the proof of Lemma 2. □

Proof of Proposition 4.

Proof. Suppose that $\hat{M}>\overline{\sigma}^{-1}(\sigma)$ and there exists an integer $\hat{M}-1$ such that

$$\overline{\sigma}^{-1}(\sigma)<\frac{(\hat{M}-1)+\hat{M}}{2},$$

so producing the good with $\hat{M}-1$ contributors leads to a good with a higher quality than having $\hat{M}$ contributors. It immediately follows that if (8) holds, then $\hat{M}$ leads to greater aggregate welfare than $\hat{M}-1$. □

Proof of Corollary 2.

Proof. The result follows from Lemma 2 and Proposition 4. □

Proof of Proposition 5.

Proof. Suppose that $\mu\in[0,1)$ and $\underline{Q}=-\infty$. Note that the indirect utility of each contributor is strictly increasing in σ. The utility of each free-rider is non-monotonic in σ. As σ grows large, $B\left(X(M,\sigma)\right)-C\left(X(M,\sigma)\right)$ experiences non-monotonic spikes, where $B(X)-C(X)$ will decrease. By (4), there exists a finite set of cutoff values $\sigma_1,\ldots,\sigma_Z$, where at each σ_z, an individual stops contributing. Between each cutoff value, $B(X)-C(X)$ is decreasing with a sharp jump at each cutoff point until there is a fixed number of contributors, which by By (4) and By (5) is at least one. As $B(X)-C(X)$ decreases, free-riders minimum payoff is zero given the availability of the outside option. Thus aggregate payoffs are bounded below by zero and can be made arbitrarily large by increasing σ and benefiting the contributors, while all non-contributors receive zero utility.

The last statement follows immediately from the argument above coupled with the added constraint on $\underline{Q}$. Because $B(X) - C(X)$ is strictly decreasing for σ large, imposing a minimum quality constraint requires that the set of feasible optimal parameters $\hat{\sigma}$ is compact. $\square$

Online Appendix

In the model developed in Section 2, contribution benefits induced a negative externality on the utility from consumption. For σ sufficiently large, the negative externality outweighed the positive externality leading to over-provision. This section offers an alternative framework in which there is no negative externality with respect to consumption. Quality is strictly increasing. The externality is instead integrated through provision. Contributors benefit not from the level of their contributions, but from their contributed share of total contributions.

Instead of building a new model from scratch, I augment the model from Section 2. There are no consumption costs: $C(X) = 0$ for all X. For simplicity, I assume that $b(x_i, X_{-i}) = b\left(x_i/(x_i + X_{-i})\right)$.

Hence utility is given by

$$u(x_i, X_{-i}; \sigma) = B(X) + \sigma b\left(\frac{x_i}{x_i + X_{-i}}\right) - c(x_i), \tag{17}$$

with arbitrary first-order condition

$$\frac{dB(X^*)}{dX} + \sigma \frac{\partial b\left(\frac{x_i}{x_i+X_{-i}}\right)}{\partial x_i}\left(\frac{X^* - x_i^*}{(X^*)^2}\right) - \frac{dc(x_i^*)}{dx_i} = 0.$$

By observation, $x_i^* = x_j^* = x^*$ for all i, j, so the above can be rewritten as

$$\frac{dB(Nx^*)}{dX} + \sigma \frac{\partial b\left(\frac{1}{N}\right)}{\partial x_i}\left(\frac{N-1}{N^2 x^*}\right) - \frac{dc(x^*)}{dx_i} = 0. \tag{18}$$

Rearranging the above yields

$$N^2 x^* \left(\frac{dc(x^*)}{dx_i} - \frac{dB(Nx^*)}{dX}\right) = \sigma(N-1).$$

In equilibrium, $\frac{dB(Nx^*)}{dX} < \frac{dc(x^*)}{dx_i}$. Note that as σ increases, the RHS increases without bound. Therefore, x^* is also increasing in σ without bound.

Next, consider welfare maximization. The welfare-maximizing program is given by

$$\begin{aligned}\sum_{k=1}^{N} u(x_k, X_{-k};\sigma) &= \sum_{k=1}^{N}\left[B\left(X\right)+\sigma\frac{x_i}{\sum_{j=1}^{N}x_j}-x_i\right] \\ &= NB\left(X\right)+\sigma\sum_{k=1}^{N} b\left(\frac{x_k}{x_k+X_{-k}}\right)-\sum_{k=1}^{N}c(x_k).\end{aligned} \tag{19}$$

The arbitrary first-order condition is given by

$$N\frac{dB(\tilde{X})}{dX}+\sigma\frac{\partial b\left(\frac{\tilde{x}_i}{\tilde{x}_i+\tilde{X}_{-i}}\right)}{\partial x_i}\frac{\tilde{X}_{-i}}{(\tilde{X})^2}-\sigma\sum_{k\neq i}\frac{\partial b\left(\frac{\tilde{x}_k}{\tilde{x}_k+\tilde{X}_k}\right)}{\partial X_{-k}}\frac{\tilde{x}_k}{(\tilde{X}_k)^2}-\frac{dc(\tilde{x}_i)}{dx_i}=0.$$

As in the equilibrium, $\tilde{x}_i=\tilde{x}_j=\tilde{x}$. Hence the above simplifies to

$$N\frac{dB(\tilde{X})}{dX}+\sigma\frac{\partial b\left(\frac{1}{N}\right)}{\partial x_i}\frac{(N-1)}{N^2\tilde{x}}-\sigma(N-1)\frac{\partial b\left(\frac{1}{N}\right)}{\partial X_{-k}}\frac{1}{N^2\tilde{x}}-\frac{dc(\tilde{x})}{dx_i}=N\frac{dB(\tilde{X})}{dX}-\frac{dc(\tilde{x})}{dx_i}=0,$$

Hence the solution is independent of σ. Thus for σ sufficiently large, $X^*>\tilde{X}$.

Contributors enter an arms race. In order to capture greater contribution benefits, each contributor seeks to contribute more than their peers. Given the increasing costs of contributing, each contributor imposes a negative externality on the other contributors. In equilibrium, all contributors contribute identical amounts and thus equal shares. Could the contributors commit to lower (but still equal) contribution levels, costs would be shaved, which increases total welfare.

As in the main text, contribution benefits determine over-provision versus under provision. In this case, the cost of the externality is borne by the contributors rather than the consumers. As a result, there are significant different policy implications for the class of public goods that exhibit consumption benefits, contribution costs, and contribution benefits that depend on shares rather than levels but no consumption costs.