Incentives for the Over-Provision of Public Goods *

Michael Sacks
Clarkson University

September 26, 2020

Abstract

A wide range of public goods such as open source software and regulatory policy involve both consumption costs and private benefits to contributing. This paper models such an environment. When there are consumption costs but no private benefits to provision, such as learning costs, increasing the number of contributors mitigates the free-rider problem, rather than exacerbating it. When there are both consumption costs and private benefits to contributing, increasing the number of contributors not only mitigates the free-rider problem, but leads to an over-provision problem in which both the number of contributors and the intensity of contributions are inefficiently high. When the population is large, every equilibrium yields over-provision. Lastly, welfare-maximizing policies involve transferring surpluses from consumers to producers, decreasing the utility from consumption and increasing the utility of contribution.

Keywords: congestion, free riding, negative externalities, over-provision, positive externalities, public goods, under-provision.

JEL classifications: C72, D29, D62, D71, H41.

*David D. Reh School of Business. *Email:* `msacks@clarkson.edu`. Thanks to Dan Bogart, Jean-Paul Carvalho, Jiawei Chen, Linda Cohen, and Stergios Skaperdas for invaluable comments. I also thank seminar participants at the Institute for Mathematical Behavioral Sciences at UC Irvine. All errors are my own. This paper was previously circulated as "Public and Club Goods when Private Costs have Negative Externalities" and "Given Enough Eyeballs, All Bugs are Shallow: Incentives for the Over-Provision of Public Goods." This version supersedes both.

1 Introduction

Linus Torvalds, the founding developer and namesake of Linux once said, "given enough eyeballs, all bugs are shallow (Raymond, 1999, p. 29)" to illustrate the benefits of having a large developer base in open source software, echoing the sentiment that increasing the number of contributors increases welfare by increasing the quality of the software. There is also an English proverb that says, "too many cooks spoil the broth," which echoes the opposite sentiment. Too many contributors can have a deleterious effect on the output. These two statements are seemingly at odds with one-another. This paper develops a model that is able to reconcile these two statements and more generally provides conditions under which there is over-provision of a public good rather than the well-studied under-provision (free-rider) result.

With many public goods, two features are true. First, the utility from consuming the public good is not strictly increasing in the total contribution level. For example, in open source software, there are often significant learning or switching costs when adopting a new program. These costs grow as the software becomes more complex (Sacks, 2015). Similarly, adopting public policies is often costly, with costly enforcement and compliance. In response to the Dodd-Frank Wall Street Reform and Consumer Protection Act of 2010, companies have experienced sharp increases in compliance costs (Hogan, 2019). During the COVID-19 pandemic, businesses and universities have heavily invested in personal protective equipment and barriers such as plexiglass in response to social distancing and lockdown policies. Second, there are often private benefits to contributing to a public good. When developing open source software, individuals benefit not just from use, but from contributing as well.[1] Politicians develop political capital in crafting public policy (Quandt, 1983).

Together, these features significantly alter the equilibrium structure, comparative statics, and welfare implications of the voluntary provision of public goods. For example, the free-rider

[1] These benefits can range from altruism (Andreoni, 1990), to signaling (Glazer and Konrad, 1996), to other common economic incentives (Lerner and Tirole, 2002). Lerner and Tirole (2002), Feller et al. (2005), Bitzer et al. (2007), Myatt and Wallace (2008), Fang and Neufeld (2009), and Athey and Ellison (2014), among others, have argued that the primary reasons individuals contribute to the development of open source software are not for consumption benefits, but for provision benefits, such as altruism, signaling skill, and the joy of coding.

problem can become less severe and even vanish entirely when the number of contributors increases. Over-provision then becomes a concern. Over-provision is particularly noticeable in innovation economics, for instance in the development of open source software and patent races.[2] This paper examines the equilibrium and welfare properties of voluntarily provided public goods exhibiting the above two features and illustrates how the typical policy prescriptions can lead to both unanticipated and undesirable outcomes. I then suggest how these policies may be altered to better suit this class of public goods. While the first feature (consumption costs) is sufficient, it is not necessary to drive over-provision. When private benefits depend on an individual's share rather than the level of contributions, as is common with impure altruism (Andreoni, 1990) or some types of signaling, and the marginal utility from consumption tends to zero as provision increases, over-provision can occur.[3] In the development of open source software, marginal utility tending to zero can occur due to technological constraints. While I do not discuss this setting in the main body of the paper, Appendix B formally illustrates over-provision in this setting.

When contributing to the public good is solely a burden, the free-rider problem persists, though its severity is decreasing in the number of contributors.[4] As the number of contributors increases, the decrease in each individual's contribution is outweighed by the increase in total contributions. In other words, provision is socially optimal in the limit. This result is in stark contrast to existing models, including the well studied congestion phenomenon, which often exhibits the non-monotonicity in consumption discussed above.[5]

Consumption costs, when coupled with private benefits to provision, can lead to over-provision. If the private benefits to provision are small, then the free-rider problem persists only when there are few contributors. If the private benefits to provision are large, then instead of free-riding, there is over-provision of the public good: the equilibrium contribution level is greater than the welfare-maximizing contribution level, which is greater than

[2]Patent races entail inefficient duplication of R&D costs.

[3]Holmström (1999) shows that signals are more effective when more visible.

[4]This result is in contrast to previous work such as Cornes and Sandler (1996), Scotchmer (2002), and the references therein.

[5]For examples, see Olson (1965), Brown, Jr. (1974), Arnott and Small (1994), and Cornes and Sandler (1996, ch. 8), among others.

the level that maximizes the utility from consumption. As the number of contributors increases, the over-provision result persists, even when the private benefits to provision are small. The private incentives for provision induce a negative externality on consumers of the public good, which is increasing in the number of contributors to the point that it dominates the positive externality of free-riding. Contributors do not internalize the impact of their own contributions on the consumption value of other contributors. While in contrast to the typical congestion problem, this phenomenon can be interpreted as a supply-side corollary to congestion and can aptly be thought of as *supply-side congestion*. Supply-side congestion occurs due to the strategic tension between individual and aggregate incentives. Each participant places value not only the total quality of the good, but on her own participation in its provision.

Stamelos et al. (2002) find such a relationship in open source software after analyzing quality characteristics of one hundred open source projects written for Linux. The authors show that the relationship between the size of a component of an open source application (aggregate investment) is negatively related to user satisfaction for the application (utility from consumption) as measured by a four-point defect severity scale developed by IBM. In cases where expert consumers reported only superficial errors at worst, the mean length of the program is strictly lower than in cases where (a) consumers reported either all major program functions are working but at least one minor function is disabled or incorrect, (b) at least one major function is disabled or incorrect, and (c) the program is inoperable. Their result is robust to other metrics, including the number of statements per module. The model developed here offers a mechanism explaining the results of Stamelos et al. (2002).

A social planner seeking to solve supply-side congestion will find the task to be quite difficult. Adjusting the outcome through changing the maximum number of contributors (e.g. by limiting the size of the committee contributing to the public good) is not likely to maximize the utility from consuming it even though maximizing the utility from consumption involves maximizing both the public benefits for everyone and the private benefits for contributors. Instead, the social planner opts to increase the number of contributors, decreasing both the public and private benefits each individual receives while increasing the number of individuals receiving private (contribution) benefits.

When a social planner is able to adjust the outcome through manipulating the value of the private benefits, the results are counterintuitive and in fact make the supply-side congestion externality worse. Private benefits can be manipulated, for example, via direct subsidies for contributing or, in the context of signaling, making the signal more visible (see Holmström (1999) for details). Welfare is maximized by increasing the private benefits without bound, sacrificing quality for the benefit of contributors. All surpluses are transferred to the contributors, leading to the worst possible outcome from the viewpoint of the non-contributing population. This outcome can be avoided only if the social planner can commit to mandating a minimal quality.

This paper contributes to several strands of literature, including the theory of public goods, public economics, industrial organization, and the economics of innovation. The modern literature on public goods can be traced back to Samuelson (1954), Olson (1965), and Buchanan (1965), with the literature on the private provision of public goods beginning with Bernheim (1986) and Bergstrom et al. (1986). Summaries of the core theories are given by Cornes and Sandler (1996), Sandler and Tschirhart (1997), and Scotchmer (2002). The current paper is the first to analyze the effects of consumption costs and private benefits in both the consumption and provision benefits, though some work has been conducted in the context of regulatory complexity.[6]

Quandt (1983) and Kearl (1983) show that there exists conditions such that there is an over-investment in regulation (a public good). This paper generalizes these results and broadens their applicability.. In a similar vein, Nitzan et al. (2013) argue that the increasing volume and velocity of complexity in regulation over time leads to inefficiencies induced by implementing barriers to entry that allow only the largest established firms to survive. The authors reference accounting regulations, corporate governance and securities regulations, and the market for corporate charters. Combining the results of Nitzan et al. (2013), Quandt (1983), Kearl (1983), and the corresponding citations therein with those of the current

[6]A general model of public goods incorporating non-monotonicity was developed in Mas-Colell (1980), where public projects are elements of a non-linear topological space. As such, it is impossible to define monotonic preferences. However, such a model is unable to capture the characteristics described above.

paper lead to a more complete understanding of the non-monotonic effect of complexity in regulation.[7] Galasso and Schankerman (2015) find that whether or not patents hinder subsequent innovation depends on the complexity of the industry and that negative effects are concentrated in more complex industries. If one accepts the claim that it is relatively more costly to implement IP in more complex industries, then Galasso and Schankerman (2015) can be viewed as evidence in support of the non-monotonicity argument presented above. Polborn (2008) also finds over-provision when there is hetergeneous individuals and competition between organizations (clubs) developing a public good. This paper shows that neither competition nor heterogeneity is necessary for over-provision.

This paper is particularly useful in understanding the structure and outcomes of industries (and more generally, economic actors) whose interactions are characterized by collaborative production, such as the software industry. Sacks (2015) argues that open source software developers will target the upper-end of the technology spectrum. I show more generically that collaborative groups will likely over-invest in technological investment and can also shed light on the nature of research joint ventures, building upon works such as Kamien et al. (1992) and those outlined in Caloghirou et al. (2003). Along a similar line, the model developed in the following sections provides a new alternative mechanism for understanding over-investment in patent races (Gilbert and Newbery, 1982; Fudenberg et al., 1983; Shapiro, 1985; Harris and Vickers, 1987; Aoki, 1991; Baye and Hoppe, 2003).

The remainder of the paper is structured as follows. Section 2 outlines the model. Section 3 presents the main results. Section 4 presents a normative analysis of how policy should be conducted. Section 5 concludes. The proofs are contained in Appendix A and the alternative specification discussed above is modeled in Appendix B.

2 The Model

In this section, I develop a model of the (voluntary) private provision of public goods. The model allows for benefits from both consumption and provision. There are also costs to both

[7] For examples of regulatory complexity in finance, see Berglund (2014). For examples in IP, see Allison and Lemley (2002). Evidence of a negative impact of the regulations is given in Murray and Stern (2007) and Williams (2013)

the provision and the implementation (or equivalently, consumption) of the public good. The model is used to illustrate the potential over-provision of public goods. In Appendix B, I offer an alternative model illustrating over-provision when there are private benefits based on the share of the public good contributed by an individual but no costly implementation (or costs of consumption). Instead, I assume marginal utility from consumption tends to zero as the size of the public good increases.

There are N individuals indexed by i and a single public good. Each individual i has the option to either (i) contribute at level $x_i > 0$ and consume, (ii) free-ride and consume $(x_i = 0)$, or (iii) not consume.[8] Denote by $\mathbf{x} = (x_1, \ldots, x_N)$ the contributions to the good by each individual. The contributions are aggregated according to the production function $X = g(\mathbf{x})$, which is assumed to be twice continuously differentiable and weakly concave. In addition, I assume that $\frac{\partial g(\mathbf{x})}{\partial x_i} = \frac{\partial g(\mathbf{x}')}{\partial x_j}$ for all i, j, where $\mathbf{x}'$ equals $\mathbf{x}$ with a permutation of elements i and j. This assumption imposes anonymity among the contributors.[9] All contribution decisions are made simultaneously and independently.

Each individual's utility consists of three additively-separable components. The first is the consumption benefit depending only on the total contributions public good. I refer to this component as the *gross quality*, denoted by $Q(g(\mathbf{x}))$. This function is assumed to be twice continuously differentiable and weakly concave with $Q(0) = 0$. The second component is the consumption (implementation/investment) cost that depends on the aggregate contributions: $I(q(\mathbf{x}))$. This function is assumed to be twice continuously differentiable and strictly convex with $I(0) = 0$.

Examples of costs depending only on aggregate contributions include learning costs, switching costs, and implementation or enforcement costs. For example, open source software often entails learning or switching costs (Sacks, 2015) as new users must learn how to navigate the software. Larger programs with more features often require greater time investments. Similarly there are often enforcement costs associated with the implementation of a new

[8] Allowing individuals to contribute but not consume only strengthens the results of the paper. Contributors will not internalize the utility from consumption, further increasing the intensity of contributing.

[9] It is straightforward to incorporate agents with varying productivity. If there is some agent j such that $\frac{\partial g(\mathbf{x})}{\partial x_i} < \frac{\partial g(\mathbf{x})}{\partial x_j}$, then sorting occurs where agents who are more productive contribute while those who are less productive free-ride, e.g. technologically savvy individuals and open source software.

public policy, such as compliance costs with the Dodd-Frank Act or the costs of PPE, social distancing measures (e.g. plexiglass), and testing supplies faced by individuals, businesses, governments, and universities during the COVID-19 pandemic.

I interpret $Q(g(\mathbf{x})) - I(g(\mathbf{x}))$ as the *net quality* of the public good. Let $\hat{X} \equiv \max_X Q(X) - I(X)$ denote the level of aggregate contributions that maximizes the net quality of the public good. This value maximizes the consumption benefits of the public good, though it need not maximize total welfare.

The third component is a private provision (contribution/production) benefit, which is unique to each individual contribution x_i. The private benefit depends on two elements: the individual's contribution x_i and an exogenous parameter $\sigma \geq 0$. If $\sigma = 0$, then contributing is viewed as a burden and any benefits from contribution arise solely from the good's quality, so the private provision benefit reduces to the typical cost function. If $\sigma > 0$, then individuals benefit not only from the good's net quality, but also from contributing to its provision. For example, when developing open source software, contributors benefit from contributing for reasons such as signaling ability.[10] Thus σ determines the relative strength of the private benefit. Denote the private benefit function by $v\left(x_i;\sigma\right)$, which is assumed to be twice continuously differentiable with $v(0;\sigma) = 0$ for all σ. In addition, I assume that $\frac{\partial v(x_i;\sigma)}{\partial \sigma} > 0$, $\frac{\partial^2 v(x_i;\sigma)}{\partial x_i \partial \sigma} > 0$, $\frac{\partial^2 v(x_i;\sigma)}{\partial x_i^2} < 0$, and if $\sigma > 0$, then there exists a value $\overline{x}(\sigma) > 0$ with $\frac{d\overline{x}(\sigma)}{d\sigma} > 0$ and $\overline{x}(0) = 0$ such that $\frac{\partial v(x_i;\sigma)}{\partial x_i} > 0$ for all $x_i < \overline{x}(\sigma)$ and $\frac{\partial v(x_i;\sigma)}{\partial x_i} < 0$ for all $x_i > \overline{x}(\sigma)$.

Therefore, individual i's utility is

$$u\left(X, x_i;\sigma\right) = \underbrace{Q\left(g\left(\mathbf{x}\right)\right) - I\left(g\left(\mathbf{x}\right)\right)}_{\textbf{utility from consumption}} + \underbrace{v\left(x_i;\sigma\right)}_{\substack{\textbf{utility from}\\ \textbf{provision}}}\ . \tag{1}$$

The utility from not consuming the public good is normalized to zero. I utilize the Nash Equilibrium solution concept.

[10]See, e.g. Lerner and Tirole (2002). Other examples of private benefits to provision include signaling wealth (Glazer and Konrad, 1996), forms of impure altruism (Andreoni, 1990), or increases in productivity via learning-by-doing (Irwin and Klenow, 1996; Bramoullé and Kranton, 2007).

3 Analysis

As individuals are anonymous, I drop the individual-indexing subscripts. Denote by x^* the individual equilibrium contribution to the public god and by $X^* = g(\mathbf{x}^*)$ the aggregate equilibrium contribution. I order the individuals by contributors followed by non-contributors, so if $M < N$ individuals contribute, individuals $1, \ldots, M$ are the contributors and individuals $M+1, \ldots, N$ are the non-contributors.

The equilibrium contributions are defined by the following system of first-order necessary conditions:

$$\begin{aligned} \left(\frac{dQ(g(\mathbf{x}^*))}{dX} - \frac{dI(g(\mathbf{x}^*))}{dX}\right)\frac{\partial g(\mathbf{x}^*)}{\partial x} + \frac{\partial v(x^*;\sigma)}{\partial x} &= 0, \quad \forall\, i = 1, \ldots, M \\ \left(\frac{dQ(g(\mathbf{x}^*))}{dX} - \frac{dI(g(\mathbf{x}^*))}{dX}\right)\frac{\partial g(\mathbf{x}^*)}{\partial x} + \frac{\partial v(0;\sigma)}{\partial x} &\leq 0, \quad \forall\, i = M+1, \ldots, N. \end{aligned} \tag{2}$$

By the implicit function theorem, $x^* = x(M,\sigma)$, $\mathbf{x}^* = \mathbf{x}(M,\sigma)$, and $X^* = X(M,\sigma)$ and by the theorem of the maximum, $\frac{\partial x(M,\sigma)}{\partial \sigma}$ exists and is well defined.[11] There are two candidate unilateral deviations to consider. Denote by $\mathbf{x}'$ the vector of contributions in which $M-1$ individuals each contribute $x(M,\sigma)$ and the remaining $N-M+1$ individuals each contribute zero. Denote by $\mathbf{x}''$ the vector of contributions in which M individuals each contribute $x(M,\sigma)$, $N-M-1$ individuals each contribute zero, and one individual contributes a value $x'' > 0$ satisfying

$$\left(\frac{dQ(g(\mathbf{x}''))}{dX} - \frac{dI(g(\mathbf{x}''))}{dX}\right)\frac{\partial g(\mathbf{x}'')}{\partial x} + \frac{\partial v(x'';\sigma)}{\partial x} = 0.$$

There exists an equilibrium in which exactly M individuals contribute if

$$Q(g(\mathbf{x}^*)) - I(g(\mathbf{x}^*)) + v(x^*;\sigma) \geq \max\{Q(g(\mathbf{x}')) - I(g(\mathbf{x}')), 0\} \tag{3}$$

$$\max\{Q(g(\mathbf{x}^*)) - I(g(\mathbf{x}^*)), 0\} \geq Q(g(\mathbf{x}'')) - I(g(\mathbf{x}'')) + v(x'';\sigma). \tag{4}$$

Inequality (3) states that there is no profitable deviation in which an individual stops contributing and (4) states that there is no profitable deviation in which a non-contributor starts contributing. Inequality (4) need only hold when $M < N$; otherwise, such a deviation does

[11] For $\sigma = 0$, define the derivative as $\lim_{\Delta \to 0^+} \frac{x(M,\Delta) - x(M,0)}{\Delta}$.

not exist. Denote by $M^*(N)$ the values of M such that $\mathbf{x}^* = \mathbf{x}(M^*(N), \sigma)$ constitutes a Nash equilibrium.[12] The second-order conditions are globally satisfied by the strict quasiconcavity of each $u(X, x_i; \sigma)$.

For comparison, it is necessary to characterize the welfare-maximizing contributions. I use $\tilde{x}$ and $\tilde{X}$ to denote the individual and aggregate welfare-maximizing contributions. As in the equilibrium case, $\tilde{x}$, $\tilde{\mathbf{x}}$, and $\tilde{X}$ can all be represented by functions of M, N, and σ. Denote by $\tilde{M}(N)$ the values of M that maximize welfare.[13] In what follows, $M^*(N)$ and $\tilde{M}(N)$ are used to denote the number of equilibrium and welfare-maximizing contributors while M will be used to represent an arbitrary number of contributors.

Proposition 1. *If $\sigma = 0$, then in equilibrium, $M^*(N) = \tilde{M}(N) = N$. For all finite $N > 1$, $X^* < \tilde{X} < \hat{X}$ and $\lim_{N\to\infty} X^* = \lim_{N\to\infty} \tilde{X} = \hat{X}$.*

The proof, and all subsequent proofs, are contained in Appendix A. When contributing is solely a burden, free-riding persists and neither the equilibrium nor the welfare maximizing contributions maximize the net quality of the public good. On the other hand both the NE and welfare-maximizing contributions entail maximizing the net quality as the size of the population grows large. Proposition 1 also illustrates the existence of a symmetric NE when contributing is solely a burden. There are two implications. First, and consistent with previous work, the typical free-rider problem persists. Second, and in contrast to previous work, as N increases the free-rider problem becomes less severe rather than more severe. The Nash equilibrium converges to maximizing both welfare and net quality. While Proposition 1 is somewhat obvious given the single-peakedness of net quality and the monotonicity of the cost of contributing, it nonetheless offers a useful benchmark for comparison because this relationship need not hold when private benefits are introduced.

Now suppose that $\sigma \geq 0$. In any equilibrium with $M^*(N)$ contributors, (2) and (3) must hold for each of the $M^*(N)$ contributors and (4) holds for the $N - M^*(N)$ free-riders. One of three mutually exclusive and exhaustive equilibrium properties must hold.

[12] Existence of $M^*(N)$ is proven in Lemma 2.
[13] Formal derivations are provided in the proof of Proposition 1.

Property 1. **(P1)** $\frac{dQ(g(\mathbf{x}(M;\sigma)))}{dX} > 0$ *and* $\frac{\partial v(x(M,\sigma);\sigma)}{\partial x} < 0$.

Property 2. **(P2)** $\frac{dQ(g(\mathbf{x}(M;\sigma)))}{dX} < 0$ *and* $\frac{\partial v(x(M,\sigma);\sigma)}{\partial x} > 0$.

Property 3. **(P3)** $\frac{dQ(g(\mathbf{x}(M;\sigma)))}{dX} = \frac{\partial v(x(M,\sigma);\sigma)}{\partial x} = 0$.

Lemma 1. *There exists a decreasing function* $\overline{\sigma}(M) : \mathbb{N} \setminus \{0\} \to \mathbb{R}_+ \cup \{0\}$ *such that if* $\sigma < \overline{\sigma}(M)$*, then* $P1$ *is satisfied; if* $\sigma > \overline{\sigma}(M)$*, then* $P2$ *is satisfied; and if* $\sigma = \overline{\sigma}(M)$*, then* $P3$ *is satisfied. As* $M \to \infty$*,* $\overline{\sigma}(M) \to 0$.

Lemma 1 shows that there exists parameter values such that either $P1$ or $P2$ can occur in equilibrium.[14] When σ is small, the typical under-provision result persists. On the other hand, over-contributing relative to the quality-maximizing contribution level occurs when the private benefit from contributing is sufficiently large ($\sigma > \overline{\sigma}(M)$). Private incentives in the provision of public goods impose a negative externality on consumption, which can be severe enough to induce over-investment. As a consequence, some individuals may choose to free-ride or consume the outside option, implying that no symmetric equilibrium exists even though the environment itself is *ex ante* symmetric.

Denote by $M^U \in \mathbb{R}_+$ the M that makes (3) hold with equality. The maximum number of contributors is given by $\lfloor M^U \rfloor$. If there are M contributors and $\sigma < \overline{\sigma}(M)$, then (4) cannot be satisfied as both $Q(\cdot)$ and $v(\cdot;\sigma)$ are increasing for small x_i. Thus the equilibrium must satisfy either $P2$ or $P3$. If $\sigma \leq \overline{\sigma}(N)$, then all N individuals contribute. If the public good is developed under $P2$ or $P3$, then the value of M that makes (3) hold with equality (M^U) indicates the upper bound on the number of contributors in equilibrium. To see this relationship, note that under $P2$ and $P3$, $v(x(M,\sigma);\sigma)$ is decreasing in M while $Q\left(g\left(\mathbf{x}'\right)\right) - Q\left(g\left(\mathbf{x}(M,\sigma)\right)\right)$ is increasing in M, so (3) is violated for all $M > M^U$. The value of M, denoted by M^L, that makes (4) hold with equality indicates the lower bound on the number of contributors in equilibrium. It follows that (4) is violated for all $M < M^L$. This result is stated formally below.

[14] $P3$ occurs only if single value of $\sigma = \overline{\sigma}(M)$, so I focus on $P1$ and $P2$.

Lemma 2. *The following statements are true.*

(i) There is at most one M^U *such that* (3) *holds with equality.*

(ii) There is at most one M^L *such that* (4) *holds with equality.*

(iii) If (3) *holds with equality, then* (4) *is strictly satisfied.*

(iv) If (4) *holds with equality, then* (3) *is strictly satisfied.*

(v) $M^U - M^L \geq 1$.

The implications of Lemma 2 are twofold. First, there can be no Nash equilibrium with $M \notin \left[M^L, M^U\right]$. Second, this range must contain an integer value, guaranteeing the existence of a Nash equilibrium. Consequently in equilibrium, the maximal number of contributors is determined by $\min\left\{\lfloor M^U \rfloor, N\right\}$. If $N > M^U$, then $N - \lfloor M^U \rfloor$ individuals do not contribute in equilibrium. The correspondence

$$M^*(N) : N \to \left\{Z \in \mathbb{N} \,|\, M^L \leq Z \leq \min\left\{M^U, N\right\}\right\}$$

defines the equilibrium number of contributors as a function of the population size.

Note that $|M^*(N)| \in \{1, 2\}$. That is, there may in fact exist two equilibria due to there being a kink in the best response function at $x_i = 0$, where an individual can choose to either free-ride, consume or not consume. Depending on the relative slopes of the three components of utility, as x_i approaches zero there may exist a situation in which there are two sequential values of M such that there is no unilateral incentive to deviate.

Proposition 2. *Suppose that* $\sigma > 0$. *For every* $M^*(N)$, *there exists a Nash equilibrium with* $M^*(N)$ *contributors, each contributing* $x^* = x(M^*(N), \sigma)$. *Furthermore,* $M^*(N) = N$ *is necessary for the public good is produced under* $P1$.

While at least one and at most two equilibria may exist, the equilibria share the same characteristics so the analysis can be conducted as if $M^*(N)$ is unique. Although the members of the population are *ex ante* symmetric, the equilibria itself need not be completely symmetric due to the fact that players can choose not only to free-ride, but not consume the good entirely.

Unlike the case with $\sigma = 0$, the equilibrium contribution $X\left(M^*(N), \sigma\right)$ converges neither to the welfare-maximizing nor quality-maximizing contribution level. The relationships between the equilibrium contributions and the quality-maximizing contributions for any $\sigma \geq 0$ are outlined in the following proposition, which is the main result of this section.

Proposition 3. *If $\sigma < \overline{\sigma}(N)$, then $X^* < \tilde{X} < \hat{X}$ and if $\sigma > \overline{\sigma}(N)$, then $x^* > \tilde{X} > \hat{X}$. Optimal provision only occurs if $M^*(N) = 1$. Furthermore, $\lim_{N\to\infty} X^* = \lim_{N\to\infty} \tilde{X} = \hat{X}$ if and only if $\sigma > \overline{\sigma}(N)$.*

When the private benefits are small ($\sigma < \overline{\sigma}(N)$), there is under-provision and the free-rider problem persists and when the private benefits are large ($\sigma > \overline{\sigma}(N)$), supply-side congestion occurs. There are "too many cooks in the kitchen" and a crowding effect where the private benefits from provision induce a significant level of contributions. These contributions induce significant costs to consumption (via $I(X)$), which are not fully internalized by the contributors. If $\sigma = \overline{\sigma}(M^*(N))$, then by $P3$ the equilibrium contribution equals the welfare-maximizing contribution, which equals the quality-maximizing contribution. The following corollary emerges as a consequence.

Corollary 1. *If $\sigma' > \sigma''$, then $g\left(\mathbf{x}\left(M^*(N), \sigma'\right)\right) > g\left(\mathbf{x}\left(M^*(N), \sigma''\right)\right)$.*

In words, the aggregate equilibrium contribution is increasing in σ, which implies that as M becomes increasingly large, the distance between the equilibrium contribution and the quality-maximizing contribution is increasing in the relative strength of the private benefit.

Regardless of σ, the welfare-maximizing contribution converges to the quality-maximizing contribution as the size of the population grows; however, in contrast to Proposition 1, when $\sigma > 0$ the equilibrium contribution does not converge to the welfare-maximizing contribution. Instead, contributions converge to a level strictly greater than the welfare-maximizing one.

This result follows from Lemma 1 as $\overline{\sigma}(M)$ tends to zero when M becomes large, which implies that the equilibrium always falls under $P2$. Thus there is always over-provision in the limit whenever $\sigma > 0$. The lack of convergence in $X\left(M^*(N), \sigma\right)$ follows from the negative externality imposed by $\sigma > 0$ dominating the positive externality (free-riding). In other words, net quality is monotonically increasing in the number of contributors whenever $\sigma = 0$. If $\sigma > 0$, then the relationship between quality and the number of contributors is non-monotonic.

4 Policy Perspectives

Depending on the context, there may be many options available to policymakers. For both tractability and concreteness, I assume that policymakers, henceforth referred to as social planners, cannot directly affect contributions, but can indirectly affect them through two channels: the number of contributors M and the private benefit parameter σ. Call the first channel the contribution channel and the second the benefits channel. Policies that manipulate the number of contributors induce changes on the intensive margin via changes on the extensive margin. Policies that manipulate the private benefit parameter induce changes on the extensive margin via altering decisions on the intensive margin. This section is divided into two subsections. The first outlines the feasible welfare-maximizing policies through the contribution channel. The second outlines the welfare-maximizing policies through the benefits channel.

4.1 Contribution Channel

Suppose that the social planner is only able to selectively restrict contributions on the extensive margin by fixing a maximum number of contributors, thereby rendering (3) and (4) obsolete. While setting the number of contributors to maximize the quality of the public good will maximize the utility of any individual contributor and any individual free-rider, a social planner interested in maximizing total welfare rarely has the incentive to maximize net quality. Instead, a social planner can shift the number of contributors and non-contributors and transfer free-rider surpluses from the net quality to contributors in the form of increasing private benefits. This outcome is a generalization of regulatory capture.

The objective of the social planner is given by

$$\max_{M}\left\{(N-M)\max\left\{Q\left(g\left(\mathbf{x}\right)\right)-I\left(g\left(\mathbf{x}\right)\right),0\right\}\right.$$
$$\left.+\sum_{j=1}^{M}\left[Q\left(g\left(\mathbf{x}\right)\right)-I\left(g\left(\mathbf{x}\right)\right)+v\left(x_j;\sigma\right)\right]\right\}, \quad (5)$$

subject to

$$\begin{aligned} x_j &= \arg\max\left\{f\left(Q\left(g\left(\mathbf{x}\right)\right)-I\left(g\left(\mathbf{x}\right)\right)\right)+v\left(x_j;\sigma\right)\right\} \quad \forall\, j\le M \\ x_j &= 0 \quad \forall\, j>M \\ \mathbf{x} &= \left(x_1,\ldots,x_N\right) \end{aligned} \quad (6)$$

where

$$f\left(Q\left(g\left(\mathbf{x}\right)\right)-I\left(g\left(\mathbf{x}\right)\right)\right)\in\left\{Q\left(g\left(\mathbf{x}\right)\right)-I\left(g\left(\mathbf{x}\right)\right),\right.$$
$$\left.(N-M)\max\left\{Q\left(g\left(\mathbf{x}\right)\right)-I\left(g\left(\mathbf{x}\right)\right),0\right\}+M(Q\left(g\left(\mathbf{x}\right)\right)-I\left(g\left(\mathbf{x}\right)\right))\right\}$$

and $M\in\{1,\ldots,N\}$. The function $f(\cdot)$ allows for comparisons with both the equilibrium contribution levels and a first-best solution along the intensive margin. Denote by $\hat{M}$ the solution to (5).

Firstly note that there can be no interior solution $\hat{M}<\overline{\sigma}^{-1}(\sigma)$ such that

$$\overline{\sigma}^{-1}(\sigma)<\frac{\hat{M}+\left(\hat{M}+1\right)}{2};$$

otherwise, adding one more contributor increases the net quality for the free-riders and both the net quality and private benefits for the original $\hat{M}$ contributors while also increasing the utility of the new contributor by the marginal change in the net quality plus the entirety of the private benefit, leading to a strict welfare gain.[15] Counter-intuitively, this logic cannot be extended to the other side of $\overline{\sigma}^{-1}(\sigma)$. Suppose that $\hat{M}>\overline{\sigma}^{-1}(\sigma)$ and there exists a value $\hat{M}-1$ such that

$$\overline{\sigma}^{-1}(\sigma)<\frac{\left(\hat{M}-1\right)+\hat{M}}{2}.$$

[15]If $\sigma=0$, then the private benefit is a net loss on the new contributor, but by (4), the marginal impact of the loss is less than the marginal increase in net quality leading to a positive change in utility.

By decreasing $\hat{M}$ to $\hat{M}-1$, the net quality increases, as does the value for each of the $\hat{M}$ contributors. Yet, this change may actually lead to a decrease in total welfare if the change in net quality, weighted by the number of consumers plus the change in private benefits, weighted by the number of contributors, is less than the loss in private benefit by the $\hat{M}^{th}$ contributor:

$$\begin{aligned}(N-M)\Big[&\max\left\{Q\left(g\left(\mathbf{x}\left(\hat{M}-1,\sigma\right)\right)\right)-I\left(g\left(\mathbf{x}\left(\hat{M}-1,\sigma\right)\right)\right),0\right\}\\&-\max\left\{Q\left(g\left(\mathbf{x}\left(\hat{M},\sigma\right)\right)\right)-I\left(g\left(\mathbf{x}\left(\hat{M},\sigma\right)\right)\right),0\right\}\Big]\\&+M\Big[Q\left(g\left(\mathbf{x}\left(\hat{M}-1,\sigma\right)\right)\right)-I\left(g\left(\mathbf{x}\left(\hat{M}-1,\sigma\right)\right)\right)\\&-\left[Q\left(g\left(\mathbf{x}\left(\hat{M},\sigma\right)\right)\right)-I\left(g\left(\mathbf{x}\left(\hat{M},\sigma\right)\right)\right)\right]\Big]\\&+\left(\hat{M}-1\right)\left[v\left(x\left(\hat{M}-1,\sigma\right);\sigma\right)-v\left(x\left(\hat{M},\sigma\right);\sigma\right)\right]\\&<v\left(x\left(\hat{M},\sigma\right);\sigma\right).\end{aligned}\tag{7}$$

Increasing the net quality of the good by decreasing the number of contributors increases the utility for each contributor and each free-rider, but may decrease welfare due to the decrease in the utility of the contributor turned free-rider. Thus depending on how welfare is defined, there are different policy protocols.

Proposition 4. *If* (7) *is satisfied, then the optimal strategy for a social planner interested in maximizing total welfare need not match the optimal strategy for a social planner interested in maximizing quality.*

Proposition 4 has bite due to the fact that maximizing net quality is akin to currently maximizing both the public and private benefits for any given contributor. A tension remains on the extensive margin, whereby adding an additional contributor may decrease both the net quality for all consumers and the private benefit for the original contributors, but increases the new contributor's payoff enough that aggregate welfare of the economy is increased. Even if there is an integer M such that $M=\overline{\sigma}^{-1}(\sigma)$, it may not be the welfare maximizer. In other words, supply-side congestion exists on both the intensive and extensive margins. It merits mentioning that the differences in outcomes described in Proposition 4 do not violate the Warr/BBV neutrality theorem, as the change is being made along the extensive margin and is thus not net-zero.[16] Thus the theorem does not apply in this type of environment.

Regardless of the type of contributor, the above shows that a social planner is willing to sacrifice the utility of a consumer to increase the utility of a contributor in order to maximize total welfare.

Corollary 2. *If the set of potential contributors is a strict subset of the population and σ is small, then for a finite population, a social planner seeking to maximize total welfare will never choose the maximum number of contributors that maximizes quality.*

Corollary 2 provides a general extension of the results in Kearl (1983), Quandt (1983), Nitzan et al. (2013), and the other associated works on regulatory complexity.

4.2 Benefits Channel

Now suppose that the social planner can only influence the individuals' benefit parameter σ. For example, that the private benefit of provision often occurs through signaling.[17] As shown in Holmström (1999), signals become more valuable as they become more observable. Thus a social planner can increase σ by making contributions more publicly visible, e.g. by publishing names and posting plaques, or alternatively decrease σ by keeping all contributions anonymous.

A social planner interested in maximizing total welfare by manipulating the benefits parameter does so by counterintuitive means. Rather than adjusting σ such that the public good is provided as close to $P3$ as is possible due to the integer nature of contributing (on the extensive margin), the social planner can unboundedly increase welfare by increasing the value of the private benefits at the expense of decreasing the net quality, transferring all surpluses to the contributors and leading all non-contributors to not consume. The remainder of the section is devoted to detailing the incentives leading to this result and providing an alternative welfare maximizing program.

[16]See Warr (1983) and Bergstrom et al. (1986) for details on the neutrality theorem.

[17]For examples, see the references in the Introduction and Section 2.

The social planner's objective function is given by

$$\max_{\sigma \geq 0} \Bigg\{ \mu \left(N - M\right) \max \left\{ Q \left(g \left(\mathbf{x}(M, \sigma)\right)\right) - I \left(g \left(\mathbf{x}(M, \sigma)\right)\right), 0 \right\} + (1 - \mu) \sum_{j=1}^{M} \left[Q \left(g \left(\mathbf{x}(M, \sigma)\right)\right) - I \left(g \left(\mathbf{x}(M, \sigma)\right)\right) + v \left(x_j(M, \sigma); \sigma\right) \right] \Bigg\}, \quad (8)$$

subject to

$$\begin{aligned} x_j(M, \sigma) &= \arg\max \left\{ f \left(Q \left(g \left(\mathbf{x} \right) \right) - I \left(g \left(\mathbf{x} \right) \right) \right) + v \left(x_j; \sigma \right) \right\} \quad \forall j \leq M \\ M &\in \left\{ M^*(N), \tilde{M}(N) \right\}, \end{aligned}$$

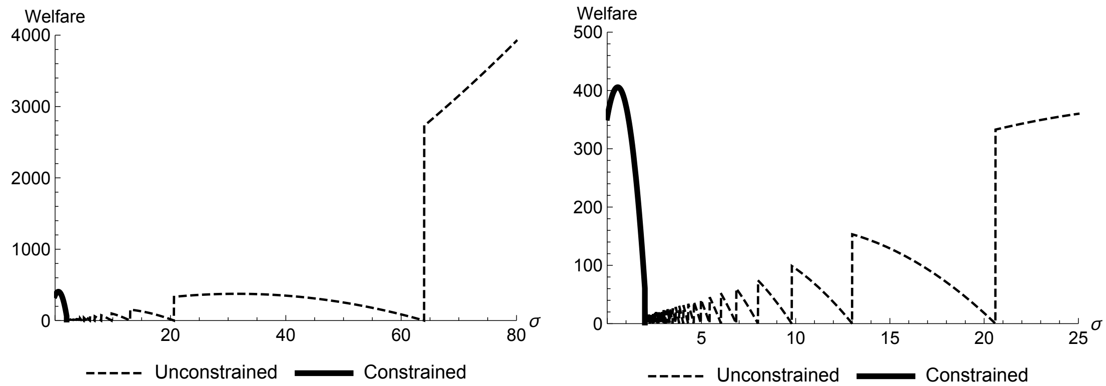
where $\tilde{M}(N)$ is the welfare-maximizing number of contributors and $\mu \in [0, 1]$ is the weight placed on the non-contributors and contributors, respectively. Although σ is continuous and $v(x; \sigma)$ is continuously differentiable with respect to σ, (8) is not continuously differentiable in σ. The objective function has discontinuous jumps in σ because $M^*(N)$ is a discontinuous function of σ (as it must be an integer).[18]

By inspection of (3) and (4), $\lceil M^L \rceil$ and $\lfloor M^U \rfloor$, are decreasing step functions of σ. As σ increases, each contributor responds by contributing a greater amount. Initially, this movement is smooth, first increasing net quality and then decreasing net quality. Increases in individual contributions cause net quality to decrease at an increasing rate and private benefits to increase at a decreasing rate. There reaches a point where an increase in σ induces a contributor to no longer contribute or consume. This change on the extensive margin decreases total contributions, leading to a discrete increase in net quality. Further increases in σ cause this process to repeat itself, inducing another contributor to stop contributing and consuming, leading to another positive jump in each remaining contributor's contribution. From (4) that there always exists the incentive for at least one contributor. Therefore, for at least one contributor,

$$\left(\frac{dQ \left(g \left(\mathbf{x}(M, \sigma) \right) \right)}{dX} - \frac{dI \left(g \left(\mathbf{x}(M, \sigma) \right) \right)}{dX} \right) \frac{\partial g \left(\mathbf{x}(M, \sigma) \right)}{\partial x} + \frac{\partial v \left(x(M, \sigma); \sigma \right)}{\partial x} = 0$$

with $x(M, \sigma) > 0$ and $M \in \left\{ M^*(N), \tilde{M}(N) \right\}$. Thus the process stabilizes to a fixed number of contributors in the limit (for σ large). As $u(X, x; \sigma)$ is strictly concave in contributions

[18]The same holds true with respect to maximizing welfare along the intensive margin.



(a) Full figure.

(b) Zoomed region.

Figure 1: Welfare maximization, Example 1.

and $v(x;\sigma)$ is strictly increasing in σ, indirect utility is also increasing in σ for σ large. Because the non-contributors' utilities are bounded below by zero, welfare can be made arbitrarily large by increasing σ. Thus the objective function in (8) takes a unique shape. It is initially smooth and hump-shaped, then taking on a sawtooth pattern with jumps representing points at which a contributor changes her decision on the extensive margin followed by smooth decreases due to increases in contributions. This pattern repeats until the process stabilizes with a final set of contributors, at which point the total welfare is strictly increasing and concave in σ. While this pattern is general, a specific example is provided below and is visualized in Figure 1.

There are two options available to the social planner that lead to a solution that does not transfer all surpluses to the contributors: force consumption or directly consider net quality in the optimization program.[19] The second solution is both more efficient and practical than the first, so I focus attention on this option.[20] A natural consideration is to ensure that the net quality is no worse than the outside option, which is accomplished by adding the condition that $Q\left(g\left(\mathbf{x}\left(M,\sigma\right)\right)\right)-I\left(g\left(\mathbf{x}\left(M,\sigma\right)\right)\right)\geq\underline{Q}$ for some $\underline{Q}\in\mathbb{R}\cup\{-\infty\}$ to (8). The most natural value is $\underline{Q}=0$, though the optimization can be conducted with any feasible $\underline{Q}$. I conclude the section with the formal statement of the result and an illustrative example.

[19]Forcing consumption will lead to a completely symmetric equilibrium with N contributors.

[20]The notion of quality control has been, for the most part, unexplored in the previous literature.

Proposition 5. *For every $\mu \in [0, 1)$ and $\underline{Q} = -\infty$, a social planner can make total welfare arbitrarily large by increasing σ. For $\mu = 1$ and $\underline{Q} = -\infty$, the optimal policy is arbitrary in the excludable case and is given by $\hat{\sigma} = \overline{\sigma}\left(M^*(N)\right)$ if contributors are individually rational and $\hat{\sigma} = \overline{\sigma}(\tilde{M}(N))$ if contributors are team players in the non-excludable case. For every finite $\underline{Q} \leq Q(\hat{X})$ and $\mu \in [0, 1]$, there exists a finite $\hat{\sigma} \geq 0$ that maximizes total welfare.*

There are two effects at work: a direct effect and an indirect effect. Increasing σ increases the benefits of all contributors. However, by Corollary 1, increasing the benefits decreases quality through increasing the aggregate contributions. Thus non-contributors are impacted negatively by such a policy. Because non-contributors and free-riders will eventually choose the outside option, a large enough increase in the private benefits will allow the net effect to grow to be arbitrarily large and positive. The following example illustrates this result.

Example 1. *Consider the utility function*

$$u(X, x_i; \sigma) = \left(\sum_{j=1}^{N} x_j\right) - \frac{1}{40}\left(\sum_{j=1}^{N} x_j\right)^2 + \sigma x_i - \frac{1}{2}x_i^2.$$

Panel (A) of Figure 1 plots the welfare objective function for $\mu = \frac{1}{2}$ and panel (B) zooms in towards the origin to provide a more detailed view. The constrained line corresponds to $\underline{Q} = 0$ and the unconstrained line corresponds to $\underline{Q} = -\infty$. Each jump corresponds to an investor switching to not-invest. The final jump, which occurs at $\sigma \approx 64$, corresponds to the point at which the number of contributors settles. In the quadratic case, there is always a minimum of five contributors.

5 Discussion and Concluding Remarks

This paper has outlined several novel and empirically relevant results. When extending the standard theories of public goods to incorporate private benefits to provision and costs of consumption (such as investment, learning, or search costs), the standard equilibrium and comparative static results may no longer apply. While inefficiencies due to free-riding may still persist, over-provision due to supply-side congestion also presents a concern, but is not dealt with in the same manner as free-riding. Over-provision is made more severe by large populations of individually rational individuals. Moreover, policy prescriptions typically used to internalize the externalities driving these inefficiencies no longer have the same desirable properties. Without further consideration, maximizing welfare by controlling the number of contributors can lead to a slight deterioration of the net quality to benefit the contributors, while maximizing welfare by controlling the private benefits leads to a severe deterioration in the net quality to benefit the contributors unless the policymaker also implements a minimum-quality requirement. The notion of quality control has thus far been under explored and is an open area for future work.

While a few assumptions were included in the model for tractability, it is worth noting that heterogeneity can be introduced into the model without qualitatively changing results. For example, suppose that heterogeneity is introduced such that the ratio $\frac{\partial g(\mathbf{x}^k)}{\partial x_i^k} / \frac{\partial g(\mathbf{x}^k)}{\partial x_j^k} = \frac{\eta_i}{\eta_j}$ for some positive constants η_i and η_j. It is straightforward to see that the only change is in the first-order conditions (for both the equilibrium and welfare-maximizing program), where the marginal effect of a change in contributions on the quality function is scaled. The resulting values are scaled accordingly, where individuals with a greater η contribute more, but their relationships remain qualitatively unchanged (i.e., the welfare-maximizing contribution is closer to the quality-maximizing contribution than the equilibrium contribution for each i). Notation would be more cumbersome but no new insights would be gained. Thus the model is fully capable of capturing various types of individuals such as high-productivity contributors and low-productivity contributors.

An interesting extension to consider is the dynamic aspects of public good provision. Regulations, statutes, and other legal guidelines such as tax codes tend to evolve over time with contributors (regulators) who have their own private and public incentives. The quality-enhancing properties of early contributions can induce a feedback loop that increases the contributions made in these regulations over time. The early contributions may lead to improvements in the quality of these regulations, but over time, these improvements diminish and eventually have deleterious effects, decaying the quality of these regulations. A similar story can be told with respect to open source software. Such a dynamic analysis focusing on the efficiency of provision over time presents an interesting avenue for future research.

References

Allison, John R. and Mark A. Lemley (2002), 'The growing complexity of the united states patent system', *Boston University Law Review* **82**(1), 77–144.

Andreoni, James (1990), 'Impure altruism and donations to public goods: A theory of warm-glow giving', *The Economic Journal* **100**(401), 464–477.

Aoki, Reiko (1991), 'R&D competition for product innovation: An endless race', *American Economic Review* **81**(2), 252–256.

Arnott, Richard and Kenneth Small (1994), 'The economics of traffic congestion', *American Scientist* **82**(5), 446–455.

Athey, Susan and Glenn Ellison (2014), 'Dynamics of open source movements', *Journal of Economics and Management Strategy* **23**(2), 294–316.

Baye, Michael and Heidrun Hoppe (2003), 'The strategic equivalence of rent-seeking, innovation, and patent-race games', *Games and Economic Behavior* **44**(2), 217–226.

Berglund, Tom (2014), 'Incentives for complexity in financial regulation', *Journal of Risk Finance* **15**(2), 102–109.

Bergstrom, Theodore, Lawrence Blume and Hal Varian (1986), 'On the private provision of public goods', *Journal of Public Economics* **29**(1), 25 – 49.

Bernheim, B. Douglas (1986), 'On the voluntary and involuntary provision of public goods', *The American Economic Review* **76**(4), pp. 789–793.

Bitzer, Jürgen, Wolfram Schrettl and Phillip J.H. Schröder (2007), 'Intrinsic motivation in open source software development', *Journal of Comparative Economics* **35**(1), 160–169.

Bramoullé, Yann and Rachel Kranton (2007), 'Public goods in networks', *Journal of Economic Theory* **135**(1), 478–494.

Brown, Jr., Gardner (1974), 'An optimal program for managing common property resources with congestion externalities', *Journal of Political Economy* **82**(1), 163–173.

Buchanan, James M. (1965), 'An economic theory of clubs', *Economica* **32**(125), 1–14.

Caloghirou, Yannis, Stavros Ionnides and Nicholas S. Vonortas (2003), 'Research joint ventures', *Journal of Economic Surveys* **17**(4), 541–570.

Cornes, Richard and Todd Sandler (1996), *The Theory of Externalities, Public Goods, and Club Goods*, 2 edn, Cambridge University Press.

Debreu, Gerard (1952), 'A social equilibrium existence theorem', *Proceedings of the National Academy of Sciences* **38**(10), 886–893.

Fan, Ky (1952), 'Fixed-point and minimax theorems in locally convex topological linear spaces', *Proceedings of the National Academy of Sciences* **38**(2), 121–126.

Fang, Yulin and Derrick Neufeld (2009), 'Understanding sustained participation in open source software projects', *Journal of Management Information Systems* **25**(4), 9–50.

Feller, Joseph, Brian Fitzgerald and Karim Lakhani, eds (2005), *Perspectives on Open Source Software*, MIT Press, London, England.

Fudenberg, Drew, Richard Gilbert, Joseph Stiglitz and Jean Tirole (1983), 'Preemption, leapfrogging and competition in patent races', *European Economic Review* **22**(1), 3–31.

Galasso, Alberto and Mark Schankerman (2015), 'Patents and cumulative innovation: Causal evidence from the courts', *Quarterly Journal of Economics* **130**(1), 317–369.

Gilbert, Richard and David Newbery (1982), 'Preemptive patenting and the persistence of monopoly', *American Economic Review* **72**(3), 514–526.

Glazer, Amihai and Kai Konrad (1996), 'A signaling explanation for private charity', *American Economic Review* **86**(4), 1019–1028.

Glicksberg, Irving (1952), 'A further generalization of the kakutani fixed point theorem, with application to nash equilibrium points', *Proceedings of the American Mathematical Society* **3**(1), 170–174.

Harris, Christopher and John Vickers (1987), 'Racing with uncertainty', *Review of Economic Studies* **54**(1), 1–21.

Hogan, Thomas L. (2019), 'Costs of compliance with the dodd-frank act', *Baker Institute for Public Policy: Issue Brief* . September 6, Available at `https://www.bakerinstitute.org/media/files/files/0febf883/bi-brief-090619-cpf-doddfrank.pdf`.

Holmström, Bengt (1999), 'Managerial incentive problems: A dynamic perspective', *Review of Economic Studies* **66**(1), 169–182.

Irwin, Douglas and Peter Klenow (1996), 'Learning-by-doing spillovers in the semiconductor industry', *Journal of Political Economy* **102**(6), 1220–1227.

Kamien, Morton I., Eitan Muller and Israel Zang (1992), 'Research joint ventures and r&d cartels', *American Economic Review* **82**(5), 1293–1306.

Kearl, J.R. (1983), 'Rules, rule intermediaries and the complexity and stability of regulation', *Journal of Public Economics* **22**(2), 215–226.

Lerner, Josh and Jean Tirole (2002), 'Some simple economics of open source', *Journal of Industrial Economics* **L**(2), 197–234.

Mas-Colell, Andreu (1980), 'Efficiency and decentralization in the pure theory of public goods', *Quarterly Journal of Economics* **94**(4), 625–641.

Murray, Fiona and Scott Stern (2007), 'Do formal intellectual property rights hinder the free flow of scientific knowledge?: An empirical test of the anti-commons hypothesis', *Journal of Economic Behavior & Organization* **63**(4), 648–687.

Myatt, David P. and Chris Wallace (2008), 'An evolutionary analysis of the volunteer's dilemma', *Games and Economic Behavior* **62**(1), 67–76.

Nitzan, Shmuel, Uriel Procaccia and Joseph Tzur (2013), 'On the political economy of complexity', *CESifo Working Paper Series No.* 4547 .

Olson, Mancur (1965), *The Logic of Collective Action: Public Goods and the Theory of Groups*, Oxford University Press.

Polborn, Mattias K. (2008), 'Competing for recognition through public good provision', *The B.E. Journal of Theoretical Economics* **8**(1), Article 22.

Quandt, Richard (1983), 'Complexity in regulation', *Journal of Public Economics* **22**(2), 199–214.

Raymond, Eric (1999), 'The cathedral and the bazaar', *Knowledge, Technology & Policy* **12**(3), 23–49.

Sacks, Michael (2015), 'Competition between open source and proprietary software: Strategies for survival', *Journal of Management Information Systems* **32**(3), 268–295.

Samuelson, Paul (1954), 'The pure theory of public expenditure', *Review of Economics and Statistics* **36**(4), 387–389.

Sandler, Todd and John Tschirhart (1997), 'Club theory: Thirty years later', *Public Choice* **93**(3/4), 335–355.

Scotchmer, Suzanne (2002), 'Chapter 29 local public goods and clubs', *Hanbook of Public Economics* **4**, 1997–2042.

Shapiro, Carl (1985), 'Patent licensing and r&d rivalry', *American Economic Review* **75**(2), 25–30.

Stamelos, Ioannis, Lefteris Angelis, Apostolos Oikonomou and Georgios L. Bleris (2002), 'Code quality analysis in open source software development', *Information Systems Journal* **12**(1), 43–60.

Warr, Peter (1983), 'The private provision of a public good is independent of the distribution of income', *Economic Letters* **13**(2-3), 207–211.

Williams, Heidi (2013), 'Intellectual property rights and innovation: Evidence from the human genome', *Journal of Political Economy* **121**(1), 1–27.

Appendix A

Proof of Proposition 1.

Proof. Set $\sigma = 0$ and suppose that $M < N$ individuals contribute. By assumption, $\frac{\partial v(x^*;0)}{\partial x} < 0$ for $x^* > 0$, so

$$\left(\frac{dQ\left(g\left(\mathbf{x}^*\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}^*\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} + \frac{\partial v\left(x^*;0\right)}{\partial x_i} \leq 0, \tag{9}$$

with equality for $i = 1, \ldots, M$. If the above holds with equality, then

$$\left(\frac{dQ\left(g\left(\mathbf{x}^*\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}^*\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} > 0.$$

Inequalities (3) and (4) require that for each i,

$$Q\left(g\left(\mathbf{x}(M,0)\right)\right) - I\left(g\left(\mathbf{x}(M,0)\right)\right) + v\left(x(M,0);0\right) \geq \max\left\{Q\left(g\left(\mathbf{x}'\right)\right) - I\left(g\left(\mathbf{x}'\right)\right), 0\right\}$$

and

$$\max\left\{Q\left(g\left(\mathbf{x}(M,0)\right)\right) - I\left(g\left(\mathbf{x}(M,0)\right)\right), 0\right\} \geq Q\left(g\left(\mathbf{x}''\right)\right) - I\left(g\left(\mathbf{x}''\right)\right) + v\left(x'';0\right).$$

Consider a deviation by individual $i = M + 1$. Inequality (4) can be rewritten as

$$\max\left\{Q\left(g\left(\mathbf{x}\right)\right) - I\left(g\left(\mathbf{x}\right)\right), 0\right\} + v(0;0) \geq \max_{x_{M+1}} Q\left(g\left(\mathbf{x}\right)\right) - I\left(g\left(\mathbf{x}\right)\right) + v\left(x_{M+1};0\right),$$

where $\mathbf{x} = (x(M,0),\ldots,x(M,0),x_{M+1},0,\ldots,0)$. At best, (4) can only be satisfied with equality, so x'' must equal zero. Differentiating the RHS and evaluating at $x_{M+1}=0$ yields

$$\left(\frac{dQ\left(g\left(\mathbf{x}\right)\right)}{dX}-\frac{dI\left(g\left(\mathbf{x}\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}\right)}{\partial x_{M+1}}>0,$$

a contradiction. Hence there is no equilibrium with $M<N$ as at least one non-contributing individual has a profitable deviation. It remains to be shown that at $M=N$, (3) is satisfied. Given $M=N$, (3) can be rewritten as

$$\max_{x}\left\{Q\left(g\left(\mathbf{x}\right)\right)-I\left(g\left(\mathbf{x}\right)\right)+v\left(x;0\right)\right\}\geq\max\left\{Q\left(g\left(\mathbf{x}'\right)\right)-I\left(g\left(\mathbf{x}'\right)\right),0\right\}+v(0;0),$$

where $\mathbf{x}$ is an $N\times 1$ vector with $N-1$ elements equaling $x(N,0)$ and the N^{th} element equaling x_N. By construction, $\mathbf{x}'$ is equal to $\mathbf{x}$ with $x_N=0$. As

$$\left(\frac{dQ\left(g\left(\mathbf{x}'\right)\right)}{dX}-\frac{dI\left(g\left(\mathbf{x}'\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}'\right)}{\partial x_{N}}>0,$$

the inequality is necessarily satisfied so no contributing individual has a unilateral incentive to deviate. Thus there exists a symmetric Nash equilibrium with N contributors. The relevant first-order condition is now

$$\left(\frac{dQ\left(g\left(\mathbf{x}(N,0)\right)\right)}{dX}-\frac{dI\left(g\left(\mathbf{x}(N,0)\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}(N,0)\right)}{\partial x_i}+\frac{\partial v\left(x(N,0);0\right)}{\partial x_i}=0,\quad\forall\, i.\qquad(10)$$

To prove that the equilibrium level of contributions converges to the quality-maximizing level, suppose to the contrary that $x^*\to z>0$ as $N\to\infty$. Convergence to a positive z implies that $X^*\to\infty$ as $N\to 0$. It follows that

$$\left(\frac{dQ\left(\infty\right)}{dX}-\frac{dI\left(\infty\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x}<0.$$

As $\frac{\partial v(z;0)}{\partial x}<0$, (10) cannot be satisfied. Thus $x^*\to 0$ as $N\to\infty$. As x^* approaches zero and $\lim_{N\to\infty}\frac{\partial v(x^*;0)}{\partial x}=0$, (10) holds only if

$$\lim_{N\to\infty}\left(\frac{dQ\left(X^*\right)}{dX}-\frac{dI\left(X^*\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x}=0,$$

which is true when $\lim_{N\to\infty} X^* = \hat{X}$.

Now, consider the welfare maximizing case. The welfare maximizing contributions are determined by the solution to

$$\max_{\mathbf{x}} \sum_{j=1}^{N} u(X, x_j; \sigma).$$

The arbitrary first-order condition with respect to contribution $x_i > 0$ is given by

$$\left(\frac{dQ\left(g\left(\tilde{\mathbf{x}}\right)\right)}{dX} - \frac{dI\left(g\left(\tilde{\mathbf{x}}\right)\right)}{dX}\right)\frac{\partial g\left(\tilde{\mathbf{x}}\right)}{\partial x_i} + \frac{1}{N}\frac{\partial v(\tilde{x}_i;\sigma)}{\partial x_i} = 0. \tag{11}$$

As $\frac{\partial g(\mathbf{x})}{\partial x_i} = \frac{\partial g(\mathbf{x})}{\partial x_j}$, $\tilde{x}_i = \tilde{x}_j$ for all i, j and $\tilde{M}(N) = N$. Taking the limit of (11) as $N \to \infty$ yields $\tilde{X} \to \hat{X}$.

Lastly to prove that $X^* < \tilde{X} < \hat{X}$ for all finite $N > 1$, note that $\frac{\partial v(x;0)}{\partial x} < 0$ for all $x > 0$. Evaluating the LHS of (10) at $x = x(N, 0)$ yields

$$\left(\frac{dQ\left(g\left(\mathbf{x}(N,0)\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}(N,0)\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}(N,0)\right)}{\partial x} + \frac{1}{N}\frac{\partial v\left(x(N,0);0\right)}{\partial x}.$$

Since $\frac{\partial v(x(N,0);0)}{\partial x} < 0$,

$$\frac{\partial v\left(x(N,0);0\right)}{\partial x} < \frac{1}{N}\frac{\partial v\left(x(N,0);0\right)}{\partial x}.$$

From (10), it follows that

$$\begin{aligned}\left(\frac{dQ\left(g\left(\mathbf{x}(N,0)\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}(N,0)\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}(N,0)\right)}{\partial x} + \frac{1}{N}\frac{\partial v\left(x(N,0);0\right)}{\partial x}&\\ > \left(\frac{dQ\left(g\left(\mathbf{x}(N,0)\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}(N,0)\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}(N,0)\right)}{\partial x} + \frac{\partial v\left(x(N,0);0\right)}{\partial x} &= 0,\end{aligned}$$

so $\tilde{x} > x^*$ and $\tilde{X} > X^*$ for all finite $N > 1$. The presence of $\frac{\partial v(x;0)}{\partial x} < 0$ for all $x > 0$ implies that both X^* and $\tilde{X}$ are less than $\hat{X}$. If $N = 1$, then there is no externality, so $x(1, 0) = \tilde{x}$ and $X^* = \tilde{X}$ by default. □

Proof of Lemma 1.

Proof. The first step is to show that $\frac{\partial x(M,\sigma)}{\partial \sigma} > 0$. Fix M and recall that, for each contributor $i = 1, \ldots, M$, it must be that

$$\left(\frac{dQ\left(g\left(\mathbf{x}^*\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}^*\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} + \frac{\partial v\left(x^*;\sigma\right)}{\partial x_i} = 0 \tag{12}$$

holds in equilibrium. Differentiating (12) with respect to σ yields

$$\begin{aligned} M\left[\underbrace{\left(\frac{d^2Q(g(\mathbf{x}(M,\sigma)))}{dX^2} - \frac{d^2I(g(\mathbf{x}(M,\sigma)))}{dX^2}\right)\left(\frac{\partial g(\mathbf{x}(M,\sigma))}{\partial x_i}\right)^2}_{A}\right]\frac{\partial x(M,\sigma)}{\partial \sigma} \\ + \left[\underbrace{\left(\frac{dQ(g(\mathbf{x}(M,\sigma))}{dX} - \frac{dI(g(\mathbf{x}(M,\sigma))}{dX}\right)\frac{\partial^2 g(\mathbf{x}(M,\sigma))}{\partial {x_i}^2}}_{B}\right]\frac{\partial x(M,\sigma)}{\partial \sigma} \\ + \underbrace{\frac{\partial^2 v(x(M,\sigma);\sigma)}{\partial {x_i}^2}}_{C}\frac{\partial x(M,\sigma)}{\partial \sigma} = -\frac{\partial^2 v(x(M,\sigma);\sigma)}{\partial x_i \partial \sigma} \end{aligned} \tag{13}$$

The RHS of (13) is strictly negative, so $\frac{\partial x(M,\sigma)}{\partial \sigma} \neq 0$ for all $\sigma \geq 0$. Therefore $x(M,\sigma)$ is monotonic in σ. Suppose that $\frac{\partial x(M,\sigma)}{\partial \sigma} < 0$. A and C from (12) are negative, so it must be that $A + B > 0$. The second term in B is non-positive, so the first term must be strictly negative for all σ including $\sigma = 0$, which is false by Proposition 1. Thus $\frac{\partial x(M,\sigma)}{\partial \sigma} > 0$.

As $\frac{\partial x(M,\sigma)}{\partial \sigma} > 0$, $g(\mathbf{x}(M,\sigma))$ is also increasing in σ. At $\sigma = 0$,

$$\frac{dQ(g(\mathbf{x}(M,0)))}{dX} - \frac{dI(g(\mathbf{x}(M,0)))}{dX} > 0,$$

so there must exist a value $\overline{\sigma}(M) > 0$ for every $M \geq 1$ such that for all $\sigma < \overline{\sigma}(M)$, property $P1$ holds and for all $\sigma > \overline{\sigma}(M)$, property $P2$ holds.

Proving that $\overline{\sigma}(M)$ is decreasing in M first requires showing that X^* is increasing in M. Fix σ and suppose, to the contrary, that X^* is decreasing in M on any subset of $\mathbb{N} \setminus \{0\}$. By

(12),

$$\left(\frac{dQ\left(g\left(\mathbf{x}(M,\sigma)\right)\right)}{dX}-\frac{dI\left(g\left(\mathbf{x}(M,\sigma)\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}(M,\sigma)\right)}{\partial x}$$
$$-\left(\frac{dQ\left(g\left(\mathbf{x}(M+1,\sigma)\right)\right)}{dX}-\frac{dI\left(g\left(\mathbf{x}(M+1,\sigma)\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}(M+1,\sigma)\right)}{\partial x}$$
$$=\frac{\partial v\left(x(M+1,\sigma);\sigma\right)}{\partial x}-\frac{\partial v\left(x(M,\sigma);\sigma\right)}{\partial x}. \quad (14)$$

As X^* is decreasing in M, x^* must also be decreasing in M, so the RHS of (14) is strictly positive. Thus the LHS must also be positive. However, note that

$$\frac{dQ\left(g\left(\mathbf{x}(M,\sigma)\right)\right)}{dX}-\frac{dI\left(g\left(\mathbf{x}(M,\sigma)\right)\right)}{dX}<\frac{dQ\left(g\left(\mathbf{x}(M+1,\sigma)\right)\right)}{dX}-\frac{dI\left(g\left(\mathbf{x}(M+1,\sigma)\right)\right)}{dX}$$

and

$$0<\frac{\partial g\left(\mathbf{x}(M,\sigma)\right)}{\partial x}\leq\frac{\partial g\left(\mathbf{x}(M+1,\sigma)\right)}{\partial x},$$

which implies the LHS of (14)is negative, a contradiction. Thus X^* is increasing in M.

Now to show that $\overline{\sigma}(M)$ is decreasing in M, suppose that $g(\mathbf{x}(M,\sigma))$ is such that

$$\frac{dQ\left(g\left(\mathbf{x}(M,\sigma)\right)\right)}{dX}-\frac{dI\left(g\left(\mathbf{x}(M,\sigma)\right)\right)}{dX}\leq 0$$

and

$$\frac{dQ\left(g\left(\mathbf{x}(M+1,\sigma)\right)\right)}{dX}-\frac{dI\left(g\left(\mathbf{x}(M+1,\sigma)\right)\right)}{dX}<0.$$

It follows that $\sigma\leq\overline{\sigma}(M)$ but $\sigma>\overline{\sigma}(M+1)$, which implies that $\overline{\sigma}(M)$ is decreasing in M. To show that as $M\to\infty$, $\overline{\sigma}(M)\to 0$ recall that by Proposition 1, $x^*\to 0$ as $M\to\infty$, while $\lim_{M\to\infty}\frac{\partial v(x(M,\sigma))}{\partial x}>0$ for all $\sigma>0$. The result follows from $P2$. □

Proof of Lemma 2.

Proof. Lemma 2 follows immediately from Kakutani's fixed point theorem and the extensions in Debreu (1952), Glicksberg (1952), and Fan (1952). □

Proof of Proposition 2.

Proof. Existence follows from Lemma 2. To prove that $M^*(N) = N$ is necessary for the public good to be produced under $P1$, suppose that $M^*(N) < N$. The first-order condition for a non-contributor is

$$\left(\frac{dQ\left(g\left(\mathbf{x}^*\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}^*\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} + \frac{\partial v\left(0;\sigma\right)}{\partial x_i} \leq 0.$$

For $\sigma > 0$, $\frac{\partial v(0;\sigma)}{\partial x} > 0$. Hence the first-order condition can only be satisfied if

$$\frac{dQ\left(g\left(\mathbf{x}^*\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}^*\right)\right)}{dX} < 0.$$

Thus $P1$ does not hold. □

Proof of Proposition 3.

Proof. The proof proceeds in two cases. Firstly suppose that $\sigma < \overline{\sigma}(N)$. By Lemma 1, $P1$ applies, and the result follows immediately from an identical argument to that of Proposition 1.

Next suppose that $\sigma > \overline{\sigma}(N)$, so by Lemma 1, $P2$ applies for $M > \overline{\sigma}^{-1}(\sigma)$. The first-order conditions for a maximum are given by

$$\left(\frac{dQ\left(g\left(\mathbf{x}^*\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}^*\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} + \frac{\partial v\left(x^*;\sigma\right)}{\partial x_i} = 0, \quad \forall\, i = 1, \ldots, M \tag{15}$$

while the welfare maximizing conditions with M contributors is given by

$$\left(\frac{dQ\left(g\left(\tilde{\mathbf{x}}\right)\right)}{dX} - \frac{dI\left(g\left(\tilde{\mathbf{x}}\right)\right)}{dX}\right)\frac{\partial g\left(\tilde{\mathbf{x}}\right)}{\partial x_i} + \frac{1}{N}\frac{\partial v\left(\tilde{x};\sigma\right)}{\partial x_i} = 0, \quad \forall\, i \leq M. \tag{16}$$

Given (15) and $\frac{\partial v(x^*;\sigma)}{\partial x_i} > 0$,

$$\left(\frac{dQ\left(g\left(\mathbf{x}^*\right)\right)}{dX} - \frac{dI\left(g\left(\mathbf{x}^*\right)\right)}{dX}\right)\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} + \frac{1}{N}\frac{\partial v\left(x^*;\sigma\right)}{\partial x_i} < 0$$

by Lemma 1. Thus $x^* > \tilde{x}$ and by extension, $X^* > \tilde{X}$. Both being greater than $\hat{X}$ follows immediately. $\tilde{X} \to \hat{X}$ as $N \to \infty$ follows from taking the limit of (16) as $N \to \infty$. By Proposition 1, $x^* \to 0$ as $N \to \infty$. As $\lim_{x \to 0^+} \frac{\partial v(x;\sigma)}{\partial x} > 0$ for $\sigma > 0$, $\frac{dQ(g(\mathbf{x}(\infty,\sigma)))}{dx} < 0$, which implies that $X^* > \hat{X}$ for all $\sigma > 0$. □

Proof of Corollary 1.

Proof. The result follows immediately from the assumptions on $v(x;\sigma)$ and Proposition 3. □

Proof of Proposition 4.

Proof. Suppose that $\hat{M} > \overline{\sigma}^{-1}(\sigma)$ and there exists an integer $\hat{M} - 1$ such that

$$\overline{\sigma}^{-1}(\sigma) < \frac{\left(\hat{M} - 1\right) + \hat{M}}{2},$$

so producing the good with $\hat{M} - 1$ investors leads to a good with a higher quality than having $\hat{M}$ investors. It immediately follows that if (7) holds, then $\hat{M}$ leads to greater aggregate welfare than $\hat{M} - 1$. □

Proof of Corollary 2.

Proof. The result follows from Lemma 1 and Proposition 4. □

Proof of Proposition 5.

Proof. Suppose that $\mu \in [0, 1)$ and $\underline{Q} = -\infty$. Note that the indirect utility of each contributor is strictly increasing in σ. The utility of each free-rider is non-monotonic in σ. As

σ grows large, $Q\left(g\left(\mathbf{x}(M,\sigma)\right)\right) - I\left(g\left(\mathbf{x}(M,\sigma)\right)\right)$ experiences non-monotonic spikes, where $Q(X)-I(X)$ will decrease. By (3), there exists a finite set of cutoff values $\sigma_1,\ldots,\sigma_Z$, where at each σ_z, an individual stops contributing. Between each cutoff value, $Q-I$ is decreasing with a sharp jump at each cutoff point until there is a fixed number of investors, which by By (3) and By (4) is at least one. As $Q(X)-I(X)$ decreases, free-riders minimum payoff is zero given the availability of the outside option. Thus aggregate payoffs are bounded below by zero and can be made arbitrarily large by increasing σ and benefiting the investors, while all non-investors receive zero utility.

Now suppose that $\mu = 1$ and $\underline{Q} = -\infty$. Then $\mu = 1$ implies quality maximization, which occurs under $P3$ and implies that $\hat{\sigma} = \overline{\sigma}(M^*(N))$.

The last statement follows immediately from the argument above coupled with the added constraint on $\underline{Q}$. Because $Q(X)-I(X)$ is strictly decreasing for σ large, imposing a minimum quality constraint requires that the set of feasible optimal parameters $\hat{\sigma}$ is compact. $\square$

Appendix B

In the model developed in Section 2, private benefits from contributing to the public good induced a negative externality on the utility from consumption. For σ sufficiently large, the negative externality outweighed the positive externality leading to over-provision. This section offers an alternative framework in which there is no negative externality with respect to consumption. The net quality is strictly increasing, though the marginal utility tends to zero as provision increases. The externality is instead integrated through the provision. Contributors benefit not from the level of their contributions, but from their contributed share of total contributions.

Instead of building a new model from scratch, I augment the model from Section 2. I assume that $Q(X)$ is twice continuously differentiable and strictly concave. In addition, $\lim_{\to\infty} \frac{dQ(X)}{dX} = 0$. There are no consumption-based costs: $I(X) = 0$ for all X. The private benefits function is adjusted to also allow the benefits to depend on X: $v(x_i, X; \sigma)$. For simplicity, I assume that $X = g(\mathbf{x}) = \sum_{j=1}^{N} x_j$ and $v(x_i, X; \sigma) = \sigma \frac{x_i}{X} - x_i$.

Hence utility is given by

$$u(x_i, X; \sigma) = Q\left(\sum_{j=1}^{N} x_j\right) + \sigma \frac{x_i}{\sum_{j=1}^{N} x_j} - x_i, \tag{17}$$

with arbitrary first-order condition

$$\frac{dQ(X^*)}{dX} + \sigma \frac{X^* - x_i^*}{(X^*)^2} - 1 = 0.$$

By observation, $x_i^* = x_j^* = x^*$ for all i, j, so the above can be rewritten as

$$\frac{dQ(Nx^*)}{dX} + \sigma \frac{N-1}{N^2 x^*} - 1 = 0. \tag{18}$$

Rearranging the above yields

$$N^2 x^* \left(1 - \frac{dQ(Nx^*)}{dX}\right) = \sigma(N-1).$$

In equilibrium, $\frac{dQ(Nx^*)}{dX} < 1$. Note that as σ increases, the RHS increases without bound. Therefore, x^* is also increasing in σ without bound.

Next, consider welfare maximization. The welfare-maximizing program is given by

$$\begin{aligned}\sum_{k=1}^{N} u(x_k, X; \sigma) &= \sum_{k=1}^{N} \left[Q\left(\sum_{j=1}^{N} x_j\right) + \sigma \frac{x_i}{\sum_{j=1}^{N} x_j} - x_i\right] \\ &= NQ\left(\sum_{j=1}^{N} x_j\right) + \sigma - \sum_{k=1}^{N} x_k,\end{aligned} \tag{19}$$

with arbitrary first-order condition

$$N \frac{dQ(\tilde{X})}{X} - 1 = 0.$$

This value is independent of σ. Thus for σ sufficiently large, $X^* > \tilde{X}$.