Incentives for the Over-Provision of Public Goods *

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Abstract

A wide range of public goods, such as open source software, possess two often-ignored features: (i) excludable and potentially rivalrous contribution benefits (e.g. status seeking) and (ii) nonexcludable and nonrival consumption costs (e.g. adoption costs). I develop a model of the voluntary provision of public goods that incorporates these features. I find that these additional features mitigate the well-known incentive problems, but introduce new ones. Costly consumption lessens the free-rider problem, leading to more efficient provision. Private benefits similarly reduce the free-rider problem, but can lead to over-provision via a negative congestion externality on the supply side. Status-seeking induces an increase in contributions to the benefit of each contributor but imposes a cost on all other consumers and contributors. Efforts to maximize welfare by a community leader or social planner often involve transferring surpluses from consumers to producers.

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JEL classifications: C72, D29, D62, D71, H41.

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1 Introduction

Over the last twenty years, there has been an explosion in collaborative and organization-backed voluntary provision of public goods, including forums (e.g. Stack Exchange), open source software (e.g. Android and the Linux kernel), and Wikipedia. The rapid growth of these public goods suggests that free-rider problems may be overwhelmed by other motivations for contributing, calling into question the notion of inefficient under-provision. Currently, Wikipedia boasts more than 6 million entries (Wikipedia, 2021a) from more than 41 million users (Wikipedia, 2021b) and the Linux kernel is buoyed by more than 20,000 contributors (Varghese, 2020).

There are many reasons that individuals voluntarily contribute to the provision of public goods. Altruism, both pure and impure, offers one explanation (Andreoni, 1990).[1] When contributing to charity, individuals seek to signal wealth (Glazer and Konrad, 1996).[2] When developing open source software, individuals contribute for a variety of reasons beyond need of use, such as altruism, the joy of coding, signaling ability, and seeking status (Lerner and Tirole, 2002; Feller, Fitzgerald, and Lakhani, 2005; Bitzer, Schrettl, and Schröder, 2007; Myatt and Wallace, 2008; Fang and Neufeld, 2009; Athey and Ellison, 2014). For-profit organizations similarly contribute to open source software for a variety of reasons. The provision of public policy and regulations allows elected officials to develop political capital and allows companies to carve out valuable exemptions (Laffont and Tirole, 1991; Dal Bó, 2006).[3]

A second feature that is especially apparent in (but not unique to) these new public goods is costly consumption. It is costly for Wikipedia readers to verify the accuracy of articles, particularly with individuals and specialized PR firms offering editing services for articles (Pinsker, 2015). Users of open source software face costs such as implementation costs,

[1]Zhang and Zhu (2011) offer greater detail on this literature and apply it to Wikipedia contributions.

[2]Bekkers and Wiepking (2011) summarize the incentives underpinning charitable giving.

[3]Public policies, which by their nature are nonexcludable and nonrival, can be viewed as public goods when the policy offers direct benefits to the target population. For example, a state-level mask mandate during COVID-19 acts as a public good offering direct nonexcludable and nonrival benefits and costs to residents, whereas tax policy offers benefits only indirectly through other public goods / policies funded via the tax.

learning costs, and switching costs (Nagy, Yassin, and Bhattacherjee, 2010; Sacks, 2015; Chen and Sacks, 2021). Public policies often impose compliance costs (Nitzan, Procaccia, and Tzur, 2013; Hogan, 2019). The misallocation of funds by charities imposes a cost on their beneficiaries. Together, these two features raise the question: should we expect these public goods to be suboptimally provided, and if so, in which direction?

To capture these additional features, we must depart from the canonical analyses of public goods.[4] I do so by developing a model of voluntary public goods provision that allows for costly consumption and private benefits to contributing. The model generates four main insights, which I discuss in further detail below. First, costly consumption can mitigate the free-rider problem and lead to optimal provision in large populations. Second, private benefits to contributors can lead to over-provision of a public good rather than under-provision. Third, the combination of costly consumption and private benefits to contributors can induce a tradeoff between the total contributions to a public good (quantity) and the net utility from consuming it (quality). Fourth, utilitarian efforts to maximize welfare often entail transferring surpluses from consumers to contributors. Traditionally, the free-rider problem suggests policy makers need to stimulate provision to improve efficiency. With the threat of both over-provision and under-provision, policy makers must be careful when determining whether to stimulate or scale back public goods provision and with applying specific studies of public goods to other contexts.

More precisely, I create a static model of the voluntary provision of a public good with a finite homogeneous population of individuals that voluntarily contribute. Individuals' utilities are a function of their own contributions and total contributions. I allow consumption of the public good to carry costs as well. Contributions may confer private benefits to the contributor that accumulate in one of two ways: by how much an individual contributes (absolute levels) or by how much an individual contributes relative to the total (relative shares). My model yields a new taxonomy of public goods in which there are four types:

(i) textbook public goods,

[4] This is not the first paper to consider private benefits to contributors (as illustrated in the above citations), though it is the first to consider how these benefits (and costly consumption) affect the relationship between the equilibrium and optimal provision levels.

(ii) public goods with costly consumption,

(iii) public goods with private benefits to contributors,

(iv) public goods with both costly consumption and private benefits.[5]

Each type of public good is unique in its equilibrium properties and welfare implications.

Beyond the textbook case, under-provision is only guaranteed in equilibrium with public goods where consumption is costly [type (ii)]. Unlike the textbook case, the free-rider problem becomes less severe as the size of the population becomes large, though this catching-up effect need not be monotonic. For small populations, increases may initially worsen the free-rider problem. This pattern offers economic support for Linus's law, which states, "given enough eyeballs, all bugs are shallow (Raymond, 1999, p. 29)."[6] This statement argues that the quality of open source software increases as more individuals contribute. While consistent with these type (ii) public goods, Linus's law need not hold in general nor even with respect to its original context of open source software.

Consider public goods that confer private benefits to contributors but no costly consumption [type (iii)]. Private benefits can mitigate the free-rider problem and even cause over-provision. If the private benefits depend on relative shares and are sufficiently strong, e.g. a strong preference to signal ability relative to peers, then over-provision occurs. On the other hand, under-provision persists when the private benefits depend on absolute levels, e.g. a preference to signal absolute ability. Much like contests, contributors have an incentive to enter into an arms race to contribute the greatest share. These efforts are negated in equilibrium while contributors still bear the full contribution costs. This relationship outlines a negative congestion externality, which unlike the traditional congestion externality found on the demand-side of club goods, is found on the supply-side. The burden of the inefficiency is borne by contributors. Free-riders benefit from over-provision with over-contributing acting as a surplus transfer via increased provision.

Previous studies have shown over-provision to be a possibility in certain contexts. Brueckner

[5] The latter two relate to the literature on impure public goods. See Cornes and Sandler (1984), Bergstrom, Blume, and Varian (1986), Cornes and Sandler (1996), Kotchen (2007), and the citations therein.

[6] This statement is from Linus Torvalds, the founder and namesake of the Linux operating system.

(1979), Van de Kragt, Orbell, and Dawes (1983), Bayindir-Upmann (1998), Bjorvatn and Schjelderup (2002), and Polborn (2008) show that over-provision can occur with heterogeneity and competition (multiple jurisdictions). Gradstein and Nitzan (1990) find over-provision to be a possibility when contributions are binary, the population is large, and marginal utility tends to zero as contributions increase. I show that these factors are not necessary for over-provision and it thus may be more prevalent than previously thought. Over-provision can occur in small homogeneous populations when those individuals have private incentives to contribute and benefit more by contributing more relative to their peers.

Private benefits that depend on levels can still cause equilibrium over-provision, but only in the presence of costly consumption as well [type (iv)]. In this case, over-provision occurs regardless of how these benefits accumulate (levels or shares) provided that these benefits are sufficiently large. While the outcome is the same as type (iii) public goods above, the welfare implications are distinct. In this case, the burden is borne by both free-riders and contributors, with the former bearing the entire burden under level-based private benefits. Letting the utility from consuming the public good signify its quality, a quality-quantity tradeoff emerges. Contributors privately benefit by increasing their contributions. Doing so imposes costs on the consumption side. For example, adding features to open source software benefits the contributors through signalling, but also induces learning costs (or implementation costs if higher quality hardware becomes necessary) on those using the software. In general, this tension leads to inefficiently complex (low quality) public goods, which are public goods where the marginal consumption costs exceed the marginal consumption benefits.

This tradeoff is observed in a variety of public goods. Stamelos, Angelis, Oikonomou, and Bleris (2002) show that the relationship between the size of a component of open source software (aggregate contributions) is negatively related to user satisfaction for the software (quality).[7] A similar tradeoff is found in Francalanci and Merlo (2008). Quandt (1983) and Kearl (1983) show that there exist conditions such that there is an over-investment in

[7] In cases where expert consumers reported only superficial errors at worst, the mean length of the program is strictly lower than in cases where (a) consumers reported either all major program functions are working but at least one minor function is disabled or incorrect, (b) at least one major function is disabled or incorrect, and (c) the program is inoperable. Their result is robust to various metrics, including the number of statements per module.

regulation. A key contribution of my work is that I develop a model that generalizes these results to a wide variety of public goods. Similarly, Nitzan, Procaccia, and Tzur (2013) argue that the increasing volume and velocity of complexity in regulation over time leads to inefficiencies induced by implementing barriers to entry so only the largest established firms survive. Combining the results of Nitzan, Procaccia, and Tzur (2013), Quandt (1983), Kearl (1983), and the citations therein with this paper leads to a more complete understanding of the effects of complexity in regulation via the quantity-quality tradeoff and shows its applicability to a variety of public goods.[8] Galasso and Schankerman (2015) find that whether or not patents hinder subsequent innovation depends on the complexity of the industry and that negative effects are concentrated in more complex industries. If it is relatively more costly to implement IP in more complex industries, then Galasso and Schankerman (2015) can be viewed as further evidence in support of the quality-quantity tradeoff.

The tension between quality and quantity is particularly strong for free-riders. This observation is important because free-riding is often unavoidable. Most individuals are not able to contribute to the development of public goods such as open source software or public policy. Hence, the majority of consumers are free-riders by necessity. I show that policies maximizing aggregate utility when offering public goods with both costly consumption and private benefits [type (iv)] often entail transferring surpluses from consumers to contributors. Thus, when equity is a concern, the formulation of the policy becomes increasingly important.

To study this policy point further, I also consider the case where the public good is coordinated by a social planner, though individual participants are still bounded by their individual rationality constraints. The social planner selects who may contribute. This setup is often the case in public goods such as open source software and public policy, where leadership teams can accept / reject contributors and their contributions. My model predicts that a planner interested in maximizing total welfare often favors transferring surpluses from consumers to contributors, which in the context of open source software entails inefficient complexity and in the context of public policy entails regulatory capture. Thus these phe-

[8]For examples of regulatory complexity in finance, see Berglund (2014). For examples in IP, see Allison and Lemley (2002). Evidence of a negative impact of regulations is given in Murray and Stern (2007) and Williams (2013).

nomena are more general than their specific contexts and this sort of over-provision applies to a wide variety of public goods.

This paper contributes to several strands of literature, including the theory of public goods, industrial organization, and the economics of innovation. The literature on the private provision of public goods began with Bernheim (1986) and Bergstrom, Blume, and Varian (1986). Summaries of the core theories are given by Cornes and Sandler (1996), Sandler and Tschirhart (1997), and Scotchmer (2002). Though some work has been conducted in the context of regulatory complexity, this paper is among the first to formally analyze the effects of consumption costs and private benefits on the efficiency of public goods.[9]

In addition to the provision of public goods, this paper is useful in both understanding and linking the structure and outcomes of industries whose interactions are characterized by collaborative production. I show that collaborative groups will likely over-invest in technological investment, shedding light on the nature of research joint ventures by building upon Kamien, Muller, and Zang (1992) and the theories outlined in Caloghirou, Ionnides, and Vonortas (2003). This paper also offers an alternative mechanism for understanding over-investment in patent races (Gilbert and Newbery, 1982; Fudenberg, Gilbert, Stiglitz, and Tirole, 1983; Shapiro, 1985; Harris and Vickers, 1987; Aoki, 1991; Baye and Hoppe, 2003).

The remainder of paper is structured as follows. Section 2 outlines the model. Section 3 presents the main results, first analyzing public goods with costly consumption, then public goods with private benefits, finishing with public goods with both costly consumption and private benefits. Section 4 presents the social planner's problem. Section 5 offers extensions related to the private benefits to contributors. Section 6 concludes.

2 The Model

There is a population of $N > 2$ homogeneous individuals, a single public good voluntarily provided by members of the population, and a (substitute) private outside option. Each

[9] A general model of public goods incorporating non-monotonicity was developed in Mas-Colell (1980), where public projects are elements of a non-linear topological space. However, such a model is unable to capture the characteristics described above.

individual i can either (1) contribute $x_i > 0$ and consume the public good, (2) contribute $x_i = 0$ (free ride) and consume the public good, or (3) contribute $x_i = 0$ and consume the outside option. Hence, if an individual contributes to the public good, then that individual must also consume it.[10] The utility from consuming the outside option is normalized to zero. All contribution and consumption decisions are made simultaneously and independently. The remainder of this section describes the utility from consuming and contributing.

Define $X = \sum_{j=1}^{N} x_j$ as the aggregate contributions to the public good and $X_{-i} = X - x_i$ as the aggregate contributions less i's input. Each individual's utility depends on both x_i and X. When consuming the public good, each individual's utility consists of four additively separable components. The first component is the consumption benefit $B(X)$, which depends only on aggregate contributions. I assume that $B(\cdot)$ is twice continuously differentiable, strictly increasing, and weakly concave with $B(0) = 0$. The second component is the consumption cost $\gamma C(X)$, where $\gamma \in \{0, 1\}$. I use γ to isolate the effects of costly consumption on provision. Like the consumption benefit, the consumption cost depends only on aggregate contributions. I assume that $C(\cdot)$ is twice continuously differentiable, strictly increasing, and strictly convex with $C(0) = 0$.

Together, $B(X) - \gamma C(X)$ represents the net utility from *consuming* the public good. I use this value to measure the public good's *quality* (relative to the outside option).[11] For example, the quality of open source software can be measured according to its features, reliability, portability, and usability (Xenos and Christodoulakis, 1997; Sacks, 2015). In this case, $B(X)$ captures the features and portability and $C(X)$ captures the difficulty of use and errors (unreliability and lack of usability). From a user's perspective, software quality increases if features can be added without sacrificing portability, reliability, or usability: $B(X) - C(X)$ is locally increasing in X. When $\gamma = 1$, let $\hat{X} \equiv \max_X \{B(X) - C(X)\}$ denote the *quality maximizer*. This value maximizes free-rider utility. When $\gamma = 0$, $\hat{X} \to \infty$ and quality is monotonically increasing in aggregate contributions: more is always better for consumption, *ceteris paribus*.

[10] Allowing individuals to contribute $x_i > 0$ while not consuming the public good strengthens the results of the paper. Contributors will not internalize any consumption-related costs, further intensifying contributions.

[11] This comparison follows from the utility of consuming the outside option being normalized to zero.

The third component captures all of the private benefits of contributing to the provision of the public good, which I call the contribution benefit. Denote this benefit by $\sigma b\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right)$, where $\lambda \in \{0,1\}$. I define σ as the relative strength of the contribution benefit. If $\sigma = 0$, then all benefits are derived via consumption. If $\sigma > 0$, then individuals privately benefit from contributing to the provision of the public good. The parameter λ allows this benefit to accumulate in two ways. If $\lambda = 1$, then the contribution benefit is given by $\sigma b(x_i)$ and is thus absolute and depends only on the *level* of individual i's contribution. If $\lambda = 0$, then the contribution benefit is given by $\sigma b\left(\frac{x_i}{X}\right)$ and is thus relative and depends on the *share* of individual i's contributions relative to the total. I assume that $b(\cdot)$ is twice continuously differentiable, strictly increasing, and weakly concave with $b(0) = 0$.[12] The fourth component is the contribution cost $c(x_i)$. I assume that $c(\cdot)$ is increasing, twice continuously differentiable, and strictly convex with $c(0) = 0$.[13] Regardless of λ, I assume that the contribution cost accumulates via the level of an individual's contributions and not the share of their contributions.[14]

Combining these four components, individual i's utility when contributing is

$$u\left(x_i, X; \sigma\right) = B\left(X\right) - \gamma C\left(X\right) + \sigma b\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - c(x_i). \tag{1}$$

When not contributing, individual i receives utility

$$u\left(0, X; \sigma\right) = \max\{B(X) - \gamma C(X), 0\}.$$

Throughout the analysis, I will be illustrating the results with a simple example that offers closed-form solutions. In the example, the utility when contributing is given by

$$u(x_i, X; \sigma) = X - \gamma\frac{1}{40}X^2 + \sigma\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - \frac{1}{2}x_i^2. \tag{2}$$

[12] I assign $b\left(\frac{0}{0}\right) = 0$.

[13] Equivalently, $c(x_i)$ can be interpreted as the opportunity cost of forgoing consumption of an unmodeled (composite) private good. Under this interpretation, each individual is endowed with a sufficiently large (but finite) stock of resources (e.g. income), and the marginal utility of consuming the composite good is infinite at zero consumption (Inada condition).

[14] Were the contribution cost to accumulate according to shares, contributing $x_i = kX$ for any $k \in (0,1)$ entails the same contribution cost for all X. Under this specification, the cost of contributing 100 lines of source code to an open source software project with a total of 1000 lines of code would be the same as the cost of contributing 100,000 lines to a project with a total of 1 million lines. The latter requires significantly more resources than the former. Therefore, I restrict the cost to levels.

Table 1: Summary of results: over-provision v. under-provision by type of public good.

	Type of public good: $\gamma = 0$	$\gamma = 1$
$\sigma = 0$	Textbook [type (i)]: Under-provision that worsens as the population grows.	Costly consumption [type (ii)]: Under-provision that is mitigated as the population grows.
$\sigma > 0$	Private benefits [type (iii)]: Over-provision if and only if $\lambda = 0$ and σ is sufficiently large. Otherwise (σ small or $\lambda = 1$), under-provision that worsens as the population grows.	Costly consumption & private benefits [type (iv)]: Over-provision if σ is sufficiently large for both $\lambda = 0$ and $\lambda = 1$. In large populations, over-provision for all $\sigma > 0$.

Two parameters, γ and σ, allow the model to capture the four types of public goods outlined in the introduction. The textbook public good emerges when $\gamma = 0$ and $\sigma = 0$ [type (i)]. When $\gamma = 1$ and $\sigma = 0$, the public good exhibits costly consumption [type (ii)]. When $\gamma = 0$ and $\sigma > 0$, the public good offers private benefits [type (iii)]. Lastly, $\gamma = 1$ and $\sigma > 0$ yields a public good with both costly consumption and private benefits [type (iv)]. Furthermore, when $\sigma > 0$, λ captures both absolute ($\lambda = 1$) and relative ($\lambda = 0$) private benefits. Hence, the results that follow are identified by three parameters: γ, σ, and λ. Table 1 summarizes how the parameters relate to the four types of public goods and offers a brief preview of the under-provision and over-provision results to follow.

3 Analysis

Denote by x_i^* individual i's (Nash) equilibrium contribution to the public good and by X^* the aggregate equilibrium contribution level.

Lemma 1. *If $x_i^* > 0$ and $x_j^* > 0$, then $x_i^* = x_j^* = x^*$ for all i and j.*

The proof, and all subsequent proofs, are contained in the appendix.

I order the individuals such that if $M < N$ individuals contribute, then individuals $1, \ldots, M$ are the contributors and individuals $M + 1, \ldots, N$ are the non-contributors. There exists an equilibrium in which exactly M individuals contribute if, for all $i = 1, \ldots, M$,

$$B\left(X^*\right) - \gamma C\left(X^*\right) + \sigma b\left(\lambda x^* + (1-\lambda)\frac{1}{M}\right) - c(x_i^*) \geq \max\{B\left(X_{-i}^*\right) - \gamma C\left(X_{-i}^*\right), 0\}, \quad (3)$$

and if $M < N$,

$$\begin{aligned} &\max\{B\left(X^*\right) - \gamma C\left(X^*\right), 0\} \geq \\ &\qquad B\left(X^* + x''\right) - \gamma C\left(X^* + x''\right) + \sigma b\left(\lambda x'' + (1-\lambda)\frac{x''}{X^* + x''}\right) - c(x'') \quad (4) \end{aligned}$$

for all $i = M+1, \ldots, N$, where $x'' > 0$ corresponds to the optimal deviation, which satisfies

$$\begin{aligned} &\frac{dB\left(X^* + x''\right)}{dX} - \gamma\frac{dC\left(X^* + x''\right)}{dX} - \frac{dc(x'')}{dx_i} \\ &\qquad + \sigma\frac{\partial b\left(\lambda x'' + (1-\lambda)\frac{x''}{X^*+x''}\right)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{X^*}{(X^* + x'')^2}\right) = 0. \end{aligned}$$

Inequality (3) states that no contributor has a profitable unilateral deviation and (4) states that no free-rider has a profitable unilateral deviation. By the implicit function theorem, $x^* = x(M, \sigma)$ and $X^* = X(M, \sigma)$ and by the theorem of the maximum, $\frac{\partial x(M,\sigma)}{\partial \sigma}$ exists and is well defined.[15] Denote by $M^*(N)$ the values of M such that $x^* = x(M^*(N), \sigma)$ for all $i = 1, \ldots, M^*(N)$ and $x^* = 0$ for all $i = M^*(N) + 1, \ldots, N$ constitutes an equilibrium.

I also characterize the welfare-maximizing contribution levels in order to determine whether over-provision or under-provision occurs in equilibrium. I use $\tilde{x}_i$ and $\tilde{X}$ to denote the individual and aggregate welfare-maximizing contributions, which are defined according to $\max_{x_1,\ldots,x_N} \sum_{j=1}^{N} u(x_j, X; \sigma)$:

$$\begin{aligned} \max_{x_1,\ldots,x_N} \Bigg\{ &(N - M)\max\{B(X) - \gamma C(X), 0\} + M\left[B(X) - \gamma C(X)\right] \\ &\qquad + \sum_{j=1}^{M}\left[\sigma b\left(\lambda x_j + (1-\lambda)\frac{x_j}{X}\right) - c(x_j)\right]\Bigg\}. \end{aligned}$$

Denote by $\tilde{M}(N)$ the value(s) of M that maximize welfare. With $M^*(N)$ and $\tilde{M}(N)$ offering the number of equilibrium and welfare-maximizing contributors, I use M when considering an arbitrary number of contributors.

[15]For $\sigma = 0$, define the derivative as $\lim_{\Delta\to 0^+} \frac{x(M,\Delta) - x(M,0)}{\Delta}$.

In what follows, I sequentially consider the latter three of the four types of public goods discussed above. I omit the textbook case from the formal analysis, as its properties are well established. This setup allows the analysis to cover the spectrum of collaboratively produced public goods and isolate how each component both independently and interactively affects the equilibrium and optimal provision.

3.1 Public Goods with Costly Consumption ($\gamma = 1$ and $\sigma = 0$)

To isolate and identify the effects of the consumption cost, I fix $\gamma = 1$ and $\sigma = 0$ (when $\sigma = 0$, λ is irrelevant). By (1), contributor i's utility is

$$u(x_i, X; 0) = B(X) - C(X) - c(x_i).$$

Recall that $\hat{X}$ denotes the level of contributions that maximizes quality: $B(X) - C(X)$.

Proposition 1. *$M^*(N) = \tilde{M}(N) = N$. Furthermore, $X^* < \tilde{X} < \hat{X}$ for all N with $\lim_{N\to\infty} X^* = \lim_{N\to\infty} \tilde{X} = \hat{X}$.*

Figure 1 diagrammatically illustrates Proposition 1, which offers an immediate departure from the textbook treatment of public goods. While free-riding persists and neither the equilibrium nor the welfare-maximizing contributions maximize quality, they do both converge to the quality maximizer as the size of the population increases. Hence, the free-rider problem becomes less severe rather than more severe. Proposition 1 provides a more general interpretation of Linus's law beyond the context of open source software.[16] While convergence to the quality maximizer is uniform, the magnitude of the free-rider problem need not converge uniformly. That is, while $\lim_{N\to\infty} \tilde{X} - X^* = 0$, convergence need not be monotonic. Example 1 illustrates this point.

Example 1. *Suppose that $\gamma = 1$ and $\sigma = 0$. Then by (2), each contributor's utility is*

$$u(x_i, X; 0) = X - \frac{1}{40}X^2 - \frac{1}{2}x_i^2.$$

[16] As I will show in Section 3.3, Linus's law need not hold for all types of open source software, specifically software with costly consumption developed by contributors with strong private incentives.

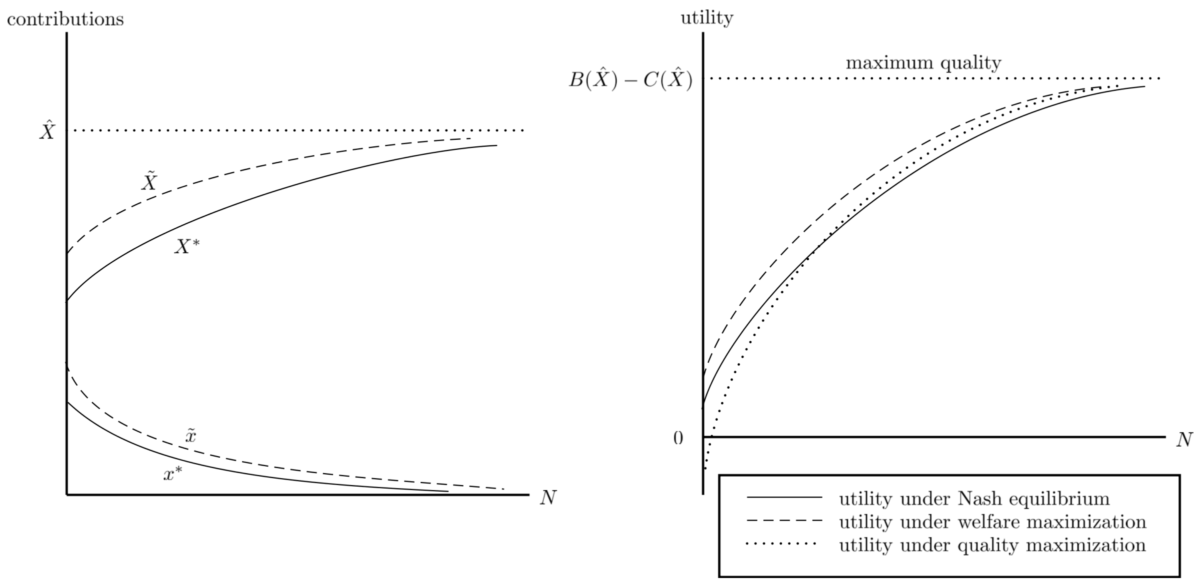

Figure 1: Contributions and utility for public goods with costly consumption.

It follows that

$$X^* = N\Bigg(\underbrace{\frac{20}{20+N}}_{x^*}\Bigg), \qquad \tilde{X} = N\Bigg(\underbrace{\frac{20N}{20+N^2}}_{\tilde{x}}\Bigg), \qquad \hat{X} = 20.$$

Whenever $N > 1$, $X^* < \tilde{X}$. *Furthermore,* $\tilde{X} - X^* = \frac{400(N-1)}{(20+N)(20+N^2)}$, *which is decreasing in* N *only for* $N \geq 5$. *Lastly, as* N *increases, both* X^* *and* $\tilde{X}$ *converge to* $\hat{X} = 20$.

In summary, costly consumption mitigates the free-rider problem in larger populations. This is not to say that an authority engaging efforts to make the consumption of public goods costly is welfare enhancing. While the gap between the welfare and utility-maximizing outcomes is smaller, the absolute levels of utility are as well; i.e., the free-rider problem is worse when $\gamma = 0$, but the raw levels of utility are greater than under $\gamma = 1$.[17]

[17]Using (2), when $\gamma = 0$, $x^* = 1$ and $\tilde{x} = N$. Thus, the respective utilities in the equilibrium and welfare maximizing case are $N - \frac{1}{2}$ and $\frac{N^2}{2}$. When $\gamma = 1$, the respective utilities are $10 - \frac{4200}{(20+N)^2}$ and $\frac{10N^2}{20+N^2}$. By inspection, each utility is higher under $\gamma = 0$.

3.2 Public Goods with Private Benefits ($\gamma = 0$ and $\sigma > 0$)

To isolate the effects of the contribution benefit, I fix $\gamma = 0$ and assume that $\sigma > 0$. By (1), contributor i receives utility

$$u(x_i, X; \sigma) = B(X) + \sigma b\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - c(x_i).$$

With costless consumption, the quality maximizer is not well defined ($\hat{X} \to \infty$), so I focus attention on the relationship between the equilibrium contributions X^* and the welfare-maximizing contributions $\tilde{X}$.

Like the previous section, the free-rider problem can be mitigated. However, private benefits can swing the pendulum in the opposite direction, where over-provision occurs rather than under-provision. The source of inefficiency (over- or under-provision) depends explicitly on both the magnitude of the contribution benefit σ and the nature of the private benefits λ.

Proposition 2. *$M^*(N) = \tilde{M}(N) = N$. Furthermore, if $\lambda = 1$, then*

(i) *$X^* < \tilde{X}$ for all N and σ.*

(ii) *Both X^* and $\tilde{X}$ are increasing in σ.*

If $\lambda = 0$, then for every N there exists a function $\overline{\sigma}(N)$ such that

(iii) *X^* is increasing in σ while $\tilde{X}$ is unaffected by σ.*

(iv) *$X^* > \tilde{X}$ if and only if $\sigma > \overline{\sigma}(N)$.*

When $\lambda = 1$, only the positive free-rider externality is present. Let $\kappa(x_i) = \sigma b(x_i) - c(x_i)$ and define $x' \equiv \arg\max \sigma b(x_i) - c(x_i)$. Given the first-order condition

$$\frac{dB\left(X^*\right)}{dX} + \sigma\frac{\partial b(x^*)}{\partial x_i} - \frac{dc(x^*)}{dx_i} = 0,$$

$x^* > x'$. Otherwise, the left-hand side would be strictly positive, as at $x^* \leq x'$, $\sigma\frac{\partial b(x^*)}{\partial x_i} - \frac{dc(x^*)}{dx_i} \geq 0$ while $\frac{dB(X^*)}{dX} > 0$. Therefore, if we replace $\sigma b(x_i) - c(x_i)$ with $\kappa(x_i)$ and use the fact that $\frac{\partial\kappa(x^*)}{\partial x_i} < 0$ for all σ, then public goods with absolute private benefits resemble the textbook public goods model. The only difference is that both x^* and $\tilde{x}$ are increasing in σ,

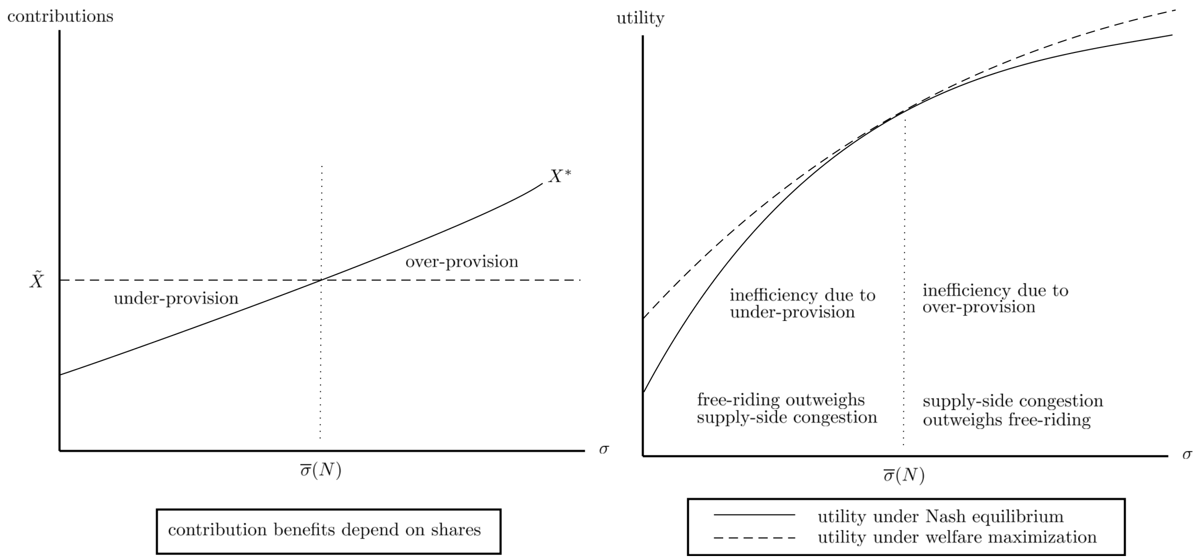

Figure 2: Contributions and utility when public goods have contribution benefits.

so contributions may be larger, but the equilibrium and welfare implications are unchanged. The severity of the inefficiency due to under-provision still increases with the population size. In summary, private incentives to contribute to public goods that accumulate based on the level an individual contributes scales up provision, but by less than what is socially optimal. The following example illustrates this relationship.

Example 2. *Suppose that $\gamma = 0$, $\sigma > 0$, and $\lambda = 1$. Then by (2), each contributor's utility is*

$$u(x_i, X, \sigma) = X + \sigma x_i - \frac{1}{2}x_i^2.$$

Aggregate contributions are given by

$$X^* = N(\underbrace{1+\sigma}_{x^*}), \qquad \tilde{X} = N(\underbrace{N+\sigma}_{\tilde{x}}).$$

Contributions are increasing in σ and free-riding persists and worsens as the population size grows: $\tilde{X} - X^* = N(N-1)$.

When $\lambda = 0$, there is a tension between two externalities. The standard positive externality is present. There is also a negative externality. Contributors enter an arms race where, in order to capture greater private benefits, each contributor seeks to contribute more than their peers. Given both the diminishing marginal returns to and the increasing marginal cost of contributing, each contributor imposes a negative congestion externality on the other contributors. Unlike the typical congestion externality found in club goods, which is observed on the demand side, this is congestion externality is observed on the supply side. In equilibrium, all contributors contribute identical amounts and thus equal shares. Could the contributors commit decreasing their contributions, contribution costs decrease and total welfare increases. When maximizing welfare, the value of σ is irrelevant. Only that each individual contributes an equal share is relevant.

If σ is small, then the free-rider externality outweighs the supply-side congestion externality and there is inefficiency due to under provision. If the contribution benefits are large, then the supply-side congestion externality outweighs the free-rider externality and there is inefficiency, but due to over-provision rather than under-provision. This tension is visualized in Figure 2. However, the effect of the population size on the cutoff value $\overline{\sigma}(N)$ depends explicitly on the shape of the benefits function.

To illustrate this dependence, suppose that $\lim_{X\to\infty} \frac{dB(X)}{dX} = 0$, as in Gradstein and Nitzan (1990). This assumption can be interpreted as technology constraints limiting the benefits of consuming a public good (such as open source software and limits to computing power). If $\frac{\partial b(1/N)}{\partial N}$ is sufficiently large and $\lim_{N\to\infty} \frac{\partial b(1/N)}{\partial N} = \infty$, then $\sigma(N)$ is decreasing in N for N large. Thus, there is over provision in large populations. If $b(\cdot)$ is linear, or the marginal utility from contributing is finite, $\lim_{N\to\infty} \frac{\partial b(1/N)}{\partial N} < \infty$, then under provision occurs in large populations.[18] This distinction is important, as this ambiguity makes it difficult to develop policy to elicit the optimal contribution levels. With over-provision, contributions should be taxed and with under-provision, they should be subsidized. The following example highlights these results.

Example 3. *Suppose that $\gamma = 0$, $\sigma > 0$, and $\lambda = 0$. Then by (2), each contributor's utility*

[18]Note that $\lim_{N\to\infty} \frac{\partial b(1/N)}{\partial N} = \infty$ is not sufficient for over-provision.

is

$$u(x_i, X; \sigma) = X + \sigma\frac{x_i}{X} - \frac{1}{2}x_i^2.$$

The Nash equilibrium and welfare-maximizing contributions are

$$X^* = N\Bigg(\underbrace{\frac{1}{2} + \sqrt{\frac{1}{4} + \sigma\frac{N-1}{N^2}}}_{x^*}\Bigg), \qquad \tilde{X} = N \times \underbrace{N}_{\tilde{x}}.$$

As in Proposition 2(iii), $\tilde{X}$ *is unaffected by* σ *while* X^* *is strictly increasing in* σ*. Eventually,* X^* *must exceed* $\tilde{X}$*. Equating* X^* *and* $\tilde{X}$ *and solving for* σ *yields the cutoff*

$$\overline{\sigma}(N) = N^3,$$

which is increasing in N *as expected given the linearity of* $b\left(\frac{x_i}{X}\right)$*.*

Before analyzing public goods with both costly consumption and private benefits, it is important to consider the case in which only a subset of the population can contribute to the public good. For example, the typical software user does not have the skills necessary to contribute to open source software and most Wikipedia users do not contribute.

Proposition 3. *Suppose that* $\lambda = 0$ *and there are* $M' < N$ *potential contributors, where* $M' \geq 2$*. Then* $M^*(N) = \tilde{M}(N) = M'$*. Furthermore, there exists a cutoff value* $\overline{\overline{\sigma}}(M, N)$ *such that* $X^* > \tilde{X}$ *if and only if* $\sigma > \overline{\overline{\sigma}}(M', N)$*.*

For a given σ, over-provision may be more likely when only a subset of the population contributes (as illustrated in Example 4 below). This point is important, as most public goods have free-riders and evidence suggests that contribution benefits are more valuable when contributions are more visible (Glazer, Niskanen, and Scotchmer, 1997; Holmström, 1999; Andreoni and Petrie, 2004; Alpizar, Carlsson, and Johansson-Stenman, 2008; Bekkers and Wiepking, 2011). There is often greater contributor visibility when there are fewer contributors (e.g. listing the contributors). When maximizing welfare, the optimal individual contribution is not a function of the number of contributors, but of the population as a whole. To the contrary, the equilibrium contribution is a decreasing function of the number of contributors. With fewer contributors, each contributor contributes a greater amount. In the presence of the supply-side congestion externality, this increase outweighs the losses in contributions from having fewer contributors. The welfare losses from over-provision are borne by the contributors. The $N - M'$ free-riders benefit from the increased provision. The contributors experience their losses through the increased contribution costs $c(\cdot)$ caused by the arms race to contribute the greatest share. The following example augments Example 3 by limiting the number of potential contributors.

Example 4. *Referring to Example 3, but with M' contributors,*

$$X^* = M' \underbrace{\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \sigma \frac{M' - 1}{(M')^2}} \right)}_{x^*}, \qquad \tilde{X} = M' \times \underbrace{N}_{\tilde{x}}.$$

Equating these two values and solving for σ yields

$$\overline{\overline{\sigma}}(M', N) = \frac{(M')^2}{M' - 1} N(N - 1).$$

Note that,

$$\overline{\overline{\sigma}}(M', N) < \overline{\sigma}(N) \Longleftrightarrow M' > \frac{N}{N - 1},$$

which is true for $2 \leq M' < N$.

Note that the relationship between the cutoff values in Examples 3 and 4 need not be true in general. That is, the minimum σ to generate over-provision when only a subset of the population can contribute need not be less than the minimum σ when the entire population can contribute. Specifically, the relationship depends on the shape of $b(1/M)$, particularly its slope as M increases.

In summary, strong relative private benefits are sufficient to induce over-provision. When the benefits accumulate absolutely there is under-provision. With under-provision, the loss in utility stemming from the inefficiency is primarily borne by the consumers. With over-provision, the loss in utility stemming from the inefficiency is borne by the contributors.

3.3 Public Goods with Costly Consumption and Private Benefits ($\gamma = 1$ and $\sigma > 0$)

I now consider public goods with both costly consumption and private benefits ($\gamma = 1$ and $\sigma > 0$). Contributor utility is as specified in (1). With costly consumption, the quality maximizer $\hat{X}$ is once again well-defined. When contributing, individual i' first-order condition is given by

$$\frac{dB\left(X_{-i}+x^*\right)}{dX} - \frac{dC\left(X_{-i}+x^*\right)}{dX} - \frac{dc(x^*)}{dx_i} + \sigma \frac{\partial b\left(\lambda x_i^* + (1-\lambda)\frac{x_i^*}{X_{-i}+x_i^*}\right)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{X_{-i}}{(X_{-i}+x_i^*)^2}\right) = 0.$$

From the above first-order condition, one of three properties will be satisfied in equilibrium.

Property 1. (P1) $\frac{dB(X^*)}{dX} - \frac{dC(X^*)}{dX} > 0 > \sigma \frac{\partial b\left(\lambda x^* + (1-\lambda)\frac{1}{M^*(N)}\right)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{M^*(N)-1}{M^*(N)^2 x^*}\right) - \frac{dc(x^*)}{dx_i}$.

Property 2. (P2) $\frac{dB(X^*)}{dX} - \frac{dC(X^*)}{dX} < 0 < \sigma \frac{\partial b\left(\lambda x^* + (1-\lambda)\frac{1}{M^*(N)}\right)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{M^*(N)-1}{M^*(N)^2 x^*}\right) - \frac{dc(x^*)}{dx_i}$.

Property 3. (P3) $\frac{dB(X^*)}{dX} - \frac{dC(X^*)}{dX} = 0 = \sigma \frac{\partial b\left(\lambda x^* + (1-\lambda)\frac{1}{M^*(N)}\right)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{M^*(N)-1}{M^*(N)^2 x^*}\right) - \frac{dc(x^*)}{dx_i}$.

Each property relates the aggregate equilibrium contributions X^* to the quality-maximizing contributions $\hat{X}$. Under Property 1, the equilibrium quality is suboptimal with too few contributions. Under Property 2, the equilibrium quality is also suboptimal, but with too many contributions. Under Property 3, quality is maximized in the equilibrium.

Property 2 illustrates an important quantity-quality tradeoff. Quality – the net utility from consumption – is decreasing in aggregate contributions. Yet each individual contributor can benefit from contributing when the contribution benefit outweighs the contribution cost. If σ is large enough, such an incentive is present. In addition, when the equilibrium quality is suboptimally low due to too little investment in the public good, there is under-provision. When the equilibrium quality is suboptimally low due to too much investment in the public good, there is over-provision. This relationship is not obvious as maximizing welfare does not correspond to maximizing quality except when $N \to \infty$.[19] This argument is formalized

[19] These statements are proven in Proposition 4.

below.

With the interaction of consumption costs and contribution benefits, $M^*(N)$ need not equal N as it does when only one is present (either $\gamma = 0$ or $\sigma = 0$). Denote by $M^U \in \mathbb{R}_+$ the M that makes (3) hold with equality. The maximum number of contributors is given by $\lfloor M^U \rfloor$. If there are M contributors and σ is sufficiently large, then (4) cannot be satisfied as both $B(x_i + X_{-i}) - C(x_i + X_{-i})$ and $\sigma b\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - c(x_i)$ are increasing for small x_i. Thus either $P2$ or $P3$ must be satisfied. If σ is sufficiently small, then all N individuals contribute. If either $P2$ or $P3$ is satisfied, then the value of M that makes (3) hold with equality (M^U) indicates the upper bound on the number of contributors in equilibrium. To see this relationship, note that under $P2$ and $P3$, $\sigma b\left(\lambda x^* + (1-\lambda)\frac{1}{M}\right) - c(x^*)$ is decreasing in M while $B\left(X^*_{-i}\right) - C\left(X^*_{-i}\right) - [B\left(X^*\right) - C\left(X^*\right)]$ is increasing in M, so (3) is violated for all $M > M^U$. The value of M, denoted by M^L, that makes (4) hold with equality indicates the lower bound on the number of contributors. It follows that (4) is violated for all $M < M^L$.

Consequently, the maximal number of contributors is determined by $\min\left\{\lfloor M^U \rfloor, N\right\}$. If $N > M^U$, then $N - \lfloor M^U \rfloor$ individuals do not contribute in equilibrium. The correspondence

$$M^*(N) : N \to \left\{Z \in \mathbb{N} \,|\, M^L \leq Z \leq \min\left\{M^U, N\right\}\right\}$$

defines the equilibrium number of contributors as a function of the population size. As welfare-maximization internalizes the externalities, $\tilde{M}(N) \leq M^*(N)$.

The relationships between the equilibrium contributions, welfare-maximizing contributions, and the quality-maximizing contributions are outlined in the following proposition.

Proposition 4. *There exists a function $\overline{\sigma}(N)$ such that the following holds.*

(i) *If $\sigma < \overline{\sigma}(N)$, then $X^* < \tilde{X} < \hat{X}$.*

(ii) *If $\sigma > \overline{\sigma}(N)$, then $X^* > \tilde{X} > \hat{X}$.*

(iii) $\lim_{N\to\infty} \overline{\sigma}(N) = 0$ *and* $\lim_{N\to\infty} \tilde{X} = \hat{X}$*, so* $\lim_{N\to\infty} X^* > \lim_{N\to\infty} \tilde{X}$ *for all $\sigma > 0$.*

Moreover, $\overline{\sigma}(N)$ is strictly decreasing in N if $\lambda = 1$, but need not be monotonic in N if $\lambda = 0$.

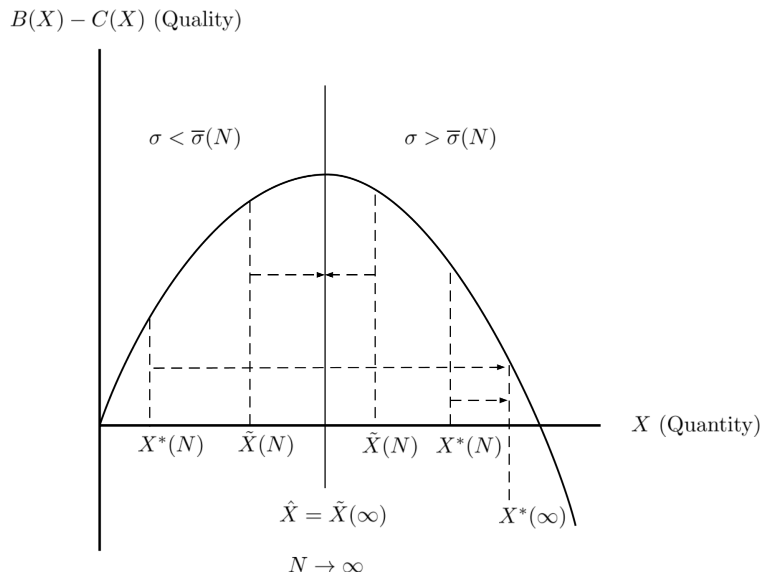

Figure 3: Equilibrium, welfare-maximizing, and quality-maximizing contributions.

Figure 3 graphically illustrates the Proposition.

Like the previous case, public goods with costly consumption and private benefits suffer from over-provision when σ is sufficiently large. However, there are a few critical differences that have substantial implications. First, as under-provision entails the aggregate equilibrium contributions falling below both the welfare maximizer and the quality maximizer and over-provision entails the aggregate equilibrium contributions landing above both the welfare maximizer and quality maximizer, consumers bear the greater share of the utility losses stemming from the inefficiency than do contributors. Second, over-provision can occur for both $\lambda = 0$ and $\lambda = 1$, whereas $\lambda = 0$ is required for over-provision when $\gamma = 0$.

When $\lambda = 1$, free-riding consumers bear the entire burden. With under-provision, consumers suffer from an inefficiently low quality public good. While the contributors suffer from this low quality as well, their suffering is mitigated via the private contribution benefit received. As that benefit rises (σ increases), contributors are incentivized to contribute even more until the public good becomes a public bad: $\frac{dB(X^*)}{dX} - \frac{dC(X^*)}{dX} < 0$. Additional contributions impose a negative externality on consumers, further depressing their utility. Over-provision becomes more likely as the population grows, as contributors do not impose a congestion externality on each other, inducing more individuals to contribute a greater amount to obtain those benefits. Thus the necessary σ is decreasing in the population size. In contrast to Linus's law, this outcome conforms with an old English proverb, "too many cooks spoil the broth."

When $\lambda = 0$, the same general pattern emerges. The key distinction is not only do additional contributions eventually impose a cost on consumers, but additional contributions also impose an additional cost on other contributors as well. For small population sizes, $\overline{\sigma}(N)$ may be increasing. Observe the first-order condition for a contributor and the welfare-maximizing condition when all individuals consume:

$$\frac{dB(X^*)}{dX} - \frac{dC(X^*)}{dX} + \sigma \frac{\partial b\left(\frac{1}{N}\right)}{\partial x_i}\left(\frac{N-1}{Nx^*}\right) - \frac{dc(x^*)}{dx_i} = 0$$
$$N\left(\frac{dB(\tilde{X})}{dX} - \frac{dC(\tilde{X})}{dX}\right) - \frac{dc(\tilde{x})}{dx_i} = 0.$$

When N is small, the free-riding externality is not significant, nor is the consumption cost $C(\cdot)$. Hence, early contributions are favorable to all, as they impart consumption and contribution benefits. That is, both X^* and $\tilde{X}$ are increasing in N and the environment is similar to the case with no consumption costs ($\gamma = 0$), which implies an increasing $\overline{\sigma}(N)$. As the peak in quality $\hat{X}$ is approached, the welfare maximizing program suggests cutting back on contributions to not exceed the quality maximizer so as to not harm consumers, particularly as N grows and consumers become relatively more important. Yet as N grows, the individual incentives are such that aggregate contributions X^* increase. The result is a smaller σ yielding over-provision, or equivalently a decreasing $\overline{\sigma}(N)$. The following example illustrates this point.

Example 5. *Suppose that $\gamma = 1$ and $\sigma > 0$. Then by (2), each contributor's utility is*

$$u(x_i, X; \sigma) = X - \frac{1}{40}X^2 + \sigma\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - \frac{1}{2}x^2,$$

leading to contributions of

$$X^* = \begin{cases} N\left(\frac{20(1+\sigma)}{20+N}\right) & \text{if } \lambda = 1 \\ N\left(\frac{2\left(5N+\sqrt{(5N)^2+5\sigma(N+20)(N-1)}\right)}{N(20+N)}\right) & \text{if } \lambda = 0, \end{cases} \qquad \tilde{X} = \begin{cases} N\left(\frac{20(N+\sigma)}{20+N^2}\right) & \text{if } \lambda = 1 \\ N\left(\frac{20N}{20+N^2}\right) & \text{if } \lambda = 0. \end{cases}$$

The welfare maximizer always converges to $20 = \hat{X}$ *as* $N \to \infty$ *while* X^* *converges to* $20 + \sigma$ *under levels and to* $10 + 2\sqrt{5}\sqrt{5+\sigma} > 20$ *under shares. Equating* X^* *and* $\tilde{X}$ *yields*

$$\overline{\sigma}(N) = \begin{cases} \frac{20}{N} & \text{if } \lambda = 1 \\ \frac{20}{N}\left(\frac{2\sqrt{5}N^2}{20+N^2}\right)^2 & \text{if } \lambda = 0. \end{cases}$$

In both cases, $\lim_{N\to\infty} \overline{\sigma}(N) = 0$*; however, when contribution benefits are derived from shares,* $\overline{\sigma}(N)$ *is increasing if* $N \leq 7$ *and decreasing if* $N \geq 8$.

When $\sigma \gg 0$, the critical distinctions between public goods with and without costly consumption ($\gamma = 1$ v. $\gamma = 0$) are the side of the market that bears the costs of over-provision and what happens when the population grows large. When $\gamma = 0$, the lost utility from over-provision is isolated among the contributors. Consumers benefit with increased utility from free-riding, as $B(X)$ is strictly increasing in X. Over-provision is also less likely as the population grows, so the environment is more likely to resemble typical under-provision, where the costs are (primarily) borne by consumers with suboptimal under-provision. When $\gamma = 1$, much of the lost utility is transferred to the consumers rather than the contributors, particularly if $\lambda = 1$. Consumers are faced with an inefficiently low-quality public good due to costly consumption, so much so that they may opt to not even free-ride and instead not consume at all. While contributors are also faced with an inefficiently low-quality public good due to over-provision, they are more than compensated through the contribution benefit. Over-provision is more likely as the population grows, making this case the most severe and concerning with large populations.

Along the entire spectrum of collaboratively provided public goods studied here, equilibrium quality is monotonically increasing in the number of contributors whenever $\sigma = 0$. If $\sigma > 0$, then the relationship between quality and quantity is non-monotonic (the tradeoff) when $\gamma = 1$. Over-provision can occur whenever a contribution benefit is present except when $\gamma = 0$ and $\lambda = 1$. Collectively, the takeaway from these points cannot be understated. When constructing a policy to optimize the provision of public goods, programs designed to increase contributions may backfire and lead to increased inefficiency. Moreover, the type of public good must be considered, particularly if a policymaker is not only interested in maximizing welfare, but also concerned with equity and the distribution of surpluses. The presence of consumption costs alters which side of the market bears a greater share of the inefficiency.

——————————— When $\sigma < \overline{\sigma}(N)$, there is under-provision and the free-rider problem persists and when $\sigma > \overline{\sigma}(N)$, supply-side congestion occurs. Hence, both Linus's law and the proverb are consistent with the collaborative provision of public goods. Which statement is applicable is determined by the presence of γ, σ, and λ. When $\gamma = 1$ and $\sigma = 0$, Linus's law is supported and when $\gamma = 1$ and $\sigma > \overline{\sigma}(N)$, the proverb is supported.

In summary, there is no globally optimal policy and both policymakers and researchers alike must be cognizant of the nature of the public good in question, carefully identifying the underlying parameters γ, σ, and λ.

4 Inefficiently Complex Public Goods

Research suggests that regulation tends to be overly complex (Kearl, 1983; Quandt, 1983; Allison and Lemley, 2002; Berglund, 2014), leading to inefficiency (Murray and Stern, 2007; Williams, 2013). Similarly, more complex open source software projects tend to have lower user-satisfaction scores (Stamelos, Angelis, Oikonomou, and Bleris, 2002) and there tends to be less subsequent innovation in more complex industries (Galasso and Schankerman, 2015). Proposition 4 predicts such an outcome for public goods with both consumption costs and contribution benefits. Yet, many public goods including R&D innovation, open source software, and public policy, are not produced by independent individually-rational utility-maximizers. Rather, individual efforts are often coordinated by a leader or leadership team.

Open source software projects are guided by a leadership team. While contributors act independently to maximize utility, they are not necessarily free to contribute as they see fit. In this section, I focus on a specific policy restriction: fixing the number of contributors. The leaders of open source software projects can choose whether or not to accept contributions. If this 'social planner' is interested in maximizing welfare, subject to contributing individuals satisfying individual rationality, how will this individual choose the number of contributors?

The objective of the social planner is given by

$$\max_M \Bigg\{ (N-M)\max\{B(X)-C(X),0\} \\ + \sum_{j=1}^{M}\left[B(X)-C(X)+\sigma b\left(\lambda x_j+(1-\lambda)\frac{x_j}{X}\right)-c(x_j)\right]\Bigg\}, \quad (5)$$

subject to

$$\begin{aligned} & x_j = x(M,\sigma) = \arg\max\left\{B(X)-C(X)+\sigma b\left(\lambda x_j+(1-\lambda)\frac{x_j}{X}\right)-c(x_j)\right\} \quad \forall\, j\le M \\ & x_j = 0 \quad \forall\, j>M \end{aligned} \quad (6)$$

Denote by $\hat{M}$ the solution to (5).

Firstly note that there can be no interior solution $\hat{M} < \overline{\sigma}^{-1}(\sigma)$ such that

$$\overline{\sigma}^{-1}(\sigma) < \frac{\hat{M}+(\hat{M}+1)}{2}.$$

Otherwise, adding one more contributor increases the both the quality and the contribution benefit for the original $\hat{M}$ contributors while also increasing the utility of the new contributor by the marginal change in the quality plus the entirety of the contribution benefit, leading to a strict welfare gain.[20] This logic cannot be extended to the other side of $\overline{\sigma}^{-1}(\sigma)$. Suppose that $\hat{M} > \overline{\sigma}^{-1}(\sigma)$ and there exists a value $\hat{M}-1$ such that

$$\overline{\sigma}^{-1}(\sigma) < \frac{(\hat{M}-1)+\hat{M}}{2}.$$

By decreasing $\hat{M}$ to $\hat{M}-1$, the quality increases, as do the contribution benefits for each of the $\hat{M}$ contributors. Yet, this change may actually lead to a decrease in total welfare if the change in the quality, weighted by the number of free riders plus the change in the contribution benefit, weighted by the number of contributors, is less than the loss in the

[20] If $\sigma = 0$, then the consumption benefit is a net loss on the new contributor, but by (4), the marginal impact of the loss is less than the marginal increase in quality leading to a positive change in utility.

contribution benefit by the $\hat{M}^{th}$ contributor:

$$
\begin{aligned}
(N-M)\big[&\max\{B(X(\hat{M}-1,\sigma))-C(X(\hat{M}-1,\sigma)),0\}\\
&-\max\{B(X(\hat{M},\sigma))-C(X(\hat{M},\sigma)),0\}\big]\\
&+M\big[B(X(\hat{M}-1,\sigma))-C(X(\hat{M}-1,\sigma))\\
&-[B(X(\hat{M},\sigma))-C(X(\hat{M},\sigma))]\big]\\
&+(\hat{M}-1)\left[\sigma b\left(\lambda x(\hat{M}-1,\sigma)+(1-\lambda)\frac{1}{\hat{M}-1}\right)-c(x(\hat{M}-1,\sigma))\right.\\
&-\left.\left[\sigma b\left(\lambda x(\hat{M},\sigma)+(1-\lambda)\frac{1}{\hat{M}}\right)-c(x(\hat{M},\sigma))\right]\right]\\
&<\sigma b\left(\lambda x(\hat{M},\sigma)+(1-\lambda)\frac{1}{\hat{M}}\right)-c(x(\hat{M},\sigma)).
\end{aligned}
\tag{7}
$$

Increasing quality by decreasing the number of contributors increases the utility for each contributor and each free rider, but may decrease welfare due to the decrease in the utility of the contributor turned free rider.

Proposition 5. *If* (7) *is satisfied, then the optimal strategy for a social planner interested in maximizing total welfare need not match the optimal strategy for a social planner interested in maximizing quality.*

This relationship illustrates a distinct equity tradeoff. Maximizing quality also yields the most equitable outcome without direct transfers, as this policy satisfies (P3) and puts each individual's utility at its peak. When inequality (7) is satisfied, welfare is increased by slightly decreasing utility from $N-1$ individuals and transferring it to a single individual. Moreover, $N-\hat{M}+1$ of the individuals experiencing the decrease are free-riders who already have lower utility than the contributors. Proposition 5 has bite due to the fact that maximizing quality is akin to concurrently maximizing both the quality and the net contribution benefit for any given contributor. For the given set of contributors and consumers, each individual's utility is at its peak. A tension still remains on the extensive margin. Adding an additional contributor may decrease both the quality for all consumers and the contribution benefits for the original set of contributors, but increases the new contributor's payoff by becoming a contributor are large enough that aggregate welfare of the population as a whole is increased.

Even if there is an integer M such that $M = \overline{\sigma}^{-1}(\sigma)$, it may not be the welfare maximizer. Supply-side congestion exists on both the intensive and extensive margins.

The above shows that a social planner is willing to sacrifice the utility of a free rider to increase the utility of a contributor in order to maximize total welfare. In the context of open source software, it may be worthwhile (from this aggregate welfare-maximizing utilitarian standpoint) to accept contributions that makes the software inefficiently complex in order to benefit the contributors at the expense of the users.

Corollary 1. *If the set of potential contributors is a strict subset of the population and σ is small, then for a finite population, a social planner seeking to maximize total welfare will never choose the maximum number of contributors that maximizes quality.*

Corollary 1 provides a general extension of the results in Kearl (1983), Quandt (1983), Nitzan, Procaccia, and Tzur (2013), and the other associated works on regulatory complexity. The economics underpinning regulatory capture are not unique to regulation, and can be applied to public goods more broadly. Under this welfare-maximization program, the social planner transfers surpluses from free riders to contributors. In order to obtain a more equitable distribution of surpluses between free riders and contributors, additionally measures need to be put in place that limit the incentives for over-provision, e.g., anonymizing contributions to lower σ.

5 Extension: Hybrid Private Benefits to Contributors

In this section, I consider the case in which the private benefits to contributors are only partially rivalrous by allowing $\lambda \in [0, 1]$. When $\lambda \in (0, 1)$, there is both an absolute and relative component to the private contributor benefits. In what follows, I show that, under certain conditions on λ, the main over-provision result carries over for all public goods with private benefits to contributors.

Proposition 6. *Suppose that $\gamma = 0$. There exists a value $\overline{\lambda}$ and function $\overline{\sigma}(N, \lambda)$ such that if $\lambda < \overline{\lambda}$ and $\sigma > \overline{\sigma}(N, \lambda, \gamma)$, then there is over-provision . If $\gamma = 1$, then there is a cutoff* *value $\overline{\sigma}(N,\lambda)$ such that over-provision occurs whenever $\sigma > \overline{\sigma}(\lambda)$.*

Proposition 6 extends the results of Propositions 2 and 4. Proposition 4 of Section 3.2 showed that over-provision can occur for both $\lambda = 0$ and $\lambda = 1$. By continuity, this over-provision carries over for all $\lambda \in (0,1)$. On the other hand, Proposition 2 in Section 3.2 showed that over-provision can only occur for $\lambda = 0$ when $\gamma = 0$. In general, some level-based benefits can still lead to over-provision in this case, but only if majority of the benefits are derived via relative shares. For example, in the context of ability or wealth, individuals must have preferences that strongly favor signaling relative ability or relative wealth over absolute ability or wealth.

I conclude this section with an example to illustrate these results.

Example 6. *Suppose that $\gamma = 0$, so by (2), contributor utility is given by*

$$u(x_i, X; \sigma) = X + \sigma\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - \frac{1}{2}x_i^2.$$

The equilibrium and welfare-maximizing contributions are given by

$$X^* = N\Bigg(\underbrace{\frac{1+\lambda\sigma}{2} + \sqrt{\frac{(1-\lambda\sigma)^2}{4} + \sigma\frac{(1-\lambda)(N-1)}{N^2}}}_{x^*}\Bigg), \quad \tilde{X} = N\underbrace{(N+\lambda\sigma)}_{\tilde{x}}.$$

To demonstrate over-provision, note that $X^* = \tilde{X}$ if and only if $\sigma = \frac{N^3}{1-\lambda(1+N^2)} \equiv \overline{\sigma}(N,\lambda)$, which is well defined (as $\sigma \geq 0$) only if $\lambda < \frac{1}{1+N^2}$. Thus, over-provision persists so long as $\lambda < \frac{1}{1+N^2}$ and $\sigma > \frac{N^3}{1-\lambda(1+N^2)}$.

Now suppose that $\gamma = 1$, so by (2), contributor utility is

$$u(x_i, X; \sigma) = X - \frac{1}{40}X^2 + \sigma\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - \frac{1}{2}x_i^2.$$

The equilibrium and welfare-maximizing contributions are given by

$$X^* = N\Bigg(\underbrace{\frac{10(1+\lambda\sigma)}{20+N} + 2\sqrt{5}\sqrt{\frac{(19N-20)(1-\lambda)\sigma}{N^2(20+N)} + \frac{5+\sigma+\lambda\sigma(9+5\lambda\sigma)}{20+N}}}_{x^*}\Bigg)$$

$$\tilde{X} = N\Bigg(\underbrace{\frac{20(N+\lambda\sigma)}{20+N^2}}_{\tilde{x}}\Bigg)$$

Equating these two values and solving for σ yields the cutoff

$$\overline{\sigma}(N,\lambda) = \frac{-400(1-\lambda)}{40\lambda^2 N^3} - \frac{1}{40\lambda N^2}\Bigg[40 - 440\lambda + N\Bigg(N + 19\lambda N - \sqrt{\frac{(20-N^2)^2\left(400(1-\lambda)^2 + N^4(1+19\lambda)^2 - 40N^2(1-\lambda)(21\lambda-1)\right)}{N^6}}\Bigg)\Bigg].$$

6 Concluding Remarks

When extending the standard theory of public goods to incorporate costly consumption and private benefits, well known incentive problems change and new issues are introduced. While inefficiencies due to free-riding may still persist, over-provision due to supply-side congestion also presents a concern. Rather than conforming to Linus's law, public goods with private benefits or both private benefits and costly consumption conform with the old English proverb, "too many cooks spoil the broth," echoing the supply-side congestion discussed above. The textbook free-rider problem, Linus's law, and the proverb are incompatible with one-another, yet they can all describe properties of public goods. Together, these statements reinforce the importance of carefully considering the heterogeneity of public goods and avoiding one-size-fits-all policy approaches to address inefficiencies in provision.

Traditional efforts, which are often designed to counter under-provision, can backfire and lead to an outcome with higher total utility in the aggregate but lower utility for consumers and higher utility for contributors than what would occur voluntarily. A policy increasing provision is beneficial for public goods with costly consumption, yet may be deleterious if the public good also provides private benefits. Therefore, policies must be evaluated on case-by-case bases, severely limiting the inferences that can be drawn from specific studies. Moreover, public goods such as open source software may themselves vary case-by-case. For example, the potential for private benefits varies across projects, as do consumption costs. Hence specific public goods may not fall under a single specific type. That is, not all open source software projects are type (iv) public goods (possessing costly consumption and private benefits). This further limits the external validity of specific policy studies. These observations open a door for policies that alter private incentives (e.g. disclosure policies).

An interesting extension to consider is the dynamic provision of public goods with costly consumption and private benefits, incorporating other forms of hybrid private benefits to contributors. Regulations, statutes, and other legal guidelines such as tax codes tend to evolve over time with contributors (regulators) who have their own private and public incentives. The quality-enhancing properties of early contributions can induce a feedback loop that increases the contributions made in these regulations over time. The early contributions may lead to improvements in the quality of these regulations, but over time, these improvements diminish and eventually have deleterious effects, decaying the quality of these regulations. A similar story can be told with respect to open source software. It is natural to consider a form of hybrid benefits in this context. Early on, private benefits to contributors are relative and accumulate via shares as the project is new and there are few contributors, making it easier to signal relative ability. As the public good grows and more individuals contribute, the benefits transition to absolute levels, as its harder to make comparisons when each individual contributes a small share relative to the total.

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Appendix

Proof of Lemma 1.

Proof. The (interior) first-order condition with respect to i is

$$\frac{dB\left(X^*\right)}{dX} - \gamma\frac{dC\left(X^*\right)}{dX} + \sigma\frac{\partial b\left(\lambda x_i^* + (1-\lambda)\frac{x_i^*}{X^*}\right)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{X_{-i}^*}{(X^*)^2}\right) - \frac{dc(x_i^*)}{dx_i} = 0.$$

Hence, for an arbitrary i and j,

$$\frac{dB\left(X^{*}\right)}{dX}-\gamma\frac{dC\left(X^{*}\right)}{dX}+\sigma\frac{\partial b\left(\lambda x_{i}^{*}+(1-\lambda)\frac{x_{i}^{*}}{X^{*}}\right)}{\partial x_{i}}\left(\lambda+(1-\lambda)\frac{X_{-i}^{*}}{(X^{*})^{2}}\right)-\frac{dc(x_{i}^{*})}{dx_{i}}$$
$$=\frac{dB\left(X^{*}\right)}{dX}-\gamma\frac{dC\left(X^{*}\right)}{dX}+\sigma\frac{\partial b\left(\lambda x_{j}^{*}+(1-\lambda)\frac{x_{j}^{*}}{X^{*}}\right)}{\partial x_{j}}\left(\lambda+(1-\lambda)\frac{X_{-j}^{*}}{(X^{*})^{2}}\right)-\frac{dc(x_{j}^{*})}{dx_{j}},$$

which simplifies to

$$\sigma\left(\frac{\partial b\left(\lambda x_{i}^{*}+(1-\lambda)\frac{x_{i}^{*}}{X^{*}}\right)}{\partial x_{i}}\left(\lambda+(1-\lambda)\frac{X_{-i}^{*}}{(X^{*})^{2}}\right)-\frac{\partial b\left(\lambda x_{j}^{*}+(1-\lambda)\frac{x_{j}^{*}}{X^{*}}\right)}{\partial x_{j}}\left(\lambda+(1-\lambda)\frac{X_{-j}^{*}}{(X^{*})^{2}}\right)\right)=\frac{dc(x_{i}^{*})}{dx_{i}}-\frac{dc(x_{j}^{*})}{dx_{j}}$$

If $x_i^* > x_j^*$, then the right-hand side is nonpositive while the left-hand side is strictly positive. Thus, $x_i^* = x_j^*$. □

Proof of Proposition 1.

Proof. Set $\sigma = 0$ and $\gamma = 1$. For $x_i^* > 0$,

$$\frac{dB\left(X^{*}\right)}{dX}-\frac{dC\left(X^{*}\right)}{dX}-\frac{dc(x_{i}^{*})}{dx_{i}}=0, \tag{8}$$

By Lemma 1, $x_i^* = x_j^* = x^*$. Moreover,

$$N\left(\frac{dB\left(X\right)}{dX}-\frac{dC\left(X\right)}{dX}\right)>0$$

for all X and N as $\frac{dc(x)}{dx_i} > 0$ when $x > 0$. Hence, there is no equilibrium with $M < N$ as at least one non-contributing individual has a profitable deviation. Thus, there exists a unique symmetric Nash equilibrium with N contributors and $M^*(N) = N$. The relevant first-order condition is now

$$\frac{dB\left(X(N,0)\right)}{dX}-\frac{dC\left(X(N,0)\right)}{dX}-\frac{dc\left(x_{i}(N,0)\right)}{dx_{i}}=0,\quad\forall\, i. \tag{9}$$

To prove that the equilibrium level of contributions converges to the quality-maximizing level, suppose to the contrary that $x_i^* \to z > 0$ as $N \to \infty$. Convergence to a positive z implies that $X^* \to \infty$ as $N \to \infty$. It follows that

$$\frac{dB\left(\infty\right)}{dX}-\frac{dC\left(\infty\right)}{dX}<0.$$

As $\frac{dc(z)}{dx_i} < 0$, (9) cannot be satisfied. Thus $x_i^* \to 0$ as $N \to \infty$. As x_i^* approaches zero and $\lim_{N\to\infty} \frac{dc\left(x_i^*\right)}{dx_i} = 0$, (9) holds only if

$$\lim_{N\to\infty} \left(\frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} \right) = 0,$$

which is true when $\lim_{N\to\infty} X^* = \hat{X}$.

Now, consider the welfare maximizing case. The welfare maximizing contributions are determined by the solution to

$$\max_{\mathbf{x}} \sum_{j=1}^{N} u(X, x_j; \sigma).$$

The arbitrary first-order condition with respect to contribution $x_i > 0$ is given by

$$\frac{dB(\tilde{X})}{dX} - \frac{dC(\tilde{X})}{dX} - \frac{1}{N}\frac{dc(\tilde{x}_i)}{dx_i} = 0. \tag{10}$$

By inspection, $\tilde{x}_i = \tilde{x}_j$ for all i, j and by an analogous argument to the above, $\tilde{M}(N) = N$. Taking the limit of (10) as $N \to \infty$ yields $\tilde{X} \to \hat{X}$.

Lastly, to prove that $X^* < \tilde{X} < \hat{X}$ for all finite $N > 1$, note that $\frac{dc(x_i)}{dx_i} > 0$ for all $x_i > 0$. Evaluating the LHS of (9) at $x_i = x_i(N, 0)$ for all i yields

$$\frac{dB\left(X(N,0)\right)}{dX} - \frac{dC\left(X(N,0)\right)}{dX} - \frac{1}{N}\frac{dc\left(x_i(N,0)\right)}{dx_i}.$$

As $\frac{dc(x_i(N,0))}{dx_i} > 0$,

$$-\frac{dc\left(x_i(N,0)\right)}{dx_i} < -\frac{1}{N}\frac{dc\left(x_i(N,0)\right)}{dx_i}.$$

From (9),

$$\begin{aligned}\frac{dB\left(X(N,0)\right)}{dX} - \frac{dC\left(X(N,0)\right)}{dX} - \frac{1}{N}\frac{dc\left(x_i(N,0)\right)}{dx_i}& \\ > \frac{dB\left(X(N,0)\right)}{dX} - \frac{dC\left(X(N,0)\right)}{dX} - \frac{dc\left(x_i(N,0)\right)}{dx_i} &= 0,\end{aligned}$$

so $\tilde{x}_i > x_i^*$ and $\tilde{X} > X^*$ for all i and j and all finite $N > 1$. The presence of $dc(x_i)/dx_i > 0$ for all $x_i > 0$ implies that both X^* and $\tilde{X}$ are less than $\hat{X}$. $\square$

Proof of Proposition 2

Proof. At $x_i = 0$,

$$\frac{dB(X)}{dx} + \sigma\frac{\partial b\,(0)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{1}{X_{-i}}\right) > 0$$
$$N\frac{dB(X)}{dx} + \sigma\frac{\partial b\,(0)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{1}{X_{-i}}\right) > 0,$$

for all $X > 0$, so $M^*(N) = \tilde{M}(N) = N$. The remainder of the proof proceeds in two cases.

Case 1: $\lambda = 1$. The first-order condition for the arbitrary contributor is given by

$$\frac{dB(X^*)}{dX} + \sigma\frac{\partial b(x^*)}{\partial x_i} - \frac{dc(x^*)}{dx_i} = 0$$

As $B(\cdot)$ and $b(\cdot)$ are weakly concave while $c(\cdot)$ is strictly convex, the first two terms are nonincreasing while the third term is strictly in increasing. Therefore, both x^* and X^* are strictly increasing in σ. The welfare-maximizing first-order condition for an arbitrary contributor is given by

$$N\frac{dB(X^*)}{dX} + \sigma\frac{\partial b(x^*)}{\partial x_i} - \frac{dc(x^*)}{dx_i} = 0,$$

so by the same argument, $\tilde{x}$ and $\tilde{X}$ are strictly increasing in σ. Now suppose that $\tilde{x} = x^*$, so $X^* = \tilde{X}$. Then

$$\frac{dB(X^*)}{dX} + \sigma\frac{\partial b(x^*)}{\partial x_i} - \frac{dc(x^*)}{dx_i} = N\frac{dB(\tilde{X})}{dX} + \sigma\frac{\partial b(\tilde{x})}{\partial x_i} - \frac{dc(\tilde{x})}{dx_i} = 0,$$

which reduces to

$$\frac{dB(X^*)}{dX} = N\frac{dB(\tilde{X})}{dX},$$

a contradiction. As the right-hand side exceeds the left and $B(\cdot)$ is weakly concave, it follows that $\tilde{X} > X^*$.

Case 2: $\lambda = 0$. The equilibrium and welfare-maximizing first order conditions for individual i are respectively given by

$$\frac{dB(X^*)}{dX} + \sigma\frac{\partial b\left(\frac{x^*}{X^*}\right)}{\partial x_i}\frac{X^*_{-i}}{(X^*)^2} - \frac{dc(x^*)}{dx_i} = 0$$
$$N\frac{dB(\tilde{X})}{dX} + \sigma\frac{\partial b\left(\frac{\tilde{x}}{\tilde{X}}\right)}{\partial x_i}\frac{\tilde{X}_{-i}}{\tilde{X}^2} - \sigma\sum_{j\neq i}\frac{\partial b\left(\frac{\tilde{x}}{\tilde{X}}\right)}{\partial x_j}\frac{\tilde{x}}{\tilde{X}^2} - \frac{dc(\tilde{x})}{dx_i} = 0.$$

By Lemma 1,

$$\frac{x^*}{X^*} = \frac{\tilde{x}}{\tilde{X}} = \frac{1}{N}, \quad \frac{X^*_{-i}}{(X^*)^2} = \frac{N-1}{N^2x^*}, \quad \text{and} \quad \frac{\tilde{X}_{-i}}{\tilde{X}^2} = \frac{N-1}{N^2\tilde{x}}.$$

Thus, the above first-order conditions can be rewritten as

$$\frac{dB(X^*)}{dX} + \sigma\frac{\partial b\left(\frac{1}{N}\right)}{\partial x_i}\frac{N-1}{N^2x^*} - \frac{dc(x^*)}{dx_i} = 0$$

$$N\frac{dB(\tilde{X})}{dX} + \underbrace{\sigma\frac{\partial b\left(\frac{1}{N}\right)}{\partial x_i}\frac{N-1}{N^2\tilde{x}} - \sigma(N-1)\frac{\partial b\left(\frac{1}{N}\right)}{\partial x_j}\frac{1}{N^2\tilde{x}}}_{=0} - \frac{dc(\tilde{x})}{dx_i} = 0.$$

It follows that $\tilde{x}$ and $\tilde{X}$ are independent of σ. Now, fix N. As

$$\sigma\frac{\partial b\left(\frac{1}{N}\right)}{\partial x_i}\frac{N-1}{N^2x^*}$$

is strictly increasing in σ, it must be that x^* and thus X^* are increasing in σ and a cutoff value $\overline{\sigma}(N)$ exists where $X^* > \tilde{X}$ if and only if $\sigma > \overline{\sigma}(N)$. □

Proof of Proposition 3

Proof. Fix $M' < N$ and $\lambda = 0$. By an analogous argument to that in Proposition 2, $M^*(N) = \tilde{M}(N) = M'$. The relevant first-order conditions for the equilibrium and welfare-maximizing program are respectively given by

$$\frac{dB(X^*)}{dX} + \sigma\frac{\partial b\left(\frac{1}{M'}\right)}{\partial x_i}\frac{M'-1}{(M')^2x^*} - \frac{dc(x^*)}{dx_i} = 0$$

$$N\frac{dB(\tilde{X})}{dX} - \frac{dc(\tilde{x})}{dx_i} = 0.$$

As in Proposition 2,

$$\sigma\frac{\partial b\left(\frac{1}{M'}\right)}{\partial x_i}\frac{M'-1}{(M')^2x^*}$$

is strictly increasing in σ and decreasing in x^*, so $\overline{\overline{\sigma}}(M', N)$ is defined analogously to $\overline{\sigma}(N)$. As $\tilde{X}$ is strictly increasing in N, $\overline{\overline{\sigma}}(M', N)$ is strictly increasing in N. □

Proof of Proposition 4.

Proof. I begin by first proving that the cutoff value $\overline{\sigma}(M)$ exists for any arbitrary number of contributors M. By lemma 1,

$$\frac{\partial x_i(M,\sigma)}{\partial \sigma} = \frac{\partial x_j(M,\sigma)}{\partial \sigma} = \frac{\partial x(M,\sigma)}{\partial \sigma}.$$

The first step is to show that $\frac{\partial x(M,\sigma)}{\partial \sigma} > 0$. Fix M and recall that, for each contributor $i = 1, \dots, M$, it must be that

$$\frac{dB\left(Mx^*\right)}{dX} - \frac{dC\left(Mx^*\right)}{dX} + \sigma \frac{\partial b\left(\lambda x^* + (1-\lambda)\frac{1}{M}\right)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{M-1}{M^2 x^*}\right) - \frac{dc(x^*)}{dx_i} = 0 \quad (11)$$

holds in equilibrium. For expositional convenience, define

$$\begin{aligned} f(x_i) &= \frac{\partial b\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right)}{\partial x_i}\left(\lambda + (1-\lambda)\frac{X_{-i}}{(X)^2}\right) \\ \frac{\partial f(x^*)}{\partial x_i} &= \frac{\partial^2 b\left(\lambda x^* + (1-\lambda)\frac{1}{M}\right)}{\partial x_i^2}\left(\lambda + (1-\lambda)\frac{M-1}{M^2 x^*}\right)^2 \\ &\qquad - \frac{\partial b\left(\lambda x^* + (1-\lambda)\frac{1}{M}\right)}{\partial x_i}(1-\lambda)\frac{M-2}{M^3 x^2} \leq 0 \end{aligned}$$

Differentiating (11) with respect to σ yields

$$\begin{aligned} &M \underbrace{\left(\frac{d^2 B(X(M,\sigma))}{dX^2} - \frac{d^2 C(X(M,\sigma))}{dX^2}\right)}_{\Omega_1} \frac{\partial x(M,\sigma)}{\partial \sigma} \\ &\qquad + \sigma M \underbrace{\left(\frac{\partial f(x(M,\sigma))}{\partial x}\right)}_{\Omega_2} \frac{\partial x(M,\sigma)}{\partial \sigma} \\ &\qquad\qquad \underbrace{-\frac{d^2 c(x^*)}{dx_i^2}}_{\Omega_3} \frac{\partial x(M,\sigma)}{\partial \sigma} = -\sigma f(x(M,\sigma)) \quad (12) \end{aligned}$$

The right-hand side of (12) is strictly negative, so $\frac{\partial x(M,\sigma)}{\partial \sigma} \neq 0$ for all $\sigma \geq 0$. Therefore, $x(M,\sigma)$ is monotonic in σ. Furthermore, $\Omega_1 < 0$, $\Omega_2 \leq 0$, and $\Omega_3 < 0$. Hence, the left-hand side is negative if and only if $\frac{\partial x(M,\sigma)}{\partial \sigma} > 0$.

As $\frac{\partial x(M,\sigma)}{\partial \sigma} > 0$, $X(M,\sigma)$ is also increasing in σ. At $\sigma = 0$,

$$\frac{dB(X(M,0))}{dX} - \frac{dC(X(M,0))}{dX} > 0,$$

so there must exist a value $\overline{\sigma}(M) > 0$ for every $M \geq 1$ such that for all $\sigma < \overline{\sigma}(M)$, property $P1$ holds and for all $\sigma > \overline{\sigma}(M)$, property $P2$ holds.

Proving the monotonicity / non-monotonicity of $\overline{\sigma}(M)$ proceeds in two cases.

Case 1: $\lambda = 1$. Proving that $\overline{\sigma}(M)$ is decreasing in M as it was without consumption costs (Proposition 2) first requires showing that X^* is increasing in M. Fix σ and suppose to the contrary that X^* is decreasing in M on any subset of $\mathbb{N} \setminus \{0,1\}$. Then,

$$\begin{aligned}\frac{dB\left(X(M,\sigma)\right)}{dX} - \frac{dC\left(X(M,\sigma)\right)}{dX} + \sigma\frac{\partial b\left(x(M,\sigma)\right)}{\partial x_i} &= \frac{dc(x(M,\sigma))}{dx_i} \\ > \frac{dc(x(M+1,\sigma))}{dx_i} &= \frac{dB\left(X(M+1,\sigma)\right)}{dX} - \frac{dC\left(X(M+1,\sigma)\right)}{dX} + \sigma\frac{\partial b\left(x(M+1)\right)}{\partial x_i},\end{aligned}$$

a contradiction. Thus X^* is increasing in M.

Now to show that $\overline{\sigma}(M)$ is decreasing in M, suppose that $X(M,\sigma)$ is such that

$$\frac{dB\left(X(M,\sigma)\right)}{dX} - \frac{dC\left(X(M,\sigma)\right)}{dX} \geq 0$$

and

$$\frac{dB\left(X(M+1,\sigma)\right)}{dX} - \frac{dC\left(X(M+1,\sigma)\right)}{dX} < 0.$$

It follows that $\sigma \leq \overline{\sigma}(M)$ but $\sigma > \overline{\sigma}(M+1)$, which implies that $\overline{\sigma}(M)$ is decreasing in M. To show that as $M \to \infty$, $\overline{\sigma}(M) \to 0$ recall that by Proposition 1, $x^* \to 0$ as $M \to \infty$, while

$$\lim_{M\to\infty} \frac{\partial b(x(M,\sigma),(M-1)x(M,\sigma))}{\partial x_i} - \frac{dc(x(M,\sigma))}{dx_i} > 0$$

for all $\sigma > 0$. The result follows from $P2$.

Case 2: $\lambda = 0$. This case is proven by a counter-example to monotonicity provided in the text.

To prove statement (i), suppose that $\sigma < \overline{\sigma}(N)$. By the above argument, $P1$ applies and the result follows immediately from an identical argument to that of Proposition 1.

To prove statement (ii), suppose that $\sigma > \overline{\sigma}(N)$, so $P2$ applies. Otherwise, there would exist an equilibrium with $M^*(N) < N$ under $P1$, a contradiction. The first-order conditions for a maximum are given by

$$\frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} + \frac{\partial b\left(1/M\right)}{\partial x_i}\frac{M-1}{M^2x^*} - \frac{dc(x^*)}{dx_i} = 0, \quad \forall\, i = 1,\ldots,M \tag{13}$$

while the welfare maximizing conditions with M contributors is given by

$$\frac{dB(\tilde{X})}{dX} - \frac{dC(\tilde{X})}{dX} + \frac{1}{N}\left(\frac{\partial b(1/M)}{\partial x_i}\frac{M-1}{M^2\tilde{x}} - \frac{dc(\tilde{x})}{dx_i}\right) = 0, \quad \forall\, i \leq M. \tag{14}$$

Given (13),

$$\frac{dB\left(X^*\right)}{dX} - \frac{dC\left(X^*\right)}{dX} + \frac{1}{N}\left(\frac{\partial b\left(1/M\right)}{\partial x_i}\frac{M-1}{M^2x^*} - \frac{dc(x^*)}{dx_i}\right) < 0.$$

Thus $x^* > \tilde{x}$ and by extension, $X^* > \tilde{X}$. Both being greater than $\hat{X}$ follows immediately.

To prove statement (iii), note that $\tilde{X} \to \hat{X}$ as $N \to \infty$ follows from taking the limit of (14) as $N \to \infty$. By Proposition 1, $x^* \to 0$ as $N \to \infty$. As

$$\lim_{x \to 0^+} \frac{\partial b(x/X)}{\partial x} \frac{X_{-i}}{X^2} - \frac{dc(x)}{dx_i} > 0$$

for $\sigma > 0$,

$$\frac{dB(X(\infty, \sigma))}{dX} - \frac{dC(X(\infty, \sigma))}{dX} < 0,$$

which implies that $X^* > \hat{X}$ for all $\sigma > 0$. □

Proof of Proposition 5.

Proof. Suppose that $\hat{M} > \overline{\sigma}^{-1}(\sigma)$ and there exists an integer $\hat{M} - 1$ such that

$$\overline{\sigma}^{-1}(\sigma) < \frac{(\hat{M} - 1) + \hat{M}}{2},$$

so producing the good with $\hat{M} - 1$ contributors leads to a good with a higher quality than having $\hat{M}$ contributors. It immediately follows that if (7) holds, then $\hat{M}$ leads to greater aggregate welfare than $\hat{M} - 1$. □

Proof of Corollary 1.

Proof. The result follows from Proposition 5. □

Proof of Proposition 6

Proof. The proof proceeds in two cases, $\gamma = 0$ and then $\gamma = 1$.

Case 1: $\gamma = 0$. In this case, the equilibrium is defined by the first-order condition

$$\frac{dB\left(X^*\right)}{dX} + \sigma \frac{\partial b\left(\lambda x^* + (1-\lambda)\frac{1}{N}\right)}{\partial x_i} \left(\lambda + (1-\lambda)\frac{N-1}{N^2 x^*}\right) - \frac{dc(x^*)}{dx_i} = 0 \tag{15}$$

for all individuals. The welfare-maximizing level of contributions is defined by

$$N\frac{dB(\tilde{X})}{dX} + \sigma \frac{\partial b\left(\lambda \tilde{x} + (1-\lambda)\frac{1}{N}\right)}{dx_i} \left(\lambda + (1-\lambda)\frac{N-1}{N^2 \tilde{x}}\right) \\ - \sigma \frac{\partial b\left(\lambda \tilde{x} + (1-\lambda)\frac{1}{N}\right)}{dx_j} \frac{(1-\lambda)(N-1)}{N^2 \tilde{x}} - \frac{dc(\tilde{x})}{dx_i} = 0,$$

which reduces to

$$\frac{dB(\tilde{X})}{dX}+\sigma\frac{\lambda}{N}\frac{\partial b\left(\lambda\tilde{x}+(1-\lambda)\frac{1}{M}\right)}{dx_i}-\frac{1}{N}\frac{dc(\tilde{x})}{dx_i}=0. \tag{16}$$

As $\lambda\to 0$, over-provision occurs for σ sufficiently large by case 2 in the proof of Proposition 2 and as $\lambda\to 1$ under-provision occurs for all σ by case 1 in the proof of Proposition 2. Take $\gamma=0$. and suppose that σ is greater than $\overline{\sigma}(N,0,0)$, so that over-provision occurs. By inspection of (16), $\tilde{x}$ is increasing in λ in the neighborhood of $\lambda=0$. Also by continuity, if $x^*>\tilde{x}$ at $\lambda=0$, then $x^*>\tilde{x}$ for small but positive values of λ. Combining this with the fact that at $\lambda=1$, $x^*<\tilde{x}$ there exists a value $\overline{\lambda}(0)$ such that at $\lambda=\overline{\lambda}(0)$, $x^*=\tilde{x}$ and thus over-provision occurs for $\sigma>\overline{\sigma}(N,\lambda,0)$ sufficiently large and $\lambda<\overline{\lambda}(0)$.

Case 2: $\gamma=1$. First, note that the same properties of the cutoff value $\overline{\sigma}(M)$ apply here with $\frac{\partial x(M,\sigma)}{\partial\sigma}>0$, as this statement was proven in Proposition 4 for an arbitrary λ. Suppose that $\sigma>\overline{\sigma}(N,\lambda,1)$. Then by (15),

$$\frac{dB\left(X^*\right)}{dX}-\gamma\frac{dC\left(X^*\right)}{dX}+\frac{1}{N}\left(\sigma\frac{\partial b\left(\lambda x^*+(1-\lambda)\frac{1}{M}\right)}{\partial x_i}\left(\lambda+(1-\lambda)\frac{M-1}{M^2x^*}\right)-\frac{dc(x^*)}{dx_i}\right)<0$$

for all M. The left-hand side of the above corresponds to the welfare-maximizing condition. Therefore, $X^*>\tilde{X}$ for $\sigma>\overline{\sigma}(N,\lambda,1)$ for all λ. □