Dynamic Price Competition and Evolutionary Behavior with Search*

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Abstract

This paper studies the short-run and long-run dynamics of price competition with boundedly-rational agents. Consumers observe one price, can engage in costly search to learn the other prices, and then purchase from the firm with the lowest observed price. Each consumer searches only if the observed price exceeds their threshold, which they dynamically update through one of two revision protocols: myopic best responses or imitation. Firms myopically optimize given the current distribution of consumer thresholds. The short run is characterized by Edgeworth cycles. Each cycle ends when the firm with the larger installed base relents and monopolizes its residual demand. Generally, price cycles persist in the long run except under restricted conditions. Convergence to the Diamond paradox occurs only if consumers adapt to search sufficiently infrequently. Similarly, the Bertrand paradox can emerge temporarily as the search cost becomes arbitrarily small, but cannot persist.

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1 Introduction

In this paper, we draw techniques from evolutionary game theory to study dynamic pricing within an undirected consumer search framework where both consumers and firms are boundedly rational.[1] Using this model, we completely characterize both the short- and long-run pricing patterns of the firms and the search behavior of the consumers. The emergent behavior is cycling prices (Edgeworth cycles), where firms sequentially undercut one-another until one firm finds it strictly more profitable to end the price war to monopolize its residual demand. We then demonstrate the fragility of the Diamond paradox (monopoly pricing with no search), which is only a stable long-run outcome under very restrictive assumptions and the Bertrand paradox (marginal cost pricing), which can only occur temporarily in the singleton case with zero search costs.

Common knowledge of prices is a frequent assumption in the industrial organization literature. This assumption relegates consumers to play a passive role in the pricing dynamics. Without researching prices, consumers cannot be expected to possess such knowledge. The literature on undirected (sequential) consumer search (Stigler, 1961) and clearinghouses (Salop and Stiglitz, 1977) was born in opposition to this assumption, studying firm and consumer behavior in markets where consumers observe only a subset of the firms' prices. Consumers may then engage in costly search, either sequentially or all at once, to learn the remaining prices.[2]

Nevertheless, the undirected search literature itself typically makes use of at least one of two quixotic assumptions: (*i*) consumers possess equilibrium knowledge and (*ii*) consumers possess knowledge of both firm cost and production processes, e.g., Salop and Stiglitz (1977), Stahl (1989), Fershtman and Fishman (1992), Benabou and Gertner (1993), Yang and Ye

[1] Undirected search refers to the scenario in which consumers do not observe all prices set by the firms and must engage in costly search to learn these prices. This is in contrast to directed search, which refers to the scenario in which consumers observe prices, but a shortage of available capacity forces consumers to "search" for a firm with remaining capacity. In the latter scenario, there is uncertainty over the decisions of other consumers, whereas uncertainty is inherent to models of undirected search. For a review of the use of bounded rationality in industrial organization, see Ellison (2006).

[2] See, e.g., Dana Jr. (1994), Bikhchandani and Sharma (1996), Anderson and Renault (1999), Baye et al. (2006), Arbatskaya (2008), Janssen et al. (2011), Garcia et al. (2017), Armstrong (2017), and Preuss (2023). Choi et al. (2018) studies the situation in which prices are known but consumers search to learn about their valuation of the good.

(2008), Tappata (2009), and Cabral and Fishman (2012). This paper jettisons these two assumptions and analyzes consumer search and the induced pricing strategies of firms when consumer behavior is governed by less burdensome informational and cognitive loads.

We develop a model that innovates upon the ideas of Stahl's (1989) undirected consumer search framework and Maskin and Tirole's (1988b) dynamic duopoly framework and apply it to a dynamic, continuous time setting in which both consumers and firms are boundedly rational. While theis paper studies the duopoly case, the results are generalizable to the arbitrary N firm case.[3] The traditional approach has consumers form rational expectations over the distributions and pricing strategies of the firms, though notable exceptions include Rothschild (1974), Benabou and Gertner (1993), Bikhchandani and Sharma (1996), Rauh (1997), and Lewis (2011).[4] The formation of such expectations is generally untenable as consumers often lack necessary knowledge of the firms' production processes. Instead, we use an approach from evolutionary game theory and assume that consumers make their decisions according to simple rules of thumb and adjust their behavior over time via processes that need not require substantial informational or cognitive burdens (Sandholm, 2010). On the demand side, consumers engage in search if the observed price exceeds some reservation price, which is specific to each consumer. The reservation price that each consumer uses to make this decision is based on an inherent threshold and a random shock specific to each consumer. Over time, the consumers adjust their inherent thresholds according to one of two common evolutionary dynamics: imitation of superior strategies or the selection of a myopic best response to the current state.[5]

On the supply side of the market, we assume that firms do not possess knowledge of the process by which the consumers decide/adjust their behavior and thus choose their prices

[3]See Online Appendix C.

[4]Instead of assuming that consumers know the probability distribution of prices from which they're searching, Rothschild (1974) assumes that searchers learn about the distribution, searchers learn about the distribution while they search it. Benabou and Gertner (1993) studies search market equilibria with Bayesian learning (adaptive search and strategic pricing). Bikhchandani and Sharma (1996) study the optimal stopping rule when the distribution of prices is unknown to searchers. Rauh (1997) studies search when agents have beliefs based on finitely many moments of the distribution of prices and their past market experiences. Lewis (2011) assumes consumers form expecatations of prices based on the observed prices during previous purchases.

[5]The qualitative results are unaffected by the choice of revision protocol. Only the rate of convergence changes.

to maximize short term profits.[6] Because the shocks to consumers' reservation prices are individual specific, each consumer searches probabilistically from the perspective of the firms, with the probability of search (weakly) increasing in the observed price. The implied tension between extracting surplus from consumers and inducing consumer search endogenously generates downward sloping demand for each firm's good, independent of the structure of each individual consumer's demand.

The primary result of this paper is a complete characterization of both the short- and long-run dynamics on both sides of the market. The short run equilibrium dynamic is characterized by Edgeworth cycles, where each firm undercuts its competitors until one firm raises its price to monopolize its residual demand, capturing only those consumers that do not search. The low price firm then raises its price to undercut the firm's relenting price. The process then repeats itself.[7] Such cycles are observed gasoline markets, e.g., Castanias and Johnson (1993), Eckert and West (2004), Noel (2007a), Noel (2008), Wang (2009), Doyle et al. (2010), Zimmerman et al. (2012), and Isakower and Wang (2014).[8]

These cycles differ from traditional Edgeworth cycles in two main ways. First, they are aperiodic and stochastic. The peak and trough of each cycle vary based on the distribution of consumer search behavior. The period length of the cycles also vary due to changes in the distribution of thresholds and the fact that prices are sticky with the timing between revisions being stochastic and exponentially distributed. While the firms are undercutting one another, both consumer revision protocols lead to consumers searching less frequently (by adopting higher reservation prices), as costly search has a negative expected value. When these cycles reset via one firm raising its price, the distribution of consumers' reservation prices shifts downward as search has a positive expected value. Second, the characteristics of the cycles differ. In Maskin and Tirole (1988b) and Eckert (2003) prices are driven down to

[6]For example, firms may be endowed with the naive assumption that consumer behavior is fixed, corresponding to firms possessing a uniform prior over all possible consumer dynamics. This assumption is discussed in more detail in Online Appendix B.

[7]Edgeworth cycles were first (informally) predicted by Edgeworth (1925), with the presence of capacity constraints driving the emergence of cycles. The notion was formally examined by Shubik (1959), who found that the equilibrium, while not characterized by cycles, involves price dispersion through mixed strategies. See Vives (1993) for a detailed discussion of the non-existence of pure-strategy equilibrium and indeterminacy of prices in Bertrand-Edgeworth games.

[8]For a more complete survey, see Eckert (2013).

marginal cost. Here, the firm with the largest installed base has the strict incentive to end the cycle first and monopolize its residual demand, which is nonzero due to the probabilistic nature of search. **?** finds in a finite-horizon symmetric duopoly model that price remains above marginal cost, with cycling in three-period intervals. Firms forgo current sales (as there is zero residual demand) and relent to earn higher future profits during their turn to undercut. Noel (2008) finds in a computational model that asymmetric forward looking firms may relent when prices are above a (stochastic) marginal cost.[9]

The potential convergence of the long-run equilibrium dynamic depends on the nature of noise in the consumer dynamic, in particular if it is possible that the aggregate probability of search can converge to (approximately) zero. If consumers can become discouraged from searching, then there is almost sure convergence to monopoly pricing and zero search (the Diamond paradox). If, on the other hand, consumers never become entirely discouraged, i.e., if consumers have a sufficiently high probability of search given the maximum reservation price and monopoly pricing, then neither the distribution of consumer thresholds nor the prices set by the firms will converge over time. Instead, the stochastic Edgeworth cycles that characterize the short run will persist indefinitely. As a consequence, marginal cost pricing (the Bertrand paradox) is unstable. If prices were to settle at marginal cost, consumers would no longer benefit from searching and thus would gradually adopt higher reservation prices. As the probability of consumer search decreases, one of the firms is able to raise its price and sell a positive quantity since there are some consumers that do not search, thereby obtaining positive profits. Relatedly, the residual profits are generally not well-behaved (nonquasiconcave), leading to multiple residual profit maximizing prices.

In addition to characterizing the equilibrium behavior, we present some comparative statics on the lower bound of these price cycles and identify the firm that resets the cycles. The greater a firm's installed base, or equivalently the firm with the most consumers initially observing its price, the greater the firm's incentive to raise its price, which corresponds to higher residual demand. This result leads to the empirically testable prediction that the firm with the greatest installed base will be the first firm to raise its price, which is consistent

[9]With future costs uncertain, the decision to relent can become a public goods game when prices are above marginal cost based on the current marginal cost draw and expectation of future draws.

with empirical findings in gasoline markets (Noel, 2007b; Atkinson, 2009; Isakower and Wang, 2014). It also offers a plausible mechanism: larger firms have larger built-in bases and thus greater residual demand. Rather than relenting as a public good, there is a strict incentive for the larger firm to monopolize its residual demand. Moreover, this result implies that a market in which firms have symmetric market shares will be characterized by more fierce competition, as the lower bound of the price cycle is increasing in the degree of asymmetry of market shares.

Lastly, we show that a uniform upward (downward) shift of the distribution of consumer thresholds results in an increase (decrease) in the lower bound of the Edgeworth cycles. In general, the upper bound of the cycles cannot be characterized, as it need not possess a monotonic relationship with the firms' installed base or the distribution of reservation prices. The relationship depends on the nature of the noise in the individual search decisions and the structure of individual demand. In special cases, such as uniformly distributed noise, the upper bound of the cycle moves with the lower bound.

Authors following the traditional approach almost all uniformly include the assumptions mentioned above (equilibrium knowledge and knowledge of both firm cost and production processes). Diamond (1971) incorporates consumer search in a dynamic market for a durable good. Consumers form cutoff prices, observe a single price in each period, and decide whether to purchase now or wait one period and observe a different firm's price. As long as the alternative price is not expected to be sufficiently lower between period $t+1$ and t is positive, the equilibrium price is the monopoly price. Stahl (1989) generalizes the Diamond model in which consumers engage in sequential search. Consumers are of two types: zero-search-cost consumers and positive-search-cost consumers. As the share of zero-search-cost consumers approaches one, the price distribution converges to the Bertrand outcome and firms set price equal to marginal cost. As the share of zero-search-cost consumers approaches zero, the price distribution converges to the Diamond outcome of monopoly pricing. The results of Stahl (1989) can be realized in our model by assuming that both consumers and firms are able to form rational expectations.[10] By endogenizing the search behavior of consumers, pricing

[10]This statement is formally proven in Online Appendix A.

will only converge to the Diamond outcome under very restrictive assumptions and never to Bertrand, though a Bertrand like-outcome can appear under restrictive conditions in which prices converge to the search cost.

The remainder of the paper is structured as follows. Section 2 outlines the formal model. The comparative statics and dynamics are presented in section 3. Discussion and concluding remarks are provided in section 4.

2 The Model

Consider a market consisting of two identical firms competing in prices and a continuum consumers with unit mass and identical consumption preferences.[11] A typical firm is indexed by i. Time flows continuously and is indexed by $t \in [0, \infty)$. Denote by $p^t = (p_1^t, p_2^t)$ the vector of firms' prices at each time t. To avoid confusion, let ξ denote a price that is not associated with any particular firm. The firms have a constant marginal cost of production normalized to zero.

Each consumer is endowed with the stationary instantaneous utility function $u(q, \xi) = v(q) - \xi q$, where q is the quantity of the good consumed, $v(\cdot)$ is strictly concave, and ξ is the price at which the good is purchased. Under this specification, there exists a continuous decreasing function $D(\xi)$ that specifies the quantity that each consumer will purchase at the price ξ at any time t. Because the mass of consumers is unity, individual demand $D(\xi)$ corresponds to market demand.

Assumption 1 (A1). *$\xi D(\xi)$ is strictly quasiconcave with unique maximizer ξ^m.*

Because $v(\cdot)$ is strictly concave, A1 implies that $\xi D(\xi)$ is uniformly continuous and strictly increasing on $[0, \xi^m]$. As the marginal cost of production is zero, the revenue function corresponds to the firms' monopoly profit, denoted by $\pi^m \equiv \xi^m D(\xi^m)$, where ξ^m is the monopoly price.

[11]As discussed in the Introduction, the model and results can be applied to the N firm case by suitably revising the definitions. See Online Appendix C for these changes.

At each time t, market activities occur in two stages. First, the prices p_i^t of each firm i are set. Second, each consumer observes a single firm's price and may search at cost $c > 0$ to learn the other firm's price.[12] The price that each consumer initially observes is random and independent of the prices observed by the other consumers. The probability of a consumer observing price p_i^t is $\alpha_i \in (0, 1)$.[13] Let $\alpha_1 = \alpha \in (0, 1)$. After the search decision, each consumer purchases quantity $q = D(\xi)$ from the firm with the lowest observed price ξ. It is useful to define the consumers' indirect utility function $v(\xi, s) = u\left(D(\xi), \xi\right) - cs$, where $s = 1$ if the consumer searches and $s = 0$ otherwise. As a convention, when both firms set the same price, a consumer that searches will purchase from the first observed price; i.e., consumers buy from firm i with probability α_i.[14]

The consumers search decision is governed by a simple rule of thumb: a consumer searches if the observed price exceeds some reservation price. Each consumer is endowed with a threshold $\tau \in \{\tau_0, \dots, \tau_L\}$. These thresholds correspond to the consumers' strategies. Define X as the unit simplex in $\mathbb{R}^L$:

$$X := \left\{(x_0, \dots, x_L) \in \mathbb{R}^{L+1} : x_k \in [0, 1] \text{ for all } k = 0, \dots, L \text{ and } \sum\nolimits_{k=0}^{L} x_k = 1\right\}.$$

Denote by $x^t = (x_0^t, \dots, x_L^t) \in X$ the mass of consumers endowed with each threshold at time t. Assume $x_k^0 > 0$ for all k, otherwise, any k such that x_k^0 can be removed from the grid and the game proceeds identically. In this context, a distribution x first-order stochastically dominates a distribution x' if, for all $k = 0, \dots, L$,

$$\sum_{\ell=0}^{k} x_\ell \leq \sum_{\ell=0}^{k} x'_\ell,$$

strictly so for at least one k.

The firms know x^t at all times t.[15] At each time t, each consumer receives an independent random shock σ^t to her threshold τ and then searches in that period if and only if the observed

[12] The structure of this game at each time t is akin to the Diamond-Stahl model of undirected search (Stahl, 1989).

[13] The following example helps illustrate why the probability of observing a given price is random. Suppose that there is a city with two gas stations, one located on each side of the city. It is unclear *ex ante* which side of the city a given driver will be on when needing to refuel. The driver observes the price of the closest station.

[14] This assumption does not influence the equilibrium outcomes.

[15] This knowledge, for example, could be obtained via a survey of consumers.

price ξ exceeds both the perturbed threshold and the search cost; i.e., if $\xi > \max\{\tau + \sigma^t, c\}$.[16] Let $\tau + \sigma^t$ denote the *minimal acceptable (reservation) price*, where any observed price $\xi > \tau + \sigma^t$ leaves that consumer dissatisfied and willing to search. However, the consumer is still self-serving and recognizes that if the price is less than the cost of search, then the savings from a lower price would not justify the search. The shock σ^t is independent across time and distributed according to the distribution φ. A consumer that observes a price ξ will therefore search with probability $\varphi(\xi - \tau)\iota_{(c,\infty)}$, where $\iota_{(c,\infty)}$ is the indicator function for $\xi \in (c, \infty)$.

Assumption 2 (A2). *$\varphi(\cdot)$ is continuous and strictly increasing on $(-\tau_L, \xi^m - \tau_0)$.*

A2 imposes two relatively mild conditions on the distribution of the shocks to consumers' search thresholds. First, $\operatorname{supp}\varphi \supseteq (-\tau_L, \xi^m - \tau_0)$, though the density remains unrestricted on this interval. Second, the magnitude of the shock may be large enough that a consumer with the highest possible threshold is dissatisfied with any positive price and a consumer with the lowest possible threshold may be satisfied with any price below the monopoly price. This is not a particularly imposing assumption, as the probability of these events may be arbitrarily small. Let

$$\bar{\varphi}(\xi, x) = \begin{cases} \sum_{k=0}^{L} \varphi\left(\xi - \tau_k\right) x_k & \text{if } \xi > c \\ 0 & \text{if } \xi \leq c. \end{cases}$$

denote the average probability that a random consumer searches after observing a price ξ. Because there are a continuum of consumers, $\bar{\varphi}(\xi, x)$ is equivalently the mass of consumers that search after observing a price ξ. The following Lemma illustrates three useful properties of $\bar{\varphi}(\xi, x)$.

Lemma 1. *Under A2, the following statements are true:*

(i) *$\bar{\varphi}(\xi, x)$ is continuous on $[0, c) \cup (c, \xi^m] \times X$,*

(ii) *$\bar{\varphi}(\xi, x)$ is strictly increasing in ξ on $[c, \xi^m]$,*

[16] The assumption that this shock is independent across consumers is unnecessary for the purposes of this paper. It is, however, very plausible and guarantees that the expected profits coincide with actual profits.

(iii) *if x first order stochastically dominates x', then $\bar{\varphi}(\xi, x) < \bar{\varphi}(\xi, x')$ for all $\xi \in [c, \xi^m]$.*

The first two statements of Lemma 1 are obvious. Statement (ii) is consistent with the empirical evidence from gasoline markets (Lewis and Marvel, 2011). The third statement shows that when consumers have higher thresholds for search, fewer consumers search. The proof, along with all subsequent proofs, can be found in the Appendix.

We now construct each firm's demand as a function of the price vector p and the distribution of consumers' thresholds x. If firm i's price is lower than its competitor's price, then i serves the consumers that initially observe p_i along with all of the searching consumers. If the two firms set the same price, then firm i will serve those consumers that initially observe p_i. Finally, if firm i does not have the lowest price, then it will serve only those consumers that initially observe p_i and do not search. Firm i's demand is thus

$$D_i(p, x) = D(p_i) \times \begin{cases} \alpha_i + (1-\alpha_i)\bar{\varphi}(p_{-i}, x) & \text{if } p_i < p_{-i} \\ \alpha_i & \text{if } p_i = p_{-i} \\ \alpha_i \left(1 - \bar{\varphi}(p_i, x)\right) & \text{if } p_i > p_{-i}. \end{cases}$$

Let the demand when $p_i < p_{-i}$ be the *front-side demand* and the demand when $p_i > p_{-i}$ be the *residual demand*. The *front-side profit* $\pi_i^F(p, x)$ and *residual profit* $\pi_i^R(\xi, x)$ are defined analogously:

$$\pi_i^F(p, x) = p_i D(p_i)\big(\alpha_i + (1-\alpha_i)\bar{\varphi}(p_{-i}, x)\big)$$
$$\pi_i^R(\xi, x) = \xi D(\xi)\alpha_i \left(1 - \bar{\varphi}(\xi, x)\right).$$

A key property of the firms' demand is that there is positive residual demand facing the firm that does not have the lowest price, even without the presence of capacity constraints. This demand is present due to the stochastic nature by which consumers search. Some consumers will not search and instead purchase at the higher price. Unlike traditional models of price competition with capacity constraints, the residual demand is independent of the low price and the front-side demand depends on the high price.

2.1 The Evolution of Strategies

The initial price vector p^0 is exogenously fixed, though the equilibrium dynamics do not depend on this starting value. Each firm's price remains fixed until that firm changes its price. Prices are sticky, so not all firms may change their price at each time t. Each firm's opportunities to adjust its price are determined independently by a Poisson process.[17] When a firm has the opportunity to revise its price, it selects a price from a finite grid $G := \{g_0, g_1, \ldots, g_M\}$ to maximize its instantaneous profits.[18] G is ordered so that $0 = g_0 < g_1 < \cdots < g_M$ and that $c, \xi^m \in G$. Denote by $\|G\| := \max_{\omega \geq 1} g_\omega - g_{\omega-1}$ the norm of G.

The distribution of the consumers' thresholds x^t evolves as the consumers update their individual strategies. Each consumer receives opportunities to update her threshold according to a Poisson process, which is independent across consumers.[19] We consider two classes of decision rules when consumers change their thresholds, which represent different informational burdens as well as degrees of sophistication in behavior: the best response dynamic and an imitation dynamic. Under the best response dynamic, consumers choose a threshold that is in the set of myopic best responses to the current state (the price vector p^t). This revision protocol does not require explicit knowledge of the prices, only of the payoffs other consumers receive for each possible threshold. Let

$$E\left[v|p^t, \tau\right] = \sum_{i=1}^{2} \alpha_i \left(\varphi\left(p_i^t, \tau\right) v \left(\min_j p_j^t, 1 \right) + \left(1 - \varphi\left(p_i^t, \tau\right)\right) v\left(p_i^t, 0\right) \right).$$

denote the expected utility of a consumer with threshold τ. The evolution of x^t under the best response dynamic is defined by the differential inclusion

$$\dot{x}^t \in B\left(p^t\right) - x^t,$$

where $B(p) \subset X$ denotes the best response correspondence for the consumers and $\dot{x}^t$ denotes the derivative of the state with respect to time.

However, this protocol still places a significant informational burden on the consumers. Contrary to the best response dynamic, the imitation dynamic imposes minimal informational burdens on the consumers. Under this dynamic, when a consumer has the opportunity to change her threshold, she is matched uniformly at random with another consumer. Conditional on being matched with a consumer with threshold τ_ℓ, a consumer with threshold

[17] This simple model of sticky prices is used by Maskin and Tirole (1988a) to motivate a dynamic model of sequential pricing. The rate at which the firms update prices is irrelevant and thus omitted.

[18] We discuss this assumption regarding the myopic nature of firms in greater detail in Online Appendix B.

[19] As with the pricing dynamic, the rate of this process is irrelevant and thus omitted.

τ_k adopts τ_ℓ with probability $r_{k\ell}\left(E\left[v|p^t,\tau_k\right],E\left[v|p^t,\tau_\ell\right]\right)$. The overall probability that a consumer switches from threshold τ_k to τ_ℓ is thus $\rho_{k\ell}=x_\ell r_{k\ell}$.

Assumption 3 (A3). *$r_{k\ell}\left(E\left[v|p^t,\tau_k\right],E\left[v|p^t,\tau_\ell\right]\right)>r_{\ell k}\left(E\left[v|p^t,\tau_\ell\right],E\left[v|p^t,\tau_k\right]\right)$ if and only if $E\left[v|p^t,\tau_k\right]>E\left[v|p^t,\tau_\ell\right]$.*

Given a pair of strategies, $A3$ requires that consumers are more likely to switch from the strategy that performs worse to one that performs better than the reverse. The evolution of x^t under the imitation dynamic is defined by the differential equation

$$\dot{x}_k^t=\sum_{\ell=0}^{L}x_\ell^t\rho_{\ell k}-x_k^t\sum_{\ell=0}^{L}\rho_{k\ell}.$$

3 Equilibrium

3.1 Residual Maximizers, Judo Prices, and Best Responses

For any distribution of thresholds x, define the set of residual maximizers as $\tilde{P}(x):=\arg\max_\xi\pi_i^R(\xi,x)$ and define the set of residual maximizers on a grid G as $\tilde{P}(x,G):=\arg\max_{g\in G}\pi_i^R(g,x)$. Subscripts on $\tilde{P}(x)$ and $\tilde{P}(x,G)$ are unnecessary as each firm's residual profit function is a constant multiple of the other.

Proposition 1. *Under A1 and A2 $\tilde{P}(x)$ is nonempty and* $\sup\tilde{P}(x)\leq\xi^m$.

With a finite grid of prices, the existence of a residual maximizer in $\tilde{P}(x,G)$ is trivial. However, we will interpret the results as the grid becomes arbitrarily fine, so it is useful to compare the results on the grid to values that are defined independent of the grid.

Given Proposition 1, all prices are henceforth restricted to be weakly below the monopoly price ($p_i^t\leq\xi^m=g_M$) as there is no justification for any firm to price above ξ^m. Even if a firm's price were set above ξ^m, that firm would reduce its price given its first opportunity to do so and would never subsequently increase its price above the monopoly level.

The equilibrium characterization is based on the *judo price* of each firm. A firm i's judo price is the highest price its competitor may set such that i prefers to monopolize its residual demand rather than undercut.[20] Formally,

$$p_i^*(x) := \sup\left\{\xi \le \xi^m : \xi D(\xi)\left(\alpha_i + (1-\alpha_i)\bar{\varphi}(\xi, x)\right) < \max_{p_i} \alpha_i p_i D(p_i)\left(1 - \bar{\varphi}(p_i, x)\right)\right\}. \quad (1)$$

The following Lemma bounds the judo price $p_i^*(x)$ between the search cost and the monopoly price.

Lemma 2. *Under A1 and A2, $c \le p_i^*(x) < \xi^m$.*

It is also to describe the judo price when constrained to the grid G:

$$p_i^*(x, G) := \max\left\{g_\omega \in G \setminus \{g_0\} : g_{\omega-1} D(g_{\omega-1})(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_\omega, x)) \right.$$
$$\left. \le \max_{p_i \in G} \alpha_i p_i D(p_i)(1 - \bar{\varphi}(p_i, x))\right\}$$

For any price $p_{-i} = g_\omega \in G$, let $R_i(g_\omega, x)$ denote firm i's best response correspondence.

Proposition 2. *Suppose A1 and A2 hold. For a sufficiently small but positive δ, if $\|G\| < \delta$, then*

$$R_i(g_\omega, x) = \begin{cases} \{g_{\omega-1}\} & \text{if } g_\omega > p_i^*(x, G) \\ \{g_{\omega-1}\} \cup \tilde{P}(x, G) & \text{if } g_\omega = p_i^*(x, G) \\ \tilde{P}(x, G) & \text{if } g_\omega < p_i^*(x, G). \end{cases}$$

Furthermore, if G^n is such that $\|G^n\| \to 0$ as $n \to \infty$, then

(i) *if $g^n \in \tilde{P}(x, G^n)$ is such that $g^n \to \xi$, then $\xi \in \tilde{P}(x)$,*

(ii) *$p_i^*(x, G^n) \to p_i^*(x)$.*

Proposition 2 demonstrates that when prices are restricted to a grid, each firm's best response correspondence mimics its "best responses" when the space of prices is unrestricted: a firm will undercut it's competitor's price unless that price is below the judo price. This

[20]The term judo price originates in a model of entry with sequential pricing developed in Gelman and Salop (1983). The authors draw an analogy between firm strategies and the martial art of judo by pointing out that an entrant firm forces accommodation from the incumbent by setting a low price and limiting its size, thereby incentivizing the incumbent to maintain a large profit margin at a higher price rather than engaging in a price war. Traditionally, a firm's judo price refers to the highest price that it may charge such that the other firm prefers to monopolize residual demand.

interpretation is only approximate as a best response to prices above the judo price does not exist when the space of prices is a continuum. Note that it is possible that a firm's best response is to set the same price as its competitor. In this case, its competitor's price will be a residual maximizer, and so it will be reflected in the term $\tilde{P}(x, G)$.[21]

As the judo price is a defining feature of the firms' best responses, it will play a large role in the equilibrium dynamics. Thus, it is useful to identify which firm has the lower judo price and how each firm's judo price changes with the distribution of consumer search thresholds.

Proposition 3. *Under A1, if $\alpha_i > \frac{1}{2}$, then $p_i^*(x) \geq p_{-i}^*(x)$. If, in addition to $\alpha_i > \frac{1}{2}$, $p_i^*(x) > c$, then $p_i^*(x) > p_{-i}^*(x)$.*

The firm with the larger installed base is less willing to engage in a price war because having a larger installed base guarantees a greater residual demand and thus higher residual profits. This result is analogous to the result found in studies of capacity constrained price competition that the firm with the larger capacity has a higher judo price.[22] The following Proposition relates the judo price to the distribution of consumer thresholds.

Proposition 4. *Under A1 and A2, if x first order stochastically dominates x', then $p_i^*(x) \geq p_i^*(x')$. For a sufficiently small and positive δ, if $\|G\| < \delta$, then $p_i^*(x, G) \geq p_i^*(x', G)$.*

As more consumers adopt a higher threshold for search, the judo price increases and as consumers search less frequently, the residual profit increases while the front-side profit decreases, so a firm has less of an incentive to undercut its competitor. While first order stochastic dominance may seem like a restrictive condition to focus on, we will demonstrate shortly that this characterization is the most natural and relevant way to characterize the judo price with respect to consumer behavior.

It merits mentioning that it is generally not possible to demonstrate the same relationship for the residual maximizers. The reason is that the relationship between $\tilde{P}(x)$ and x depends

[21] This statement is formally proven in Lemma 4.

[22] See, e.g., Osborne and Pitchik (1986), Deneckere and Kovenock (1992), and Allison and Lepore (2016).

heavily on the functional form of φ. In special cases, the upper bound of the cycles move with the lower bound.[23]

3.2 Equilibrium Dynamics

Consumers will gradually adopt higher search thresholds whenever the prices are such that search has a negative expected profit, and will otherwise gradually adopt lower search thresholds.

Proposition 5. *Suppose that A2 and A3 hold and that $p^t = p$ for all $t \in [T, T+\varepsilon)$ for any time T and $\varepsilon > 0$. Define $c^*(p)$ by*

$$c^*(p) := \sup \{c \geq 0 : v(\min p, 1) > \alpha v(p_1, 0) + (1-\alpha) v(p_2, 0)\}.$$

For all $t, t' \in [T, T+\varepsilon)$ with $t > t'$, it follows that

(i) if $c < c^*(p)$, then $x^{t'}$ first order stochastically dominates x^t,

(ii) if $c > c^*(p)$, then x^t first order stochastically dominates $x^{t'}$.[24]

The fact that the distribution of consumer thresholds is always increasing or decreasing (under the ordering induced by first order stochastic dominance) is particularly useful because it implies that Proposition 5 characterizes the motion of the firms' judo prices over time (by Proposition 4).

3.2.1 Edgeworth Cycles

Equilibrium pricing takes the approximate form of cycles of price wars in which firms drive down the price to the point that one firm relents and raises its price, starting the cycle anew. Due to the constantly changing consumer thresholds and stochastic nature of price stickiness, the actual pattern of pricing is not quite cyclical in the classical sense since

[23] An example offered in Online Appendix A demonstrates that it may be the case that as consumers adopt higher search thresholds, the firms' residual maximizers increase for some distributions of consumer thresholds while these maximizers decrease given other distributions of thresholds. A subsequent example in Online Appendix A shows that when φ is uniformly distributed with $\operatorname{supp} \varphi \supseteq (-\tau_L, \xi^m - \tau_0)$.

[24] The case in which $c = c^*(p)$ can be ignored, as the grid may always be perturbed such that no prices satisfy this relationship.

the bounds of the price war are not constant. However, the general pattern repeats and maintains the characteristics of an Edgeworth cycle. By Proposition 2, the cycles can be formally described using the judo prices as defined on the unconstrained price space. These price cycles will be qualitatively identical on the constrained space for a sufficiently fine grid, the only difference being that the judo prices and residual maximizers may differ by some arbitrarily small amount (bounded by $\|G\|$).

Define $p^*(x) \equiv \max\{p_1^*(x), p_2^*(x)\}$ as the *critical judo price*. The equilibrium prices are described by the following pattern, which resembles an Edgeworth cycle. Consider an initial price vector such that $p_i^0 > p_{-i}^*(x)$ for both firms and suppose that the distribution of consumers' thresholds were to remain fixed. Without loss of generality, assume that firm 2 has a weakly larger initial share of consumers, and thus by Proposition 3 the higher judo price. Hence, firm 2's judo price is also the critical judo price. The Edgeworth cycle proceeds as follows:

1. Firm 1 sets a price just below p_2 and will not adjust it until p_2 changes.

2. Firm 2 sets a price just below p_1 and will not adjust it until p_1 changes.

3. Steps 1 and 2 repeat until the prices are reduced to the critical judo price $p_2^*(x)$.

4. Firm 2 relents and sets a price in $\tilde{P}(x)$ to maximize its residual profit.

5. Repeat this process from step 1.

During the first three steps of the cycle, the prices of the firms will be close enough that search will not be beneficial. Thus, Proposition 5 implies that during steps 1-3 of the cycle, consumers will be adopting higher search thresholds, which by Proposition 4 implies that the firms' judo prices will be increasing. If these judo prices increase enough, then a firm that currently has the lowest price (just below the other firm's price) may skip to step 4 of the cycle and monopolize its residual demand. While Proposition 3 guarantees that the firm with the larger initial share of consumers will always relent first for a fixed distribution of consumer thresholds, changes in the distribution can lead to the other firm relenting first. Because the consumer dynamic is continuous, as consumers raise their thresholds and judo prices increase, the critical judo price may increase to a value above the current price before the nonbinding judo price. As such, if the firm with the smaller installed base relents first, then the other firm would also have relented if it had received the opportunity to do so.

The range of equilibrium pricing and potential convergence depend on the parameters and functional forms of the model. This section considers two conditions and the subsequent section analyzes their complements. Denote by

$$e_k = (\underbrace{0, \ldots, 0}_{k \text{ zeros}}, 1, 0, \ldots, 0).$$

the $k+1^{th}$ basis vector in X. I.e., e_k corresponds to the distribution of consumer thresholds in which all consumers have threshold τ_k.

Condition 1 (C1). *At $p = (p_i, p_{-i})$, where $p_i = \inf \tilde{P}(e_L)$ and $p_{-i} = p_i^*(e_L) = p^*(e_L)$, $c < c^*(p)$.*

C1 states that the cost of search is sufficiently low so that search is beneficial when all consumers have the highest possible threshold, one firm charges the critical judo price, and the other firm charges the smallest residual maximizer. C1 not only provides an explicit condition on the cost of search, but also implicitly puts some structure on the profit functions in that it requires $p^*(e_L) < \inf \tilde{P}(e_L)$.

The following Theorem demonstrates that under $A1$-$A3$ and C1, the equilibrium pricing dynamic does not converge and that the range of prices in the cycles is large enough to induce search. Before stating the result, define $\underline{p} \equiv \max\{p_1^*(e_0), p_2^*(e_0)\}$ and $\hat{p} = \inf \tilde{P}(e_L)$. Thus, $\underline{p}$ represents the highest judo price when all consumers have the lowest search threshold and $\hat{p}$ represents the smallest residual maximizer when all consumers have the highest search threshold.

Theorem 1. *Under $A1 - A3$ and $C1$, for all $\varepsilon > 0$, there exists a $\delta > 0$ and $T > 0$ such that if $\|G\| < \delta$, then*

(i) the firms' prices p^t and distribution of consumer threshold x^t do not converge as $t \to \infty$,

(ii) for all times $t > T$, $p_i^t \in \left[\underline{p} - \varepsilon, \xi^m\right]$,

(iii) there exists sequences t_n, t'_n such that $x^{t_n} \to e_0$, $x^{t'_n} \to e_L$, $p_i^{t_n} \to \xi \leq \underline{p} + \varepsilon$, and $p_i^{t'_n} \to \xi' \geq \hat{p} - \varepsilon$.

Thus, the range of prices is at least $[\underline{p}, \hat{p}]$, and the distribution of consumer thresholds varies between the two extremes in which all consumers have the lowest and highest thresholds e_0 and e_L. Proposition 2 implies that some firm will eventually choose a price that is just below its competitor's price. Given such prices, if the grid is sufficiently fine, then the difference in prices will be less than the cost of search so it cannot be beneficial to search. By proposition 5, consumers gradually adopt higher search thresholds. Price stickiness implies that, over time, prices will almost surely be stuck close together or that the time until a firm relents from the price war will be sufficiently long such that the distribution of consumer thresholds approaches e_L. At this point, the best response correspondence implies that prices should cycle with a maximum price of at least $\hat{p}$. C1 implies that immediately after a firm relents, when the prices are approximately $p^*(e_L)$ and $\xi \in \tilde{P}(e_L)$, the cost of search is sufficiently low so search is beneficial. Again, this will almost surely occur for long enough that the distribution of consumer thresholds approaches e_0, at which point the prices will cycle with a lower bound of approximately $\underline{p}$. Thus, the process cannot converge and these bounds must be approached infinitely many times.

Condition 2 (C2). *At $p = (p_i, p_{-i})$, where $p_i = \sup \tilde{P}(e_L)$ and $p_{-i} = p_i^*(e_L) = p^*(e_L)$, $c < c^*(p)$.*

Replacing C1 with C2 yields the following corollary to Theorem 1.

Corollary 1. *Under $A1 - A3$ and $C2$, for all $\varepsilon > 0$, there exists a grid G such that there exists an equilibrium in which*

(i) the firms' prices p^t and the distribution of consumer thresholds x^t do not converge as $t \to \infty$,

(ii) for all times $t > T$, $p_i^t \in [\underline{p} - \varepsilon, \xi^m + \varepsilon]$, and

(iii) there exists sequences t_n, t'_n *such that* $x^{t_n} \to e_0$, $x^{t'_n} \to e_L$, $p^{t_n} \to \xi \le \underline{p} + \varepsilon$, *and* $p_i^{t'_n} \to \xi' \ge \hat{p} - \varepsilon$.

Note that C2 is weaker than C1, with the two coinciding if and only if $\tilde{P}(e_L)$ is a singleton.[25] This result is informative, as Section 3.2.2 shows that under an equilibrium exists in which the distribution of consumer thresholds converges under the complement of C1 and all equilibria have this property under the complement of C2. Hence, multiple equilibrium dynamics may exist for a range of search costs if $\tilde{P}(e_L)$ is not a singleton.

The results of this section correspond closely with those of Maskin and Tirole (1988b) regarding Edgeworth Cycles, with several key differences. First and foremost, the mechanism by which the cycles emerge is different. In Maskin and Tirole, cycles emerge as a best response to a dynamic, forward-looking strategy, whereas cycles in this model emerge due to the presence of boundedly rational consumers which creates positive residual demand for the firm with a higher price. Thus, the cycles in our paper are driven by the demand side of the market, as opposed to Maskin and Tirole, where cycles are driven by the supply side. Second, the characteristics of the cycles differ. In Maskin and Tirole, prices are driven down to marginal cost, and firms relent randomly since relenting yields zero profits in the short run. That is, once price hits marginal cost, firms make zero profits. Because residual demand is zero for the firm with the higher price, relenting leads to zero profits as well. Each firm prefers that its competition relents first, so it can benefit by undercutting. Due to the public nature of relenting, firms employ a mixed strategy once price hits marginal cost. In this setting, relenting is optimal due to non-zero residual demand for the firm with the higher price, allowing identification of not only when the firms will relent, but also which firm will relent.

[25] The arguments made to prove this corollary are nearly identical to those made in Theorem 1 with one key difference: the grid needs to be chosen such that for some neighborhood $\mathcal{N}(e_L)$ of e_L and all $x \in \mathcal{N}(e_L)$, $\sup \tilde{P}(e_L, G) \in \tilde{P}(x, G)$ and the equilibrium needs to dictate that when a firm chooses a price $p^t \in \tilde{P}(x, G)$, it will choose $p^t = \sup \tilde{P}(e_L, G)$.

3.2.2 Kinked Demand Curve Equilibria and the Bertrand and Diamond Paradoxes

When the conditions presented in the previous subsection are violated, the equilibrium converges over time. There are two notions in which the dynamic may converge, depending on the parameters of the model. First, Edgeworth cycles may persist indefinitely, but with the range of the cycles shrinking in the limit. Second, there may be convergence of both prices to the search cost in finite time, though occurrence of this dynamic requires severely restrictive conditions. In either case, the distribution of consumer thresholds converges to e_L. Formally, the conditions considered here are as follows.

Condition 1′ (*C*1′). *At $p = (p_i, p_{-i})$, where $p_i = \inf \tilde{P}(e_L)$ and $p_{-i} = p_i^*(e_L) = p^*(e_L)$, $c > c^*(p)$.*

Condition 2′ (*C*2′). *At $p = (p_i, p_{-i})$, where $p_i = \sup \tilde{P}(e_L)$ and $p_{-i} = p_i^*(e_L) = p^*(e_L)$, $c > c^*(p)$.*

Condition 3 (C3)**.** *There exists a neighborhood $\mathcal{N}(e_L)$ of e_L such that $\tilde{P}(x) = \{c\}$ for all $x \in \mathcal{N}(e_L)$.*

C2′ is the complement of C2 and C1′ is the complement of C1. While not immediately obvious, C3 is a subcase of C2′. To see why, note that since $p_i^* \leq \inf \tilde{P}(x)$, under C3, $p_i^*(x) = c$ for each firm i and all $x \in \mathcal{N}(e_L)$. It follows that $c^*(p^*(e_L), \sup \tilde{P}(e_L)) = 0$, so $c > 0$ implies C2′.

The following theorem characterizes the first type of convergence in which the distribution of consumer thresholds converges to e_L and price cycles indefinitely under a small range.

Theorem 2. *Under $A1 - A3$ and C2′, for all $\varepsilon > 0$, there exists a $\delta > 0$ and $T > 0$ such that if $\|G\| < \delta$, then*

(i) the distribution of consumer thresholds $x^t \to e_L$,

(ii) for all times $t > T$, $p_i^t \in [p^*(e_L) - \varepsilon, \sup \tilde{P}(e_L) + \varepsilon]$, and

(iii) there exists sequences t_n and t'_n such that $p_i^{t_n} \to \xi \leq p^*(e_L) + \varepsilon$, and $p_i^{t'_n} \to \xi' \geq \hat{p} - \varepsilon$.

Theorem 2 shows that in the long run, prices cycle indefinitely with a lower bound of $p^*(e_L)$ and an upper bound between $\inf \tilde{P}(e_L)$ and $\sup \tilde{P}(e_L)$. If the firms' prices are close ($|p_1 - p_2| < c$), then consumers do not benefit from searching, so the distribution of consumer search thresholds will tend towards e_L. Given price stickiness and the fact that firms will set prices that are close until one firm relents and raises its price, the firms' prices will almost surely remain close for a sufficiently long period of time such that the distribution of consumer thresholds approaches e_L. At that point, the consumers search sufficiently infrequently such that the gap that emerges following one firm relenting will not be large enough to induce search. As such, the distribution of consumers' thresholds will continue to converge towards e_L and the firms' cycles will remain fixed.

Corollary 2. *Under $A1-A3$ and $C1'$, for all $\varepsilon > 0$, there exists a grid G and a time $T > 0$ such that there exists an equilibrium in which*

(i) the distribution of consumer thresholds $x^t \to e_L$,

(ii) for all times $t > T$, $p_i^t \in [p^*(e_L) - \varepsilon, \sup \tilde{P}(e_L) + \varepsilon]$, and

(iii) there exists sequences t_n, t'_n such that $p_i^{t_n} \to \xi \leq p^*(e_L) + \varepsilon$, and $p_i^{t_n'} \to \xi' \geq \hat{p} - \varepsilon$.

Thus C1$'$ is a sufficient condition for this type of convergence to occur. If $\tilde{P}(e_L)$ contains is not a singleton, then there is a range of search costs in which both C1$'$ and C2$'$ are satisfied. For search costs in that range, there exists both convergent and nonconvergent equilibrium paths. This multiplicity of equilibria can be ruled out by assuming φ is such that the residual profit given $x = e_L$, $\xi D(\xi)(1 - \varphi(\xi - \tau_L))$, is strictly quasiconcave in ξ. The following proposition demonstrates some limiting properties as the grid of consumer thresholds becomes large.

Proposition 6. *Under $A1-A3$, as $\tau_L \to \xi^m - \inf \operatorname{supp} \varphi$, $p_i^*(e_L) \to \xi^m$ and thus $\sup \tilde{P}(e_L) = p_i^*(e_L)$. Consequently, $C2$ holds for sufficiently large τ_L. Furthermore for all $\varepsilon > 0$, there* *exists a $\bar{\tau} > 0$, $\delta > 0$, and $T > 0$ such that if $\tau_L > \bar{\tau}$ and $\|G\| < \delta$, then all equilibria are such that $p_i^t > \xi^m - \varepsilon$ for both firms i and all $t > T$.*

This result follows as a corollary of Theorem 2 and is very similar to the Diamond paradox - equilibrium monopoly pricing with any number of firms - that emerges in the dynamic model of undirected consumer search in Diamond (1971) and the static model of Stahl (1989). The intuition for the result is similar, though the mechanisms by which the equilibrium emerges is different. In this model, the firms will price in such a way that consumers are induced to search less often, and eventually search with sufficiently low probability such that monopoly pricing is optimal. Once reached, search is not beneficial, as both firms set the monopoly price, and so the outcome is stable. In the Diamond-Stahl model of consumer search, the consumers form rational expectations of the firms' pricing distributions, and the only possible equilibrium involves monopoly pricing by the firms and no search by the consumers.[26] With this mechanism, the Diamond paradox emerges because of simultaneous anticipation of this outcome by both firms and consumers.

Proposition 6 posits an equilibrium of similar character to the Diamond paradox (Stahl, 1989) and the kinked demand curve equilibrium of Maskin and Tirole (1988b). The difference is in the mechanisms driving the result. In Stahl (1989), convergence to monopoly pricing and no search occurs when the exogenously determined proportion of searchers (with a zero search cost) tends to zero. In Maskin and Tirole (1988b), convergence follows from the firm side of the market. As firms become infinitely patient, a Markov perfect equilibrium in which both firms choose the monopoly price can be sustained, as the firms are able to internalize the cost of engaging in a price war. In this paper, the equilibrium follows from demand side characteristics. Moreover, the process by which consumers stop searching is endogenous to the firms' pricing strategies. Thus, a feedback loop occurs where the proximity of the firms' prices during the cycles induce consumers to search less. As consumers search less, the lower bound of the cycles increases (Proposition 4). Furthermore under $C2$, the difference in pricing as the cycles reset is not enough to decrease the consumers thresholds. Thus the process continues until prices approach the monopoly level and the distribution of thresholds

[26] See Online Appendix A for a full derivation of this result.

approaches e_L.

Lastly, the following theorem demonstrates that under condition $C3$, all equilibria are such that the firms' prices converge in finite time, with the distribution of consumers' converging over time to e_L.

Theorem 3. *Under A1, A2, A3, and C3, for all $\varepsilon > 0$, there exists a $\delta > 0$ and $T > 0$ such that if $\|G\| < \delta$, then $x^t \to e_L$ and firms will undercut one another until $p_1^t = p_2^t = \xi = c$, and will remain at that price thereafter.*

Under C3, given any distribution of thresholds, the consumers search at any price $\xi > c$ with sufficiently high probability, so the residual profit is always maximized by setting the price equal to the search cost to induce consumers not to search. Thus given some initial prices, the firms engage in a price war until the price is driven to the search cost, and the firms never have the incentive to increase their prices. This result can be seen as a Bertrand-like outcome; if we allow the search cost to tend to zero, then the equilibrium will converge to marginal cost pricing, but will not remain as there will be random shifts in the search thresholds, allowing firms to raise their prices and receive positive profits.

4 Discussion and Concluding Remarks

By developing a model of undirected consumer search with firm competition, this paper has characterized the market outcomes and developed several empirically testable predictions – specifically the short- and long-run dynamics – resulting from the bounded rationality of agents. The short-run dynamics are characterized by stochastic Edgeworth cycles. For a given distribution of consumer search thresholds, the firm with the larger installed base has the greater incentive to monopolize residual demand rather than continuing the price war. Hence, in situations in which the firms' opportunities to update prices occur more frequently than the consumers' opportunities to reevaluate search decisions, larger firms are more likely to end the reset the cycle by monopolizing residual demand, a prediction consistent with gasoline markets (Noel, 2007b; Atkinson, 2009; Zimmerman et al., 2012; Isakower and Wang, 2014). These cycles persist (aperiodically) in the long run unless consumers learn to search with sufficient infrequency. Hence, convergence to the kinked demand equilibria of Maskin and Tirole (1988b) and the Diamond paradox occurs only under very special circumstances.

The bounded rationality of consumers has important implications. The model does not coincide with Maskin and Tirole (1988b), Eckert (2003), and Noel (2008) even if search costs are zero or firms are forward looking. Whenever prices equalize, consumers will randomly shift toward higher price thresholds, thereby creating an incentive for firms to raise prices. Both types of equilibria that Maskin and Tirole characterize involve firms setting identical prices at least for some period of time. Thus, the underlying consumer dynamic will induce some change in Maskin and Tirole's results that does not disappear in the limit.

We now conclude with a discussion of the implications of the model and the assumptions surrounding the search costs and decisions. First, a major implication of the results is that the Bertrand paradox does not emerge as an equilibrium outcome as the cost of search tends to zero. This result demonstrates the instability of the Bertrand paradox as it only occurs under highly specific assumptions and is upset by arbitrary perturbations. Marginal cost pricing cannot emerge with even arbitrarily small search costs in our model because consumers do not benefit from search when prices approach this level. Over time, this will lead to random shifts in the distribution of consumer price thresholds (Proposition 5) that firms can take advantage to obtain positive profits (by A3). Thus, the Bertrand paradox is only a temporary outcome in the singleton unrealistic case with exactly zero search costs.

Second, the cost of search does not influence any of the qualitative results of the model as long as that cost is nonzero. The cost is only relevant insofar as it determines whether search has a positive or negative expected value, and as long as the price grid is sufficiently fine, it can take on both of these values. The only effect that the cost has on the process is on the rate at which the process evolves. The reason is that the probability that a consumer adopts a strategy is increasing in the payoff from that strategy and decreasing in the payoff of the incumbent strategy. A higher search cost reduces the expected value of search in all states, and thus increases the rate at which consumers are discouraged from search during the price cycles and decreases the rate at which consumers are encouraged to search when the cycle resets.

Though the search cost does not influence the qualitative results, the model offers important insights into the role of the search cost (and the endogenizing of the search cost) in pricing behavior. If higher search costs increase the rate of convergence to higher thresholds and reduce the rate of convergence to lower thresholds, then firms benefit by raising the search cost so as to make it so that consumers search less for greater periods of time. Furthermore, both the critical judo price and smallest residual maximizer are bounded below by the search cost. Hence, firms can potentially increase their profits via obfuscation (Ellison and Ellison, 2009; Ellison and Wolitsky, 2012) to increase the search cost. When binding (i.e., when the judo price is the search cost or under C3 and the conditions of Theorem 3), firms can strictly benefit from actions that increase the search cost c, provided that the cost of such actions are not too large.

Third, a component that most models of undirected search incorporate is a fraction of "shoppers," consumers that either have zero search cost (and thus optimally search) or are exogenously informed as to the prices set by the firms. As this paper's primary objective is to weaken the assumptions that are often present in these models and examine the impact of bounded rationality on firm and consumer behavior, such an inclusion would be inappropriate for this paper. Nevertheless, it is straightforward to deduce the impact that a fraction of shoppers would have on the model's predictions. These shoppers increase the fraction of consumers that search given a fixed observed price. Thus, the inclusion of shoppers in the model is equivalent to creating a negative bias to the noise in each consumer's price threshold. If the fraction of shoppers is large enough, this will induce the conditions of Theorem 1 and thus perpetual cycling of prices and consumer price thresholds. Otherwise, if this fraction is sufficiently small, then the conditions of Theorem 2 and Proposition 6 may still be satisfied, resulting in a long run convergence to the Diamond paradox.

This result stands in contrast to Stahl (1989), who finds in a static model where consumers form rational expectations about the firms' strategies that the distribution of firm's prices varies continuously between marginal cost pricing (Bertrand paradox) and monopoly pricing (Diamond paradox) as the fraction of shoppers varies between zero and one.[27] Similarly, this

[27]Equilibrium pricing in Stahl's model is in mixed strategies when the fraction of shoppers is strictly between zero and one.

result is in contrast to the theoretical results of Pennerstorfer et al. (2020), who find similar results to Stahl (1989). However, this result (persistence of price dispersion) is consistent with the semiparametric empirical evidence of Pennerstorfer et al. (2020), whereby the degree of price dispersion decreases, but does not disappear in the limits (share of shoppers tending to zero or one).

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Appendix

Proofs of the Results in Sections 2 and 3

Proof of Lemma 1.

Proof. Statements (i) and (ii) follow immediately from A2 and the definition of $\bar{\varphi}(\xi, x)$. Item (iii) is trivially true if $\xi \leq c$. We now prove statement (iii). Suppose that $\xi > c$ and x first order stochastically dominates x'. By Abel's lemma, $\bar{\varphi}(\xi, x) = \sum_{k=0}^{L} \varphi(\xi - \tau_k) x_k$ can be rewritten as

$$\varphi(\xi - \tau_L) - \sum_{k=0}^{L-1} \left(\sum_{\ell=0}^{k} x_\ell \right) (\varphi(\xi - \tau_{k+1}) - \varphi(\xi - \tau_k))$$

Hence, for distributions x and x', $\bar{\varphi}(\xi, x) < \bar{\varphi}(\xi, x')$ if and only if

$$\varphi(\xi - \tau_L) - \sum_{k=0}^{L-1} \left(\sum_{\ell=0}^{k} x_\ell \right) (\varphi(\xi - \tau_{k+1}) - \varphi(\xi - \tau_k)) \\ < \varphi(\xi - \tau_L) - \sum_{k=0}^{L-1} \left(\sum_{\ell=0}^{k} x'_l \right) (\varphi(\xi - \tau_{k+1}) - \varphi(\xi - \tau_k)),$$

which simplifies to

$$\sum_{k=0}^{L-1} \left(\sum_{\ell=0}^{k} (x_\ell - x'_\ell) \right) (\varphi(\xi - \tau_{k+1}) - \varphi(\xi - \tau_k)) > 0.$$

By A2, $(\varphi(\xi - \tau_{k+1}) - \varphi(\xi - \tau_k)) < 0$ for all $k = 0, \ldots, L$, so it is sufficient that, for each $k = 0, \ldots, L - 1$,

$$\sum_{k=0}^{L-1} \sum_{\ell=0}^{k} (x_\ell - x'_\ell) = \sum_{k=0}^{L} \sum_{\ell=0}^{k} (x_\ell - x'_\ell) < 0,$$

which is true if x exhibits first order stochastic dominance over x'. $\square$

Proof of Proposition 1.

Proof. To demonstrate that $\tilde{P}(x)$ is nonempty, it sufficient to show that for all $x \in X$, $\pi_i^R(\xi, x)$ is upper semicontinuous in ξ. By Lemma 1(i), $\bar{\varphi}(\xi, x)$ is continuous on $[0, c) \cup (c, \xi^m) \times$

X. Therefore, it is sufficient verify upper semicontinuity at $\xi = c$. For all $\xi \leq c$, $\bar{\varphi}(\xi, x) = 0$, so $\pi_i^R(\xi, x) = \alpha_i \xi D(\xi)$. For any $\xi > c$, $\pi_i^R(\xi, x) = \alpha_i \xi D(\xi)(1 - \bar{\varphi}(\xi, x)) \leq \alpha_i \xi D(\xi)$. Thus,

$$\limsup_{\xi \to c} \pi_i^R(\xi, x) = \pi_i^R(c, x),$$

so $\pi_i^R(\xi, x)$ is upper semicontinuous and possesses a maximizer on $[0, \xi^m]$. Hence, $\tilde{P}(x)$ is nonempty.

To show that $\sup \tilde{P}(x) \leq \xi^m$, suppose to the contrary that $\xi \in \tilde{P}(x)$ but $\xi > \xi^m$. Then, under A1, $\xi D(\xi) < \xi^m D(\xi^m)$. Judiciously adding zero by adding and subtracting

$$\xi^m D(\xi^m)(1 - \bar{\varphi}(\xi^m, x)) + \xi D(\xi) \bar{\varphi}(\xi^m, x)$$

to this candidate maximal residual profits yields

$$\begin{aligned} \xi D(\xi)(1 - \bar{\varphi}(\xi, x)) = \xi^m D(\xi^m)(1 - \bar{\varphi}(\xi^m, x)) - \xi D(\xi)(\underbrace{\bar{\varphi}(\xi, x) - \bar{\varphi}(\xi^m, x)}_{>0 \text{ by Lemma 1}}) \\ - (\underbrace{\xi^m D(\xi^m) - \xi D(\xi)}_{>0 \text{ by } A1})(1 - \bar{\varphi}(\xi^m, x)) < \xi^m D(\xi^m)(1 - \bar{\varphi}(\xi^m, x)), \end{aligned}$$

a contradiction of ξ as a residual maximizer. Hence, $\sup \tilde{P}(x) \leq \xi^m$. □

Proof of Lemma 2.

Proof. For $\xi < c$, observe that the front-side profits given prices $p = (\xi, \xi)$ is $\pi^F((\xi, \xi), x) = \alpha_i \xi D(\xi)$. $A1$ implies that the front-side profit is strictly increasing in ξ. Since firm i's residual profits at $p_i = c$ are

$$\pi_i^R(c, x) = \alpha_i c D(c) > \alpha_i \xi D(\xi) = \pi^F((\xi, \xi), x),$$

it must be that $c \leq p_i^*(x)$ as the firm strictly prefers maximizing its residual demand rather than undercutting at any $\xi < c$.

By $A2$, $\bar{\varphi}(\xi^m, x) > 0$. Consider $\xi \in \tilde{P}(x)$ and note that

$$\begin{aligned}
\pi_i^F((\xi^m, \xi^m), x) = \xi^m D(\xi^m)(\alpha_i + (1-\alpha_i)\bar{\varphi}(\xi^m, x)) &> \alpha_i \xi^m D(\xi^m) \\
&\geq \alpha_i \xi D(\xi) \\
&\geq \max_{p_i} \alpha_i p_i D(p_i)(1 - \bar{\varphi}(p_i, x)) \\
&= \max_{p_i}, \pi^R(p_i, x),
\end{aligned}$$

where the second inequality follows from $\xi \leq \xi^M$ being a residual maximizer and last inequality follows from Proposition 1. Hence, $p_i^*(x) < \xi^m$. □

The proof of Proposition 2 relies on the following three lemmas.

Lemma 3. *Let $g_{\omega^*(x)} = p_i^*(x, G)$. There exists a $\delta > 0$ such that if $\|G\| < \delta$, then $g_{\omega^*(x)} < g_M = \xi^m$.*

Proof. Suppose to the contrary that $g_{\omega^*(x)} = g_M = \xi^m$ and consider $\xi \in \tilde{P}(x, G)$. The proof proceeds in two cases: $\xi < \xi^m$ and $\xi = \xi^m$.

<u>Case 1</u>: $\xi < \xi^m$. If $\xi < \xi^m$, then $\xi^m = g_M > g_{M-1} \geq \xi$. Under this supposition,

$$g_{M-1} D(g_{M-1})(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_M, x)) < \alpha_i \xi D(\xi)(1 - \bar{\varphi}(\xi, x)), \tag{2}$$

by the definition of the restricted judo price. $A1$ and Lemma 1(ii) imply that $g_{\omega-1} D(g_{\omega-1})(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_\omega, x))$ is weakly increasing in both ω and g_ω, so

$$\begin{aligned}
g_{M-1} D(g_{M-1})(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_M, x)) &\geq g_{M-1} D(g_{M-1})(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_{M-1}, x)) \\
&\geq \xi D(\xi)(\alpha_i + (1-\alpha_i)\bar{\varphi}(\xi, x)) \\
&\geq \alpha_i \xi D(\xi)(1 - \bar{\varphi}(\xi, x)). \qquad (3)
\end{aligned}$$

By (2) and (3),

$$\alpha_i \xi D(\xi)(1 - \bar{\varphi}(\xi, x)) < \alpha_i \xi D(\xi)(1 - \bar{\varphi}(\xi, x)),$$

a contradiction.

Case 2: $\xi = \xi^m$. Because $g_{M-1} < g_{\omega^*(x)} = g_M = \xi^m$ by hypothesis, it follows from the definition of the restricted judo price that

$$g_{M-1}D(g_{M-1})(\alpha_i + (1-\alpha_i)\bar{\varphi}(\xi^m, x)) < \alpha_i \xi^m D(\xi^m)(1-\bar{\varphi}(\xi^m, x)),$$

which when rearranged yields

$$\xi^m D(\xi^m) - g_{M-1}D(g_{M-1}) > g_{M-1}D(g_{M-1})\frac{\bar{\varphi}(\xi^m, x)}{\alpha(1-\bar{\varphi}(\xi^m, x))}. \tag{4}$$

Suppose that $\|G\| < \delta$ for some $\delta > 0$. Then, $g_{M-1} \geq \xi^m - \delta$, so by $A1$, $g_{M-1}D(g_{M-1}) \geq (\xi^m - \delta)D(\xi^m - \delta)$. Hence, if (4) is satisfied, then

$$\xi^m D(\xi^m) - (\xi^m - \delta)D(\xi^m - \delta) > (\xi^m - \delta)D(\xi^m - \delta)\frac{\bar{\varphi}(\xi^m, x)}{\alpha(1-\bar{\varphi}(\xi^m, x))}. \tag{5}$$

By $A1$, the LHS of (5) is decreasing in δ while the RHS of (5) is increasing in δ. Taking $\delta \to \xi^m$ yields $\xi^m D(\xi^m) > 0$ and taking $\delta \to 0$ yields

$$0 > (\xi^m)D(\xi^m)\frac{\bar{\varphi}(\xi^m, x)}{\alpha(1-\bar{\varphi}(\xi^m, x))}.$$

a contradiction. By continuity, there exists a δ' such that there is a contradiction for all $\delta \leq \delta'$, so for $\|G\| < \delta'$, $g_{\omega^*(x)} = p_i^*(x, G) < g_M$. □

Lemma 4. *There exists a small positive δ such that if $\|G\| < \delta$ and $g_\omega \in R_i(g_\omega, x)$, then $g_\omega \in \tilde{P}(x, G)$.*

Proof. Suppose $g_\omega \in R_i(g_\omega, x)$. Then,

$$\max_{p_i \in G} \alpha_i p_i D(p_i)(1-\bar{\varphi}(p_i, x)) \leq \alpha_i g_\omega D(g_\omega)$$

and

$$g_{\omega-1}D(g_{\omega-1})(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_\omega, x)) \leq \alpha_i g_\omega D(g_\omega), \tag{6}$$

otherwise it must be that $g_\omega \notin R_i(g_\omega, x)$. Define $g_\psi \in G \setminus \{g_M\}$ such that $g_\psi \leq c < g_{\psi+1}$, noting that $\bar{\varphi}(\xi, x) = 0$ for all $\xi \leq g_\psi$. There are two cases to consider: (i) $g_\omega \leq g_\psi$ and (ii) $g_\omega > g_\psi$.

Case 1: $g_\omega \le g_\psi$. Then, $\bar{\varphi}(g_\omega, x) = 0$, so

$$\begin{aligned}\max_{p_i \in G} \alpha_i p_i D(p_i)(1 - \bar{\varphi}(p_i, x)) \le \alpha_i g_\omega D(g_\omega) &= \alpha_i g_\omega D(g_\omega)(1 - \bar{\varphi}(g_\omega, x)) \\ &\le \alpha_i g_\psi D(g_\psi)(1 - \bar{\varphi}(g_\psi, x)) \\ &\le \max_{p_i \in G} \alpha_i p_i D(p_i)(1 - \bar{\varphi}(p_i, x)),\end{aligned}$$

confirming the supposition. Thus, if $g_\omega \in R_i(g_\omega, x)$, then $g_\omega \in \tilde{P}(x, G)$.

Case 2: $g_\omega > g_\psi$. Rearranging (6) yields

$$\bar{\varphi}(g_\omega, x) \le \frac{\alpha_i}{1 - \alpha_i}(g_\omega D(g_\omega) - g_{\omega-1} D(g_{\omega-1})). \tag{7}$$

If $\|G\| < \delta$, then

$$g_\omega D(g_\omega) - g_{\omega-1} D(g_{\omega-1}) \le g_\omega D(g_\omega) - (g_\omega - \delta) D(g_\omega - \delta)$$

By the uniform continuity of $\xi D(\xi)$ on $[0, \xi^m]$, choose δ such that

$$|\xi D(\xi) - (\xi - \delta) D(\xi - \delta)| < \frac{1 - \alpha_i}{\alpha_i} \varphi(c - \tau_L).$$

Hence, if $\|G\| < \delta$, then (7) implies that $\bar{\varphi}(g_\omega, x) < \varphi(c - \tau_L)$, a contradiction. Thus, it cannot be that $g_\omega > g_\psi$, so the lemma holds. □

Lemma 5. *If $g_\omega = p_i^*(x, G)$, then $g_{\omega-1} \in R_i(g_\omega, x)$.*

Proof. Suppose that $g_\omega = p_i^*(x, G)$. Because $p_i^*(x, G) \in G$, it must be that

$$g_{\omega-1} D(g_{\omega-1})(\alpha_i + (1 - \alpha_i)\bar{\varphi}(g_\omega, x)) = \max_{p_i \in G} \alpha_i D(p_i)(1 - \bar{\varphi}(p_i, x)),$$

or the construction of $p_i^*(x, G)$ would have been perturbed such that $p_i^*(x, G) \notin G$ (see the proof of Proposition 2). By Lemma 4, $g_\omega \in R_i(g_\omega, x)$ implies that $g_\omega \in \tilde{P}(x, G)$, so $g_{\omega-1} \in R_i(g_\omega, x)$. □

Proof of Proposition 2.

Proof. We first prove that, under $A1$ and $A2$, if $\|G\| < \delta$, then $R_i(g_\omega, x)$ is the best response correspondence. Suppose that $\|G\| < \delta < c$. That $p_i^*(x, G) \in G$ follows from $g_0 = 0$, which implies that

$$g_0 D(g_0)(1 - \bar{\varphi}(g_1, x)) = 0,$$

while there is some $g \in G$ with $\in (0, c)$ guaranteeing that

$$\max_{p_i \in G} \alpha_i p_i D(p_i)(1 - \bar{\varphi}(p_i, x)) \geq gD(g) > 0.$$

If

$$g_{\omega^*(x)-1} D(g_{\omega^*(x)-1})(\alpha_i + (1 - \alpha_i)\bar{\varphi}(g_{\omega^*(x)}, x)) < \max_{p_i \in G} \alpha_i p_i D(p_i)(1 - \bar{\varphi}(p_i, x)),$$

then perturb the judo price by setting $p_i^*(x, G) = (g_{\omega^*(x)} + g_{\omega^*(x)+1})/2$. This, or any perturbation in $(g_{\omega^*(x)}, g_{\omega^*(x)+1})$ is necessary, as the proposition states that $g_{\omega^*(x)-1}$ is a best response to $p_i^*(x, G)$. In order for this perturbation to be well defined, it must be $\omega^*(x) < M$, which follows from Lemma 3.

$A1$ implies that $\xi D(\xi)(\alpha_i + (1 - \alpha_i)\bar{\varphi}(g_\omega, x))$ is strictly increasing in ξ. Thus,

$$R_i(g_\omega, x) \supset \{g_{\omega-1}, g_\omega\} \cup \tilde{P}(x, G).$$

For $R_i(g_\omega, x)$ to be as in the statement of the proposition, two properties must be satisfied: (i) if $g_\omega \in R_i(g_\omega, x)$, then $g_\omega \in \tilde{P}(x, G)$, and (ii) if $g_\omega = p_i^*(x, G)$, then $g_{\omega-1} \in R_i(g_\omega, x)$. The fact that $\tilde{P}(x, G) \subset R_i(g_\omega, x)$ for $g_\omega \leq p_i^*(x, G)$ and $g_{\omega-1} \in R_i(g_\omega, x)$ for $g_\omega > p_i^*(x, G)$ follows directly from the construction of $p_i^*(x, G)$ and the fact that $g_{\omega-1} D(g_{\omega-1})(\alpha_i + (1 - \alpha_i)\bar{\varphi}(g_\omega, x))$ is strictly increasing in ω. Lemmas 4 and 5 prove properties (i) and (ii), respectively.

We now prove the two limiting statements. Let G^n be such that $\|G^n\| \to 0$ as $n \to \infty$. We first show that $\max_{g \in G^n} \pi_i^R(g, x) \to \max_\xi \pi_i^R(\xi, x)$, i.e., the constrained maximal residual profits converge to the unconstrained maximal residual profits as the grid becomes arbitrarily fine.

Define $g^n \in G^n$ such that $g^n \to g \in \tilde{P}(x)$ and $g^n < g$. The continuity of π_i^R on $[0, c) \cup (c, \infty) \times X$, coupled with the fact that $\lim_{\xi \to c^-} \pi_i^R(\xi, x) = \pi_i^R(c, x)$ implies that $\pi_i^R(g^n, x) \to \pi_i^R(g, x)$.

Thus,

$$\lim_{n\to\infty}\max_{g\in G^n}\pi_i^R(g,x)\geq\max_{\xi}\pi_i^R(\xi,x),$$

while

$$\max_{g\in G^n}\pi_i^R(g,x)\leq\max_{\xi}\pi_i^R(\xi,x)$$

as the unconstrained maximum must be weakly greater than the constrained maximum. Therefore, $\max_{g\in G^n}\pi_i^R(g,x)\to\max_\xi\pi_i^R(\xi,x)$.

Next, define $g^n\in\tilde{P}(x,G^n)$ such that $g^n\to\xi$. By definition, $\pi_i^R(g^n,x)=\max_{g\in G^n}\pi_i^R(g,x)$ and by the preceding argument, $\pi_i^R(g^n,x)\to\max_{p_i}\pi_i^R(p_i,x)$. If $\xi\neq c$, then the continuity of π_i^R guarantees that $\xi\in\tilde{P}(x)$. Now suppose that $\xi=c$. As $\pi_i^R(c,x)=\limsup_{p_i\to c}\pi_i^R(p_i,x)$ and $\max_{g\in G^n}\pi_i^R(g,x)\to\max_{p_i}\pi_i^R(p_i,x)$ implies that $\pi_i^R(c,x)\geq\max_{p_i}\pi_i^R(p_i,x)$, $\xi\in\tilde{P}(x)$, proving statement (i).

We now prove statement (ii). Let $p_i^*(x,G^n)\to\xi^*$. We will show that $\xi^*=p_i^*(x)$. Define $\rho=(\xi^*+p_i^*(x))/2$. The remainder of the proof proceeds in two cases.

<u>Case 1</u>: $\xi^*\geq c$.

Case 1a: $\xi^*<p_i^*(x)$. It follows that there exists a value n^* such that $p_i^*(x,G^n)<\rho$ for all $n>n^*$. Choose such an n^* and note that for all $n>n^*$ and all $g_\omega^n\in G^n\cap(\rho,p_i^*(x))$,

$$\begin{aligned}g_\omega^nD(g_\omega^n)(\alpha_i+(1-\alpha_i)\bar{\varphi}(g_\omega^n,x))&>\rho D(\rho)(\alpha_i+(1-\alpha_i)\bar{\varphi}(\rho,x))\\&>\max_{g\in G^n}\pi_i^R(g,x).\end{aligned}\tag{8}$$

The first inequality follows from the fact that $\xi D(\xi)(\alpha_i+(1-\alpha_i)\bar{\varphi}(\xi,x))$ is increasing in ξ by $A1$ and Lemma 1 and the second inequality follows from $\rho>p_i^*(x,g)$. Assign $g_\omega^n\in G^n\cap(\rho,p_i^*(x))$ with $g^n\to g<p_i^*(x)$, where this set is nondegenerate for δ sufficiently small. Because $\lim_n\max_{g\in G^n}\pi_i^R(g,x)=\max_\xi\pi_i^R(\xi,x)$, it follows from (8) that

$$gD(g)(\alpha_i+(1-\alpha_i)\bar{\varphi}(g,x))\geq\max_{\xi}\pi_i^R(\xi,x).\tag{9}$$

However, as $g<p_i^*(x)$,

$$gD(g)(\alpha_i+(1-\alpha_i)\bar{\varphi}(g,x))<p_i^*(x)D(p_i^*(x))(\alpha_i+(1-\alpha_i)\bar{\varphi}(p_i^*(x),x)).$$

and the definition of the judo price $p_i^*(x)$ implies that,

$$p_i^*(x)D(p_i^*(x))(\alpha_i + (1-\alpha_i)\bar{\varphi}(p_i^*(x), x)) \leq \max_{\xi} \pi_i^R(\xi, x), \tag{10}$$

Collectively, (8)-(10) imply that

$$\begin{aligned}\max_{\xi} \pi_i^R(\xi, x) &\leq gD(g)(\alpha_i + (1-\alpha_i)\bar{\varphi}(g, x)) \\ &< p_i^*(x)D(p_i^*(x))(\alpha_i + (1-\alpha_i)\bar{\varphi}(p_i^*(x), x)) \leq \max_{\xi} \pi_i^R(\xi, x),\end{aligned}$$

a contradiction. Thus, $\xi^* \geq p_i^*(x)$.

Case 1b: $\xi^* > p_i^*(x)$. Let n^* be such that $p_i^*(x, G^n) < \rho$ and note that (by an analogous argument to case 1a) for all $n > n^*$ and all $g_\omega^n \in G^n \cap (p_i^*(x), \rho)$,

$$\begin{aligned}g_\omega^n D(g_\omega^n)(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_\omega^n, x)) &< \rho D(\rho)(\alpha_i + (1-\alpha_i)\bar{\varphi}(\rho, x)) \\ &< \max_{g \in G^n} \pi_i^R(g, x).\end{aligned}$$

Now let $g_\omega^n \in G^n \cap (\rho, p_i^*(x))$ with $g^n \to g > p_i^*(x)$. It follows that

$$gD(g)(\alpha_i + (1-\alpha_i)\bar{\varphi}(g, x)) \leq \max_{\xi} \pi_i^R(\xi, x).$$

The definition of $p_i^*(x)$ implies that

$$gD(g)(\alpha_i + (1-\alpha_i)\bar{\varphi}(g, x)) > \max_{g \in G^n} \pi_i^R(g, x),$$

which further implies that $\max_\xi \pi^R(\xi, x) > \max_\xi \pi^R(\xi, x)$, a contradiction.

<u>Case 2</u>: $\xi^* < c$. Recall that $p_i^*(x) \geq c$. Define n^* such that $p_i^*(x, G^n) < \rho$ for all $n > n^*$. Then let $g_\omega^n \in G^n \cap (\rho, p_i^*(x))$ and note that

$$g_{\omega-1}^n D(g_{\omega-1}^n)(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_\omega^n, x)) \geq \max_{g \in G^n} \pi_i^R(g, x).$$

Because $g_\omega^n < c$,

$$\begin{aligned}g_{\omega-1}^n D(g_{\omega-1}^n)(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_\omega^n, x)) &= \alpha_i g_{\omega-1}^n D(g_{\omega-1}^n) \\ &< \alpha_i c D(c) \\ &= \pi_i^R(c, x) \\ &\leq \max_{g \in G^n} \pi_i^R(g, x),\end{aligned}$$

a contradiction. Thus, $\xi^* \geq c$ and $\xi^* = p_i^*(x)$, completing the proof. □

Proof of Proposition 3.

Proof. As each $p_i^*(x) \geq c$, the result holds trivially if $p_{-i}^*(x) = c$. Suppose that $p_i^*(x) \in (c, \xi^m)$. By the continuity of $D(\xi)$ and $\bar{\varphi}(\xi, x)$ on $(c, \infty) \times X$, $\pi_{-i}^F(p_{-i}^*(x), x) = \pi_{-i}^R(x)$. That is,

$$p_{-i}^*(x)D(p_{-i}^*(x))(\alpha_{-i} + (1 - \alpha_{-i})\bar{\varphi}(p_{-i}^*(x), x)) = \max_{p_{-i}} \alpha_{-i}D(p_{-i})(1 - \bar{\varphi}(p_{-i}, x)).$$

Define the function

$$f(\xi, \alpha) = \xi D(\xi)(\alpha + (1 - \alpha)\bar{\varphi}(\xi, x)) - \max_{\zeta} \alpha\zeta D(\zeta)(1 - \bar{\varphi}(\zeta, x)),$$

so that $p_{-i}^*(x)$ is defined by $f(p_{-i}^*(x), \alpha_{-i}) = 0$. Note that $f(\xi, \alpha)$ is strictly increasing in ξ. It therefore suffices to show (by the implicit function theorem) that $f(\xi, \alpha)$ is decreasing in α:

$$\frac{\partial}{\partial\alpha} f(\xi, \alpha) = \xi D(\xi)(1 - \bar{\varphi}(\xi, x)) - \max_{\zeta} \zeta D(\zeta)(1 - \bar{\varphi}(\zeta, x)) \leq 0.$$

If $\alpha_{-i}p_{-i}^*(x)D(p_{-i}^*(x))(1 - \bar{\varphi}(p_{-i}^*(x), x)) = \max_\zeta \alpha\zeta D(\zeta)(1 - \bar{\varphi}(\zeta, x))$, then by definition, $p_{-i}^*(x) \in \tilde{P}(x)$. Thus

$$p_{-i}^*(x)D(p_{-i}^*(x))(\alpha_{-i} + (1 - \alpha_{-i})\bar{\varphi}(p_{-i}^*(x), x)) = \alpha_{-i}p_{-i}^*(x)D(p_{-i}^*(x))(1 - \alpha_{-i}\bar{\varphi}(p_{-i}^*(x), x)),$$

which holds only if $\bar{\varphi}(p_{-i}^*(x), x) = 0$, which requires $p_{-i}^*(x) < c$. Because $p_{-i}^*(x) > c$, $\frac{\partial}{\partial\alpha} f(\xi, \alpha) < 0$. Hence, $p_i^*(x) > p_{-i}^*(x)$. □

Proof of Proposition 4.

Proof. Suppose that x first order stochastically dominates x'. Lemma 1(iii) implies that $\bar{\varphi}(\xi, x) \leq \bar{\varphi}(\xi, x')$ for all ξ. Thus,

$$\xi D(\xi)(\alpha_i + (1 - \alpha_i)\bar{\varphi}(\xi, x)) \leq \xi D(\xi)(\alpha_i + (1 - \alpha_i)\bar{\varphi}(\xi, x'))$$

and

$$\max_{p_i} \alpha_i p_i D(p_i)(1 - \bar{\varphi}(p_i, x)) \geq \max_{p_i} \alpha_i p_i D(p_i)(1 - \bar{\varphi}(\xi, x')).$$

If

$$\xi D(\xi)(\alpha_i + (1-\alpha_i)\bar{\varphi}(\xi, x')) \leq \max_{p_i} \alpha_i p_i D(p_i)(1-\bar{\varphi}(p_i, x')), \tag{11}$$

then

$$\xi D(\xi)(\alpha_i + (1-\alpha_i)\bar{\varphi}(\xi, x)) \leq \max_{p_i} \alpha_i p_i D(p_i)(1-\bar{\varphi}(p_i, x)). \tag{12}$$

Therefore

$$\begin{aligned}
&\sup\left\{\xi \leq \xi^m : \xi D(\xi)\left(\alpha_i + (1-\alpha_i)\bar{\varphi}(\xi, x')\right) < \max_{p_i} \alpha_i p_i D(p_i)\left(1-\bar{\varphi}(p_i, x')\right)\right\} \\
&\qquad \leq \sup\left\{\xi \leq \xi^m : \xi D(\xi)\left(\alpha_i + (1-\alpha_i)\bar{\varphi}(\xi, x)\right) < \max_{p_i} \alpha_i p_i D(p_i)\left(1-\bar{\varphi}(p_i, x)\right)\right\},
\end{aligned}$$

and so by definition, $p_i^*(x) \geq p_i^*(x')$.

Now consider the case in which prices are constrained to a grid G with $\|G\| < \delta$, where $\delta > 0$ is sufficiently small such that each firm's best response correspondence is as in Proposition 2. Since $\bar{\varphi}(\xi, x) \leq \bar{\varphi}(\xi, x')$, it follows that for all $g_\omega \in G$,

$$g_{\omega-1} D(g_{\omega-1})(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_\omega, x)) \leq g_{\omega-1} D(g_{\omega-1})(\alpha_i + (1-\alpha_i)\bar{\varphi}(g_\omega, x'))$$

and

$$\max_{p_i \in G} \alpha_i p_i D(p_i)(1-\bar{\varphi}(p_i, x)) \geq \max_{p_i \in G} \alpha_i p_i D(p_i)(1-\bar{\varphi}(\xi, x')).$$

Thus, by an identical argument, the unperturbed critical price $p_i^*(x, G) \geq p_i^*(x', G)$. Furthermore, the same argument is valid given strict inequalities in (11) and (12). Therefore, if $p_i^*(x', G)$ is perturbed as in the proof of Proposition 2, then either $p_i^*(x, G) > p_i^*(x', G)$ or $p_i^*(x, G)$ is also perturbed and the inequality holds. $\square$

Proof of Proposition 5.

Proof. First, note that the case in which $c = c^*(p)$ can be ignored, as the grid may be perturbed so that no prices satisfy this relationship. Let $p^t = p$ for all $t \in [T, T+\varepsilon)$. By the definition of $c^*(p)$, search has a negative expected payoff if and only if $c > c^*(p)$.

Consequently, a consumer's expected payoff is strictly increasing in her threshold τ when $c > c^*(p)$ as the probability of search is decreasing in the threshold τ.

Under the best response dynamic, if $c > c^*(p)$, then x_L^t is increasing and x_k^t is decreasing for all $k \neq L$. By an analogous argument, if $c < c^*(p)$, then x_0^t is increasing and x_k^t is decreasing for all $k \neq 0$. Thus, the result holds under the best response dynamic.

Now, consider the imitation dynamic. Suppose $c > c^*(p)$. Let $r_{k\ell} = r_{k\ell}(E[\upsilon|p^t, \tau_k], E[\upsilon|p^t, \tau_\ell])$. Then by $A3$, $r_{k\ell} > r_{\ell k}$ if and only if $k < \ell$ because search is not profitable when $c > c^*(p)$. Recall that the net flow of x^t under the imitation dynamic is

$$\begin{aligned} \dot{x}_k^t &= \sum\nolimits_\ell x_\ell^t \rho_{\ell k} - x_k^t \sum\nolimits_\ell \rho_{k\ell} \\ &= \sum\nolimits_\ell x_\ell^t x_k^t r_{\ell k} - x_k^t \sum\nolimits_\ell x_\ell^t r_{k\ell} \\ &= x_k^t \sum\nolimits_\ell x_\ell^t (r_{\ell k} - r_{k\ell}). \end{aligned} \tag{13}$$

We now show that for all $t, t' \in [T, T+\varepsilon)$ with $t > t'$,

$$\sum\nolimits_{\ell=0}^k x_\ell^t \leq \sum\nolimits_{\ell=0}^k x_\ell^{t'}$$

for all $k = 0 : L$. It is sufficient to show that, for all $a = 0 : L$,

$$\sum\nolimits_{k=0}^a \dot{x}_k^t \leq 0.$$

For all $a = 0 : L$ and by (13),

$$\begin{aligned} \sum\nolimits_{k=0}^a \dot{x}_k^t &= \sum\nolimits_{k=0}^a x_k^t \sum\nolimits_\ell x_\ell^t (r_{\ell k} - r_{k\ell}) \\ &= \underbrace{\sum\nolimits_{k=0}^a \sum\nolimits_{\ell=0}^a x_k^t x_\ell^t (r_{\ell k} - r_{k\ell})}_{=0} + \sum\nolimits_{k=0}^a \sum\nolimits_{\ell=a+1}^L x_k^t x_\ell^t (r_{\ell k} - r_{k\ell}) \\ &= \sum\nolimits_{k=0}^a \sum\nolimits_{\ell=a+1}^L x_k^t x_\ell^t (r_{\ell k} - r_{k\ell}). \end{aligned}$$

As argued above, $r_{\ell k} > r_{k\ell}$ for all $\ell = a+1 : L$ and all $k = 0 : a < \ell$. Hence,

$$\sum\nolimits_{k=0}^a \dot{x}_k^t = \sum\nolimits_{k=0}^a \sum\nolimits_{\ell=a+1}^L x_k^t x_\ell^t (r_{\ell k} - r_{k\ell}) < 0,$$

completing this case. The proof for $c < c^*(p)$ proceeds identically. $\square$

The following Lemma is used to prove Theorem 1.

Lemma 6. *Under A1, A2, and C1, there exists a $\delta > 0$ such that if $\|G\| < \delta$, then there exists a neighborhood $\mathcal{N}(e_L)$ of e_L such that $c < c^*(p)$ for all $x \in \mathcal{N}(e_L)$, where $p_i = \inf \tilde{P}(x, G)$ and $p_{-i} = p_i^*(x, G) = p^*(x, G)$ is the critical grid-constrained judo price.*

Proof. First, we prove that there exists a neighborhood $\mathcal{N}(e_L)$ of e_L such that $\tilde{P}(x, G) = \tilde{P}(e_L, G)$ for all $x \in \mathcal{N}(e_L)$. Take $g \in \tilde{P}(e_L, G)$ and define

$$\Delta = \pi_i^R(g, e_L) - \max_{g' \in G \setminus \tilde{P}(e_L, G)} \pi_i^R(g', e_L)$$

as the difference between the maximal constrained residual profits and the second-best. Because π_i^R is continuous in x, there exists a neighborhood $\mathcal{N}(e_L)$ of e_L such that

$$\begin{aligned} \pi_i^R(g, x) >& \pi_i^R(g, e_L) - \frac{\Delta}{2} \\ \pi_i^R(g', x) <& \pi_i^R(g', e_L) + \frac{\Delta}{2} \end{aligned}$$

for all $g' \in G \setminus \tilde{P}(e_L, G)$ and all $x \in \mathcal{N}(e_L)$. Therefore, $\pi_i^R(g, x) > \pi_i^R(g', x)$ for all $x \in \mathcal{N}(e_L)$ and $g' \in G \setminus \tilde{P}(e_L, G)$. Hence, $g \in \tilde{P}(x, G)$ and $\tilde{P}(x, G) = \tilde{P}(e_L, G)$. As e_L first order stochastically dominates all $x \in X$, $p^*(x, G) \leq p^*(e_L, G)$ for all $x \in X$ by Proposition 4.

We now prove that, for all $x \in \mathcal{N}(e_L)$, $c < c^*(p)$ when $p_i = \inf \tilde{P}(x, G)$ and $p_{-i} = p_i^*(x, G) = p^*(x, G)$. By $C1$, $c < \min\{c^*((p^*(e_L), \inf \tilde{P}(e_L))), c^*((\inf \tilde{P}(e_L), p^*(e_L)))\}$. Recall that $c^*(p)$ is defined by

$$u(D(\min p), \min p) - c^*(p) = \alpha u(D(p_1), p_1) + (1 - \alpha) u(D(p_2), p_2). \tag{14}$$

The continuity of u and D imply that (14) is decreasing in $\min p$ and increasing in $\max p$. To illustrate this relationship, suppose that $p_1 \leq p_2$. Then, (14) can be rearranged as

$$c^*(p) = (1 - \alpha)(u(D(p_1), p_1) - u(D(p_2), p_2)),$$

which by inspection is decreasing in $p_1 = \min p$ and increasing in $p_2 = \max p$.

Set $\varepsilon' = c^*(p) - c$ for the prices given in the statement of the Lemma and define $\underline{x}$ in the closure of $\mathcal{N}(e_L)$ such that $\underline{x}$ is first order stochastically dominated by all $x \in \mathcal{N}(e_L)$. Fix $\varepsilon > 0$.

Define $\delta(\varepsilon)$ such that if both $|p_i - p_i'| \leq \delta(\varepsilon)$ and $|p_{-i} - p_{-i}| \leq \delta(\varepsilon)$, then $|c^*(p) - c^*(p')| < \varepsilon$. By Proposition 2, δ' can be chosen such that if $\|G\| < \delta'$, then $|p^*(\underline{x}, G) - p^*(e_L)| \leq \delta(\varepsilon)$ and $\min \tilde{P}(e_L, G) \geq \inf \tilde{P}(e_L) - \delta(\varepsilon)$. By Proposition 4, if $|p^*(\underline{x}, G) - p^*(e_L)| \leq \delta(\varepsilon)$, then $|p^*(e_L, G) - p^*(e_L)| < \delta(\varepsilon)$. Because $c^*(p)$ is decreasing in $\min p$ and $\min \tilde{P}(\underline{x}, G) = \min \tilde{P}(e_L, G) \geq \inf \tilde{P}(e_L) - \delta(\varepsilon)$, a well-defined neighborhood $\mathcal{N}(e_L)$ exists for every ε' such that

$$\min\{c^*(p^*(\underline{x}, G), \min \tilde{P}(\underline{x}, G)), c^*(\min \tilde{P}(\underline{x}, G), p^*(\underline{x}, G))\} \\ > \min\{c^*((p^*(e_L), \inf \tilde{P}(e_L))), c^*((\inf \tilde{P}(e_L), p^*(e_L)))\} - \varepsilon. \quad (15)$$

Assigning

$$\varepsilon < \min\{c^*((p^*(e_L), \inf \tilde{P}(e_L))), c^*((\inf \tilde{P}(e_L), p^*(e_L)))\} - c,$$

which is well defined by $C1$, implies that

$$c < \min\{c^*(p^*(x, G), \min \tilde{P}(x, G)), c^*(\min \tilde{P}(x, G), p^*(x, G))\}$$

for all $x \in \mathcal{N}(e_L)$ and G such that $\|G\| < \delta'$. □

Proof of Theorem 1.

Proof. Suppose that $\|G\| < \delta < c$, where δ is such that best response correspondence for each firm is as stated in Proposition 2.

First, we prove statement (i) (the prices do not converge). To the contrary, suppose that $p^t \to (g_\omega, g_{\omega'})$. Without loss of generality, assume that $g_\omega \leq g_{\omega'}$. By Proposition 2, for some time $T > 0$, it must that (i) $g_\omega \leq p_2^*(x^t, G)$ for all $t > T$, (ii) $g_{\omega'} \in \tilde{P}(x, G)$, and (iii) $g_\omega \in \{g_{\omega'}, g_{\omega'-1}\}$. As $|p_1^t - p_2^t| \leq \delta < c$ for all $t > T$, Proposition 5 implies that $x^t \to e_L$ as $t \to \infty$ because at such prices, $\dot{x}_L^t \to 0$ if and only if $x_L \to 1$. Consider two cases.

<u>Case 1</u>: $g_\omega = g_{\omega'}$. In this case, it must be that $g_\omega \leq p_1^*(x^t, G)$ for all $t > T$. Because $p_i^*(x, G) \leq \min \tilde{P}(x, G)$ for all $x \in X$, $p^*(x^t, G) = \min \tilde{P}(x^t, G)$ for all $t > T$. Hence,

$$\min\{c^*((p^*(x^t, G), \min \tilde{P}(x^t, G))), c^*((\min \tilde{P}(x^t, G), p^*(x^t, G)))\} = 0 < c$$

for all $t > T$, which contradicts Lemma 6.

Case 2: $g_\omega = g_{\omega'-1}$. In this case, it must be that $g_{\omega'} \geq p_1^*(x^t, G)$ for all $t > T$. It follows that either $p^*(x^t) = g_\omega$ or that $p^*(x^t) = g_{\omega'}$ for all $t > T$. Either way,

$$\min\{c^*((p^*(x^t, G), \min \tilde{P}(x^t, G))), c^*((\min \tilde{P}(x^t, G), p^*(x^t, G)))\} \leq \delta < c$$

for all $t > T$, which contradicts Lemma 6. Therefore, the prices do not converge.

Next, we prove statement (ii) (in the limit, prices are bounded). By Proposition 2, given a distribution x^t, neither firm i will ever choose a price $p_i^t < g_{\omega-1}$ when $g_\omega = p_i^*(x, G)$. Thus, by Propositions 2 and 4, neither firm i will ever choose a price below $p_i^t < g_{\omega-1}$ when $g_\omega = p_i^*(e_0, G)$. Set $g_\omega = p_i^*(e_0, G)$ and suppose that $p_i^*(e_0, G) = p^*(e_0, G)$. If $p_i^t = g_{\omega'}$, then $p_{-i}^t \geq g_{\omega'-1}$. As firm i will never choose a price $p_i^t < g_{\omega-1}$, then there will be some time T such that firm i eventually chooses a price $p_i^t \geq g_{\omega-1}$. It follows that for all times $t > T$, $p_i^t \geq g_{\omega-1}$ and $p_{-i}^t \geq g_{\omega-2}$. Therefore, Proposition 2 guarantees that for sufficiently small $\delta > 0$, if $\|G\| < \delta$, then $|p_i^*(e_0, G) - p_i^*(e_0)| < \varepsilon$, and so given such a δ, $p_i^t \geq \underline{p} - \varepsilon$ for both firms i and all $t > T$. That $p_i^t \leq \xi^m$ for all $t > T$ follows directly from $A1$ (and $\max G = \xi^m$).

Lastly, we prove statement (iii) (infinite cycles between e_0 and e_L). To prove this statement, it is sufficient to show that for any $\varepsilon' > 0$ and any neighborhoods $\mathcal{N}(e_0)$ of e_0 and $\mathcal{N}(e_L)$ of e_L, for all T, there is a positive probability that (a) $x^t \in \mathcal{N}(e_L)$ for some $t > T$, (b) $p_i^{t'_n} \to \xi' \geq \hat{p} - \varepsilon$ for some $t > T$, (c) $x^t \in \mathcal{N}(e_0)$ for some $t > T$, and (d) $p_i^{t_n} \to \xi \leq \underline{p} + \varepsilon$ for some $t > T$. We will jointly demonstrate (a) and (b) followed by (c) and (d).

Proposition 2 dictates that at some time t, firms will set their prices such that $p_i^t = g_\omega$ and $p_{-i}^t = g_{\omega-1}$ for some ω. Given such prices, Proposition 5 implies that the distribution of consumer thresholds will be shifting toward e_L. Under the best response dynamic

$$\dot{x}_L^t = 1 - x_i^t,$$

while under the imitation dynamic

$$\dot{x}_L^t = x_L^t \sum_{\ell=0}^{L} x_\ell^t (r_{\ell L} - r_{L\ell}).$$

Given either dynamic, if the prices are fixed at $p_i^t = g_\omega$ and $p_{-i}^t = g_{\omega-1}$ for some ω, then $\dot{x}_L^t \to 0$ if and only if $x_L \to 1$. If these prices were to remain fixed, then $x^t \to e_L$.

Let $\mathcal{N}(e_L)$ be a neighborhood of e_L. Define

$$\tilde{T} = \sup_{p \in G^2} \sup_{x^T \in X} \inf\{t \geq 0 : x^{T+t} \in \mathcal{N}(e_L) \text{ given } p\}.$$

Given any time T, any prices g_ω and $g_{\omega-1}$, and any $x^T \in X$, it follows that if $p_i^t = g_\omega$ and $p_{-i}^t = g_{\omega-1}$ for all $t > T$, then $x^{T+t} \in \mathcal{N}(e_L)$ for all $t > \tilde{T}$. Given the stickiness of pricing, for any prices $p_i^T = g_\omega$, $p_{-i}^T = g_{\omega-1}$, and any $t' > \tilde{T}$, there is a positive probability that $p_i^{T+t} = g_\omega$ and $p_{-i}^{T+t} = g_{\omega-1}$ for all $t < t'$. Thus, there is a positive probability that $x^t \in \mathcal{N}(e_L)$ for some $t > T$.

Choose $\mathcal{N}(e_L)$ such that for all $x \in \mathcal{N}(e_L)$, $p_i^*(x, G) = p_i^*(e_L, G)$ for each firm i and $\tilde{P}(x, G) \subseteq \tilde{P}(e_L, G)$ (By Lemma 6, this neighborhood is well defined). If $x^T \in \mathcal{N}(e_L)$, define

$$\tilde{T}(x^T) = \inf_{p \in G^2} \inf\{t \geq 0 : x^{T+t} \notin \mathcal{N}(e_L) \text{ given } p\}.$$

Then, given any time T such that $x^T \in \mathcal{N}(e_L)$, let Q denote the maximum number of sequential price changes by the firms according to the best response correspondence that are necessary to reach a pair of prices such that $p_i \in \tilde{P}(x, G)$ and $p_{-i} = g_{\omega-1}$, where $g_\omega = p_{-i}$. There is a positive probability that the firms are able to make Q sequential price changes in the time interval $(T, T + \tilde{T}(x^T))$. If δ is chosen sufficiently small so that $\inf \tilde{P}(e_L, G) \geq \inf \tilde{P}(e_L) - \varepsilon$, then there is a positive probability that for either firm i, for some time $t > T$, $p_i^t \geq \hat{p} - \varepsilon$.

Next, using Lemma 6, choose δ and $\mathcal{N}(e_L)$ such that if $\|G\| < \delta$ and $x \in \mathcal{N}(e_L)$, then $c < c^*(p)$, where $p_i = \inf \tilde{P}(x, G)$ and $p_{-i} = p_i^*(x, G) = p^*(x, G)$. Given $x^T \in \mathcal{N}(e_L)$, there will be some $t > T$ where p^t is such that $c < c^*(p^t)$. Let $\mathcal{N}(e_0)$ be a neighborhood of e_0 and define

$$\tilde{T}_0 = \sup_{p \in G^2} \sup_{x^T \in X} \inf\{t \geq 0 : x^{T+t} \in \mathcal{N}(e_0) \text{ given } p\}.$$

Then given $c < c^*(p^T)$, for any $t' > \tilde{T}_0$ there is a positive probability that the firms prices remain fixed for all $t \in [T, t)$, and thus that $x^{T+t'} \in \mathcal{N}(e_0)$.

Finally, choose $\mathcal{N}(e_0)$ and δ such that for all $x \in \mathcal{N}(e_0)$, $p^*(x, G) = p^*(e_0, G) < p^*(e_0) + \varepsilon$.

Then define for all $x^T \in \mathcal{N}(e_0)$

$$\tilde{T}_0(x^T) = \inf_{p \in G^2} \inf\{t \geq 0 : x^{T+t} \notin \mathcal{N}(e_L) \text{ given } p\}$$

and let Q_0 be the maximum number of sequential price changes by the firms according to the best response correspondence that are necessary to reach a pair of prices such that $p_i = g_\omega = p^*(x, G)$ and $p_{-i} \in \{g_{\omega-1}, g_{\omega+1}\}$. Given any $x^T \in \mathcal{N}(e_0)$ there is a positive probability that the firms are able to make Q_0 sequential price changes in the time interval $(T, T + \tilde{T})_0(x^T))$, and thus that $p_i^t < p^*(e_L, G) + \varepsilon$ for each firm i for some $t > T$. $\square$

The following Lemma is used to prove Theorem 2.

Lemma 7. *Under A1, A2, and C2', there exists a* $\delta > 0$ *such that if* $\|G\| < \delta$*, then there exists a neighborhood* $\mathcal{N}(e_L)$ *of* e_L *such that* $c > c^*(p)$ *for all* $x \in \mathcal{N}(e_L)$*, where* $p_i = \sup \tilde{P}(x, G)$ *and* $p_{-i} = p_i^*(x, G) = p^*(x, G)$*.*

Proof. The proof of this lemma mirrors that of Lemma 6. By an analogous argument to Lemma 6 there exists a neighborhood $\mathcal{N}(e_L)$ of e_L such that $\tilde{P}(x, G) = \tilde{P}(e_L, G)$. As in the proof of Lemma 6, e_L first order stochastically dominates all $x \in X$, so $p^*(x, G) \leq p^*(e_L, G)$ for all $x \in X$.

We now prove that, for all $x \in \mathcal{N}(e_L)$, $c > c^*(p)$ when $p_i = \max \tilde{P}(x, G)$ and $p_{-i} = p_i^*(x, G) = p^*(x, G)$. By $C2$, $c > \min\{c^*((p^*(e_L), \sup \tilde{P}(e_L))), c^*((\sup \tilde{P}(e_L), p^*(e_L)))\}$. Fix $\varepsilon > 0$. Again, define $\delta(\varepsilon)$ such that if both $|p_i - p_i'| \leq \delta(\varepsilon)$ and $|p_{-i} - p_{-i}| \leq \delta(\varepsilon)$, then $|c^*(p) - c^*(p')| < \varepsilon$. By Proposition 2, δ' can be chosen such that if $\|G\| < \delta'$, then $\max \tilde{P}(e_L, G) \geq \sup \tilde{P}(e_L) - \delta(\varepsilon)$. Because $c^*(p)$ is decreasing in $\min p$ and thus $p^*(x, G)$ while $\max \tilde{P}(x, G) = \max \tilde{P}(e_L, G)$ for all $x \in \mathcal{N}(e_L)$,

$$\begin{aligned} &\min\{c^*(p^*(x, G), \max \tilde{P}(x, G)), c^*(\max \tilde{P}(x, G), p^*(x, G))\} \\ &\qquad\qquad < \{c^*((p^*(e_L), \sup \tilde{P}(e_L))), c^*((\sup \tilde{P}(e_L), p^*(e_L)))\} + \varepsilon. \end{aligned}$$

Assigning

$$\varepsilon < c - \min\{c^*((p^*(e_L), \sup \tilde{P}(e_L))), c^*((\sup \tilde{P}(e_L), p^*(e_L)))\},$$

which is well defined by $C2'$, implies that

$$c > \min\{c^*(p^*(x,G), \max \tilde{P}(x,G)), c^*(\max \tilde{P}(x,G), p^*(x,G))\}$$

for all $x \in \mathcal{N}(e_L)$ and G such that $\|G\| < \delta'$. $\square$

Proof of Theorem 2.

Proof. Suppose that $\|G\| < \delta < c$, where δ is such that the best response correspondences are as stated in Proposition 2. We first show that $x^t \to e_L$. By Proposition 2, there exists a time $T \geq 0$ such that $p_i = g_\omega$ and $p_{-i} \in \{g_\omega, g_{\omega\pm1}\}$. Hence, $|p_i - p_{-i}| < c$. Given price stickiness, there is positive probability that these prices will remain in this interval until some time $T' > T$. By proposition 5, for a sequence of times $\{t_n\} \subset [T', T)$, x^{t_n} first order stochastically dominates $x^{t_{n'}}$ for all $n' > n$. As e_L first order stochastically dominates all $x \in X$, there exists a $t' \in [T', T)$ such that a state $x^{t'} \in \mathcal{N}(e_L)$ is reached with positive probability, where $\mathcal{N}(e_L)$ is chosen such that Lemma 7 applies. Lemma 7 and Proposition 5 imply that e_L is an absorbing state. Hence, $x^t \to e_L$.

Statement (ii) follows from an identical argument to the proof of statement (ii) of Theorem 1 but fixing $x^t \in \mathcal{N}(e_L)$. Statement (iii) then follows from the best response correspondences of Proposition 2. $\square$

Proof of Proposition 6.

Proof. As $\tau_L \to \xi^m - \inf \operatorname{supp} \varphi$, $\xi^m \leq \tau_L + \sigma^t$ for all σ^t. Hence, $\varphi(\xi^m - \tau_L) \to 0$ as $\tau_L \to \xi^m - \inf \operatorname{supp} \varphi$. Given state $x = e_L$ and $\tau_L \to \xi^m - \inf \operatorname{supp} \varphi$,

$$\begin{aligned}
\pi^R(\xi, e_L) &= \alpha_i \xi D(\xi)\left(1 - \varphi(\xi - \tau_L)\right) \\
&\leq \alpha_i \xi^m D(\xi^m)\left(1 - \varphi(\xi - \tau_L)\right) \\
&\leq \alpha_i \xi^m D(\xi^m) \\
&= \pi_i^R(\xi^m, e_L).
\end{aligned}$$

Hence, $\tilde{P}(e_L) = \{\xi^m\}$. Therefore, the judo price is

$$p_i^*(e_L) = \sup\{\xi \leq \xi^m : \xi D(\xi)\,(\alpha_i + (1-\alpha_i)\varphi(\xi - \tau_L)) < \alpha_i \xi^m D(\xi^m)\}$$

Evaluating firm i's front-side profits as $p_i \to \xi^m$ and $p_{-i} = \xi^m$ yields

$$\alpha_i \xi^m D(\xi^m) = \xi^m D(\xi^m)(\alpha_i + (1-\alpha_i)\varphi(\xi^m - \tau_L)).$$

Hence, $p_i^*(e_L) = \xi^m = \sup \tilde{P}(e_L)$. That $C2$ holds for sufficiently large τ_L follows immediately.

Suppose that $\|G\| < \delta < c$ so that each firm's best response correspondence is as in Proposition 2. By continuity, for every $\varepsilon' > 0$, There exists a neighborhood $\mathcal{N}(\xi^m - \inf \operatorname{supp}\varphi)$ such that if $\tau_L \in \mathcal{T}(\xi^m - \inf \operatorname{supp}\varphi)$, then $p_i^*(e_L) > \xi^m - \varepsilon$. Let $\bar{\tau} = \inf \mathcal{N}(\xi^m - \inf \operatorname{supp}\varphi)$. Then, for all $\tau > \bar{\tau}$, $p_i^*(e_L) > \xi^m - \varepsilon$. For a sufficiently small δ, $|p_i^*(e_L, G) - p_i^*(e_L)|$ is sufficiently small such that $p^*(e_L, G) \geq p_i^*(e_L, G) > \xi^m - \varepsilon$.

As $A1$-$A3$ and $C2'$ are all satisfied, Theorem 2 implies that $x^t \to e_L$. Hence, there exists a time $T \geq 0$ such that for all $t > T$, $p_i^t > \xi^m - \varepsilon$. $\square$

Proof of Theorem 3.

Proof. Suppose $A1$-$A3$ and $C3$ are satisfied and set δ sufficiently small such that the best response correspondences are as in Proposition 2. By $C3$, given a neighborhood $\mathcal{N}(e_L)$, $p^*(x) = \sup \tilde{P}(x) = c$ for all $x \in \mathcal{N}(e_L)$. As $C3$ implies $C2$, there exists a time $T \geq 0$ such that $x^t \in \mathcal{N}(e_L)$ for all $t \geq T$. For any prices $p^t = (p_i^t, p_{-i}^t) \geq (c, c)$, Proposition 2 implies that the two firms will undercut each other at each revision opportunity until $p^t = (c, c)$. Because $\tilde{P}(x) = \{c\}$ for all $x \in \mathcal{N}(e_L)$, there exists a time $T' \geq T$ such that $p^t = (c, c)$ for all $t > T'$. $\square$