Problem 1

20 points
Problem **2.30** page 157 in the textbook.
Use Pumping Lemma for CFLs to prove the following language is not context-free.

$$A = \{ \, 0^n\#0^{2n}\#0^{3n} \mid n \geq 0 \, \}$$

Note: Alphabet $\Sigma = \{0, \#\}$. That is, # is a character in a string just like 0 or 1 in other languages discussed in this course.

Assume A is context-free. So, for some pumping length p, the pumping lemma for CFLs applies.

Let s = $0^p\#0^{2p}\#0^{3p}$ be a string in A, where |s| = 6p+2 > p

Then, there are 2 cases for dividing s into 5 parts uvxyz:

Case 1: v and y both contain only 0s.

The string s contains 3 substrings of 0s separated by "#" symbols. Since v and y do not contain any "#" symbols, then they can be part of at most 2 of the 3 substrings. So, pumping down causes at least one substring to have less 0s than in s and at least one substring to have the same amount of 0s as in s. Therefore, at least one substring of 0s will not satisfy the proportion 1:2:3, so the string uxz cannot be in A.

Case 2: v or y contains a # symbol.

By pumping down (i=0), the resulting string s=uxz will contain only 1 "#" symbol, so the string uxz cannot be in A.

Since there are no ways to divide s and satisfy the pumping lemma, then there is a contradiction, so A is not context-free.

1 Let $S = 0^p \# 0^{2p} \# 0^{3p}$ —1 $\rightarrow uvxyz$
must satisfy the below Condition
1. $\forall i \geq 0$, $uv^ixy^iz \in A$
2. $|vy| > 0$
3. $|vxy| \leq p$

Let's put $vxy = 0^p$ in the middle of 1
which gives $v = 0^2$, $x = 0^2$, $y = 0^2$
Contradicting 1. ~~say~~ indicating v & y have same index(2) and also x has the same index(2) to i
$\rightarrow$ So Not CFL

Problem 2

20 points

Is the following language B context-free? If yes, show a context-free grammar (CFG) that generates B. If no, please prove it using Pummping Lemma for CFLs.

$$B = \{ \ 0^n 0^{2n} \# 0^{3n} \mid n \geq 0 \ \}$$

Note: B is the same as A in Problem 1, except that there is one less # in each string of B (compared to the strings in A).

(2) $B = \{0^n 0^{2n} \# 0^{3n} \mid n \geq 0\} = \{0^{3n} \# 0^{3n}\}$

$S \Rightarrow ASA \mid \#$ $\qquad$ $S \rightarrow \{\underbrace{000}_{A} \underbrace{\#}_{S} \underbrace{000}_{S}\}$

$A \rightarrow 000$ $\qquad$ Yes, CFL

Problem 3

20 points

Problem 2.31 (page 157) in the textbook.

2.31 Let B be the language of all palindromes over $\{0,1\}$ containing equal numbers of 0s and 1s. Show that B is not context free.

Assume B is context-free. So, for some pumping length p, the pumping lemma for CFLs applies.

Let $s = 1^p0^p0^p1^p$ be a string in B, where $|s| = 4p > p$.

Then, there are 2 cases for dividing s into 5 parts uvxyz:

Case 1: v and y (combined) contain an equal number of 0s and 1s

If v and y contain an equal number of 1s and 0s, then the substring vxy must be either in the 1st half or the 2nd half of s. Pumping down (i=0) produces a string uxz that is no longer a palindrome, since at least one 1 was removed from either the first or second half of the string, but the other half of the string remains unchanged. Since uxz is not a palindrome, it is not in B.

Case 2: v and y (combined) do not contain an equal number of 0s and 1s.

If v or y do not contain an equal number of 0s and 1s, then pumping down (i=0) produces a string uxz with either more 0s than 1s or more 1s than 0s. Since the number of 0s is not equal to the number of 1s, then the string uxz is not in B.

Since there are no ways to divide s and satisfy the pumping lemma, then there is a contradiction, so B is not context-free.

= 111 0000 0000 111 → Palindrome

(3) Let s = 111 | 00 | 0 | 00 | 000111

u | v^2 | x | y^2 | z

follows 1st rule of Pumping lemma

$s = 1^3 0^3 0^3 1^3$

However,

#1 = 6

#0 = 8

nbr of Ones & nbr of Zeros are not equal

so, no CFL

Problem 4

20 points

Problem 2.32 (page 157) in the textbook.

2.32 Let $\Sigma = \{1, 2, 3, 4\}$ and $C = \{w \in \Sigma^* \mid$ in w, the number of 1s equals the number of 2s, and the number of 3s equals the number of 4s$\}$. Show that C is not context free.

2.32 Assume C is context-free and get its pumping length p from the pumping lemma. Let $s = 1^p 3^p 2^p 4^p$. Because $s \in C$, it can be split $s = uvxyz$ satisfying the conditions of the lemma. By condition 3, vxy cannot contain both 1s and 2s, and cannot contain both 3s and 4s. Hence uv^2xy^2z doesn't have equal number of 1s and 2s or of 3s and 4s, and therefore won't be a member of C, so s cannot be pumped and contradiction is reached. Therefore C isn't context-free.

(4) s = 1 2 3 2 14

u = 1, v = 2, x = 3, y = 2, z = 14

#1 → 2
#2 → 2

#3 → 1
#4 → 1

Let $i = 2$

uv^2xy^2z

= 122322 14

Gives #1 → 2, #2 → 4 → fails here, so no CFL

#3 → 1, #4 → 1 ✓

2.30 **a.** Assume A is context-free. Let p be the pumping length given by the pumping lemma. We show that $s = 0^p1^p0^p1^p$ cannot be pumped. Let $s = uvxyz$. If either v or y contain more than one type of alphabet symbol, uv^2xy^2z does not contain the symbols in the correct order and cannot be a member of A. If both v and y contain (at most) one type of alphabet symbol, uv^2xy^2z contains runs of 0's and 1's of unequal length and cannot be a member of A. Because s cannot be pumped without violating the pumping lemma conditions, A is not context free.

d. Assume A is context-free. Let p be the pumping length from the pumping lemma. Let $s = a^pb^p\#a^pb^p$. We show that $s = uvxyz$ cannot be pumped. Use the same reasoning as in part (c).

2.31 Assume B is context-free and get its pumping length p from the pumping lemma. Let $s = 0^p1^{2p}0^p$. Because $s \in B$, it can be split $s = uvxyz$ satisfying the conditions of the lemma. We consider several cases.

i) If both v and y contain only 0's (or only 1's), then uv^2xy^2z has unequal numbers of 0s and 1s and hence won't be in B.

ii) If v contains only 0s and y contains only 1s, or vice versa, then uv^2xy^2z isn't a palindrome and hence won't be in B.

iii) If both v and y contain both 0s and 1s, condition 3 is violated so this case cannot occur.

iv) If one of v and y contain both 0s and 1s, then uv^2xy^2z isn't a palindrome and hence won't be in B.

Hence s cannot be pumped and contradiction is reached. Therefore B isn't context-free.