COMP 4200 - Formal Languages: Solution to Homework #1

Due on Wed, Aug 31, 2018 at 2:00pm

Instructor: Dr. Anh Nguyen

Note: Late assignments will not be graded. You should submit your work on a separate sheet of paper in the order the questions are asked. You will not only be graded on your mathematics, but also on your organization, proper use of English, spelling, punctuation, and logic.

Problem 1

Answer the following exercises/problems in the book:

1. Exercise **0.1f** (page 25)
2. Exercise **0.1e** (page 25)
3. Exercise **0.6d** (page 25)
4. Exercise **0.6e** (page 25)

Solution:

1. The empty set.
2. The set of palindromes over the binary alphabet.
3. The range is $\{6, 7, 8, 9, 10\}$
   The domain is $\{(1, 6), (1, 7), (1, 8), (1, 9), (1, 10), (2, 6)(2, 7), (2, 8), (2, 9), (2, 10), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (4, 6), (4, 7), (4, 8), (4, 9), (4, 10), (5, 6), (5, 7), (5, 8), (5, 9), (5, 10)\}$
4. $g(4, f(4)) = g(4, 7) = 8$

Problem 2

Prove the following formulas by mathematical induction:

1. For all $n \in \mathbb{Z}^+$:
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$
2. For all $n \in \mathbb{Z}^+$:
$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Solution:

1. We will prove by induction that, for all $n \in \mathbb{Z}^+$
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \tag{1}$$
   Base case: Prove that the statement holds when n = 1.
$$\sum_{i=1}^{i} \frac{1}{i(i+1)} = \frac{1}{2}$$
$$\frac{n}{n+1} = \frac{1}{2}$$

When $n = 1$, the left side of equation is $\frac{1}{1 \cdot 2} = \frac{1}{2}$, and the right side is $\frac{1}{2}$, so both sides are equal and the (1) is true for $n = 1$

Induction Hypothesis: Assume that the statement holds when $n = k$.

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Inductive Step: Prove that the statement holds when when n = k+1 using the assumption above. Let $k \in \mathbb{Z}^+$ be given and suppose (1) is true for $n = k$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{(by induction hypothesis)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} && \text{(by algebra)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} && \text{(by algebra)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

Thus, (1) holds for $n = k+1$, and the proof of the induction step is complete.Therefore, the proposition holds for all $n \in \mathbb{Z}^+$.

2. It uses the fact that for any positive integer n:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Base case: Prove that the statement holds when $n = 1$.

$$1^3 = 1^2$$

Induction Hypothesis: Assume that the statement holds when $n = k$.

$$1^3 + 2^3 + 3^3 + ... + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

Inductive step: : Prove that the statement holds when when $n = k + 1$ using the assumption above. Assume the result is true for $n = k$, that is

$$\begin{aligned}
1^3 + 2^3 + 3^3 + 4^3 + \ldots + k^3 &= (\frac{k(k+1)}{2})^2 \\
&= (1 + 2 + 3 + \ldots + k)^2
\end{aligned}$$

Let $n = k + 1$ then

$$\begin{aligned}
&1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 \\
=&1^3 + 2^3 + 3^3 + 4^3 + \ldots + (k+1)^3 \\
=&(1 + 2 + 3 + \ldots + k)^2 + (k+1)^3 \\
=&(\frac{k(k+1)}{2})^2 + (k+1)^3 && \text{(by induction hypothesis)} \\
=&\frac{k^2(k+1)^2}{4} + (k+1)^3 \\
=&\frac{(k+1)^2}{4}(k^2 + 4k + 4) \\
=&\frac{((k+1)^2(k+2)^2)}{4} \\
=&(\frac{(k+1)(k+2)}{2})^2 \\
=&(1 + 2 + 3 + \ldots + (k+1))^2
\end{aligned}$$

Thus $1^3 + 2^3 + 3^3 + 4^3 + ...n^3 = (1 + 2 + 3 + ...n)^2$ for all positive integers n.