Tomographic reconstruction
Lecture 1: Radon transform, Filtered backprojection and Inverse Problems

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[1]based on a previous lecture by Ozan Öktem

Formalizing the notion of an image

What is an image?

- User's viewpoint: A quantity spatially distributed in 2D/3D.
- Mathematical viewpoint: An image is a function in 2D/3D:
  - Grey-scale images: Real valued function in 2D/3D, *i.e.* $f : \mathbb{R}^n \to \mathbb{R}$ where $n = 2$ for 2D images and $n = 3$ for 3D images.
  - Colour images: Vector valued function in 2D/3D, *i.e.* $f : \mathbb{R}^n \to \mathbb{R}^k$ where $n = 2$ or 3 and k is the number of colour channels (*e.g.* $k = 3$ for red, green & blue).

  We will only work with grey-scale images.
- Support of an image: The set $\Omega \subset \mathbb{R}^n$ where the image is defined, usually a rectangular region.
- Dynamic range of an image: The range of values in $\mathbb{R}^k$ that an image $f : \Omega \to \mathbb{R}^k$ can attain.

Image reconstruction

In many imaging applications, the image we seek is only indirectly observed. This is especially the case when we seek to observe a 3D image.

- 3D/2D tomography in medical imaging
- Geophysical exploration
- 3D microscopy
- Radar imaging

Image reconstruction

Image reconstruction as an inverse problem

Recover the image $f_{\text{true}} \in \mathscr{X}$ from measured data $\boldsymbol{g} \in \mathbb{R}^m$ assuming

$$\boldsymbol{g} = \mathcal{S}\big(\mathcal{A}(f_{\text{true}})\big) + \boldsymbol{g}_{\text{noise}}.$$

- $f_{\text{true}} \colon \Omega \to \mathbb{R}$ is the image that is to be recovered and $\mathscr{X}$ (reconstruction space) is the set of feasible images.
- $\boldsymbol{g} \in \mathbb{R}^m$ is the measured data, *i.e.*, the m numeric quantities recorded by the imaging device.
- $\mathcal{A} \colon \mathscr{X} \to \mathscr{Y}$ is the forward operator and $\mathscr{Y}$ (data space) is the set of possible continuum data. $\mathcal{A}$ models the imaging device for continuum data in absence of noise and measurement errors.
- $\mathcal{S} \colon \mathscr{Y} \to \mathbb{R}^m$ models how continuum data is digitized (sampling operator) during the imaging.
- $\boldsymbol{g}_{\text{noise}} \in \mathbb{R}^m$ is the noise component of measured data.

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Images

The infinite dimensional setting

Reconstruction space $\mathscr{X}$

- Set of possible functions $f : \Omega \to \mathbb{R}$ (images)
- Usually some (infinite dimensional) vector space over $\mathbb{R}$, *i.e.*,

$$f + h \in \mathscr{X} \text{ and } \alpha f \in \mathscr{X} \quad \text{for any } f, h \in \mathscr{X} \text{ and } \alpha \in \mathbb{R}.$$

Square integrable functions: $\mathscr{L}^2(\Omega)$ is infinite dimensional and

$$\int_\Omega \big|f(x)\big|^2 \, \mathrm{d}x < \infty \quad \text{whenever } f \in \mathscr{L}^2(\Omega).$$

$\mathscr{L}^2(\Omega)$ has inner-product and norm (Hilbert space):

$$\langle f, h \rangle_{\mathscr{L}^2(\Omega)} := \int_\Omega f(x) h(x) \, \mathrm{d}x \quad \text{for } f, h \in \mathscr{L}^2(\Omega)$$

$$\|f\|_2 := \sqrt{\langle f, f \rangle^2_{\mathscr{L}^2(\Omega)}} := \Big(\int_\Omega f(x)^2 \, \mathrm{d}x \Big)^{1/2}.$$

Hilbert spaces: Generalizes the notion of an inner product to infinite dimensional vector spaces.

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Hilbert spaces: Generalizes the notion of an inner product to infinite dimensional vector spaces.

Discretization: Computers cannot handle elements in infinite dimensional vector spaces. Need to replace the infinite dimensional vector space with a finite dimensional counterpart.

Basis expansion

Assume $\mathscr{X}$ is a Hilbert space with norm $\|\cdot\|_{\mathscr{X}}$ and $\{\phi_j\}_j \subset \mathscr{X}$ is a fixed set (dictionary/frame). Define $\mathscr{X}_n \subset \mathscr{X}$ as the linear span of $\{\phi_1, \ldots, \phi_n\}$, so for any element $f \in \mathscr{X}_n$ then there exists real numbers $\alpha_j \in \mathbb{R}$ (that depend on f) such that

$$f(x) = \sum_{j=1}^{n} \alpha_j \phi_j(x) \quad \text{for all } x \in \Omega.$$

If $\{\phi_j\}_j$ are a basis (linearly independent), then $\mathscr{X}_n$ is an n-dimensional vector space over $\mathbb{R}$.

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How to construct α_j's from a given f

Given $f \in \mathscr{X}$, it is natural to seek an element $\tilde{f} \in \mathscr{X}_n$ that is the "best approximation" to f.

Stated equivalently, given f we seek real numbers $\alpha_j \in \mathbb{R}$ (that depend on f) that minimizes the "difference" between f and

$$\tilde{f}(x) := \sum_{j=1}^{n} \alpha_j \phi_j(x) \quad \text{for } x \in \Omega.$$

Solution: Choose $\tilde{f}$ that minimizes $\|f - \tilde{f}\|_{\mathscr{X}}$. If $\{\phi_j\}_j$ is an orthonormal basis ($\langle \phi_i, \phi_j \rangle_{\mathscr{X}} = 0$ if $i \neq j$ and $= 1$ if $i = j$), then this corresponds to choosing $\tilde{f} \in \mathscr{X}_n$ as the orthogonal projection of f onto $\mathscr{X}_n$, so

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Images
Discretization – Voxel basis

Sampling (uniform): Sub-divide (the rectangular) region $\Omega \subset \mathbb{R}^3$ into n non-overlapping (rectangular) sub-regions, voxels, of equal size that cover Ω.

Voxel basis: Given a sampling, define the voxel basis as

$$\phi_j(x) := \begin{cases} 1 & \text{if } x \text{ is in the } j\text{:th voxel,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\phi_j\}_j$ is a basis and the orthogonal projection $\tilde{f}$ of f onto $\mathscr{X}_n$ is given by

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where $\alpha_j := \langle f, \phi_j \rangle_{\mathscr{X}}$ is the average of f in the j:th voxel.

Note: In many applications, it is common to choose α_j as the value of f at the midpoint of the j:th voxel.

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Basic assumptions

- $\mathscr{X} \subset \mathscr{L}^2(\Omega)$ is a sub-space
- A sampling of the domain Ω into voxels
- $\{\phi_j\}_j$ is a voxel basis based on the sampling

Discretization: To each image $f \in \mathscr{X}$ we can associate a unique vector (a finite dimensional representation) $\boldsymbol{\alpha} \in \mathbb{R}^n$ as follows:

$$\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \quad \text{where} \quad \alpha_j := \langle f, \phi_j \rangle_{\mathscr{X}}.$$

Digital image

- A vector representing grey-scale intensities corresponding to a voxel.
- Grey-scale values are usually mapped to integers $\{0, \ldots, 2^p - 1\}$ where 0 is white and 1 is black (p-bit image).
  - 2-bit image (binary images) corresponds to two values.
  - 8-bit corresponds to 256 different values.
  - 16-bit corresponds to 65 536 different values.
  - 32-bit image corresponds to 4 294 967 296 different values.

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- Accuracy: For inverse problems, the discretization must be accurate in both reconstruction and data spaces, *i.e.*,

$$f \approx \sum_{j=1}^{n} \alpha_j \phi_j \quad \text{and} \quad \mathcal{A}(f) \approx \mathcal{A}\Big(\sum_{j=1}^{n} \alpha_j \phi_j\Big)$$

  for any $f \in \mathscr{X}$ where $\alpha_j := \langle f, \phi_j \rangle_{\mathscr{X}}$.

- Discretized forward operator: Define $A \colon \mathbb{R}^n \to \mathbb{R}^m$ as

$$A(\boldsymbol{\alpha}) := \mathcal{S}\bigg(\mathcal{A}\Big(\sum_{j=1}^{n} \alpha_j \phi_j\Big)\bigg) \quad \text{for } \boldsymbol{\alpha} \in \mathbb{R}^n.$$

- Linearity & measurement matrix: Assume $A \colon \mathbb{R}^n \to \mathbb{R}^m$ is linear (*e.g.*, both $\mathcal{A} \colon \mathscr{X} \to \mathscr{Y}$ and $\mathcal{S} \colon \mathscr{Y} \to \mathbb{R}^m$ are linear). There exists an $(m \times n)$-matrix $\mathbf{A}$ (measurement matrix) so that $A(\boldsymbol{\alpha}) = \mathbf{A} \cdot \boldsymbol{\alpha}$.

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- Elements of the measurement matrix: When $A\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, then

$$A(\boldsymbol{\alpha}) = \sum_{j=1}^{n} \alpha_j \mathcal{S}\bigl(\mathcal{A}(\phi_j)\bigr).$$

Hence, the measurement matrix $\mathbf{A}$ is given as

$$\mathbf{A} := \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

where

$$a_{i,j} := i\text{:th component of } \mathcal{S}\bigl(\mathcal{A}(\phi_j)\bigr) = \mathcal{S}\bigl(\mathcal{A}(\phi_j)\bigr)_i \in \mathbb{R}$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$.

Remember: n is number of voxels and m is number of data points.

The reconstruction problem
Different formulations of the reconstruction problem

Continuum noise-free formulation

Recover the image $f_{\text{true}} \in \mathscr{X}$ from continnum data $g \in \mathscr{Y}$ assuming $g = \mathcal{A}(f_{\text{true}})$. Here, $\mathcal{A} \colon \mathscr{X} \to \mathscr{Y}$ is the forward operator.

Original formulation (sampled noisy data)

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Fully discretized formulation

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The reconstruction problem
Tomography

- The data space: $\mathscr{Y}$ is an infinite dimensional vector space of real-valued functions defined on a manifold M of lines in $\mathbb{R}^3$, so continuum data $g \in \mathscr{Y}$ is a function
$$g\colon M \to \mathbb{R}.$$
- The manifold of lines M: A line ℓ in $\mathbb{R}^3$ is uniquely determined by its direction ω (directional vector) and a point $\boldsymbol{x} \in \omega^{\perp}$ that lies on ℓ ($\omega^{\perp}$ is the 2D plane orthogonal to ω):
$$\ell : t \mapsto \boldsymbol{x} + t\omega.$$
Hence, the pair $(\omega, \boldsymbol{x})$ corresponds to a unique line ℓ and vice versa, so M can be considered as a vector space of such pairs. Note: Each $\boldsymbol{x} \in \omega^{\perp}$ corresponds to a unique point on the detector surface.
- Data acquisition geometry: The arrangement of the m lines in M that correspond to the measurements, *i.e.*, the sampling of ω and $\boldsymbol{x} \in \omega^{\perp}$.

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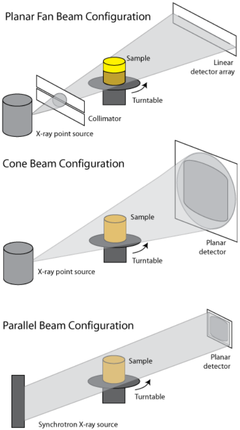
Tomography

Common data acquisition geometries

- Parallel beam tomography
- Fan-beam
- 3D cone-beam tomography
- 3D helical/spiral tomography



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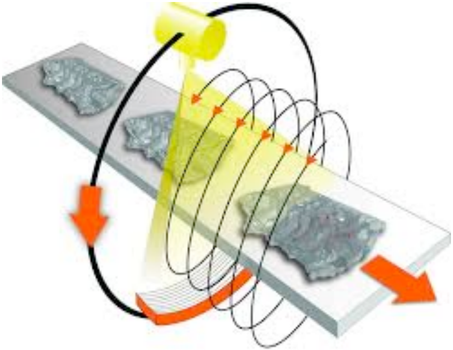
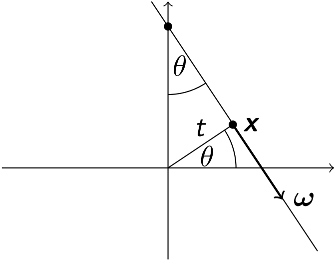
- Parallel beam tomography
- Fan-beam
- 3D cone-beam tomography
- 3D helical/spiral tomography



The reconstruction problem
Tomography

- Transmission tomography: Forward operator given by the ray transform:
$$\mathcal{A}(f)(\boldsymbol{\omega}, \boldsymbol{x}) = \int_{-\infty}^{\infty} f(\boldsymbol{x} + t\boldsymbol{\omega})\,\mathrm{d}t \quad \text{for a line } (\boldsymbol{\omega}, \boldsymbol{x}) \in M.$$
Note that $g := \mathcal{A}(f)$ is a real-valued function defined on M a set of lines in $\mathbb{R}^3$.
- Exact reconstruction in 2D: Write $\boldsymbol{x}$ and $\boldsymbol{\omega}$ in polar coordinates:



$$\boldsymbol{x} = \begin{pmatrix} t\cos\theta \\ t\sin\theta \end{pmatrix}$$
$$\boldsymbol{\omega} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Filtered backprojection
Ingredients

- Forward operator in polar coordinates:

$$\mathcal{A}(f)(t,\theta) = \int_{-\infty}^{\infty} f\left[t\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + s\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right] \mathrm{d}s$$

  for $t \in \mathbb{R}$ and $0 < \theta < \pi$

- Backprojection = average over all lines through one point:

$$\mathcal{B}(g)(x,y) = \frac{1}{\pi}\int_0^{\pi} g(x\cos\theta + y\sin\theta, \theta)\,\mathrm{d}\theta.$$

  The backprojection takes a sinogram (=image on M) and maps it back to an image in $\mathbb{R}^2$. But it is **not** the inverse mapping of $\mathcal{A}$.

Filtered backprojection
Ingredients

- Fourier transform: We transform only with respect to the first variable

$$\mathcal{F}(g)(t,\theta) = \int_{-\infty}^{\infty} g(s,\theta)\exp(-ist)\,\mathrm{d}s$$
$$\mathcal{F}^{-1}(g)(t,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s,\theta)\exp(+ist)\,\mathrm{d}s$$

  Each of these transformations maps a function on M to another function on M, and $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$. There exist fast implementations for both CPU (e.g. (py)FFTW) and GPU (cuFFT).

Filtered backprojection

The formula

$$f(x,y) = \frac{1}{2}\mathcal{B}(\mathcal{F}^{-1}(|t|\mathcal{F}(\mathcal{A}f)(t,\theta)))(x,y)$$

Step by step:

1. Start with the sinogram $\mathcal{A}f$, which you obtain from the scanner.
2. Calculate the Fourier transform $\mathcal{F}(\mathcal{A}f)$.
3. Multiply by the absolute value of the first (radial) component. (This is the filtering).
4. Calculate the inverse Fourier transform. You still have a sinogram.
5. Apply the (nonfiltered) backprojection. Here you obtain an image.
6. Divide by 2.

Filtered backprojection

Challenges

- Requires full knowledge of all line integrals to give an exact reconstruction.
- The multiplication with the abolute value amplifies errors, in particular high frequencies. Solution: Use a different filter, which does not grow so rapidly.
- But it is fast in comparison with iterative and variational methods.

- Sampling operator: Given m data sampling points $(\boldsymbol{\omega}_1, \boldsymbol{x}_1), \dots, (\boldsymbol{\omega}_m, \boldsymbol{x}_m)$ (which are lines) that correspond to actual measured data, define

$$\mathcal{S}(g) = \big(g(\boldsymbol{\omega}_1, \boldsymbol{x}_1), \dots, g(\boldsymbol{\omega}_m, \boldsymbol{x}_m)\big) \in \mathbb{R}^m \quad \text{for } g \in \mathscr{Y}.$$

In particular, the i:th measurement corresponding to the i:th line $(\boldsymbol{\omega}_i, \boldsymbol{x}_i)$ is modeled by

$$\mathcal{S}\big(\mathcal{A}(f)\big)_i = \mathcal{A}(f)(\boldsymbol{\omega}_i, \boldsymbol{x}_i) = \int_{-\infty}^{\infty} f(\boldsymbol{x}_i + t\boldsymbol{\omega}_i)\, \mathrm{d}t.$$

The reconstruction problem
Tomography

- Measurement matrix: Fix a basis $\{\phi_i\}_i$ of $\mathscr{X}$ and a data acquisition geometry $(\boldsymbol{\omega}_1, \boldsymbol{x}_1), \dots, (\boldsymbol{\omega}_m, \boldsymbol{x}_m)$. Then, the elements in the $(m \times n)$-measurement matrix

$$\mathbf{A} := \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

  are given as

$$a_{i,j} = \mathcal{A}(\phi_i)(\boldsymbol{\omega}_j, \boldsymbol{x}_j) = \int_{-\infty}^{\infty} \phi_i(\boldsymbol{x}_j + t\boldsymbol{\omega}_j)\,\mathrm{d}t$$

  for $i = 1, \dots, n$ and $j = 1, \dots, m$.
- Voxel basis: The ray transform assumes the x-ray beam is one-dimensional (pencil beam), so

$$a_{i,j} = \text{length of part of } i\text{:th line in } j\text{:th voxel.}$$

- Voxel basis and x-ray with 2D-dimensional cross section: The lines are replaced by 3D strips, so

$$a_{i,j} = \text{volume of part of } i\text{:th strip in } j\text{:th voxel.}$$

More complicated beam profiles: Can be included by weighting different parts of the strips differently. The calculation of $a_{i,j}$ requires a lot of work and much of the literature on iterative methods discuss effects of using various schemes to approximate the measurement matrix.

Historical note: In the early approaches for 2D tomography by Hounsfield, one used

$$a_{i,j} = \begin{cases} 1 & \text{if center of } j\text{:th pixel is in } i\text{:th strip,} \\ 0 & \text{otherwise.} \end{cases}$$

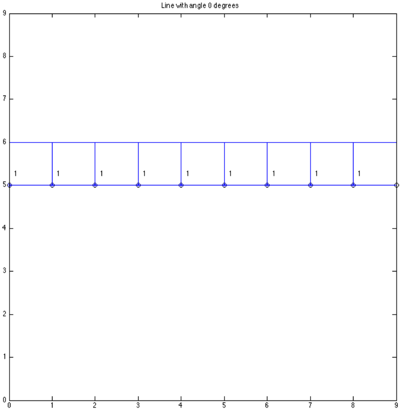
Then, values of $a_{i,j}$ can be computed at run time even with slow computers and do not have to be stored.

The reconstruction problem
Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 0^\circ$ with horizontal axis



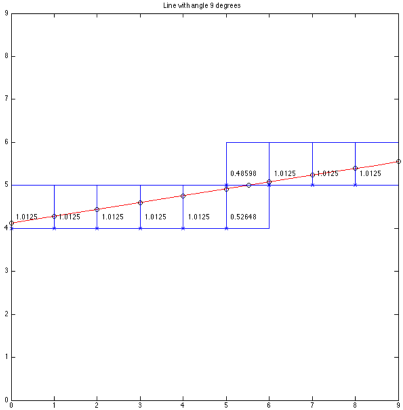
The reconstruction problem

Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 9^\circ$ with horizontal axis



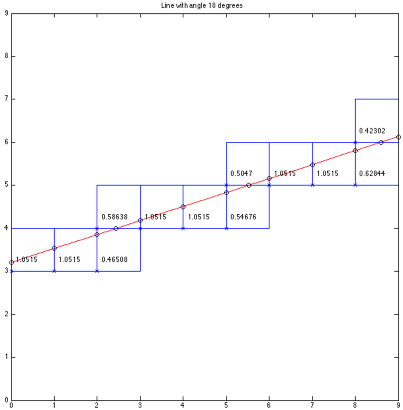
The reconstruction problem

Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 18^\circ$ with horizontal axis



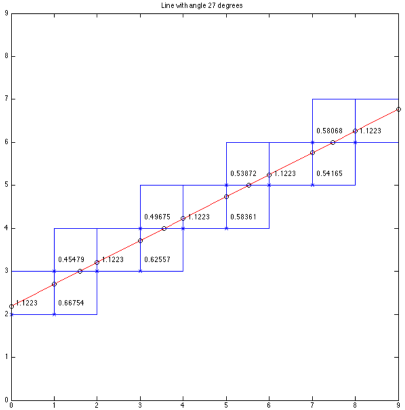
The reconstruction problem

Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 27^\circ$ with horizontal axis



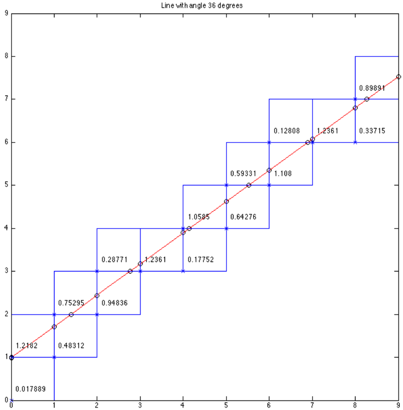
The reconstruction problem

Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 36^\circ$ with horizontal axis



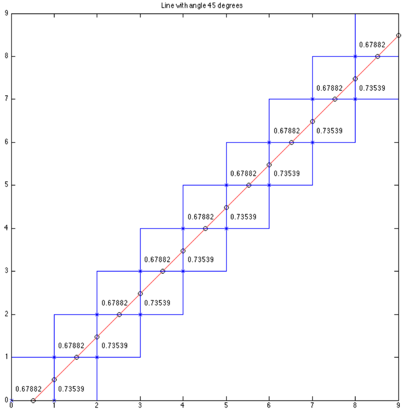
The reconstruction problem

Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 45^\circ$ with horizontal axis



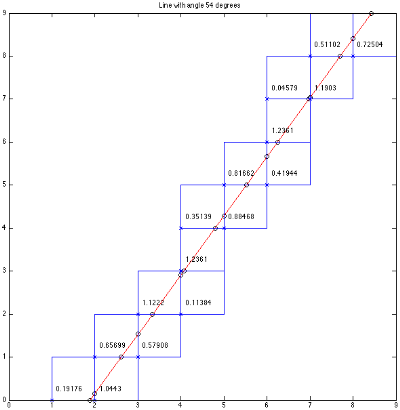
The reconstruction problem

Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 54^\circ$ with horizontal axis



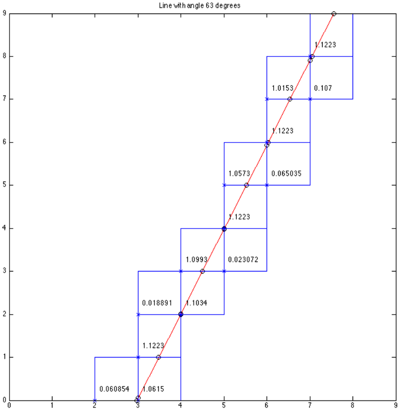
The reconstruction problem

Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 63^\circ$ with horizontal axis

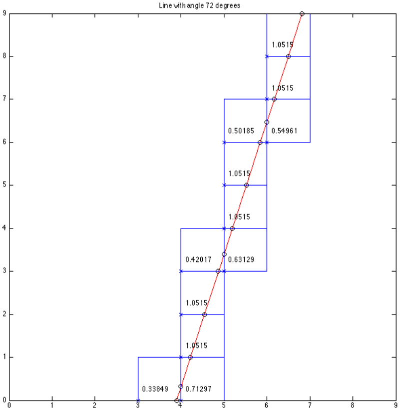


The reconstruction problem
Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 72^\circ$ with horizontal axis



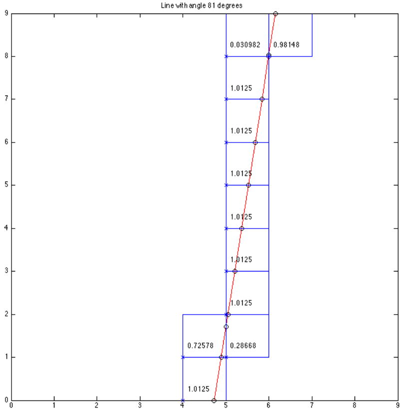
The reconstruction problem

Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 81^\circ$ with horizontal axis



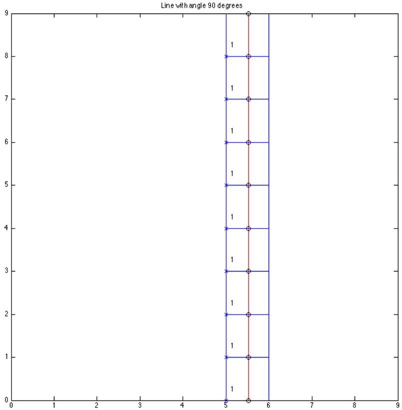
The reconstruction problem

Tomography: Computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: Calculating i:th row $\boldsymbol{a}_i$ for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 90^\circ$ with horizontal axis

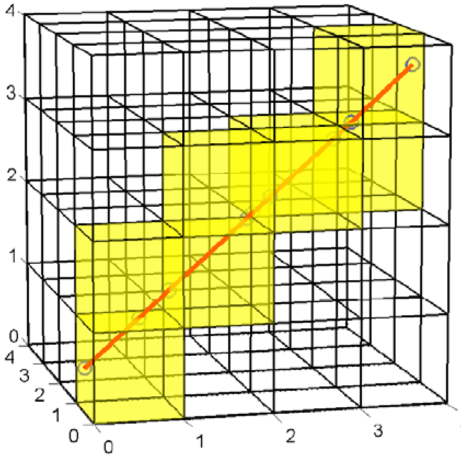


The reconstruction problem

Tomography: Computing the measurement matrix for lines in 3D

Matrix element $a_{i,j}$ is the length of i:th line in j:th voxel.

Calculate the intersection of i:th line ℓ_i with j:th voxel.



Matrix element $a_{i,j}$ is the length of i:th line in j:th voxel.

Calculate the intersection of i:th line ℓ_i with j:th voxel.

- Intersection with x_1-planes: Consider the intersection point in $\mathbb{R}^3$ between a line ℓ and a hyperplane parallel to the x_1-axis:

$$(x_1, x_2, x_3) \mapsto (p, x_2, x_3) \quad \text{where } p \text{ is fix.}$$

  Assume ℓ is given by $(\boldsymbol{\omega}, \boldsymbol{x}^0)$, *i.e.*,

$$\ell_i : t \mapsto \boldsymbol{x}^0 + t\boldsymbol{\omega}.$$

  Let $\boldsymbol{x}^* = (x_1^*, x_2^*, x_3^*)$ denote this intersection, so $x_1^* = p$ by definition and

$$x_2^* = x_2^0 + \frac{p - x_1^0}{\omega_1}\omega_2 \quad \text{and} \quad x_3^* = x_3^0 + \frac{p - x_1^0}{\omega_1}\omega_3.$$

The reconstruction problem

Tomography: Computing the measurement matrix for lines in 3D

Matrix element $a_{i,j}$ is the length of i:th line in j:th voxel.

Calculate the intersection of i:th line ℓ_i with j:th voxel.

- Intersection with x_2- and x_3-planes: Similar equations for the planes $x_2 = q$ and $x_3 = r$.
- From these intersections it is easy to compute the ray length in voxel j and there are fast method for these computations:
  - R. L. Siddon. *Fast calculation of the exact radiological path for a 3-dimensional CT array*. Medical Physics, vol. 12, no. 2, pages 252–255, 1985.
  - H. Gao. *Fast parallel algorithms for the X-ray transform and its adjoint*. Medical Physics, vol. 39, no. 11, pages 7110–7120, 2012.

- For most tomography problems, the measurement matrix **A** is very large and it cannot be stored in computer memory. Problem size in a modern 64-slice helical CT:
  - A single 2D cross-section
    - About 1 000 detector elements arranged along 64 (detector) slices, each roughly corresponding to a 2D slice through the object. Radiation source attains 800 different positions each second (two rotations/second) $\implies 1000 \cdot 800 \approx 800\,000$ data points.
    - 2D cross-section is 512×512 pixels in size $\implies 260\,000$ pixels.

    Total: 260 000 unknowns and 800 000 equations.
  - A single 2D cross-section corresponds to a $1\ \mathrm{mm}$ thick slice in 3D, so 1 000 cross sections needed to cover $1\ \mathrm{m}$: $\implies n \sim 260 \cdot 10^6 \quad \text{and} \quad m \sim 800 \cdot 10^6$.
- **A** is sparse since each ray only intersects a few voxels.

The reconstruction problem
Tomography: Properties of the measurement matrix

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The reconstruction problem

Naive inversion and concept of ill-posedness

Recover the digital image $\boldsymbol{\alpha}_{\text{true}} \in \mathbb{R}^n$ from measured data $\boldsymbol{g} \in \mathbb{R}^m$ assuming

$$\boldsymbol{g} = \mathbf{A} \cdot \boldsymbol{\alpha}_{\text{true}} + \boldsymbol{g}_{\text{noise}}. \tag{1}$$

Here, $\mathbf{A}$ is the $(m \times n)$-measurement matrix.

Existence: Naive inversion $\boldsymbol{\alpha}_{\text{naive}} := \mathbf{A}^{-1} \cdot \boldsymbol{g}$ is not possible when $\mathbf{A}$ is not invertible (no solutions to (1)).

Generalized solution: Relax the notion of a solution to (1) to enforce existence. Define a data error function $E \colon \mathbb{R}^m \to \mathbb{R}_+$ and look for a vector $\boldsymbol{\alpha}^* \in \mathbb{R}^n$ that solves

$$\min_{\boldsymbol{\alpha}} E(\mathbf{A} \cdot \boldsymbol{\alpha} - \boldsymbol{g}).$$

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Here, $\mathbf{A}$ is the $(m \times n)$-measurement matrix.

- Least-squares solutions: Choose E as the 2-norm of the residual, *i.e.*, solve

$$\min_{\boldsymbol{\alpha}} \|\mathbf{A} \cdot \boldsymbol{\alpha} - \boldsymbol{g}\|_2^2.$$

- Maximum likelihood solution: Choosing E as the neglog of the data likelihood gives maximum likelihood solutions to (1).

The reconstruction problem
Naive inversion and concept of ill-posedness

Recover the digital image $\boldsymbol{\alpha}_{\text{true}} \in \mathbb{R}^n$ from measured data $\boldsymbol{g} \in \mathbb{R}^m$ assuming

$$\boldsymbol{g} = \mathbf{A} \cdot \boldsymbol{\alpha}_{\text{true}} + \boldsymbol{g}_{\text{noise}}. \tag{1}$$

Here, $\mathbf{A}$ is the $(m \times n)$-measurement matrix.

Ill-posedness: Generalized solutions handle the issue of existence (there is always a generalized solution to (1)). Two serious problems remain:

- Non-uniqueness: Infinitely many generalized solutions.
- Instability: Generalized solutions do not depend continuously on the data $\boldsymbol{g}$.

The inverse problem (1) is ill-posed if any of these issues occur.

The reconstruction problem
Naive inversion and concept of ill-posedness

Recover the digital image $\boldsymbol{\alpha}_{\text{true}} \in \mathbb{R}^n$ from measured data $\boldsymbol{g} \in \mathbb{R}^m$ assuming

$$\boldsymbol{g} = \mathbf{A} \cdot \boldsymbol{\alpha}_{\text{true}} + \boldsymbol{g}_{\text{noise}}. \tag{1}$$

Here, $\mathbf{A}$ is the $(m \times n)$-measurement matrix.

Condition number and ill-posedness:

Assume (1) has a solution for any $\boldsymbol{g}_{\text{noise}}$, let $\boldsymbol{\alpha}_{\text{naive}} := \mathbf{A}^{-1} \cdot \boldsymbol{g}$.
Classical perturbation theory leads to the bound

$$\frac{\|\boldsymbol{\alpha}_{\text{true}} - \boldsymbol{\alpha}_{\text{naive}}\|_2}{\|\boldsymbol{\alpha}_{\text{true}}\|_2} \leq \text{cond}(\mathbf{A}) \frac{\|\boldsymbol{g} - \boldsymbol{g}_{\text{exact}}\|_2}{\|\boldsymbol{g}_{\text{exact}}\|_2}$$

where $\boldsymbol{g}_{\text{exact}} := \mathbf{A} \cdot \boldsymbol{\alpha}_{\text{true}}$ and $\text{cond}(\mathbf{A})$ is the condition number (ratio of the maximal and minimal singular values).
$\text{cond}(\mathbf{A})$ large $\Longrightarrow$ naive inversion not useful since $\boldsymbol{\alpha}_{\text{naive}}$ can be very far from $\boldsymbol{\alpha}_{\text{true}}$.

Regularization: Mathematical theory for handling Ill-posed problems. Stability and uniqueness is enforced by accounting for a priori knowledge about the true (unknown) image.

The main idea of regularization

Replace the original ill-posed reconstruction problem by a well-posed reconstruction problem (*i.e.*, it has a unique solution that depends continuously on the data) that is convergent as the data error goes to zero.

Well-posedness: Guarantees stability

Convergence: The reconstructions obtained converge to a least squares solution when the data error approaches zero and the parameters in the reconstruction method (regularization parameters) are chosen appropriately.

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