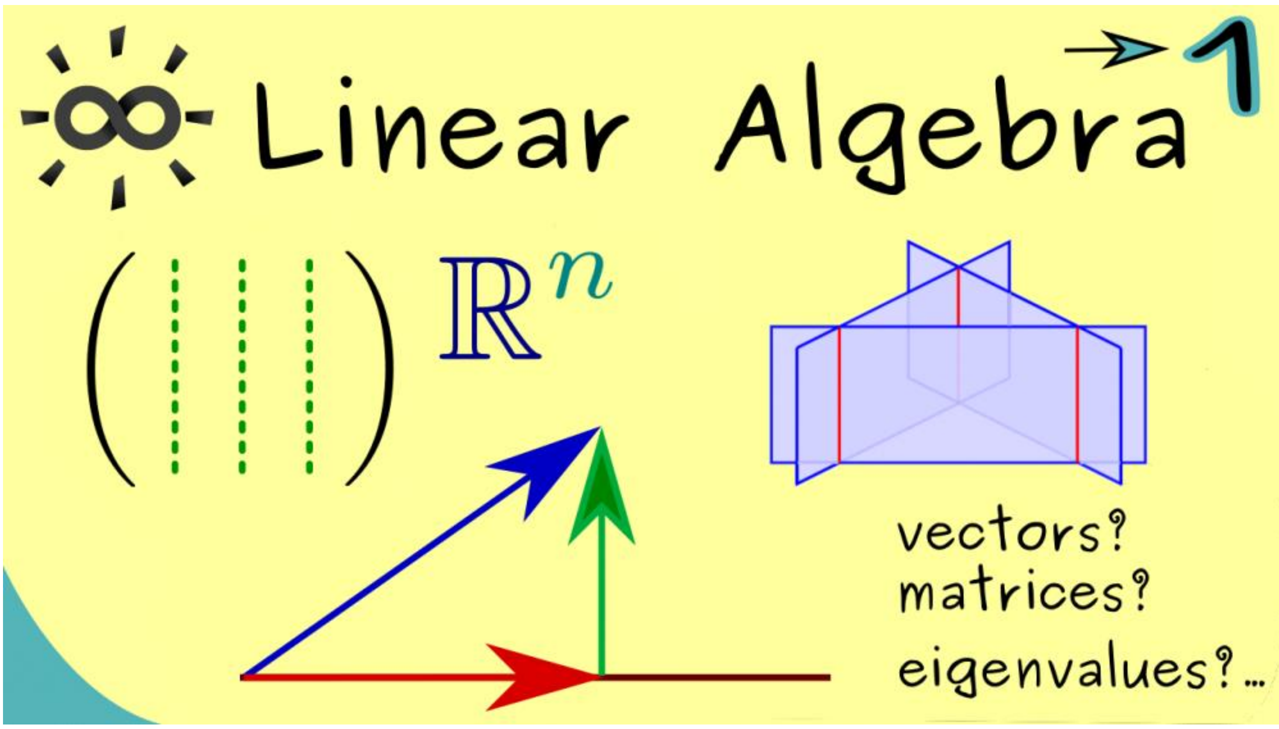
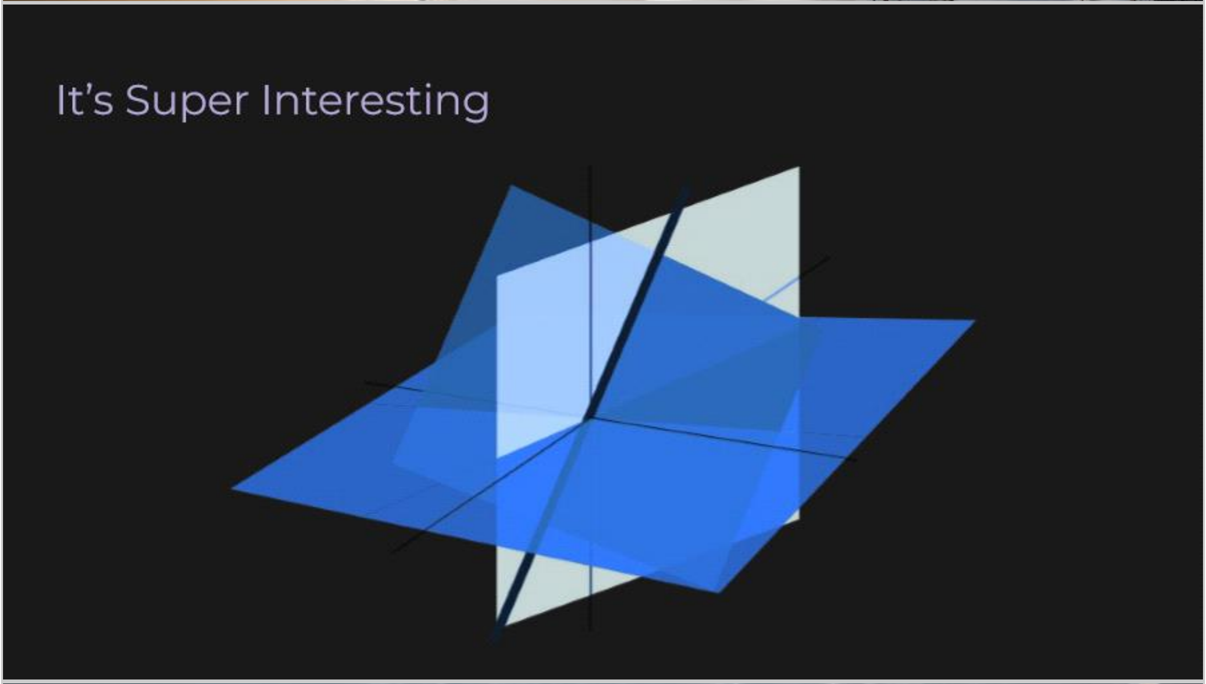
Linear Algebra - 1

Tuesday, 27 January 2026 10:52 pm

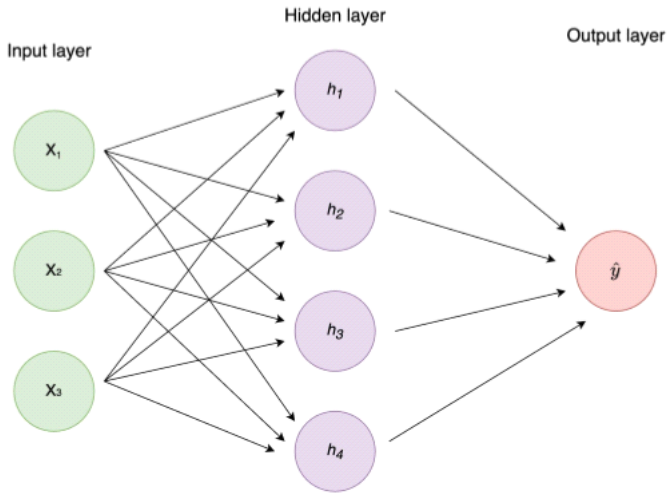



It's Useful

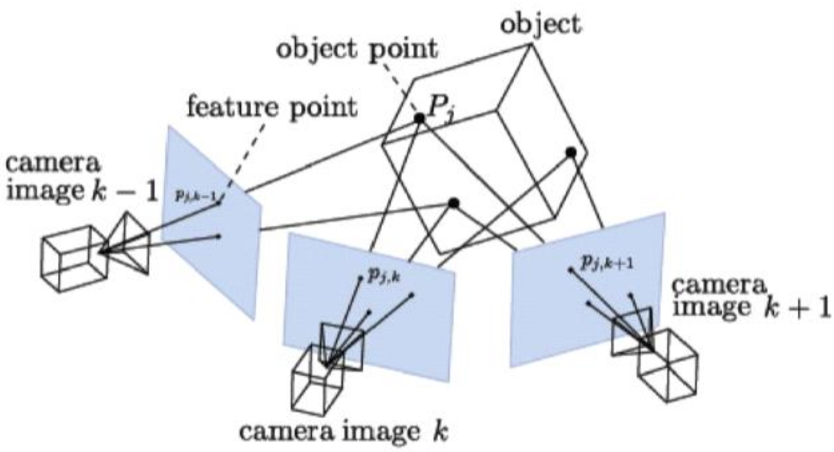

It's Super Interesting


It's Everywhere

Artificial Intelligence

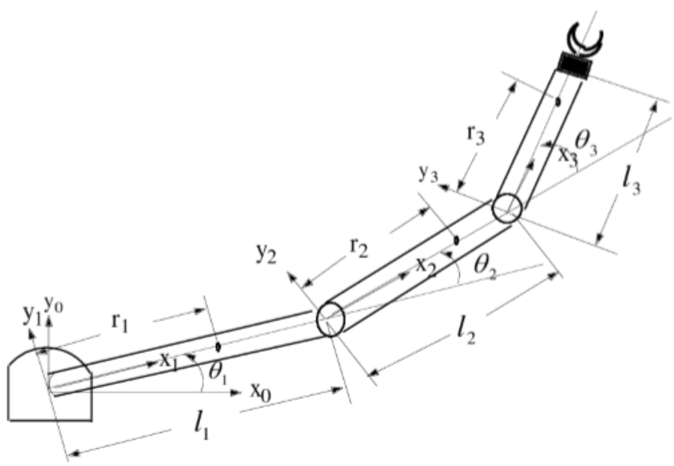


Computer Vision

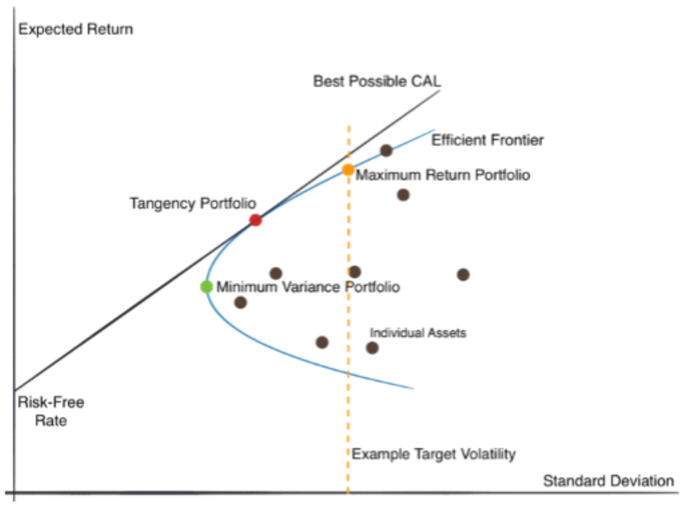


Robotics

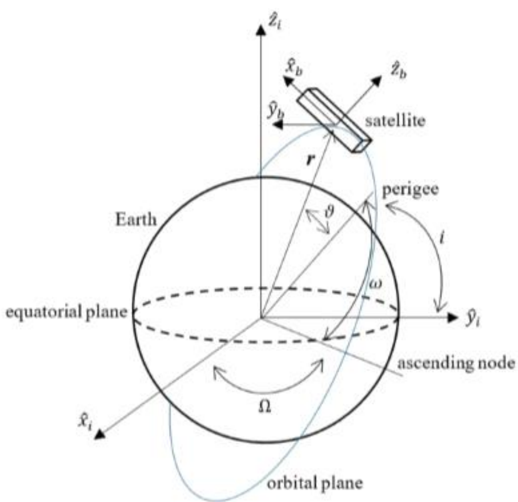
$$M(q)\,\ddot{q} + h(\dot{q}, q) + g(q) = F(t)$$



Finance



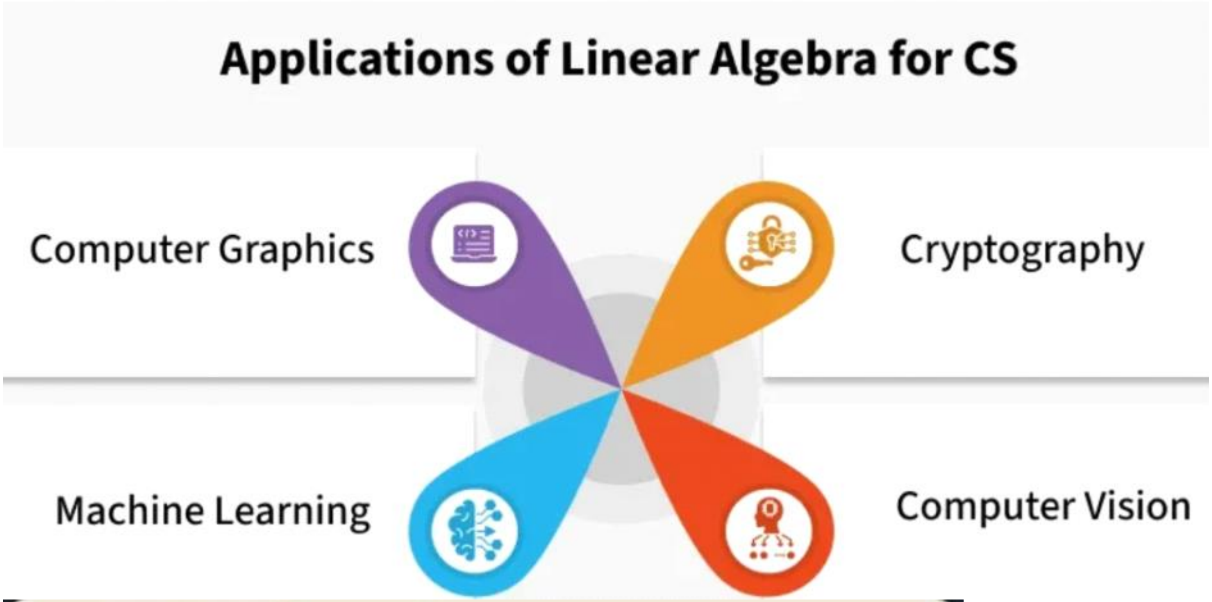
Aircraft and Spacecraft Control

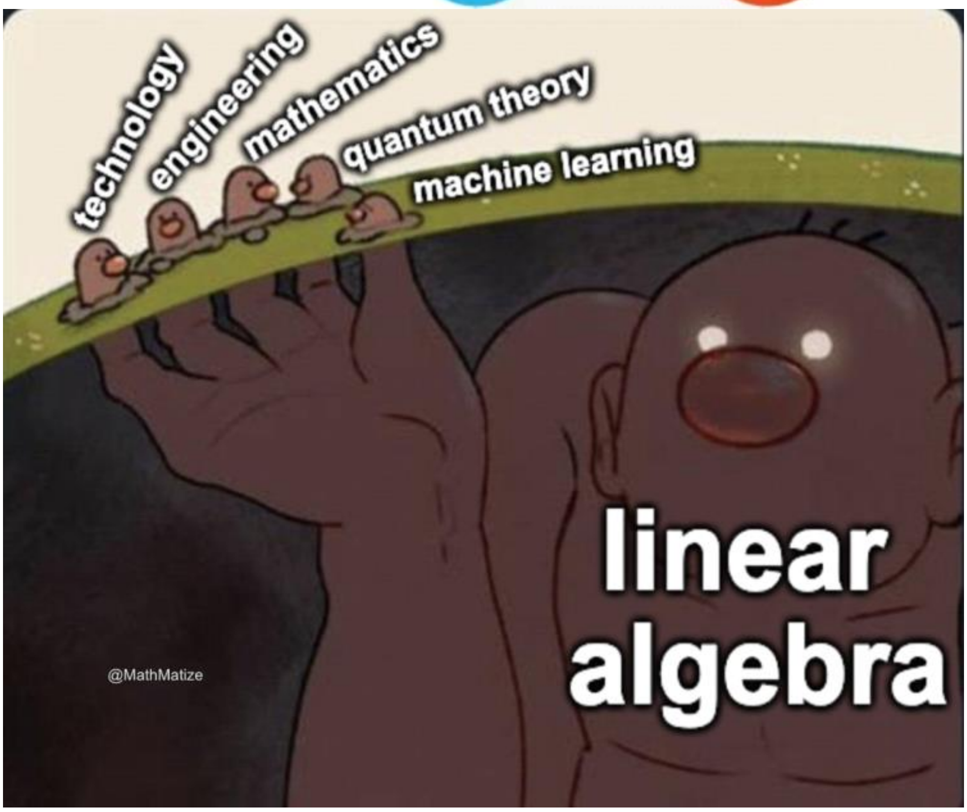


Linear Algebra in Computer Science

Last Updated : 5 Aug, 2025

Linear algebra is a core mathematics discipline that has a fundamental role in many areas of computer science. It forms the mathematical foundation for numerous algorithms and methods used in fields such as computer graphics, machine learning, data analysis, and more. Linear algebra deals with vectors, matrices, and linear transformations - all essential tools for expressing and manipulating data, solving equations, and calculating geometric transformations.





1.1 What is a system of linear equations?

Definition 1.1: A **system of** m **linear equations** in n unknown variables $x_1, x_2, \ldots, x_n$ is a collection of m equations of the form

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{1.1}$$

The numbers a_{ij} are called the **coefficients** of the linear system; because there are m equations and n unknown variables there are thefore $m \times n$ coefficients. The main problem with a linear system is of course to solve it:

Problem: Find a list of n numbers $(s_1, s_2, \ldots, s_n)$ that satisfy the system of linear equations (1.1).

system consisting of $m = 2$ equations and $n = 3$ unknowns:

$$\begin{aligned} x_1 - 5x_2 - 7x_3 &= 0 \\ 5x_2 + 11x_3 &= 1 \end{aligned}$$

Here is a linear system consisting of $m = 3$ equations and $n = 2$ unknowns:

$$\begin{aligned} -5x_1 + x_2 &= -1 \\ \pi x_1 - 5x_2 &= 0 \\ 63x_1 - \sqrt{2}x_2 &= -7 \end{aligned}$$

And finally, below is a linear system consisting of $m = 4$ equations and $n = 6$ unknowns:

$$\begin{aligned} -5x_1 + x_3 - 44x_4 - 55x_6 &= -1 \\ \pi x_1 - 5x_2 - x_3 + 4x_4 - 5x_5 + \sqrt{5}x_6 &= 0 \\ 63x_1 - \sqrt{2}x_2 - \tfrac{1}{5}x_3 + \ln(3)x_4 + 4x_5 - \tfrac{1}{33}x_6 &= 0 \\ 63x_1 - \sqrt{2}x_2 - \frac{1}{5}x_3 - \tfrac{1}{8}x_4 - 5x_6 &= 5 \end{aligned}$$

Example 1.2. Verify that $(1, 2, -4)$ is a solution to the system of equations

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 2 \\ x_1 + 3x_2 - x_3 &= 11. \end{aligned}$$

Is $(1, -1, 2)$ a solution to the system?

$$\text{INCONSISTENT} \Leftrightarrow \text{NO SOLUTION}$$

A linear system is called **consistent** if it has at least one solution:

$$\text{CONSISTENT} \Leftrightarrow \text{AT LEAST ONE SOLUTION}$$

Example 1.3. Show that the linear system does not have a solution.

$$\begin{aligned} -x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1. \end{aligned}$$

Solution. If we add the two equations we get

$$0 = 4$$

which is a contradiction. Therefore, there does not exist a list (s_1, s_2) that satisfies the system because this would lead to the contradiction $0 = 4$. □

Example 1.4. Let t be an arbitrary real number and let

$$\begin{aligned} s_1 &= -\tfrac{3}{2} - 2t \\ s_2 &= \tfrac{3}{2} + t \\ s_3 &= t. \end{aligned}$$

Show that for any choice of the parameter t, the list (s_1, s_2, s_3) is a solution to the linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 3x_2 - x_3 &= 3. \end{aligned}$$

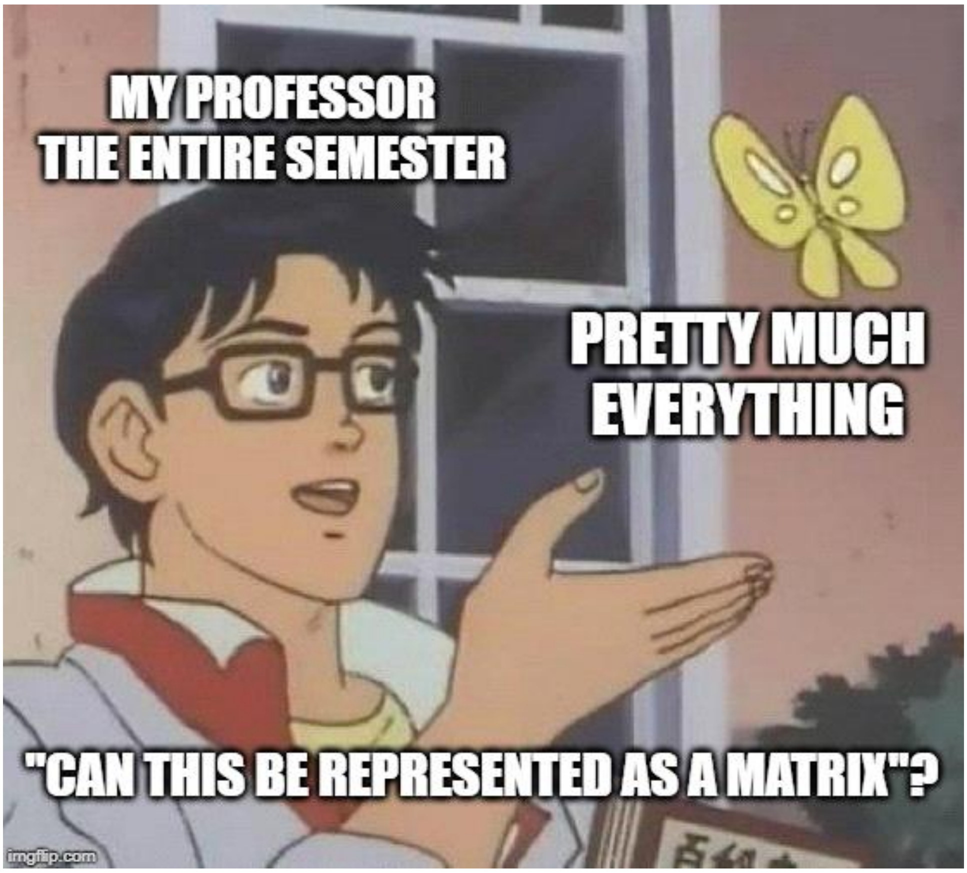
Solution. Substitute the list (s_1, s_2, s_3) into the left-hand-side of the first equation

$$\left(-\tfrac{3}{2} - 2t\right) + \left(\tfrac{3}{2} + t\right) + t = 0$$

and in the second equation

$$\left(-\tfrac{3}{2} - 2t\right) + 3\left(\tfrac{3}{2} + t\right) - t = -\tfrac{3}{2} + \tfrac{9}{2} = 3$$

Both equations are satisfied for any value of t. Because we can vary t arbitrarily, we get an infinite number of solutions parameterized by t. For example, compute the list (s_1, s_2, s_3) for $t = 3$ and confirm that the resulting list is a solution to the linear system. □



Equation to Matrix

$$\begin{aligned} 5x_1 - 3x_2 + 8x_3 &= -1 \\ x_1 + 4x_2 - 6x_3 &= 0 \\ 2x_2 + 4x_3 &= 3 \end{aligned}$$

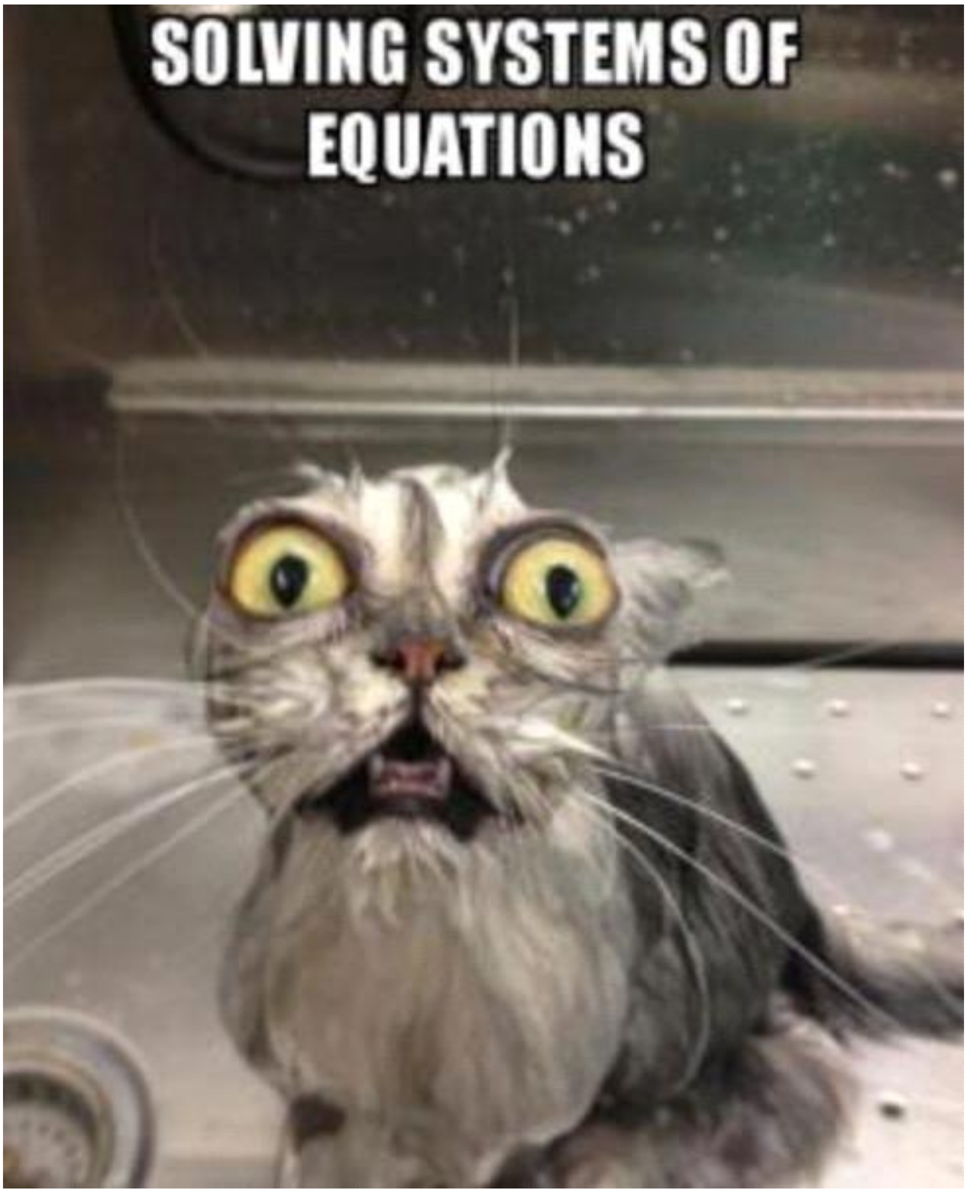
$$\mathbf{A} = \begin{bmatrix} 5 & -3 & 8 \\ 1 & 4 & -6 \\ 0 & 2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \quad [\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 5 & -3 & 8 & -1 \\ 1 & 4 & -6 & 0 \\ 0 & 2 & 4 & 3 \end{bmatrix}.$$

Matrix to Equation

$$\begin{bmatrix} 1 & 4 & -2 & 8 & 12 \\ 0 & 1 & -7 & 2 & -4 \\ 0 & 0 & 5 & -1 & 7 \end{bmatrix}$$

$$\begin{aligned} x_1 + 4x_2 - 2x_3 + 8x_4 &= 12 \\ x_2 - 7x_3 + 2x_4 &= -4 \\ 5x_3 - x_4 &= 7. \end{aligned}$$

Solving linear systems



1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of one equation to another.

Example 1.6. Solve the linear system using elementary row operations.

$$\begin{aligned} -3x_1 + 2x_2 + 4x_3 &= 12 \\ x_1 - 2x_3 &= -4 \\ 2x_1 - 3x_2 + 4x_3 &= -3 \end{aligned}$$

Solution. Our goal is to perform elementary row operations to obtain a triangular structure and then use back substitution to solve. The augmented matrix is

$$\begin{bmatrix} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{bmatrix}.$$

Interchange Row 1 (R_1) and Row 2 (R_2):

$$\begin{bmatrix} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{bmatrix}$$

As you will see, this first operation will simplify the next step. Add $3R_1$ to R_2:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{bmatrix} \xrightarrow{3R_1+R_2} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{bmatrix}$$

Add $-2R_1$ to R_3:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix}$$

Multiply R_2 by $\frac{1}{2}$:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix}$$

Add $3R_2$ to R_3:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix} \xrightarrow{3R_2+R_3} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

Multiply R_3 by $\frac{1}{5}$:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix} \xrightarrow{\frac{1}{5}R_3} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Example 1.7. Using elementary row operations, show that the linear system is inconsistent.

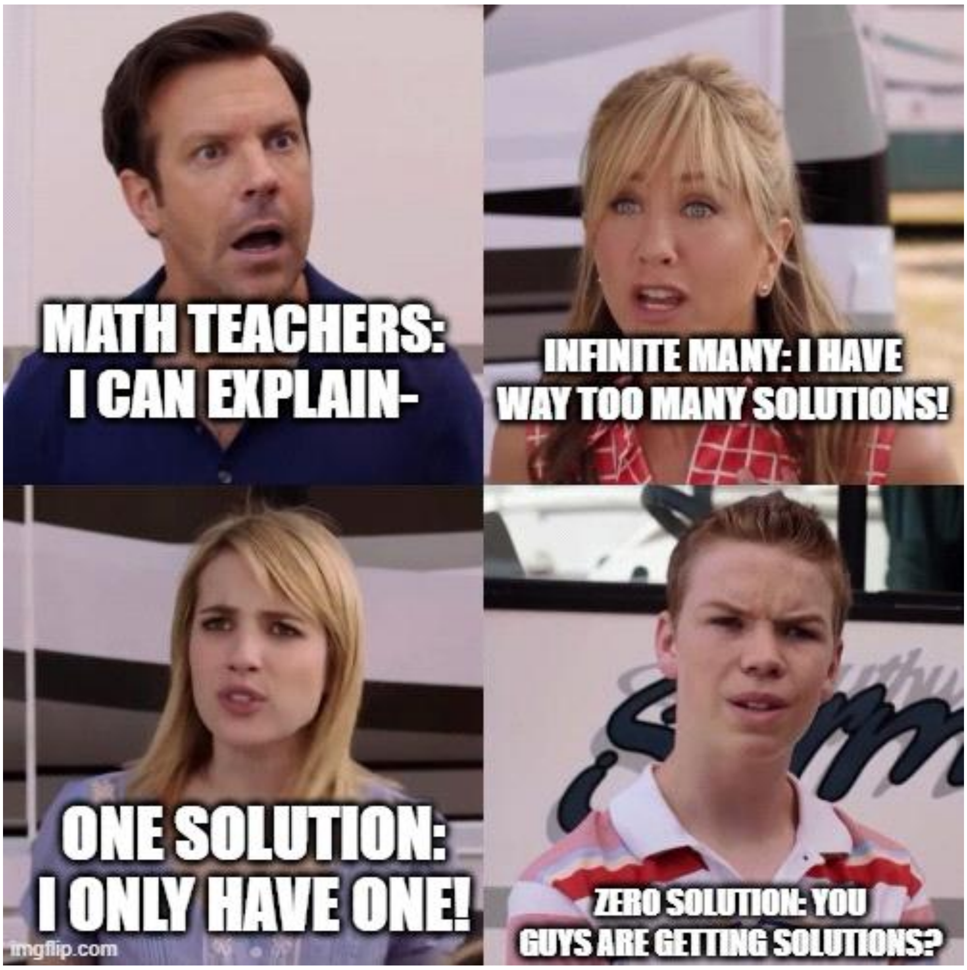
$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 + x_3 &= 0 \\ 2x_1 + 4x_3 &= 1 \end{aligned}$$

Solution. The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}$$

Perform the operation $-2R_1 + R_3$:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$



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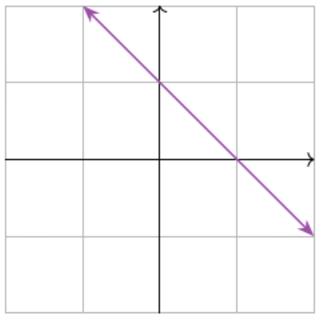
Geometric interpretation of the solution set

1.1.2 Pictures of Solution Sets

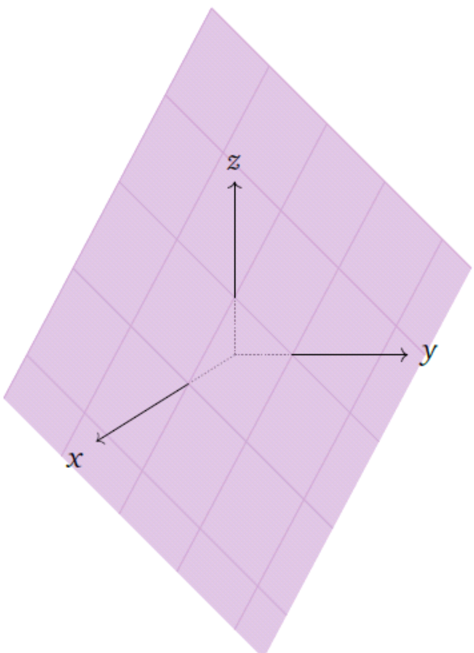
Before discussing how to solve a system of linear equations below, it is helpful to see some pictures of what these solution sets look like geometrically.

One Equation in Two Variables. Consider the linear equation $x + y = 1$. We can rewrite this as $y = 1 - x$, which defines a line in the plane: the slope is -1, and the x-intercept is 1.



Definition (Lines)**.** For our purposes, a **line** is a ray that is *straight* and *infinite* in both directions.

One Equation in Three Variables. Consider the linear equation $x + y + z = 1$. This is the **implicit equation** for a plane in space.



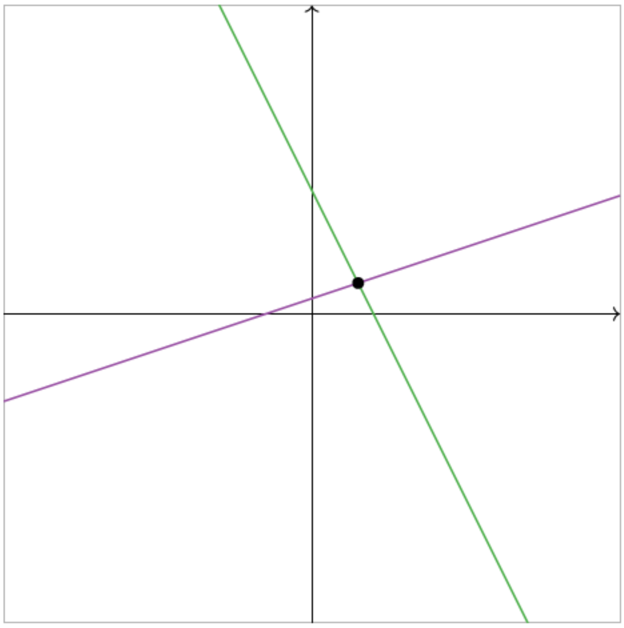
Definition (Planes)**.** A **plane** is a flat sheet that is infinite in all directions.

Remark. The equation $x + y + z + w = 1$ defines a "3-plane" in 4-space, and more generally, a single linear equation in n variables defines an "$(n-1)$-plane" in n-space. We will make these statements precise in Section 2.7.

Two Equations in Two Variables. Now consider the system of two linear equations

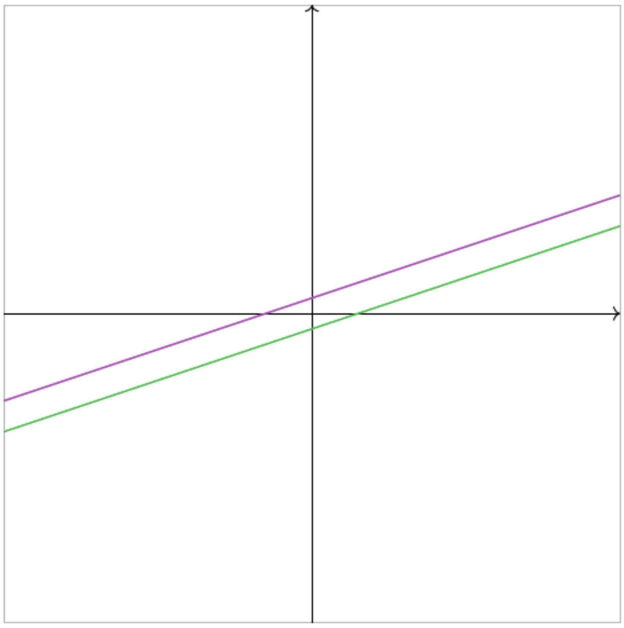
$$\begin{cases} x - 3y = -3 \\ 2x + y = 8. \end{cases}$$

Each equation individually defines a line in the plane, pictured below.



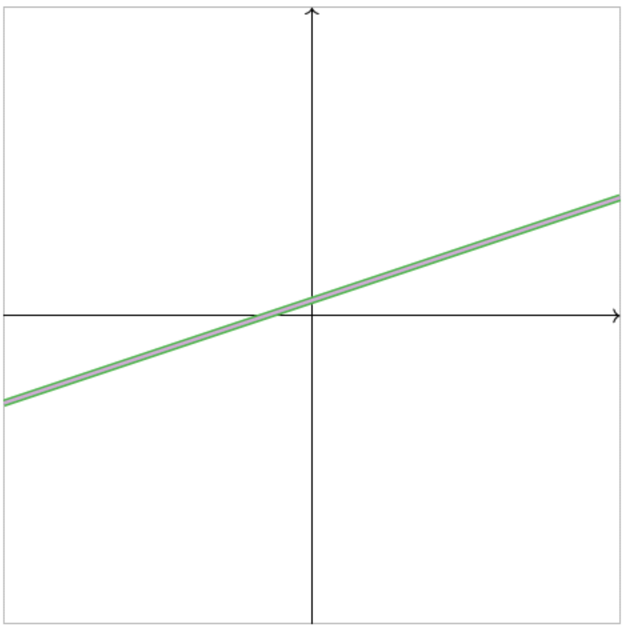
$$\begin{cases} x - 3y = -3 \\ x - 3y = 3. \end{cases}$$

These define *parallel* lines in the plane.



$$\begin{cases} x - 3y = -3 \\ 2x - 6y = -6. \end{cases}$$

The second equation is a multiple of the first, so these equations define the *same* line in the plane.



The set of points (x_1, x_2) that satisfy the linear system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \tag{1.2}$$

is the intersection of the two lines determined by the equations of the system. The solution for this system is $(3, 2)$. The two lines intersect at the point $(x_1, x_2) = (3, 2)$, see Figure 1.1.

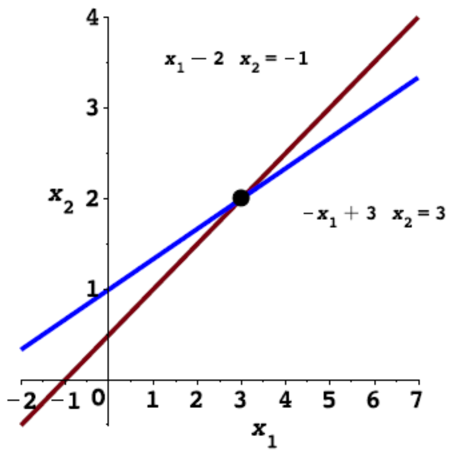
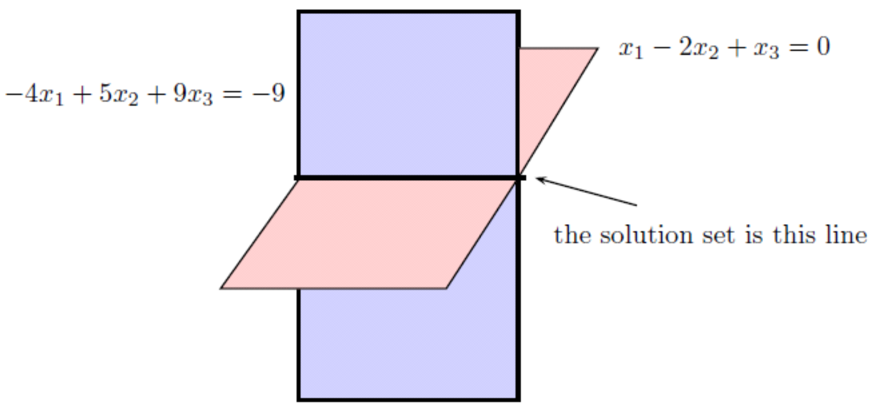


Figure 1.1: The intersection point of the two lines is the solution of the linear system (1.2)

Similarly, the solution of the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$



After this lecture you should know the following:

- what a linear system is
- what it means for a linear system to be consistent and inconsistent
- what matrices are
- what are the matrices associated to a linear system
- what the elementary row operations are and how to apply them to simplify a linear system
- what it means for two matrices to be row equivalent
- how to use the method of back substitution to solve a linear system
- what an inconsistent row is
- how to identify using elementary row operations when a linear system is inconsistent
- the geometric interpretation of the solution set of a linear system