

# Integration - I (Cartesian)

Saturday, 17 January 2026 5:09 pm



## Double Integrals over Rectangles

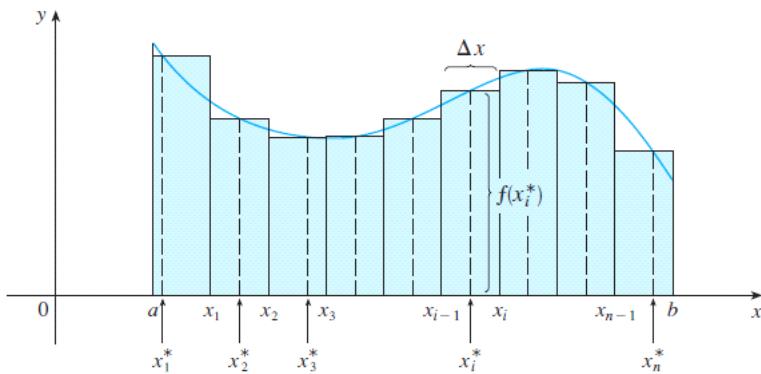
### Review of the Definite Integral

First let's recall the basic facts concerning definite integrals of functions of a single variable. If  $f(x)$  is defined for  $a \leq x \leq b$ , we start by dividing the interval  $[a, b]$  into  $n$  subintervals  $[x_{i-1}, x_i]$  of equal width  $\Delta x = (b - a)/n$  and we choose sample points  $x_i^*$  in these subintervals. Then we form the Riemann sum

$$1 \quad \sum_{i=1}^n f(x_i^*) \Delta x$$

and take the limit of such sums as  $n \rightarrow \infty$  to obtain the definite integral of  $f$  from  $a$  to  $b$ :

$$2 \quad \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$



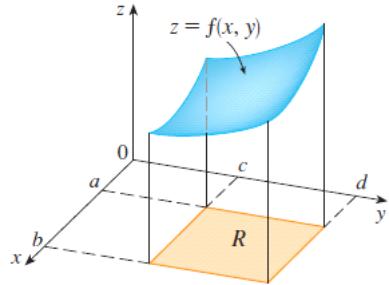
## Volumes and Double Integrals

In a similar manner we consider a function  $f$  of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

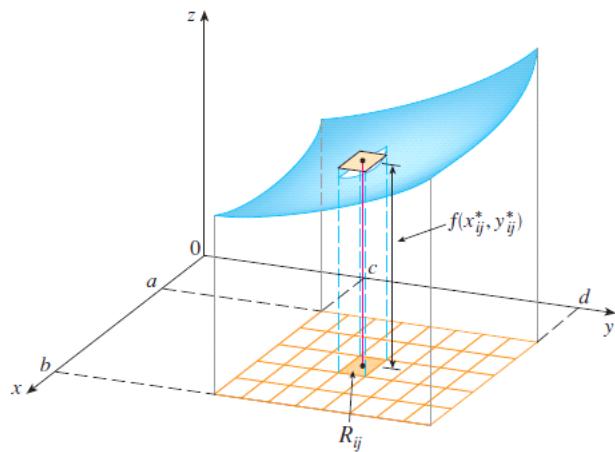
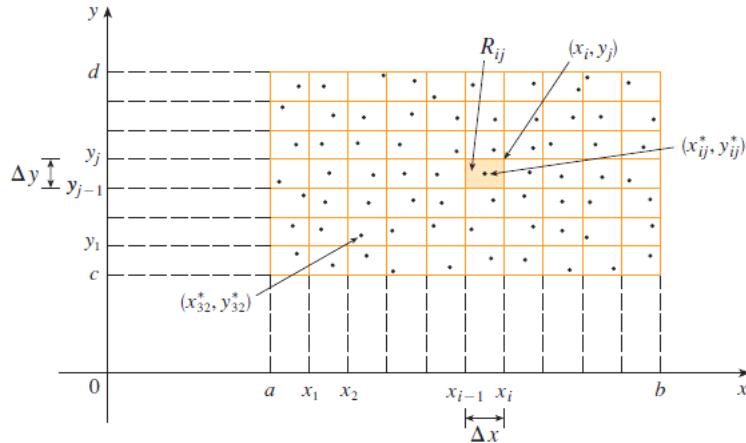
and we first suppose that  $f(x, y) \geq 0$ . The graph of  $f$  is a surface with equation  $z = f(x, y)$ . Let  $S$  be the solid that lies above  $R$  and under the graph of  $f$ , that is,

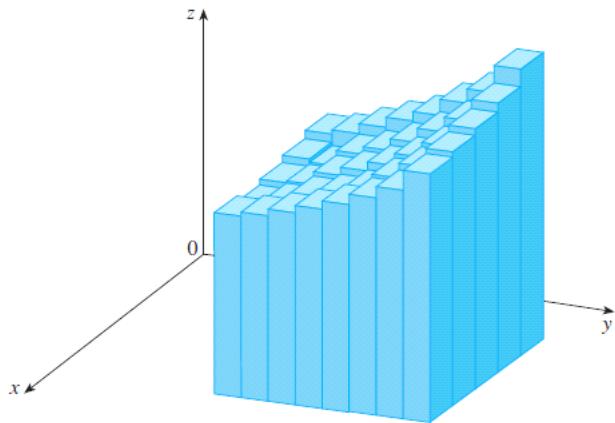
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y), (x, y) \in R\}$$



$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

each with area  $\Delta A = \Delta x \Delta y$ .



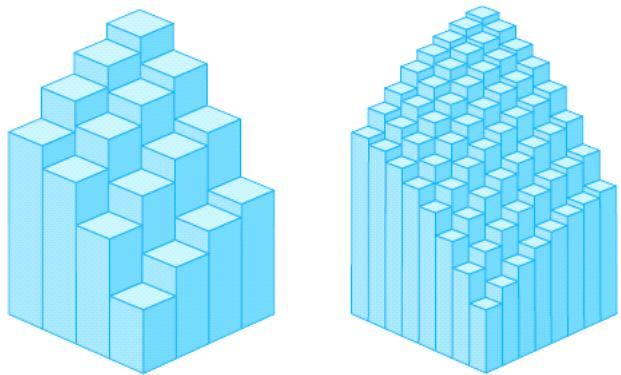
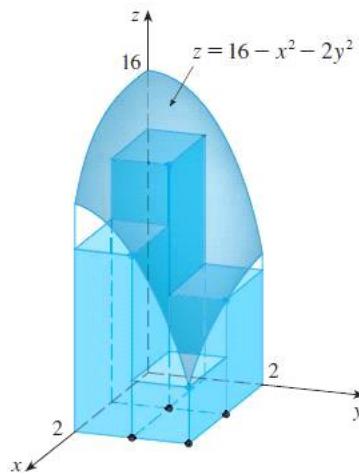


$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_j^*) \Delta A$$

**5 Definition** The double integral of  $f$  over the rectangle  $R$  is

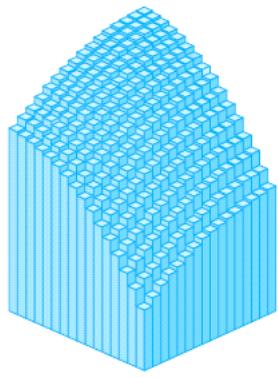
$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_j^*) \Delta A$$

if this limit exists.



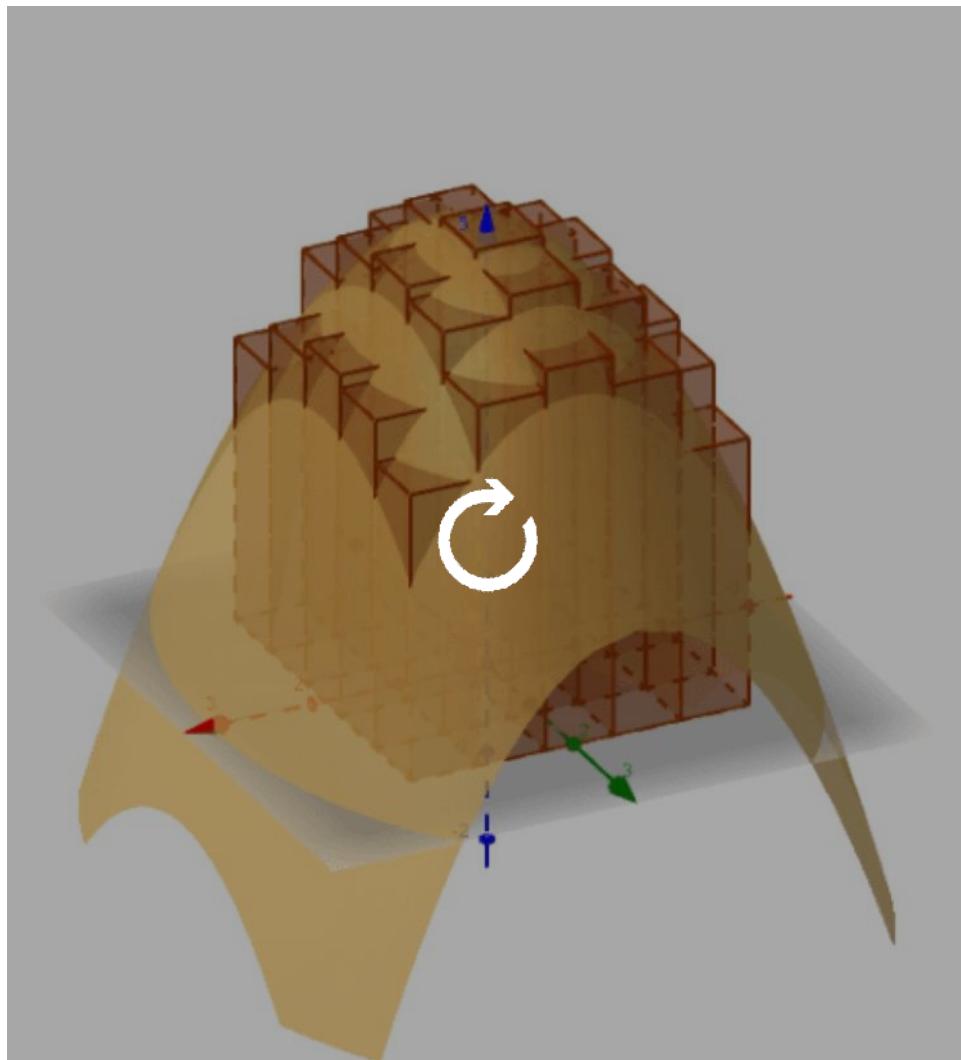
(a)  $m = n = 4, V \approx 41.5$

(b)  $m = n = 8, V \approx 44.875$



(c)  $m = n = 16$ ,  $V \approx 46.46875$

[Definition of the double integral](#)



**EXAMPLE 1** Evaluate the iterated integrals.

$$(a) \int_0^3 \int_1^2 x^2 y \, dy \, dx$$

$$(b) \int_1^2 \int_0^3 x^2 y \, dx \, dy$$

### SOLUTION

(a) Regarding  $x$  as a constant, we obtain

$$\int_1^2 x^2 y \, dy = \left[ x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left( \frac{2^2}{2} \right) - x^2 \left( \frac{1^2}{2} \right) = \frac{3}{2} x^2$$

Thus the function  $A$  in the preceding discussion is given by  $A(x) = \frac{3}{2} x^2$  in this example. We now integrate this function of  $x$  from 0 to 3:

$$\begin{aligned} \int_0^3 \int_1^2 x^2 y \, dy \, dx &= \int_0^3 \left[ \int_1^2 x^2 y \, dy \right] dx \\ &= \int_0^3 \frac{3}{2} x^2 \, dx = \left[ \frac{x^3}{2} \right]_0^3 = \frac{27}{2} \end{aligned}$$

(b) Here we first integrate with respect to  $x$ :

$$\begin{aligned} \int_1^2 \int_0^3 x^2 y \, dx \, dy &= \int_1^2 \left[ \int_0^3 x^2 y \, dx \right] dy = \int_1^2 \left[ \frac{x^3}{3} y \right]_{x=0}^{x=3} dy \\ &= \int_1^2 9y \, dy = 9 \left[ \frac{y^2}{2} \right]_1^2 = \frac{27}{2} \end{aligned}$$

**4 Fubini's Theorem** If  $f$  is continuous on the rectangle

$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ , then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that  $f$  is bounded on  $R$ ,  $f$  is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

**V EXAMPLE 2** Evaluate the double integral  $\iint_R (x - 3y^2) \, dA$ , where  $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$ . (Compare with Example 3 in Section 15.1.)

**SOLUTION 1** Fubini's Theorem gives

$$\begin{aligned} \iint_R (x - 3y^2) \, dA &= \int_0^2 \int_1^2 (x - 3y^2) \, dy \, dx = \int_0^2 \left[ xy - y^3 \right]_{y=1}^{y=2} dx \\ &= \int_0^2 (x - 7) \, dx = \left[ \frac{x^2}{2} - 7x \right]_0^2 = -12 \end{aligned}$$

**V EXAMPLE 3** Evaluate  $\iint_R y \sin(xy) \, dA$ , where  $R = [1, 2] \times [0, \pi]$ .

**SOLUTION 1** If we first integrate with respect to  $x$ , we get

$$\begin{aligned} \iint_R y \sin(xy) \, dA &= \int_0^\pi \int_1^2 y \sin(xy) \, dx \, dy = \int_0^\pi \left[ -\cos(xy) \right]_{x=1}^{x=2} dy \\ &= \int_0^\pi (-\cos 2y + \cos y) \, dy \\ &= -\frac{1}{2} \sin 2y + \sin y \Big|_0^\pi = 0 \end{aligned}$$

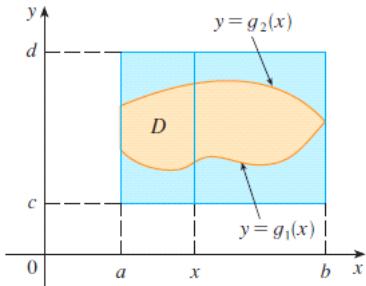
**V EXAMPLE 4** Find the volume of the solid  $S$  that is bounded by the elliptic paraboloid  $x^2 + 2y^2 + z = 16$ , the planes  $x = 2$  and  $y = 2$ , and the three coordinate planes.

**SOLUTION** We first observe that  $S$  is the solid that lies under the surface  $z = 16 - x^2 - 2y^2$  and above the square  $R = [0, 2] \times [0, 2]$ . (See Figure 5.) This solid was considered in Example 1 in Section 15.1, but we are now in a position to evaluate the double integral using Fubini's Theorem. Therefore

$$\begin{aligned} V &= \iint_R (16 - x^2 - 2y^2) dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2) dx dy \\ &= \int_0^2 \left[ 16x - \frac{1}{3}x^3 - 2y^2x \right]_{x=0}^{x=2} dy \\ &= \int_0^2 \left( \frac{88}{3} - 4y^2 \right) dy = \left[ \frac{88}{3}y - \frac{4}{3}y^3 \right]_0^2 = 48 \end{aligned}$$

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## Double Integrals over General Regions

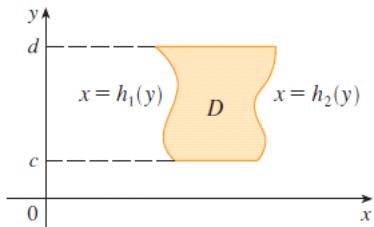


**3** If  $f$  is continuous on a type I region  $D$  such that

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

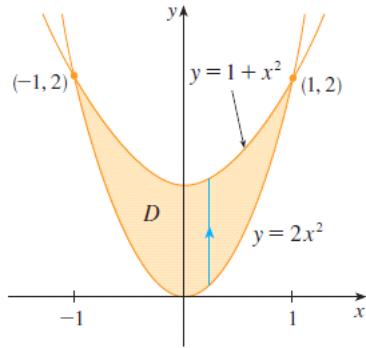
$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$



**5**

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

**V EXAMPLE 1** Evaluate  $\iint_D (x + 2y) dA$ , where  $D$  is the region bounded by the parabolas  $y = 2x^2$  and  $y = 1 + x^2$ .



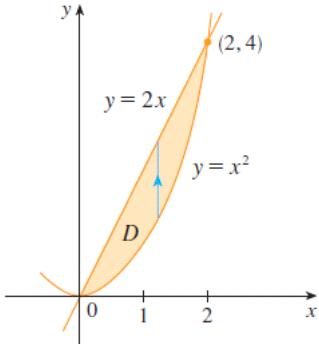
**SOLUTION** The parabolas intersect when  $2x^2 = 1 + x^2$ , that is,  $x^2 = 1$ , so  $x = \pm 1$ . We note that the region  $D$ , sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

Since the lower boundary is  $y = 2x^2$  and the upper boundary is  $y = 1 + x^2$ , Equation 3 gives

$$\begin{aligned} \iint_D (x + 2y) dA &= \int_{-1}^1 \int_{2x^2}^{1+x^2} (x + 2y) dy dx \\ \iint_D (x + 2y) dA &= \int_{-1}^1 \left[ xy + y^2 \right]_{y=2x^2}^{y=1+x^2} dx \\ &= \int_{-1}^1 [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2] dx \\ &= \int_{-1}^1 (-3x^4 - x^3 + 2x^2 + x + 1) dx \\ &= \left[ -3 \frac{x^5}{5} - \frac{x^4}{4} + 2 \frac{x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^1 = \frac{32}{15} \end{aligned}$$

**EXAMPLE 2** Find the volume of the solid that lies under the paraboloid  $z = x^2 + y^2$  and above the region  $D$  in the  $xy$ -plane bounded by the line  $y = 2x$  and the parabola  $y = x^2$ .



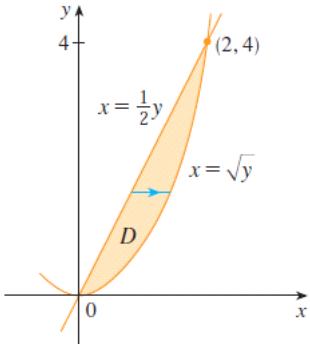
**FIGURE 9**  
 $D$  as a type I region

**SOLUTION 1** From Figure 9 we see that  $D$  is a type I region and

$$D = \{(x, y) \mid 0 \leq x \leq 2, x^2 \leq y \leq 2x\}$$

Therefore the volume under  $z = x^2 + y^2$  and above  $D$  is

$$\begin{aligned} V &= \iint_D (x^2 + y^2) dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx \\ V &= \int_0^2 (x^2 + y^2) dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx \\ &= \int_0^2 \left[ x^2y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx \\ &= \int_0^2 \left[ x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3} \right] dx \\ &= \int_0^2 \left( -\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx \\ &= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \Big|_0^2 = \frac{216}{35} \end{aligned}$$



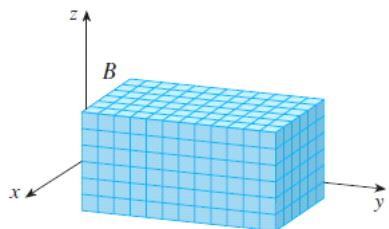
**FIGURE 10**  
 $D$  as a type II region

**SOLUTION 2** From Figure 10 we see that  $D$  can also be written as a type II region:

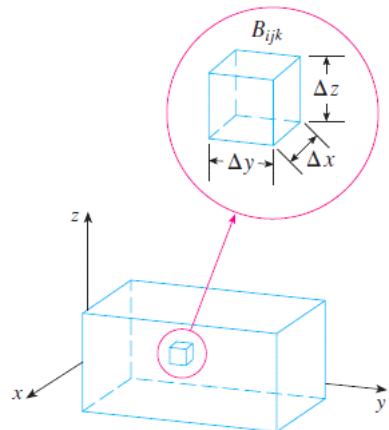
$$D = \{(x, y) \mid 0 \leq y \leq 4, \frac{1}{2}y \leq x \leq \sqrt{y}\}$$

Therefore another expression for  $V$  is

$$\begin{aligned} V &= \iint_D (x^2 + y^2) dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2) dx dy \\ &= \int_0^4 \left[ \frac{x^3}{3} + y^2x \right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy = \int_0^4 \left( \frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \right) dy \\ &= \frac{2}{15}y^{5/2} + \frac{2}{7}y^{7/2} - \frac{13}{96}y^4 \Big|_0^4 = \frac{216}{35} \end{aligned}$$



$$B = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$



$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1. Each sub-box has volume  $\Delta V = \Delta x \Delta y \Delta z$ .

Then we form the **triple Riemann sum**

$$2 \quad \sum_{i=1}^I \sum_{j=1}^M \sum_{k=1}^N f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

**3 Definition** The **triple integral** of  $f$  over the box  $B$  is

$$\iiint_B f(x, y, z) dV = \lim_{I, M, N \rightarrow \infty} \sum_{i=1}^I \sum_{j=1}^M \sum_{k=1}^N f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

if this limit exists.

**4 Fubini's Theorem for Triple Integrals** If  $f$  is continuous on the rectangular box  $B = [a, b] \times [c, d] \times [r, s]$ , then

$$\iiint_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

**V EXAMPLE 1** Evaluate the triple integral  $\iiint_B xyz^2 dV$ , where  $B$  is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$$

**SOLUTION** We could use any of the six possible orders of integration. If we choose to integrate with respect to  $x$ , then  $y$ , and then  $z$ , we obtain

$$\begin{aligned} \iiint_B xyz^2 dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dx dy dz = \int_0^3 \int_{-1}^2 \left[ \frac{x^2 yz^2}{2} \right]_{x=0}^{x=1} dy dz \\ &= \int_0^3 \int_{-1}^2 \frac{yz^2}{2} dy dz = \int_0^3 \left[ \frac{y^2 z^2}{4} \right]_{y=-1}^{y=2} dz \\ &= \int_0^3 \frac{3z^2}{4} dz = \left[ \frac{z^3}{4} \right]_0^3 = \frac{27}{4} \end{aligned}$$

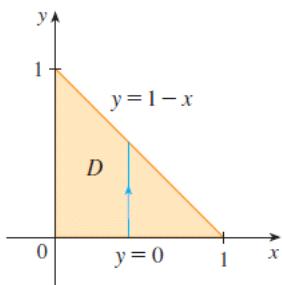
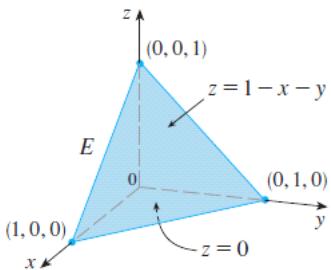
Evaluate the integral in Example 1, integrating first with respect to  $y$ , then  $z$ , and then  $x$ .

Evaluate the integral  $\iiint_E (xy + z^2) dV$ , where

$$E = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 3\}$$

using three different orders of integration.

**EXAMPLE 2** Evaluate  $\iiint_E z dV$ , where  $E$  is the solid tetrahedron bounded by the four planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ , and  $x + y + z = 1$ .



**9**  $E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}$

$$\begin{aligned} \iiint_E z dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z dz dy dx = \int_0^1 \int_0^{1-x} \left[ \frac{z^2}{2} \right]_{z=0}^{z=1-x-y} dy dx \\ &= \frac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2 dy dx = \frac{1}{2} \int_0^1 \left[ -\frac{(1 - x - y)^3}{3} \right]_{y=0}^{y=1-x} dx \\ &= \frac{1}{6} \int_0^1 (1 - x)^3 dx = \frac{1}{6} \left[ -\frac{(1 - x)^4}{4} \right]_0^1 = \frac{1}{24} \end{aligned}$$

