

Integration - I (Cartesian)

Saturday, 17 January 2026 5:09 pm



Double Integrals over Rectangles

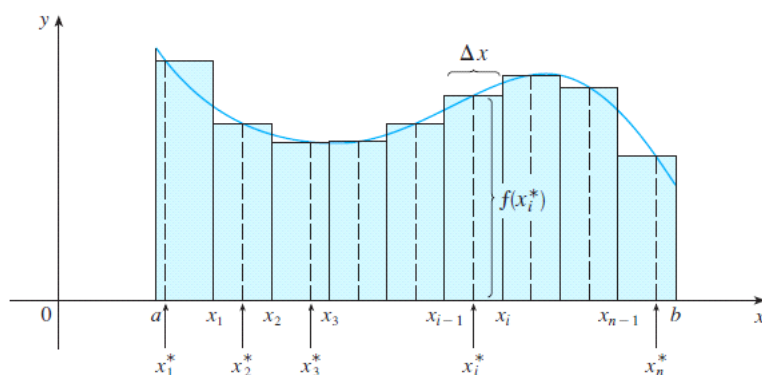
Review of the Definite Integral

First let's recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ is defined for $a \leq x \leq b$, we start by dividing the interval $[a, b]$ into n sub-intervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

$$1 \quad \sum_{i=1}^n f(x_i^*) \Delta x$$

and take the limit of such sums as $n \rightarrow \infty$ to obtain the definite integral of f from a to b :

$$2 \quad \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$



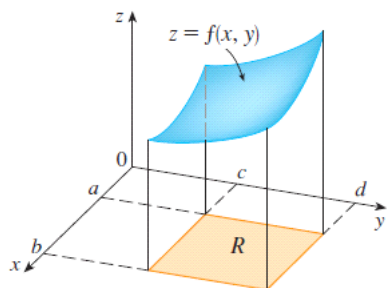
Volumes and Double Integrals

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

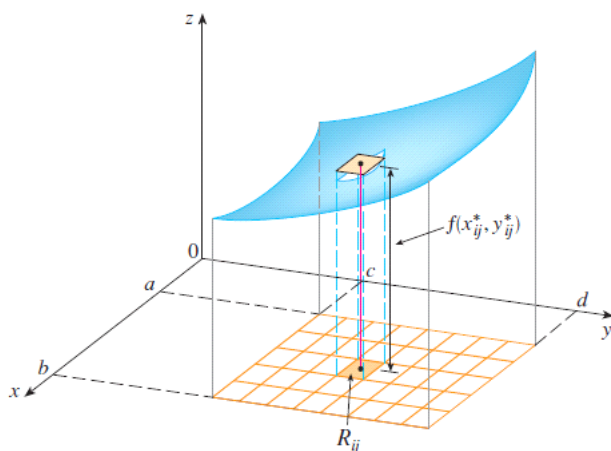
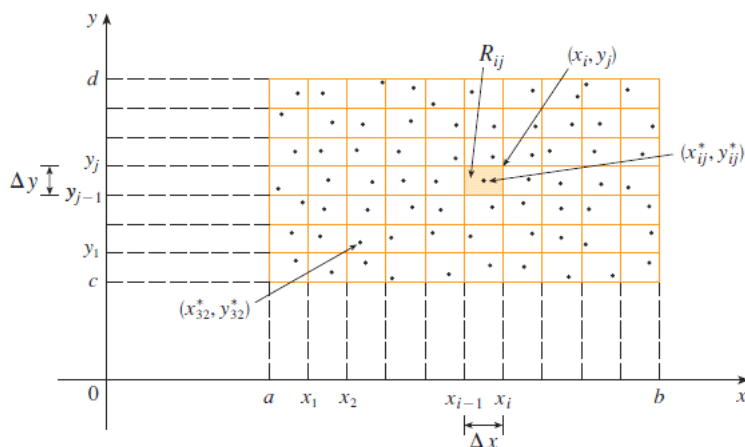
and we first suppose that $f(x, y) \geq 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f , that is,

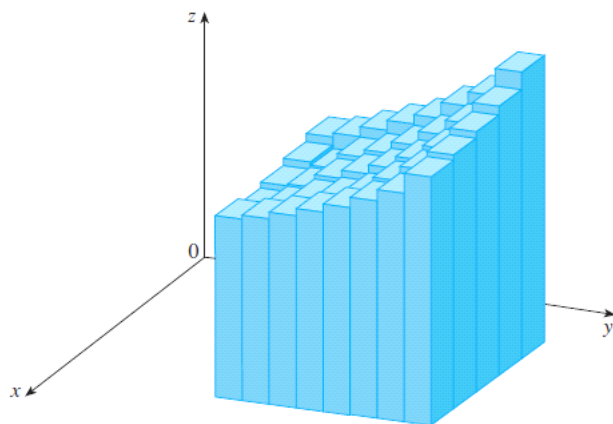
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y), (x, y) \in R\}$$



$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

each with area $\Delta A = \Delta x \Delta y$.



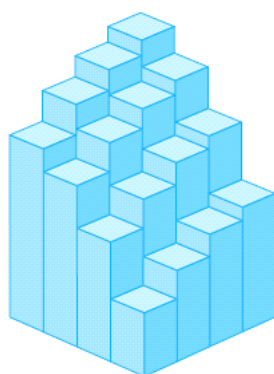
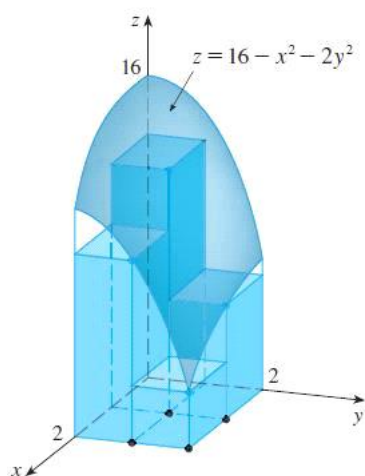


$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

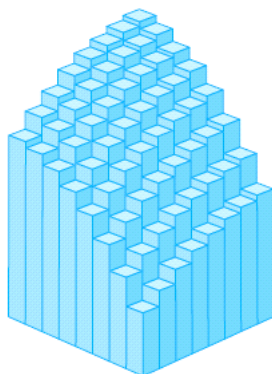
5 Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x, y) \, dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

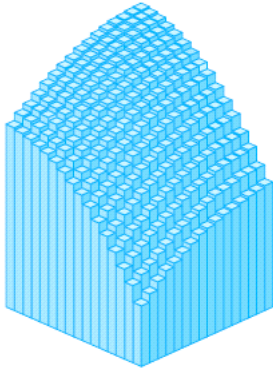
if this limit exists.



(a) $m = n = 4$, $V \approx 41.5$

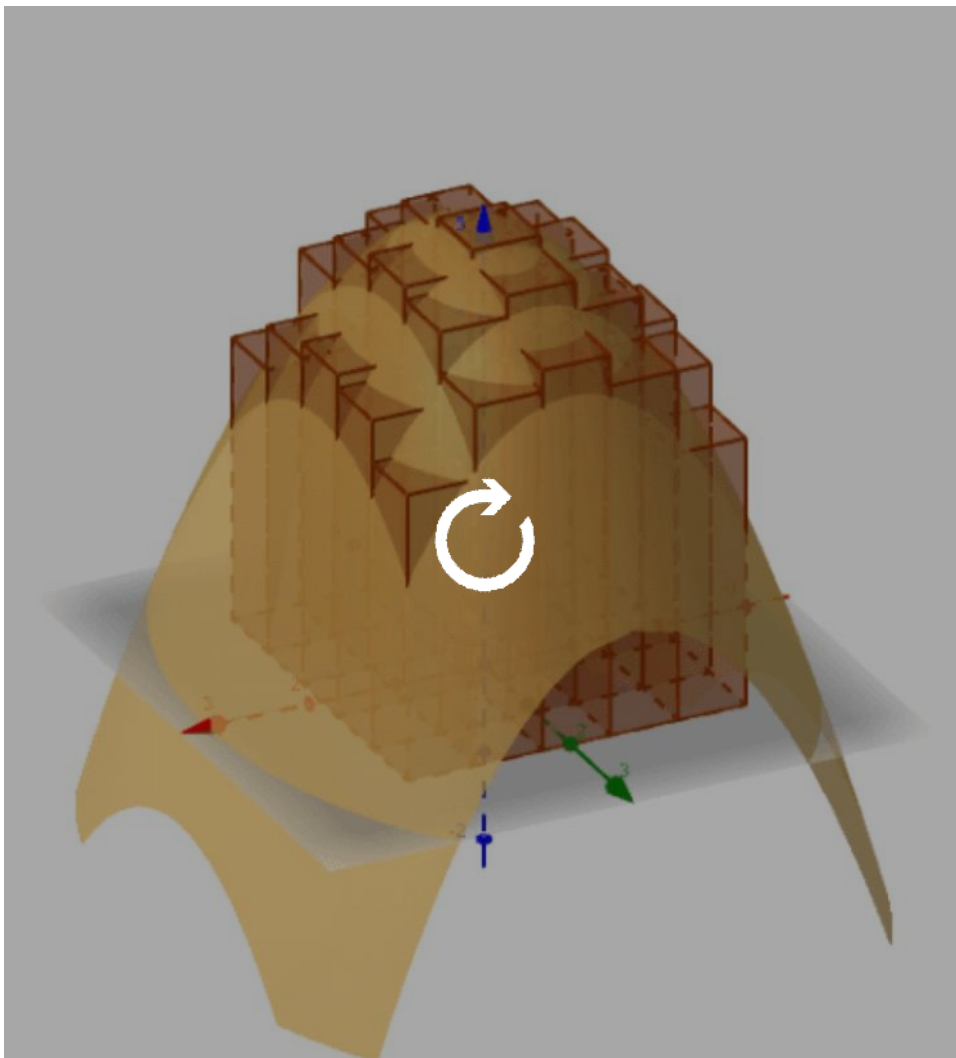


(b) $m = n = 8$, $V \approx 44.875$



(c) $m = n = 16$, $V \approx 46.46875$

[Definition of the double integral](#)



EXAMPLE 1 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

SOLUTION

(a) Regarding x as a constant, we obtain

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \frac{3}{2} x^2$$

Thus the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\begin{aligned} \int_0^3 \int_1^2 x^2 y \, dy \, dx &= \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx \\ &= \int_0^3 \frac{3}{2} x^2 \, dx = \left[\frac{x^3}{2} \right]_0^3 = \frac{27}{2} \end{aligned}$$

(b) Here we first integrate with respect to x :

$$\begin{aligned} \int_1^2 \int_0^3 x^2 y \, dx \, dy &= \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy \\ &= \int_1^2 9y \, dy = 9 \left[\frac{y^2}{2} \right]_1^2 = \frac{27}{2} \end{aligned}$$

4 Fubini's Theorem If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that f is bounded on R , f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

V EXAMPLE 2 Evaluate the double integral $\iint_R (x - 3y^2) \, dA$, where $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$. (Compare with Example 3 in Section 15.1.)

SOLUTION 1 Fubini's Theorem gives

$$\begin{aligned} \iint_R (x - 3y^2) \, dA &= \int_0^2 \int_1^2 (x - 3y^2) \, dy \, dx = \int_0^2 \left[xy - y^3 \right]_{y=1}^{y=2} dx \\ &= \int_0^2 (x - 7) \, dx = \left[\frac{x^2}{2} - 7x \right]_0^2 = -12 \end{aligned}$$

V EXAMPLE 3 Evaluate $\iint_R y \sin(xy) \, dA$, where $R = [1, 2] \times [0, \pi]$.

SOLUTION 1 If we first integrate with respect to x , we get

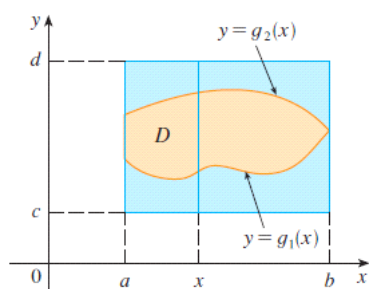
$$\begin{aligned} \iint_R y \sin(xy) \, dA &= \int_0^\pi \int_1^2 y \sin(xy) \, dx \, dy = \int_0^\pi \left[-\cos(xy) \right]_{x=1}^{x=2} dy \\ &= \int_0^\pi (-\cos 2y + \cos y) \, dy \\ &= -\frac{1}{2} \sin 2y + \sin y \Big|_0^\pi = 0 \end{aligned}$$

V EXAMPLE 4 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

SOLUTION We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0, 2] \times [0, 2]$. (See Figure 5.) This solid was considered in Example 1 in Section 15.1, but we are now in a position to evaluate the double integral using Fubini's Theorem. Therefore

$$\begin{aligned} V &= \iint_R (16 - x^2 - 2y^2) dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2) dx dy \\ &= \int_0^2 \left[16x - \frac{1}{3}x^3 - 2y^2x \right]_{x=0}^{x=2} dy \\ &= \int_0^2 \left(\frac{88}{3} - 4y^2 \right) dy = \left[\frac{88}{3}y - \frac{4}{3}y^3 \right]_0^2 = 48 \end{aligned}$$

Double Integrals over General Regions

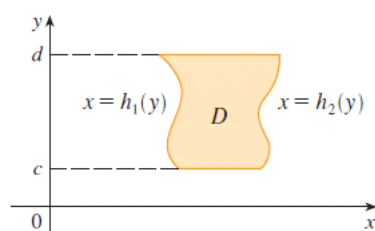


3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

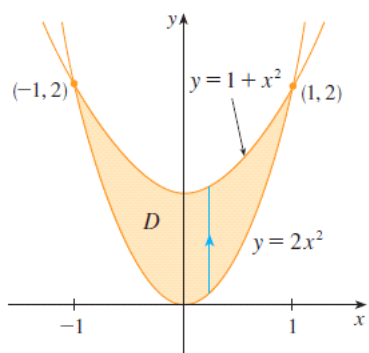
$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$



5

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

V EXAMPLE 1 Evaluate $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D , sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\begin{aligned} \iint_D (x + 2y) \, dA &= \int_{-1}^1 \int_{2x^2}^{1+x^2} (x + 2y) \, dy \, dx \\ \iint_D (x + 2y) \, dA &= \int_{-1}^1 \int_{2x^2}^{1+x^2} (x + 2y) \, dy \, dx \\ &= \int_{-1}^1 [xy + y^2]_{y=2x^2}^{y=1+x^2} \, dx \\ &= \int_{-1}^1 [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2] \, dx \\ &= \int_{-1}^1 (-3x^4 - x^3 + 2x^2 + x + 1) \, dx \\ &= -3 \frac{x^5}{5} - \frac{x^4}{4} + 2 \frac{x^3}{3} + \frac{x^2}{2} + x \bigg|_{-1}^1 = \frac{32}{15} \end{aligned}$$

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy -plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

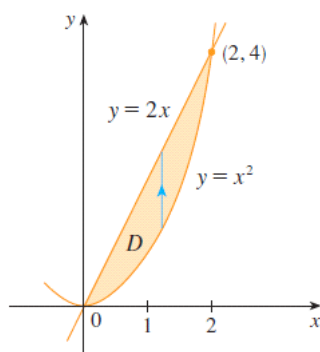


FIGURE 9
 D as a type I region

SOLUTION 1 From Figure 9 we see that D is a type I region and

$$D = \{(x, y) \mid 0 \leq x \leq 2, x^2 \leq y \leq 2x\}$$

Therefore the volume under $z = x^2 + y^2$ and above D is

$$V = \iint_D (x^2 + y^2) dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx$$

$$\begin{aligned} V &= \iint_D (x^2 + y^2) dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx \\ &= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx \\ &= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3} \right] dx \\ &= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx \\ &= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \bigg|_0^2 = \frac{216}{35} \end{aligned}$$

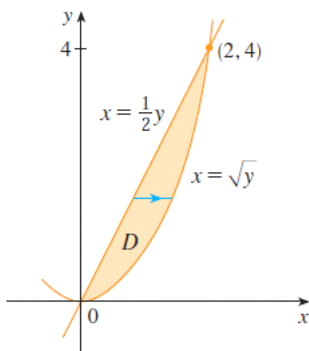


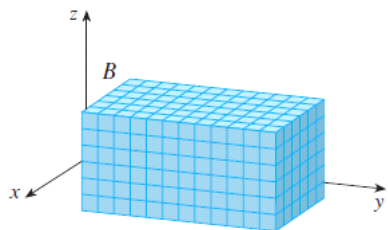
FIGURE 10
 D as a type II region

SOLUTION 2 From Figure 10 we see that D can also be written as a type II region:

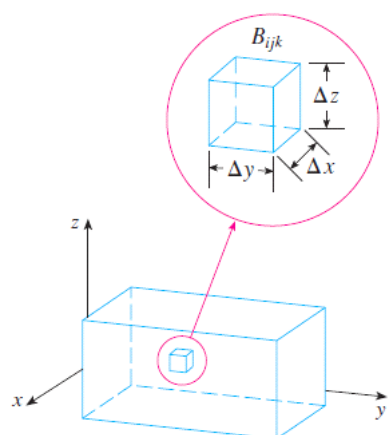
$$D = \{(x, y) \mid 0 \leq y \leq 4, \frac{1}{2}y \leq x \leq \sqrt{y}\}$$

Therefore another expression for V is

$$\begin{aligned} V &= \iint_D (x^2 + y^2) dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2) dx dy \\ &= \int_0^4 \left[\frac{x^3}{3} + y^2 x \right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \right) dy \\ &= \frac{2}{15} y^{5/2} + \frac{2}{7} y^{7/2} - \frac{13}{96} y^4 \bigg|_0^4 = \frac{216}{35} \end{aligned}$$



$$B = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$



$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1. Each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$.

Then we form the **triple Riemann sum**

$$\boxed{2} \quad \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z) dV = \lim_{l, m, n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

if this limit exists.

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

V EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2 dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x , then y , and then z , we obtain

$$\begin{aligned} \iiint_B xyz^2 dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dx dy dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2 y z^2}{2} \right]_{x=0}^{x=1} dy dz \\ &= \int_0^3 \int_{-1}^2 \frac{y z^2}{2} dy dz = \int_0^3 \left[\frac{y^2 z^2}{4} \right]_{y=-1}^{y=2} dz \\ &= \int_0^3 \frac{3z^2}{4} dz = \left[\frac{z^3}{4} \right]_0^3 = \frac{27}{4} \end{aligned}$$

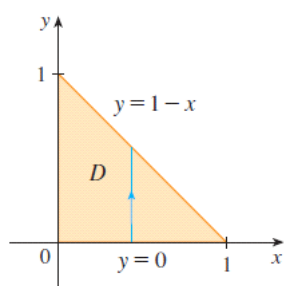
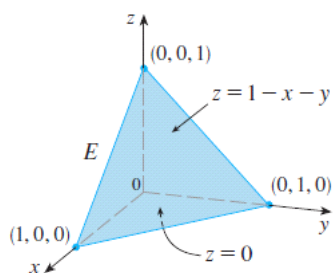
Evaluate the integral in Example 1, integrating first with respect to y , then z , and then x .

Evaluate the integral $\iiint_E (xy + z^2) dV$, where

$$E = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 3\}$$

using three different orders of integration.

EXAMPLE 2 Evaluate $\iiint_E z dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.



9 $E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}$

$$\begin{aligned} \iiint_E z dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z dz dy dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2} \right]_{z=0}^{z=1-x-y} dy dx \\ &= \frac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2 dy dx = \frac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3} \right]_{y=0}^{y=1-x} dx \\ &= \frac{1}{6} \int_0^1 (1 - x)^3 dx = \frac{1}{6} \left[-\frac{(1 - x)^4}{4} \right]_0^1 = \frac{1}{24} \end{aligned}$$

