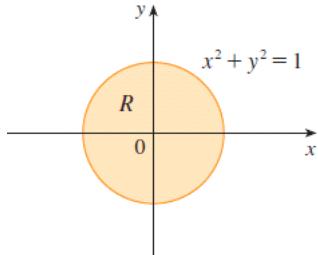


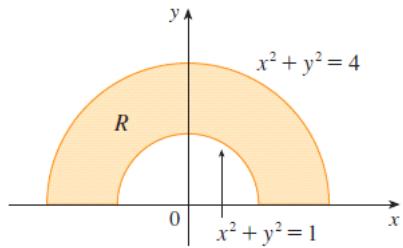
Integration - II (Non-Cartesian)

Sunday, 18 January 2026 1:53 pm

Double Integrals in Polar Coordinates



$$(a) R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$$



$$(b) R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$$

$$r^2 = x^2 + y^2 \quad x = r \cos \theta \quad y = r \sin \theta$$

The regions in Figure 1 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

2 Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

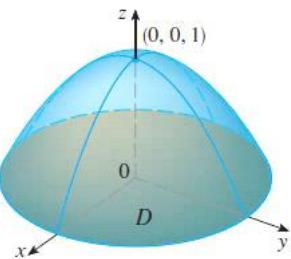
SOLUTION The region R can be described as

$$R = \{(x, y) \mid y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$$

It is the half-ring shown in Figure 1(b), and in polar coordinates it is given by $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$. Therefore, by Formula 2,

$$\begin{aligned}
 \iint_R (3x + 4y^2) dA &= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta) r dr d\theta \\
 &= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta) dr d\theta \\
 &= \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta \right]_{r=1}^{r=2} d\theta = \int_0^\pi (7\cos\theta + 15\sin^2\theta) d\theta \\
 &= \int_0^\pi \left[7\cos\theta + \frac{15}{2}(1 - \cos 2\theta) \right] d\theta \\
 &= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta \Big|_0^\pi = \frac{15\pi}{2} \quad \blacksquare
 \end{aligned}$$

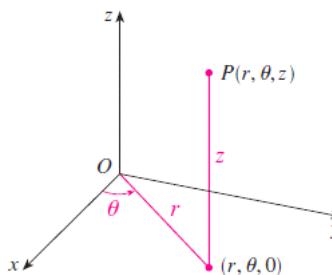
V EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.



SOLUTION If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leq 1$ [see Figures 6 and 1(a)]. In polar coordinates D is given by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$\begin{aligned}
 V &= \iint_D (1 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta \\
 &= \int_0^{2\pi} d\theta \int_0^1 (r - r^3) dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2}
 \end{aligned}$$

Triple Integrals in Cylindrical Coordinates



$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

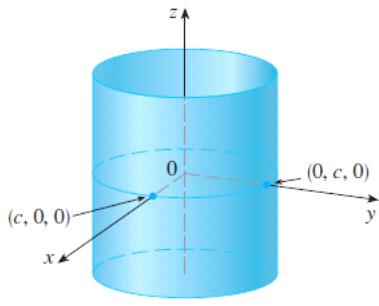


FIGURE 4
 $r = c$, a cylinder

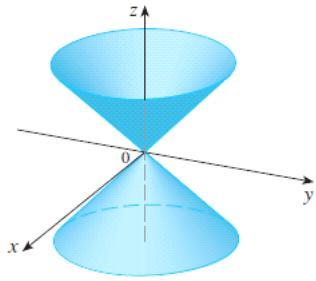
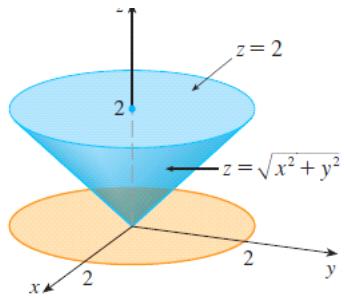


FIGURE 5
 $z = r$, a cone

$$\iiint_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta$$

EXAMPLE 4 Evaluate $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz \, dy \, dx$.



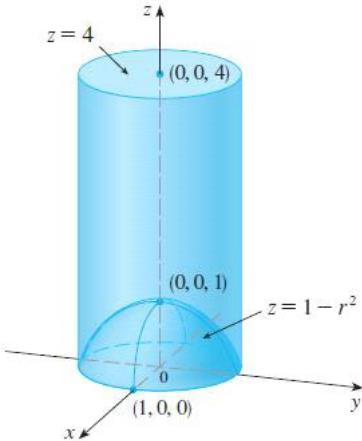
$$E = \{(x, y, z) \mid -2 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}, \sqrt{x^2+y^2} \leq z \leq 2\}$$

$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, r \leq z \leq 2\}$$

$$\begin{aligned} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz \, dy \, dx &= \iiint_E (x^2 + y^2) \, dV \\ &= \int_0^{2\pi} \int_0^2 \int_r^2 r^2 r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^2 r^3 (2 - r) \, dr \\ &= 2\pi \left[\frac{1}{2}r^4 - \frac{1}{5}r^5 \right]_0^2 = \frac{16}{5}\pi \end{aligned}$$

EXAMPLE 3 A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E .

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

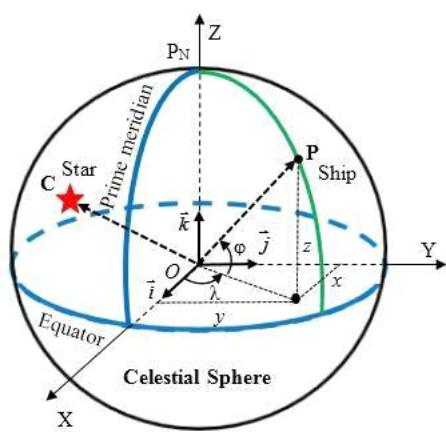


$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 - r^2 \leq z \leq 4\}$$

$$\begin{aligned} m &= \iiint_E K\sqrt{x^2 + y^2} dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr) r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 Kr^2 [4 - (1 - r^2)] dr d\theta = K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4) dr \\ &= 2\pi K \left[r^3 + \frac{r^5}{5} \right]_0^1 = \frac{12\pi K}{5} \end{aligned}$$

$$\int_0^2 \int_0^{2\pi} \int_0^r r dz d\theta dr$$

Triple Integrals in Spherical Coordinates



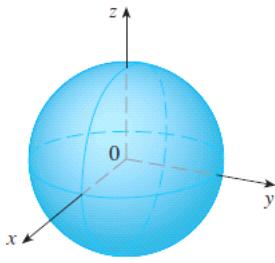


FIGURE 2 $\rho = c$, a sphere

$$\rho^2 = x^2 + y^2 + z^2$$

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

$$\boxed{3} \quad \begin{aligned} & \iiint_E f(x, y, z) \, dV \\ &= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \end{aligned}$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

V EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

$$B = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$$

In addition, spherical coordinates are appropriate because

$$x^2 + y^2 + z^2 = \rho^2$$

Thus $\boxed{3}$ gives

$$\begin{aligned} \iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV &= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^\pi \sin \phi \, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3} \, d\rho \\ &= [-\cos \phi]_0^\pi (2\pi) \left[\frac{1}{3} e^{\rho^3}\right]_0^1 = \frac{4}{3}\pi(e - 1) \end{aligned}$$

$$\int_0^{2\pi} \int_{\pi/2}^\pi \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$