Echelon Form & Row Reduced Echelon Form

Wednesday, 7 January 2026 4:31 pm

Lecture 2

Row Reduction and Echelon Forms

2.1 Row echelon form (REF)

Consider the linear system

$$\begin{aligned} x_1 + 5x_2 - 2x_4 - x_5 + 7x_6 &= -4 \\ 2x_2 - 2x_3 + 3x_6 &= 0 \\ -9x_4 - x_5 + x_6 &= -1 \\ 5x_5 + x_6 &= 5 \\ 0 &= 0 \end{aligned}$$

having augmented matrix

$$\begin{bmatrix} 1 & 5 & 0 & -2 & -1 & 7 & -4 \\ 0 & 2 & -2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -9 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 5 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

P1. All nonzero rows are above any rows of all zeros.

P2. The leftmost nonzero entry of a row is to the right of the leftmost nonzero entry of the row above it.

Any matrix satisfying properties P1 and P2 is said to be in **row echelon form** (**REF**). In REF, the leftmost nonzero entry in a row is called a **leading entry**:

$$\begin{bmatrix} \mathbf{1} & 5 & 0 & -2 & -1 & 7 & -4 \\ 0 & \mathbf{2} & -2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \mathbf{-9} & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \mathbf{5} & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A consequence of property P2 is that every entry below a leading entry is zero:

$$\begin{bmatrix} \mathbf{1} & 5 & 0 & -2 & -4 & -1 & -7 \\ \mathbf{0} & \mathbf{2} & -2 & 0 & 0 & 3 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{-9} & -1 & 1 & -1 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{5} & 1 & 5 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & 0 & 0 \end{bmatrix}$$

We can perform elementary row operations, or **row reduction**, to transform a matrix into REF.

Reduced row echelon form (RREF)

A leading 1 in the RREF of a matrix is called a **pivot**. For example, the following matrix in RREF:

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

has three pivots:

$$\begin{bmatrix} \mathbf{1} & 6 & \mathbf{0} & 3 & \mathbf{0} & 0 \\ \mathbf{0} & 0 & \mathbf{1} & -4 & \mathbf{0} & 5 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{1} & 7 \end{bmatrix}$$

Example 2.2. Use row reduction to transform the matrix into RREF.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution. The first step is to make the top leftmost entry nonzero:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Now create a leading 1 in the first row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Create zeros under the newly created leading 1:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Create a leading 1 in the second row:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Create zeros under the newly created leading 1:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Create zeros above the leading 1 in the second row:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{3R_2+R_1} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This completes the row reduction algorithm and the matrix is in RREF. □

Example 2.3. Use row reduction to solve the linear system.

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 8 \\ x_1 + 2x_2 + 4x_3 &= 8 \\ 3x_1 + 6x_2 + 9x_3 &= 12 \end{aligned}$$

Create a leading 1 in the first row:

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Create zeros under the first leading 1:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, however, there are only 2 nonzero rows but 3 unknown variables. This means that the solution set will contain $3 - 2 = 1$ **free parameter**. The second row in the augmented matrix is equivalent to the equation:

$$\begin{aligned} x_1 &= -8 - 2t \\ x_2 &= t \\ x_3 &= 4 \end{aligned}$$

In general, if a linear system has n unknown variables and the row reduced augmented matrix has r leading entries, then the number of free parameters d in the solution set is

$$d = n - r.$$

$$d = n - r = 3 - 2 = 1.$$

$$r = \text{rank}(\mathbf{A}).$$

Example 2.4. Solve the linear system represented by the augmented matrix

$$\begin{bmatrix} 1 & -7 & 2 & -5 & 8 & 10 \\ 0 & 1 & -3 & 3 & 1 & -5 \\ 0 & 0 & 0 & 1 & -1 & 4 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -89 - 31t + 19s \\ x_2 &= -17 - 4t + 3s \\ x_3 &= s \\ x_4 &= 4 + t \\ x_5 &= t \end{aligned}$$

2.3 Existence and uniqueness of solutions

The REF or RREF of an augmented matrix leads to three distinct possibilities for the solution set of a linear system.

Theorem 2.5: Let $[\mathbf{A}\ \mathbf{b}]$ be the augmented matrix of a linear system. One of the following distinct possibilities will occur:

1. The augmented matrix will contain an inconsistent row.
2. All the rows of the augmented matrix are consistent and there are no free parameters.
3. All the rows of the augmented matrix are consistent and there are $d \geq 1$ variables that must be set to arbitrary parameters

After this lecture you should know the following:

- what the REF is and how to compute it
- what the RREF is and how to compute it
- how to solve linear systems using row reduction (**Practice!!!**)
- how to identify when a linear system is inconsistent
- how to identify when a linear system is consistent
- what is the rank of a matrix
- how to compute the number of free parameters in a solution set
- what are the three possible cases for the solution set of a linear system (Theorem 2.5)