Cal2

Monday, 3 June 2024 12:07 pm

14.4 GRADIENTS AND DIRECTIONAL DERIVATIVES IN THE PLANE

The Rate of Change in an Arbitrary Direction: The Directional Derivative

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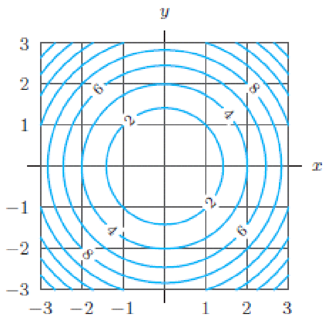

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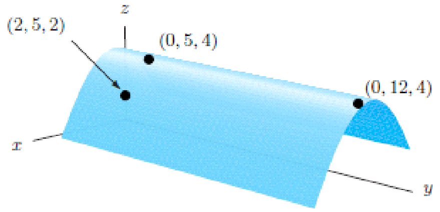

Figure 14.36

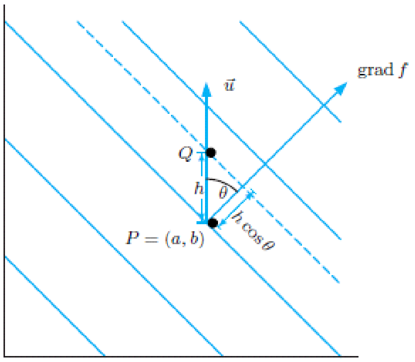

Figure 14.40

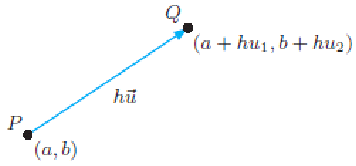

Figure 14.29: Displacement of $h\vec{u}$ from the point (a, b)

Taking the limit as $h \to 0$ gives the instantaneous rate of change and the following definition:

Directional Derivative of f at (a, b) in the Direction of a Unit Vector $\vec{u}$

If $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, we define the directional derivative, $f_{\vec{u}}$, by

$$f_{\vec{u}}(a,b) = \begin{array}{c}\text{Rate of change}\\ \text{of } f \text{ in direction}\\ \text{of } \vec{u} \text{ at } (a,b)\end{array} = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h},$$

provided the limit exists.

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Example 2 For each of the functions f, g, and h in Figure 14.30, decide whether the directional derivative at the indicated point is positive, negative, or zero, in the direction of the vector $\vec{v} = \vec{i} + 2\vec{j}$, and in the direction of the vector $\vec{w} = 2\vec{i} + \vec{j}$.

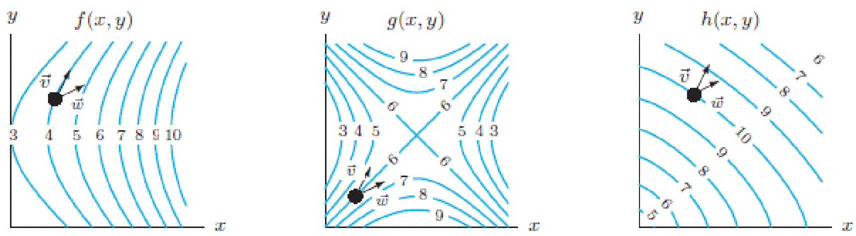

Figure 14.30: Contour diagrams of three functions with direction vectors $\vec{v} = \vec{i} + 2\vec{j}$ and $\vec{w} = 2\vec{i} + \vec{j}$ marked on each

The Gradient Vector of a differentiable function f at the point (a, b) is

$$\operatorname{grad} f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$$

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The Directional Derivative and the Gradient

If f is differentiable at (a, b) and $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, then

$$f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = \operatorname{grad} f(a,b) \cdot \vec{u}.$$

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Example 3 Calculate the directional derivative of $f(x,y) = x^2 + y^2$ at $(1, 0)$ in the direction of the vector $\vec{i} + \vec{j}$.

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Example 5 Find the gradient vector of $f(x,y) = x + e^y$ at the point $(1, 1)$.

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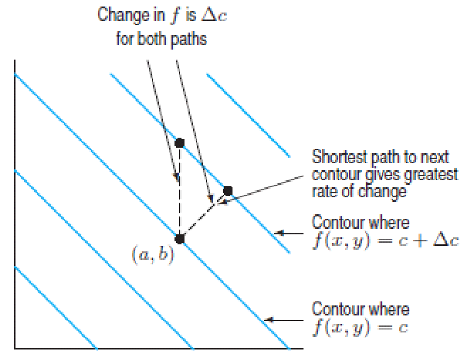

Figure 14.32: Close-up view of the contours around (a,b), showing the gradient is perpendicular to the contours

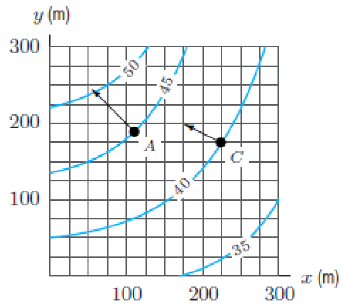

Figure 14.33: A temperature map showing directions and relative magnitudes of two gradient vectors

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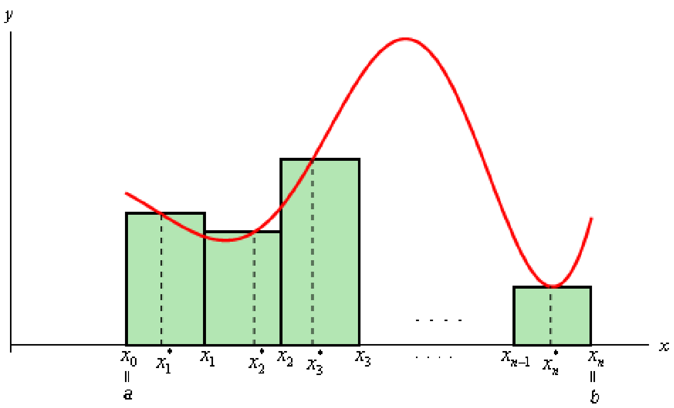
Example 7 Use the gradient to find the directional derivative of $f(x,y) = x + e^y$ at the point $(1,1)$ in the direction of the vectors $\vec{i} - \vec{j}, \vec{i} + 2\vec{j}, \vec{i} + 3\vec{j}$.

Double Integral:

$$\int_a^b f(x)\, dx$$



$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

To get the exact area we then took the limit as n goes to infinity and this was also the definition of the definite integral.

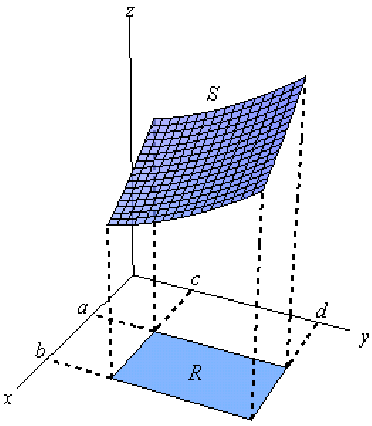
$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

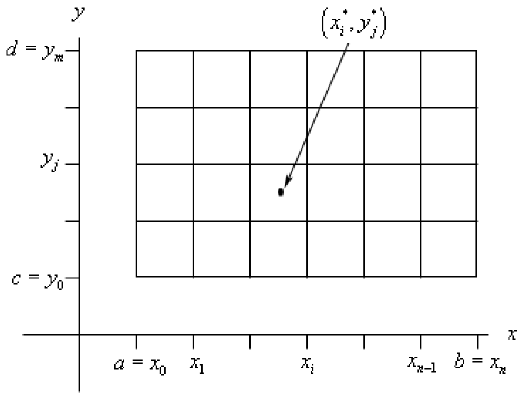
We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

$$R = [a,b] \times [c,d]$$

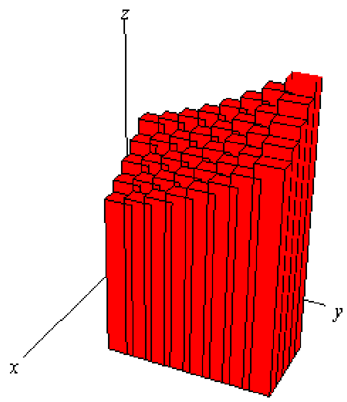
This means that the ranges for x and y are $a \le x \le b$ and $c \le y \le d$.

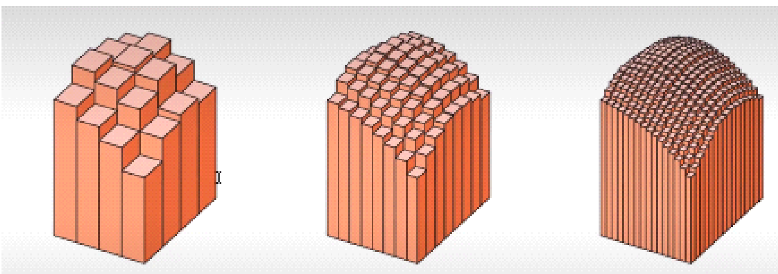
Also, we will initially assume that $f(x,y) \ge 0$ although this doesn't really have to be the case. Let's start out with the graph of the surface S given by graphing $f(x,y)$ over the rectangle R.





Now, over each of these smaller rectangles we will construct a box whose height is given by $f\left(x_i^*, y_j^*\right)$. Here is a sketch of that.





(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

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$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \, \Delta A$$

$$V = \lim_{n,\ m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \, \Delta A$$

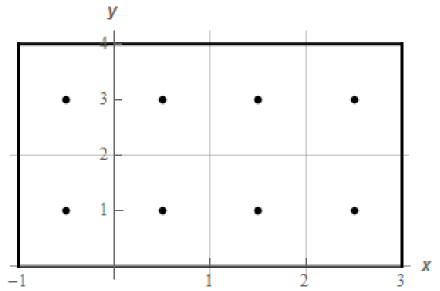
$$\iint\limits_R f(x, y) \, dA = \lim_{n,\ m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \, \Delta A$$

$$\text{Volume} = \iint\limits_R f(x, y) \, dA$$

The sum $\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$ is called DOUBLE RIEMANN SUM and is used as an approximation to the value of double integral.

EXAMPLE: Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into 4 equal squares & chose the sample point to be the upper right-hand corner of each square R_{ij}.

1. Use the Midpoint Rule to estimate the volume under $f(x, y) = x^2 + y$ and above the rectangle given by $-1 \le x \le 3$, $0 \le y \le 4$ in the xy-plane. Use 4 subdivisions in the x direction and 2 subdivisions in the y direction.



$$V = \iint\limits_R f(x, y)\, dA$$

$$\iint\limits_R f(x, y)\, dA \approx \sum_{i=1}^{4} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\, \Delta A \qquad f(x, y) = x^2 + y$$

$\Delta A =$

$$V \approx \sum_{i=1}^{4} \sum_{j=1}^{2} 2f\left(\overline{x}_i, \overline{y}_j\right) \qquad f(x, y) = x^2 + y$$

$$i = 1 \quad : \quad \sum_{j=1}^{2} 2f\left(\overline{x}_1, \overline{y}_j\right) =$$

$$i = 2 \quad : \quad \sum_{j=1}^{2} 2f\left(\overline{x}_2, \overline{y}_j\right) =$$

$$i = 3 \quad : \quad \sum_{j=1}^{2} 2f\left(\overline{x}_3, \overline{y}_j\right) =$$

$$i = 4 \quad : \quad \sum_{j=1}^{2} 2f\left(\overline{x}_4, \overline{y}_j\right) =$$

For reference purposes we will eventually be able to verify that the exact volume is

Fubini's Theorem

If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$ then,

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

These integrals are called **iterated integrals**.

Choosing order wisely!

Example 4:

$$\iint\limits_R x\mathbf{e}^{xy}\, dA,\ R = [-1, 2] \times [0, 1]$$

$$\begin{aligned}
\iint\limits_R x\mathbf{e}^{xy}\, dA &= \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \int \frac{1}{y}\mathbf{e}^{xy}\, dx \right)\Bigg|_{-1}^{2} dy \\
&= \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \frac{1}{y^2}\mathbf{e}^{xy} \right)\Bigg|_{-1}^{2} dy \\
&= \int_0^1 \left(\frac{2}{y}\mathbf{e}^{2y} - \frac{1}{y^2}\mathbf{e}^{2y} \right) - \left(-\frac{1}{y}\mathbf{e}^{-y} - \frac{1}{y^2}\mathbf{e}^{-y} \right) dy
\end{aligned}$$

Ex $\int_0^3 \int_1^2 x^2 y\, dy\, dx$

Ex $\iint\limits_R (x - 3y^2)\, dA$ where $R = \{(x,y) \mid 0 \le x \le 2, 1 \le y \le 2\}$

Ex $\iint\limits_R y \sin(xy)\, dA$ where $R = [1, 2] \times [0, \pi]$

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