

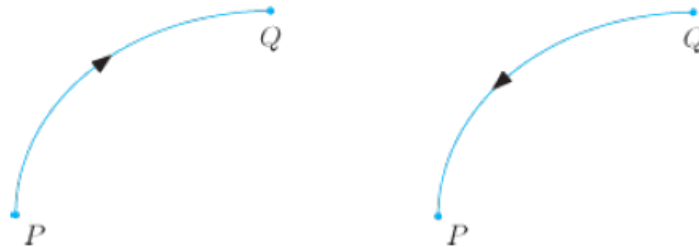
Line Integral, Green & Stoke Theorem

Friday, 28 June 2024 10:51 pm

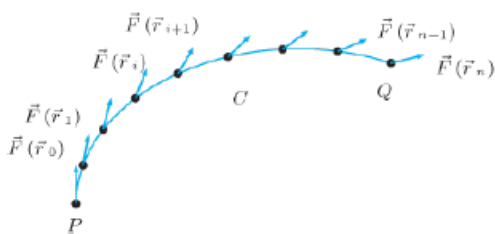
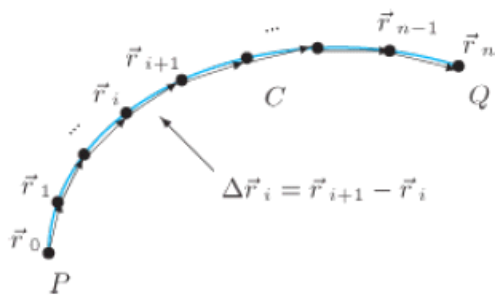
Orientation of a Curve

The concept of orientation of a curve is simple enough to understand. A curve can be traced out in one of two directions. Choosing one of those directions determines an *orientation* of the curve.

A curve is said to be *oriented* if we have chosen a direction of travel on it.



Definition of the Line Integral



$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i$$

The *line integral* of a vector field \vec{F} along an oriented curve C is

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\vec{r}_i\| \rightarrow 0} \sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i.$$

Work Done by $\vec{F} = \int_C \vec{F} \cdot d\vec{r}$.

If C is an oriented, closed curve, the line integral of a vector field F around C is called the *circulation* of \vec{F} around C .

We will often use the notation $\oint_C \vec{F} \cdot d\vec{r}$ to refer to circulations.

If $\vec{r}(t)$, for $a \leq t \leq b$, is a smooth parametrization of the oriented curve C and \vec{F} is a vector field that is continuous on C , then

$$(1) \quad \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt.$$

In words: To compute the line integral of \vec{F} over C , take the dot product of \vec{F} evaluated on C with the velocity vector, $\vec{r}'(t)$, of the parametrization C , then integrate along the curve.

EXAMPLE 2 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ along the curve C given by $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\mathbf{k}$, $0 \leq t \leq 1$.

EXAMPLE 3 Evaluate the line integral $\int_C -y dx + z dy + 2x dz$, where C is the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 2\pi$.

EXAMPLE 4 Find the work done by the force field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \leq t \leq 1$, from $(0, 0, 0)$ to $(1, 1, 1)$ (Figure 16.18).

EXAMPLE 7 Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$ (Figure 16.19).

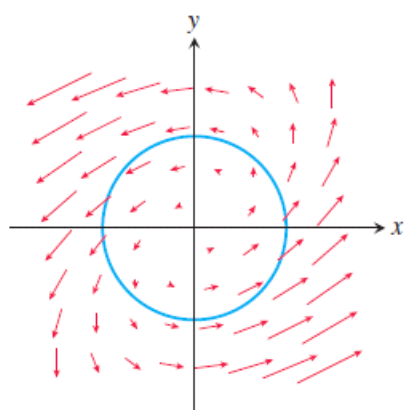


FIGURE 16.19 The vector field \mathbf{F} and curve $\mathbf{r}(t)$ in Example 7.

THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R . Then the counterclockwise circulation of \mathbf{F} around C equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over R .

$$\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \oint_C M \, dx + N \, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy \quad (3)$$

Counterclockwise circulation Curl integral

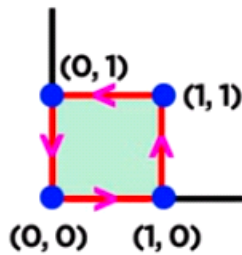
EXAMPLE 3 Verify both forms of Green's Theorem for the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region R bounded by the unit circle

$$C: \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi.$$

$$\int_C \mathbf{P} dx + \mathbf{Q} dy = \iint_D \left(\frac{\partial \mathbf{Q}}{\partial x} - \frac{\partial \mathbf{P}}{\partial y} \right) dx dy$$



$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

$$\vec{\mathbf{F}} = \langle xy, x^2 \rangle$$

THEOREM 6—Stokes' Theorem Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C . Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S . Then the circulation of \mathbf{F} around C in the direction counterclockwise with respect to the surface's unit normal vector \mathbf{n} equals the integral of the curl vector field $\nabla \times \mathbf{F}$ over S :

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma \quad (4)$$

Counterclockwise
circulation
Curl integral

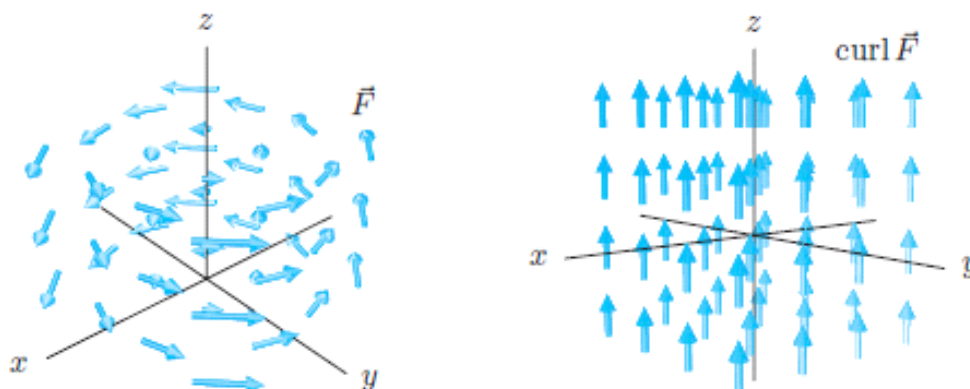


Figure 20.14: The vector fields $\vec{\mathbf{F}}$ and $\text{curl } \vec{\mathbf{F}}$

EXAMPLE 2 Evaluate Equation (4) for the hemisphere $S: x^2 + y^2 + z^2 = 9, z \geq 0$, its bounding circle $C: x^2 + y^2 = 9, z = 0$, and the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

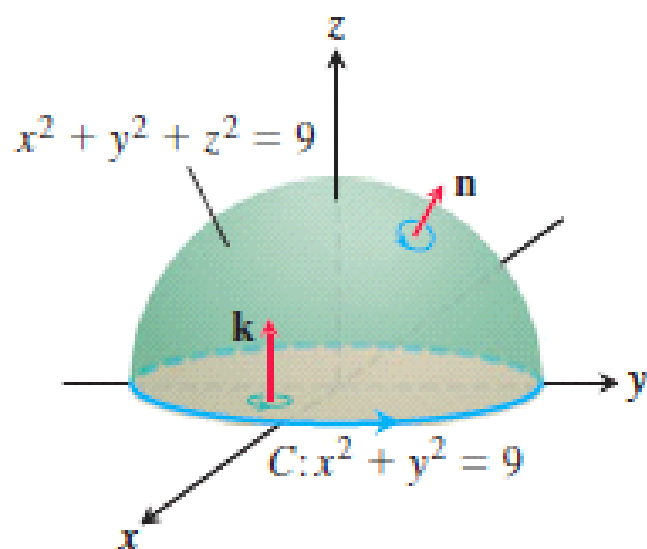



FIGURE 16.58 A hemisphere and a disk, each with boundary C (Examples 2 and 3).

$$\oint_C \vec{F} \cdot d\vec{P} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma$$

Example: $x^2 + y^2 + z^2 = 1, z \geq 0$
 $\vec{F} = y\hat{i} - x\hat{j} + z\hat{k}$



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