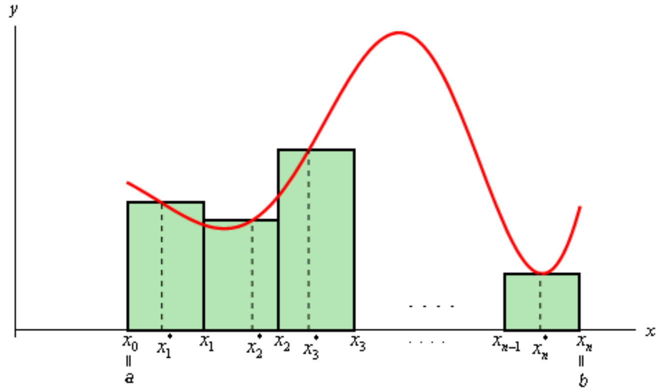
Double Integral:

$$\int_a^b f(x)\,dx$$



$$A \approx f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_i^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x$$

To get the exact area we then took the limit as n goes to infinity and this was also the definition of the definite integral.

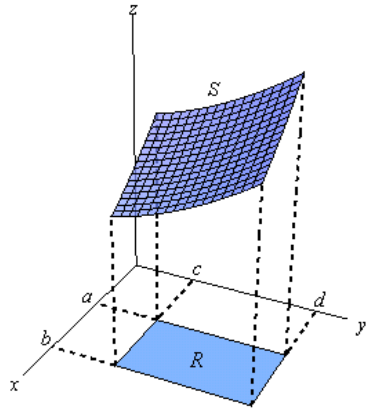
$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

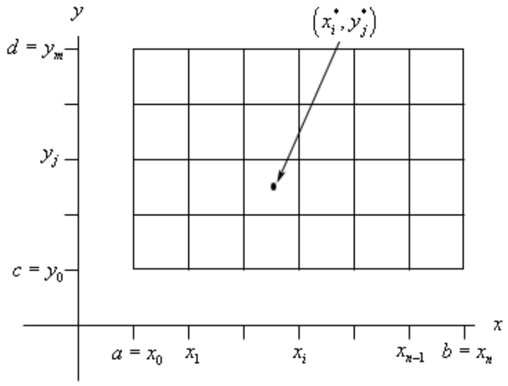
We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

$$R = [a,b] \times [c,d]$$

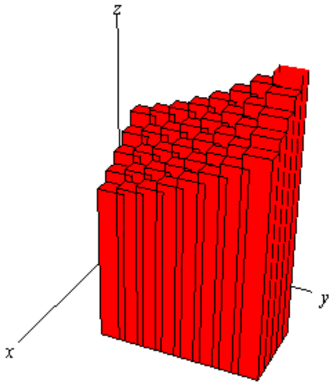
This means that the ranges for x and y are $a \le x \le b$ and $c \le y \le d$.

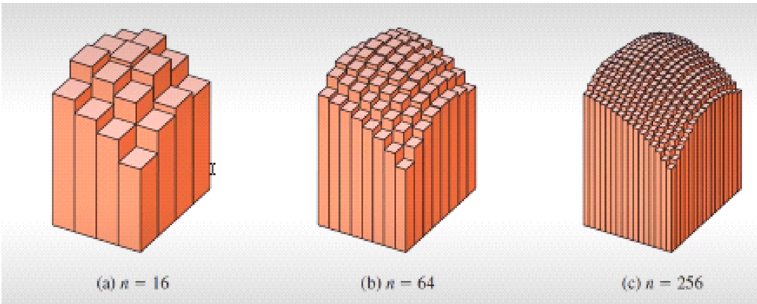
Also, we will initially assume that $f(x,y) \ge 0$ although this doesn't really have to be the case. Let's start out with the graph of the surface S given by graphing $f(x,y)$ over the rectangle R.





Now, over each of these smaller rectangles we will construct a box whose height is given by $f\left(x_i^*, y_j^*\right)$. Here is a sketch of that.





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$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A$$

$$V = \lim_{n,\ m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A$$

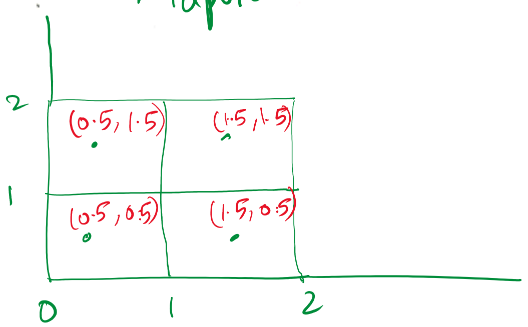
$$\iint_R f(x, y)\, dA = \lim_{n,\ m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A$$

$$\text{Volume} = \iint_R f(x, y)\, dA$$

The sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$ is called DOUBLE RIEMANN SUM and is used as an approximation to the value of double integral.

EXAMPLE: Estimate the volume of the solid that lies above the square $R = [0,2]\times[0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into 4 equal squares & chose the Sample Point to be the corner of each square R_{ij}.

Midpoint



$$\sum_{i=1}^{4} f(x_i)\,\Delta x$$

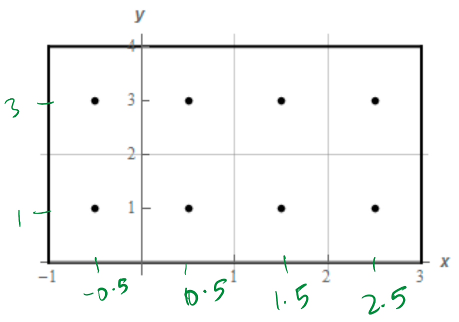
$\Delta x = 1$

$$\Rightarrow f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$\Rightarrow (16 - (0.5)^2 - 2(0.5)^2) \times 0.1 + (16 - (1.5)^2 - 2(0.5)^2) \times 0.1$$
$$+ (16 - (0.5)^2 - 2(1.5)^2) \times 0.1 + (16 - (1.5)^2 - 2(1.5)^2) \times 0.1$$

$\Rightarrow$

1. Use the Midpoint Rule to estimate the volume under $f(x,y) = x^2 + y$ and above the rectangle given by $-1 \le x \le 3$, $0 \le y \le 4$ in the xy-plane. Use 4 subdivisions in the x direction and 2 subdivisions in the y direction.



$(-0.5, 3), (0.5, 3), (1.5, 1), (2.5, 1)$

$(-0.5, 1)\ (0.5, 1), (1.5, 1), (2.5, 1)$

$$V = \iint_R f(x,y)\, dA$$

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{4}\sum_{j=1}^{2} f\left(\bar{x}_i, \bar{y}_j\right)\Delta A \qquad f(x,y) = x^2 + y$$

$\Delta A =$

$$V \approx \sum_{i=1}^{4}\sum_{j=1}^{2} 2f\left(\bar{x}_i, \bar{y}_j\right) \qquad f(x,y) = x^2 + y$$

$$i=1 \quad : \quad \sum_{j=1}^{2} 2f\left(\bar{x}_1, \bar{y}_j\right) = \sum_{j=1}^{2} 2f\left(-\frac{1}{2}, \bar{y}_j\right) = 2\left[f\left(-\frac{1}{2}, 1\right) + f\left(-\frac{1}{2}, 3\right)\right] = 9$$

$$i=2 \quad : \quad \sum_{j=1}^{2} 2f\left(\bar{x}_2, \bar{y}_j\right) = \sum_{j=1}^{2} 2f\left(\frac{1}{2}, \bar{y}_j\right) = 2\left[f\left(\frac{1}{2}, 1\right) + f\left(\frac{1}{2}, 3\right)\right] = 9$$

$$i=3 \quad : \quad \sum_{j=1}^{2} 2f\left(\bar{x}_3, \bar{y}_j\right) = \sum_{j=1}^{2} 2f\left(\frac{3}{2}, \bar{y}_j\right) = 2\left[f\left(\frac{3}{2}, 1\right) + f\left(\frac{3}{2}, 3\right)\right] = 17$$

$$i=4 \quad : \quad \sum_{j=1}^{2} 2f\left(\bar{x}_4, \bar{y}_j\right) = \sum_{j=1}^{2} 2f\left(\frac{5}{2}, \bar{y}_j\right) = 2\left[f\left(\frac{5}{2}, 1\right) + f\left(\frac{5}{2}, 3\right)\right] = 33$$

$$V \approx \sum_{i=1}^{4} \sum_{j=1}^{2} 2f\left(\bar{x}_i, \bar{y}_j\right) = 9 + 9 + 17 + 33 = \boxed{68}$$

For reference purposes we will eventually be able to verify that the exact volume is

$$\int_0^4 \int_{-1}^3 (x^2+y)\,dx\,dy \Rightarrow \int_0^4 \left(\frac{x^3}{3} + xy\right)\Big|_{-1}^{3} dy$$

$$\Rightarrow \int_0^4 \left(\left(\frac{3^3}{3} + 3y\right) - \left(\frac{(-1)^3}{3} + (-1)\,y\right)\right) dy \Rightarrow \int_0^4 \left(\frac{28}{3} + 4y\right) dy$$

$$\frac{28}{3}y + 4\frac{y^2}{2}\Big|_0^4 \Rightarrow \frac{28(4)}{3} + 4\frac{(4)^2}{2} = 69.333$$

Fubini's Theorem

If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$ then,

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

These integrals are called **iterated integrals**.

Choosing order wisely!

Example 4:

$$\iint\limits_R x\mathbf{e}^{xy}\,dA, \; R = [-1,2] \times [0,1]$$

$$\iint\limits_R x\mathbf{e}^{xy}\,dA = \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \int \frac{1}{y}\mathbf{e}^{xy}\,dx\right)\Big|_{-1}^{2} dy$$

$$= \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \frac{1}{y^2}\mathbf{e}^{xy}\right)\Big|_{-1}^{2} dy$$

$$= \int_0^1 \left(\frac{2}{y}\mathbf{e}^{2y} - \frac{1}{y^2}\mathbf{e}^{2y}\right) - \left(-\frac{1}{y}\mathbf{e}^{-y} - \frac{1}{y^2}\mathbf{e}^{-y}\right) dy$$

$$\int_{-1}^{2}\int_0^1 x e^{xy}\,dy\,dx$$

$$\int_{-1}^{2} x\left(\int_0^1 e^{xy}\,dy\right)dx$$

$$\Rightarrow \int_{-1}^{2} \cancel{x}\left(\frac{e^{xy}}{\cancel{x}}\right)\Big|_0^1 dx \Rightarrow \int_{-1}^{2} e^{xy}\Big|_0^1 dx$$

$$\Rightarrow \int_{-1}^{2} (e^x - 1)\,dx \Rightarrow e^x - x\Big|_{-1}^{2} \Rightarrow (e^2 - 2) - (e^{-1} - (-1))$$

Ex ① $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

Ex ② $\iint_R (x - 3y^2) \, dA$ where $R = \{(x,y) \mid 0 \le x \le 2, 1 \le y \le 2\}$

Ex $\iint_R y \sin(xy) \, dA$ where $R = [1,2] \times [0,\pi]$

Fact

If $f(x,y) = g(x)\,h(y)$ and we are integrating over the rectangle $R = [a,b] \times [c,d]$ then,

$$\iint_R f(x,y)\, dA = \iint_R g(x)\,h(y)\, dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$$

Let's do a quick example using this integral.

Example 2 Evaluate $\iint_R x\cos^2(y)\, dA$, $R = [-2,3] \times \left[0, \frac{\pi}{2}\right]$.

Hide Solution ▾

Since the integrand is a function of x times a function of y we can use the fact.

$$\begin{aligned}\iint_R x\cos^2(y)\, dA &= \left(\int_{-2}^{3} x\, dx\right)\left(\int_0^{\frac{\pi}{2}} \cos^2(y)\, dy\right)\\ &= \left(\frac{1}{2}x^2\right)\Big|_{-2}^{3}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos(2y)\, dy\right)\\ &= \left(\frac{5}{2}\right)\left(\frac{1}{2}\left(y + \frac{1}{2}\sin(2y)\right)\Big|_0^{\frac{\pi}{2}}\right)\\ &= \frac{5\pi}{8}\end{aligned}$$