Cal2(Directional Derivative & Definite Double Integral)

Monday, 3 June 2024 12:07 pm

14.4 GRADIENTS AND DIRECTIONAL DERIVATIVES IN THE PLANE

The Rate of Change in an Arbitrary Direction: The Directional Derivative

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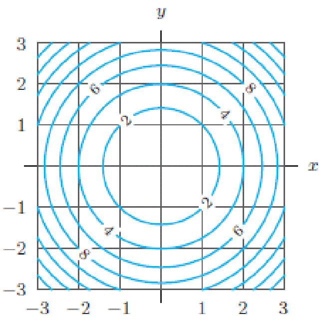

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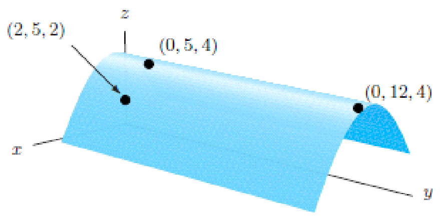

Figure 14.36

Example 2 For each of the functions f, g, and h in Figure 14.30, decide whether the directional derivative at the indicated point is positive, negative, or zero, in the direction of the vector $\vec{v} = \vec{i} + 2\vec{j}$, and in the direction of the vector $\vec{w} = 2\vec{i} + \vec{j}$.

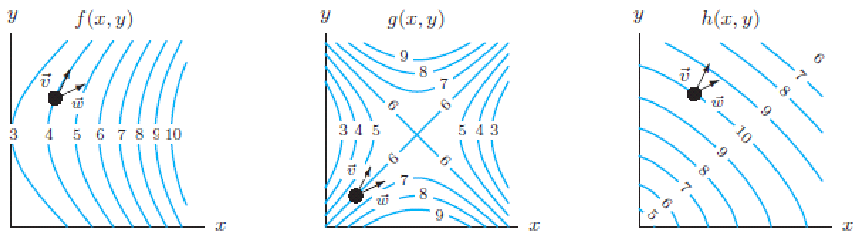

Figure 14.30: Contour diagrams of three functions with direction vectors $\vec{v} = \vec{i} + 2\vec{j}$ and $\vec{w} = 2\vec{i} + \vec{j}$ marked on each

The Gradient Vector of a differentiable function f at the point (a, b) is

$$\operatorname{grad} f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$$

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The Directional Derivative and the Gradient

If f is differentiable at (a,b) and $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, then

$$f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = \operatorname{grad} f(a,b)\cdot\vec{u}\,.$$

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Example 3 Calculate the directional derivative of $f(x,y) = x^2+y^2$ at $(1,0)$ in the direction of the vector $\vec{i}+\vec{j}$.

Solution First we have to find the unit vector in the same direction as the vector $\vec{i}+\vec{j}$. Since this vector has magnitude $\sqrt{2}$, the unit vector is

$$\vec{u} = \frac{1}{\sqrt{2}}(\vec{i}+\vec{j}) = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\,.$$

Thus,

$$f_{\vec{u}}(1,0) = \lim_{h\to 0}\frac{f(1+h/\sqrt{2}, h/\sqrt{2}) - f(1,0)}{h} = \lim_{h\to 0}\frac{(1+h/\sqrt{2})^2 + (h/\sqrt{2})^2 - 1}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{2}h + h^2}{h} = \lim_{h\to 0}(\sqrt{2}+h) = \sqrt{2}.$$

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Example 5 Find the gradient vector of $f(x,y) = x + e^y$ at the point $(1,1)$.

Solution Using the definition, we have

$$\operatorname{grad} f = f_x\vec{i} + f_y\vec{j} = \vec{i} + e^y\vec{j}\,,$$

so at the point $(1,1)$

$$\operatorname{grad} f(1,1) = \vec{i} + e\vec{j}\,.$$

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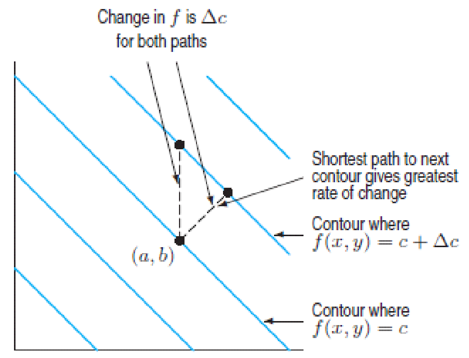

Figure 14.32: Close-up view of the contours around (a,b), showing the gradient is perpendicular to the contours

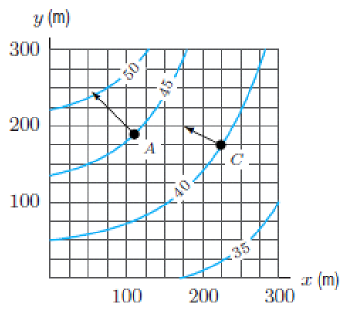

Figure 14.33: A temperature map showing directions and relative magnitudes of two gradient vectors

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Example 7 Use the gradient to find the directional derivative of $f(x,y) = x + e^y$ at the point $(1,1)$ in the direction of the vectors $\vec{i} - \vec{j}$, $\vec{i} + 2\vec{j}$, $\vec{i} + 3\vec{j}$.

Solution In Example 5 we found

$$\text{grad}\, f(1,1) = \vec{i} + e\vec{j}\,.$$

A unit vector in the direction of $\vec{i} - \vec{j}$ is $\vec{s} = (\vec{i} - \vec{j})/\sqrt{2}$, so

$$f_{\vec{s}}(1,1) = \text{grad}\, f(1,1) \cdot \vec{s} = (\vec{i} + e\vec{j}) \cdot \left(\frac{\vec{i} - \vec{j}}{\sqrt{2}}\right) = \frac{1-e}{\sqrt{2}} \approx -1.215.$$

A unit vector in the direction of $\vec{i} + 2\vec{j}$ is $\vec{v} = (\vec{i} + 2\vec{j})/\sqrt{5}$, so

$$f_{\vec{v}}(1,1) = \text{grad}\, f(1,1) \cdot \vec{v} = (\vec{i} + e\vec{j}) \cdot \left(\frac{\vec{i} + 2\vec{j}}{\sqrt{5}}\right) = \frac{1+2e}{\sqrt{5}} \approx 2.879.$$

A unit vector in the direction of $\vec{i} + 3\vec{j}$ is $\vec{w} = (\vec{i} + 3\vec{j})/\sqrt{10}$, so

$$f_{\vec{w}}(1,1) = \text{grad}\, f(1,1) \cdot \vec{w} = (\vec{i} + e\vec{j}) \cdot \left(\frac{\vec{i} + 3\vec{j}}{\sqrt{10}}\right) = \frac{1+3e}{\sqrt{10}} \approx 2.895.$$

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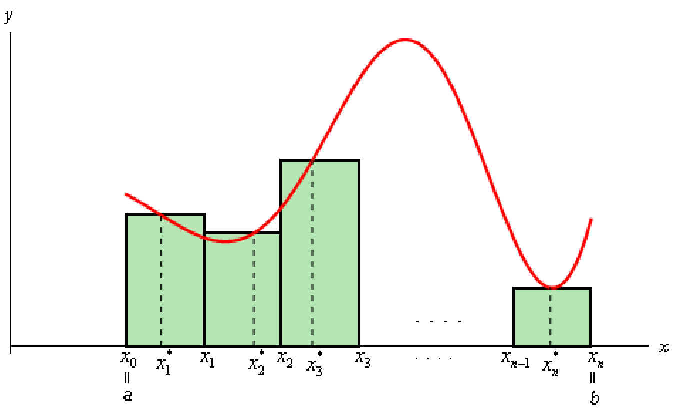
Double Integral:

$$\int_a^b f(x)\, dx$$



$$A \approx f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_i^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x$$

To get the exact area we then took the limit as n goes to infinity and this was also the definition of the definite integral.

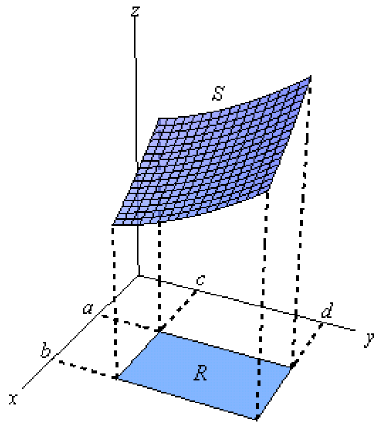
$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

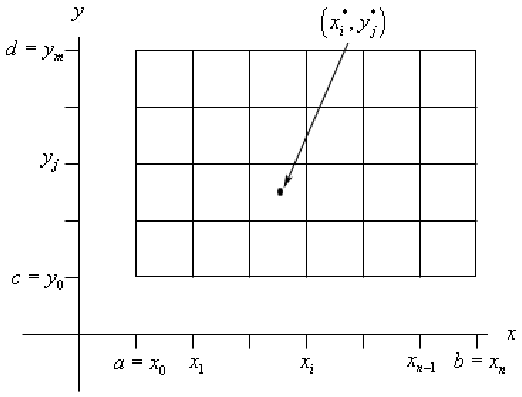
We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

$$R = [a,b] \times [c,d]$$

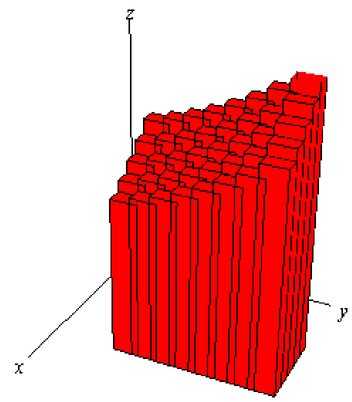
This means that the ranges for x and y are $a \le x \le b$ and $c \le y \le d$.

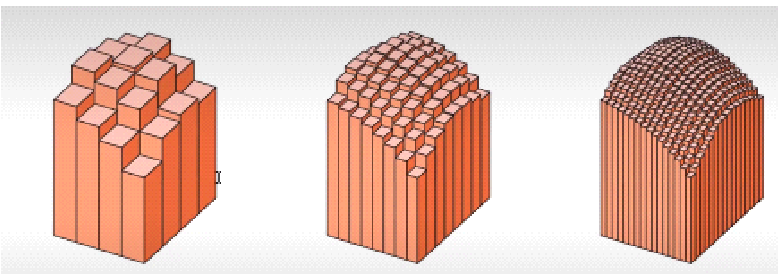
Also, we will initially assume that $f(x,y) \geq 0$ although this doesn't really have to be the case. Let's start out with the graph of the surface S given by graphing $f(x,y)$ over the rectangle R.





Now, over each of these smaller rectangles we will construct a box whose height is given by $f\left(x_i^*, y_j^*\right)$. Here is a sketch of that.




(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

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$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\,\Delta A$$

$$V = \lim_{n,\ m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\,\Delta A$$

$$\iint\limits_R f(x,y)\,dA = \lim_{n,\ m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\,\Delta A$$

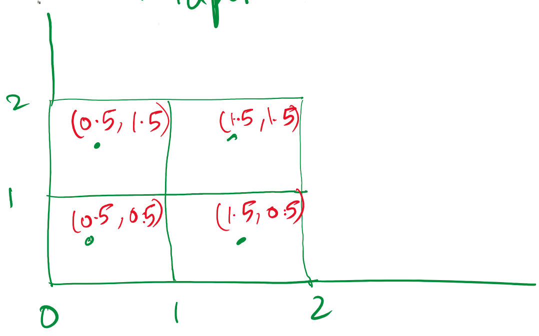
$$\text{Volume} = \iint\limits_R f(x,y)\,dA$$

The sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$ is called an DOUBLE RIEMANN SUM and is used as approximation to the value of double integral.

EXAMPLE: Estimate the volume of the solid that lies above the square $R = [0,2]\times[0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into 4 equal squares & chose the Sample Point to be the corner of each square R_{ij}.

Midpoint

$$\sum_{i=1}^{n} f(x_i)\,\Delta x$$



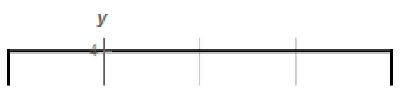
$$\Rightarrow f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

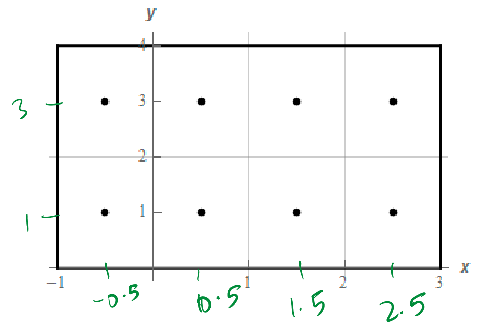
$\Delta x = 1$

$$\Rightarrow (16-(0.5)^2-2(0.5)^2)\times 0.1 + (16-(1.5)^2-2(0.5)^2)\times 0.1 + (16-(0.5)^2-2(1.5)^2)\times 0.1 + (16-(1.5)^2-2(1.5)^2)\times 0.1$$

$\Rightarrow$

1. Use the Midpoint Rule to estimate the volume under $f(x,y) = x^2 + y$ and above the rectangle given by $-1 \le x \le 3$, $0 \le y \le 4$ in the xy-plane. Use 4 subdivisions in the x direction and 2 subdivisions in the y direction.





(-0.5, 3), (0.5,3), (1.5, 1), (2.5, 1)

(-0.5, 1) (0.5,1), (1.5, 1), (2.5,1)

$$V = \iint\limits_R f(x,y)\,dA$$

$$\iint\limits_R f(x,y)\,dA \approx \sum_{i=1}^{4}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\,\Delta A \qquad f(x,y) = x^2 + y$$

$$\Delta A =$$

$$V \approx \sum_{i=1}^{4}\sum_{j=1}^{2} 2f\left(\overline{x}_i, \overline{y}_j\right) \qquad f(x,y) = x^2 + y$$

$$i = 1 \;:\; \sum_{j=1}^{2} 2f\left(\overline{x}_1, \overline{y}_j\right) = \sum_{j=1}^{2} 2f\left(-\frac{1}{2}, \overline{y}_j\right) = 2\left[f\left(-\frac{1}{2}, 1\right) + f\left(-\frac{1}{2}, 3\right)\right] = 9$$

$$i = 2 \;:\; \sum_{j=1}^{2} 2f\left(\overline{x}_2, \overline{y}_j\right) = \sum_{j=1}^{2} 2f\left(\frac{1}{2}, \overline{y}_j\right) = 2\left[f\left(\frac{1}{2}, 1\right) + f\left(\frac{1}{2}, 3\right)\right] = 9$$

$$i = 3 \;:\; \sum_{j=1}^{2} 2f\left(\overline{x}_3, \overline{y}_j\right) = \sum_{j=1}^{2} 2f\left(\frac{3}{2}, \overline{y}_j\right) = 2\left[f\left(\frac{3}{2}, 1\right) + f\left(\frac{3}{2}, 3\right)\right] = 17$$

$$i = 4 \;:\; \sum_{j=1}^{2} 2f\left(\overline{x}_4, \overline{y}_j\right) = \sum_{j=1}^{2} 2f\left(\frac{5}{2}, \overline{y}_j\right) = 2\left[f\left(\frac{5}{2}, 1\right) + f\left(\frac{5}{2}, 3\right)\right] = 33$$

$$V \approx \sum_{i=1}^{4}\sum_{j=1}^{2} 2f\left(\overline{x}_i, \overline{y}_j\right) = 9 + 9 + 17 + 33 = \boxed{68}$$

For reference purposes we will eventually be able to verify that the exact volume is

$$\int_0^4\int_{-1}^{3} (x^2+y)\,dx\,dy \Rightarrow \int_0^4 \left(\frac{x^3}{3}+xy\right)\Big|_{-1}^{3} dy$$

$$\Rightarrow \int_0^4 \left(\left(\frac{3^3}{3}+3y\right) - \left(\frac{(-1)^3}{3}+(-1)\,y\right)\right)dy \Rightarrow \int_0^4 \left(\frac{28}{3}+4y\right)dy$$

$$\frac{28}{3}y + 4\frac{y^2}{2}\Big|_0^4 \Rightarrow \frac{28(4)}{3} + 4\frac{(4)^2}{2} = 69.333$$

Fubini's Theorem

If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$ then,

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

These integrals are called **iterated integrals**.

Choosing order wisely!

Example 4:

$$\iint_R x\mathbf{e}^{xy}\, dA,\ R = [-1, 2] \times [0, 1]$$

$$\iint_R x\mathbf{e}^{xy}\, dA = \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \int \frac{1}{y}\mathbf{e}^{xy}\, dx\right)\Bigg|_{-1}^{2} dy$$

$$= \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \frac{1}{y^2}\mathbf{e}^{xy}\right)\Bigg|_{-1}^{2} dy$$

$$= \int_0^1 \left(\frac{2}{y}\mathbf{e}^{2y} - \frac{1}{y^2}\mathbf{e}^{2y}\right) - \left(-\frac{1}{y}\mathbf{e}^{-y} - \frac{1}{y^2}\mathbf{e}^{-y}\right) dy$$

$$\int_{-1}^{2}\int_0^1 x e^{xy}\, dy\, dx$$

$$\int_{-1}^{2} x\left(\int_0^1 e^{xy}\, dy\right) dx$$

$$\Rightarrow \int_{-1}^{2} \cancel{x}\left(\frac{e^{xy}}{\cancel{x}}\right)\Bigg|_0^1 dx \Rightarrow \int_{-1}^{2} e^{xy}\Big|_0^1 dx$$

$$\Rightarrow \int_{-1}^{2} (e^x - 1)\, dx \Rightarrow e^x - x\Big|_{-1}^{2} \Rightarrow (e^2 - 2) - (e^{-1} - (-1))$$

Ex $\int_0^3\int_1^2 x^2 y\, dy\, dx$

Ex $\iint_R (x - 3y^2)\, dA$ where $R = \{(x,y) \mid 0 \le x \le 2, 1 \le y \le 2\}$

Ex $\iint_R y \sin(xy)\, dA$ where $R = [1, 2] \times [0, \pi]$

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