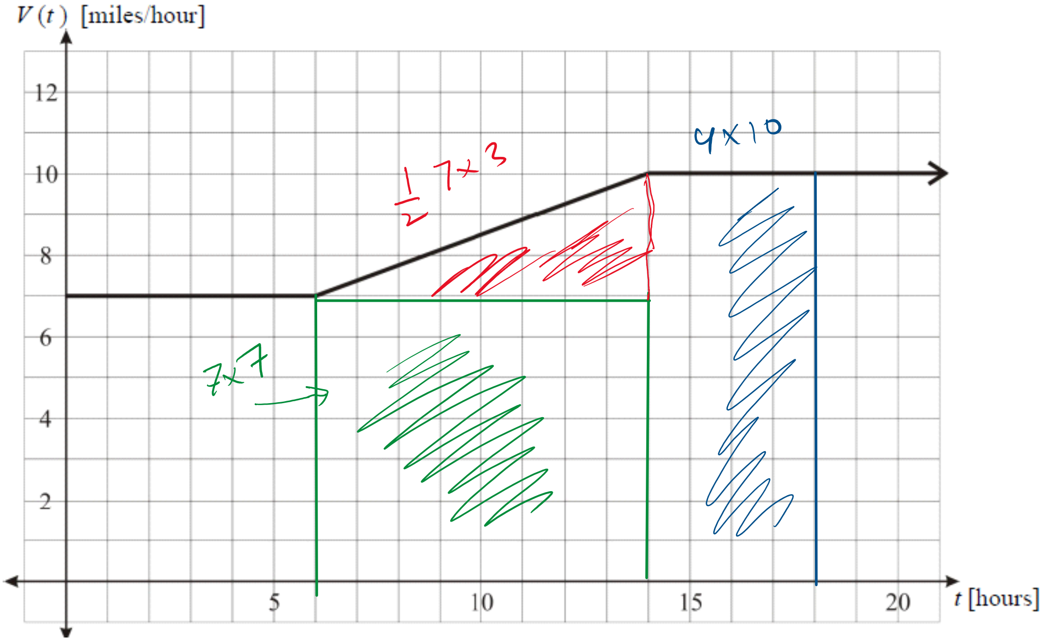
Cal1(Definite Integral)

Monday, 3 June 2024 12:07 pm

Area under a Curve

Below is a hypothetical graph of the velocity of a tractor traveling on a straight road. Find the distance traveled by the tractor between $t = 6$ and $t = 18$.



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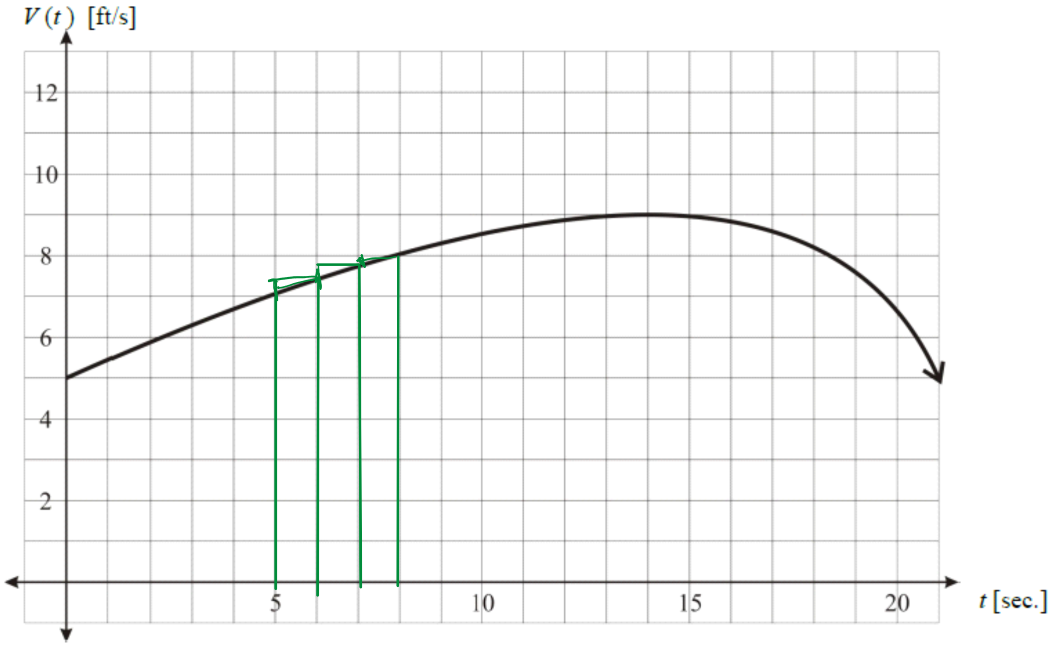
Riemann Sum

Next, let's consider the velocity function of a Victorian horse-carriage (graphed below). It is hard to envision being able to get an exact area, judging from the shape of the curve, but you can surely get an approximation. Use your creativity with various shapes and find the area under the curve over the interval [8, 20].



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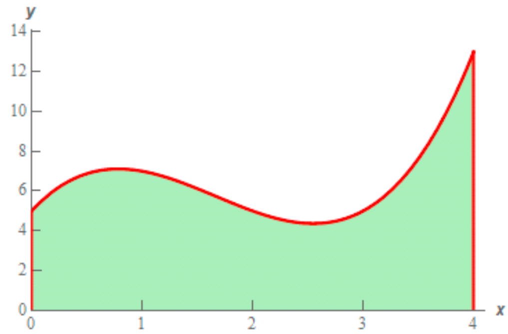
Example 1 Estimate the area between $f(x) = x^3 - 5x^2 + 6x + 5$ and the x-axis on $[0, 4]$ using $n = 5$ subintervals and all three cases above for the heights of each rectangle.

Hide Solution

First, let's get the graph to make sure that the function is positive.



So, the graph is positive and the width of each subinterval will be,

$$\Delta x = \frac{4}{5} = 0.8$$

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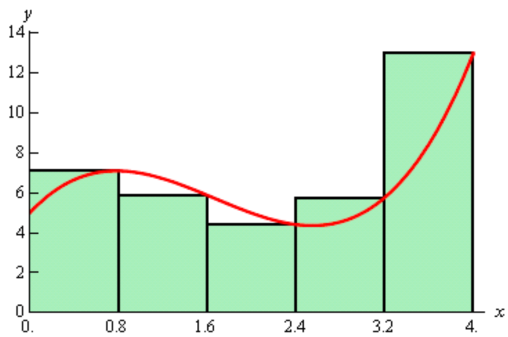
Let's first look at using the right endpoints for the function height. Here is the graph for this case.



Notice, that unlike the first area we looked at, the choosing the right endpoints here will both over and underestimate the area depending on where we are on the curve. This will often be the case with a more general curve that the one we initially looked at. The area estimation using the right endpoints of each interval for the rectangle height is,

$$\begin{aligned} A_r &= 0.8f(0.8) + 0.8f(1.6) + 0.8f(2.4) + 0.8f(3.2) + 0.8f(4) \\ &= 28.96 \end{aligned}$$

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Definite Integral

Given a function $f(x)$ that is continuous on the interval $[a, b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of $f(x)$ from a to b** is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

$$\int_0^2 x^2 + 1\, dx \Rightarrow \frac{x^3}{3} + x \Big|_0^2 \Rightarrow \frac{8}{3} + 2$$

Properties

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\int_a^a f(x)\,dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$. We can break up definite integrals across a sum or difference.

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5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a, c]$ and $[c, b]$. Note however that c doesn't need to be between a and b.

6. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$. The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

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Example 2 Use the results from the first example to evaluate each of the following.

(a) $\int_2^0 x^2 + 1\,dx$

(b) $\int_0^2 10x^2 + 10\,dx$

(c) $\int_0^2 t^2 + 1\,dt$

(a) $\int_2^0 x^2 + 1\,dx$ *Hide Solution* ▾

In this case the only difference between the two is that the limits have interchanged. So, using the first property gives,

$$\begin{aligned}\int_2^0 x^2 + 1\,dx &= -\int_0^2 x^2 + 1\,dx \\ &= -\frac{14}{3}\end{aligned}$$

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(b) $\displaystyle\int_0^2 10x^2 + 10\,dx$ *Hide Solution* ▼

For this part notice that we can factor a 10 out of both terms and then out of the integral using the third property.

$$\begin{aligned}\int_0^2 10x^2 + 10\,dx &= \int_0^2 10\left(x^2+1\right)\,dx \\ &= 10\int_0^2 x^2+1\,dx \\ &= 10\left(\frac{14}{3}\right) \\ &= \frac{140}{3}\end{aligned}$$

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(c) $\displaystyle\int_0^2 t^2 + 1\,dt$ *Hide Solution* ▼

In this case the only difference is the letter used and so this is just going to use property 6.

$$\int_0^2 t^2+1\,dt = \int_0^2 x^2+1\,dx = \frac{14}{3}$$

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Example 3 Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2+1}\,dx$$

Hide Solution ▼

There really isn't anything to do with this integral once we notice that the limits are the same. Using the second property this is,

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2+1}\,dx = 0$$

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Example 4 Given that $\int_{6}^{-10} f(x)\, dx = 23$ and $\int_{-10}^{6} g(x)\, dx = -9$ determine the value of

$$\int_{-10}^{6} 2f(x) - 10g(x)\, dx$$

Hide Solution ▾

We will first need to use the fourth property to break up the integral and the third property to factor out the constants.

$$\begin{aligned}\int_{-10}^{6} 2f(x) - 10g(x)\, dx &= \int_{-10}^{6} 2f(x)\, dx - \int_{-10}^{6} 10g(x)\, dx \\ &= 2\int_{-10}^{6} f(x)\, dx - 10\int_{-10}^{6} g(x)\, dx\end{aligned}$$

Now notice that the limits on the first integral are interchanged with the limits on the given integral so switch them using the first property above (and adding a minus sign of course). Once this is done we can plug in the known values of the integrals.

$$\begin{aligned}\int_{-10}^{6} 2f(x) - 10g(x)\, dx &= -2\int_{6}^{-10} f(x)\, dx - 10\int_{-10}^{6} g(x)\, dx \\ &= -2(23) - 10(-9) \\ &= 44\end{aligned}$$

edu/Classes/CalcI/ComputingDefiniteIntegrals.aspx

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More Properties

7. $\int_{a}^{b} c\, dx = c(b-a)$, c is any number.

8. If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_{a}^{b} f(x)\, dx \geq 0$.

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Heart & Soul of Calculus

Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a,b]$ then,

$$g(x) = \int_{a}^{x} f(t)\ dt$$

is continuous on $[a,b]$ and it is differentiable on (a,b) and,

$$g'(x) = f(x)$$

An alternate notation for the derivative portion of this is,

$$\frac{d}{dx}\int_{a}^{x} f(t)\ dt = f(x)$$

(a) $g(x) = \int_{-4}^{x} \mathbf{e}^{2t}\cos^2(1-5t)\ dt$ *Hide Solution* ▾

This one is nothing more than a quick application of the Fundamental Theorem of Calculus.

$$g'(x) = \mathbf{e}^{2x}\cos^2(1-5x)$$

Section 5.7 : Computing Definite Integrals

In this section we are going to concentrate on how we actually evaluate definite integrals in practice. To do this we will need the Fundamental Theorem of Calculus, Part II.

Fundamental Theorem of Calculus, Part II

Suppose $f(x)$ is a continuous function on $[a, b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

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