Triple Integration: Cartesian

Thursday, 20 June 2024   4:13 pm

one integral

gives a ~~two dimensional~~

two integrals

gives a ~~three dimensional~~ ...

three integrals

gives a ~~four dimensional~~ ...

four integrals

Section 16.3: Triple Integrals

A function of two-variables is integrated over a two-dimensional region $\mathbb{R}^2$. A function of three variables, then, will be integrated over a three-dimensional solid in $\mathbb{R}^3$. We will begin by considering the case where we are integrating a function $f(x, y, z)$ over a rectangular prism W.

We first slice the box up into subdivisions with volume $\Delta V = \Delta x \, \Delta y \, \Delta z$
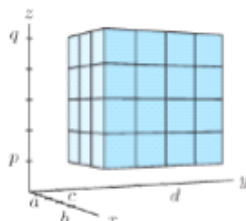
We pick a point $(u_{ijk}, v_{ijk}, w_{ijk})$ in the ijk-th box, and we form the sum

$$\sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk}) \, \Delta V.$$

Then, much as before, we take the limit as Δx, Δy and $\Delta z \to 0$. If f is continuous, the sum converges to the *triple integral* of f over W:

$$\int_W f \, dV = \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk}) \, \Delta x \, \Delta y \, \Delta z.$$

TRIPLE INTEGRAL AS AN ITERATED INTEGRAL :

$$\int_W f \, dV = \int_p^q \left(\int_c^d \left(\int_a^b f(x, y, z) \, dx \right) dy \right) dz,$$

where y and z are treated as constants in the innermost (dx) integral, and z is treated as a constant in the middle (dy) integral. Other orders of integration are possible.
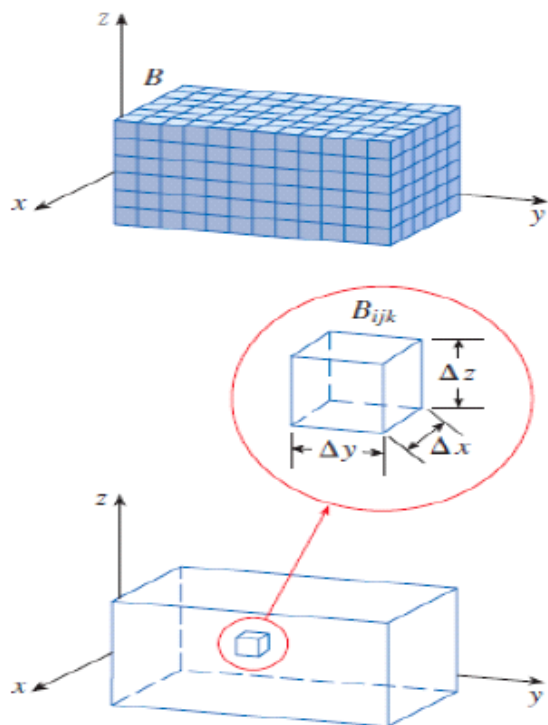
Triple Integrals

Fubini's Theorem

Let $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

Note:

There are five other possible orders in which we can integrate, all of which give the same value, that is,

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_a^b \int_c^d f(x, y, z)\, dy\, dx\, dz.$$

Example 1   A cube C has sides of length 4 cm and is made of a material of variable density. If one corner is at the origin and the adjacent corners are on the positive x, y, and z axes, then the density at the point (x, y, z) is $\delta(x, y, z) = 1 + xyz$ gm/cm^3. Find the mass of the cube.

2. $h(x, y, z) = ax + by + cz$, W is the rectangular box
$0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 2$.

Limits on Triple Integrals

- The limits for the outer integral are constants.
- The limits for the middle integral can involve only one variable (that in the outer integral).
- The limits for the inner integral can involve two variables (those on the two outer integrals).

Example: Set up an iterated integral to compute mass of a solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, if the density is given by $\delta(x, y, z) = z$.
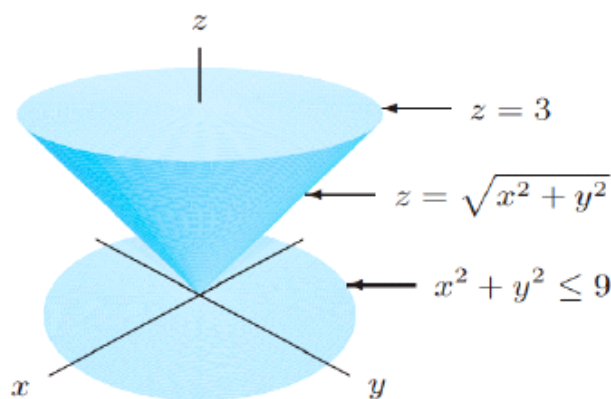

Figure 16.25