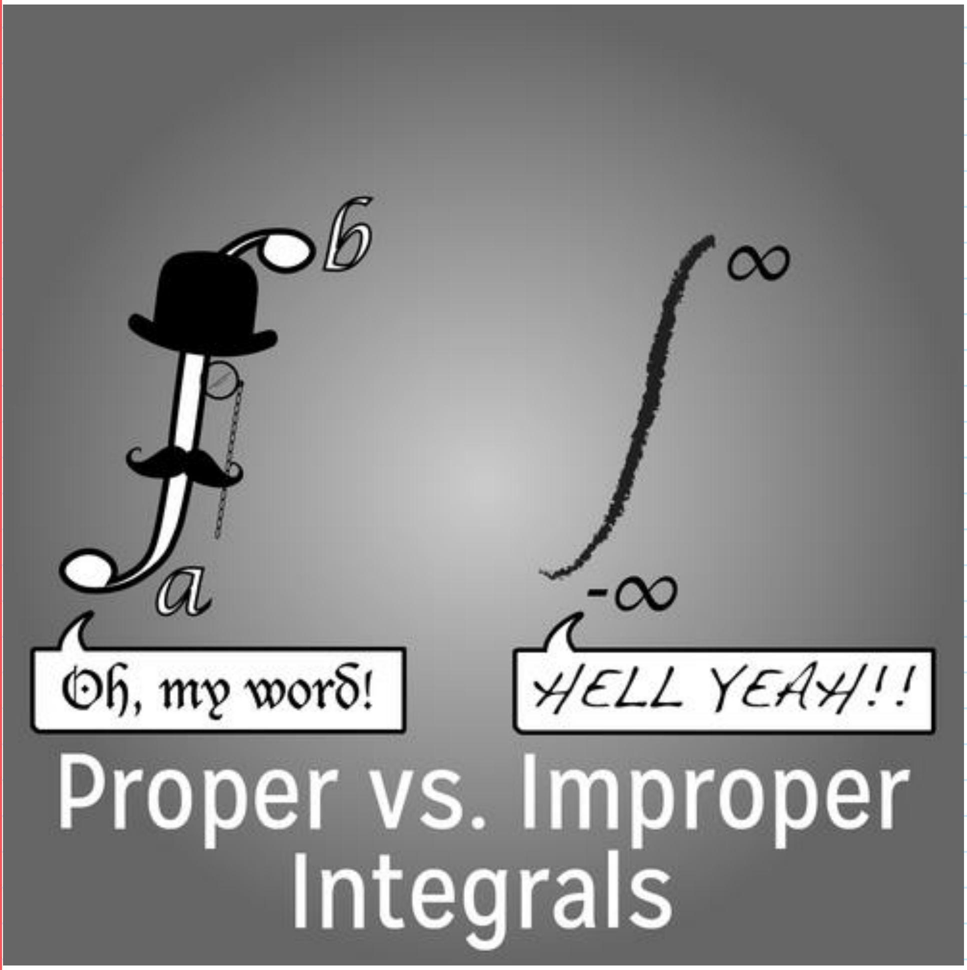
Improper Integral

Sunday, 1 June 2025 5:00 pm



Indefinite Integral	Proper Definite Integral	Improper Definite Integral
$\int e^{-t}\,dt$	$\int_0^1 e^{-t}\,dt$	$\int_0^\infty e^{-t}\,dt$

7.8 Improper Integrals

Definition:

The definite integral $\int_a^b f(x)dx$ is called an improper integral if

(a) At least one of the limits of integration is infinite, or

(b) The integrand $f(x)$ has one or more points of discontinuity on the interval $[a, b]$.

WHAT IS AN IMPROPER INTEGRAL?

- AN INTEGRAL WITH UPPER AND LOWER LIMITS THAT GO TO INFINITY IN ONE DIRECTION OR BOTH

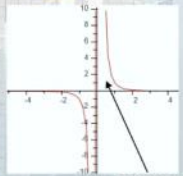
$$\int_{-2}^{\infty} \sin x \, dx$$

$$\int_{-\infty}^{\infty} x e^{-x^2} \, dx$$

$$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}} \, dx$$

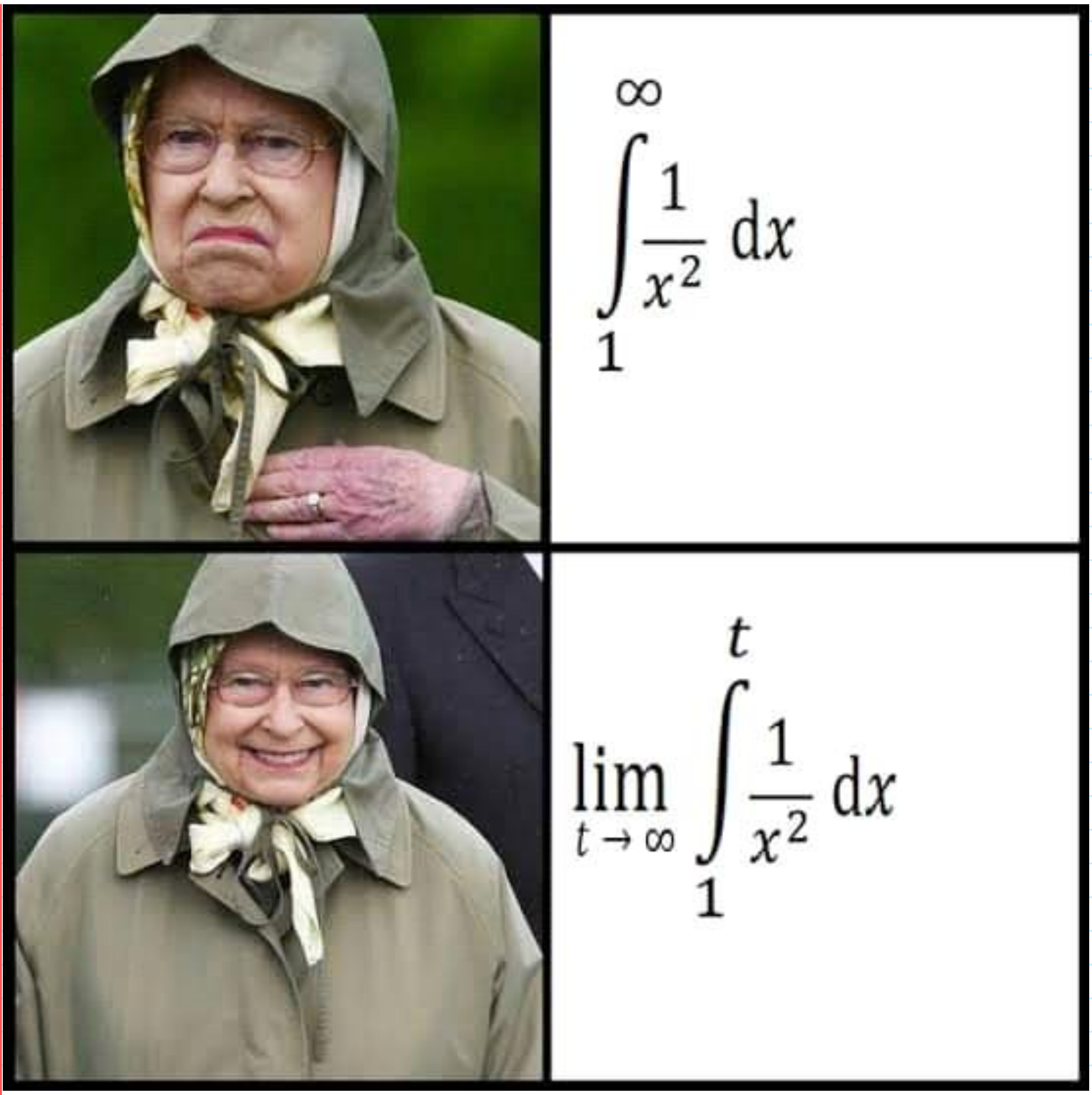
- TYPE II: LOOKS NORMAL BUT CANNOT BE EVALUATED WITH FTC II BECAUSE OF A DISCONTINUITY.

$$\int_{-2}^{3} \frac{1}{x^3} \, dx$$



VERTICAL ASYMPTOTE

How do we solve this?



Type 1: Infinite Intervals

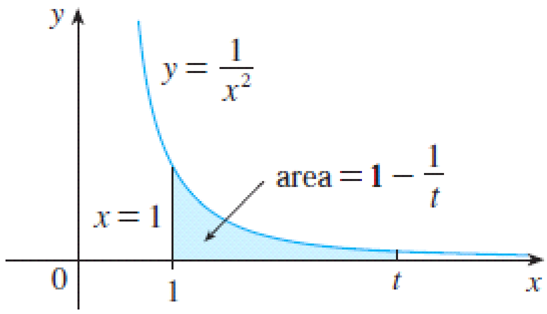
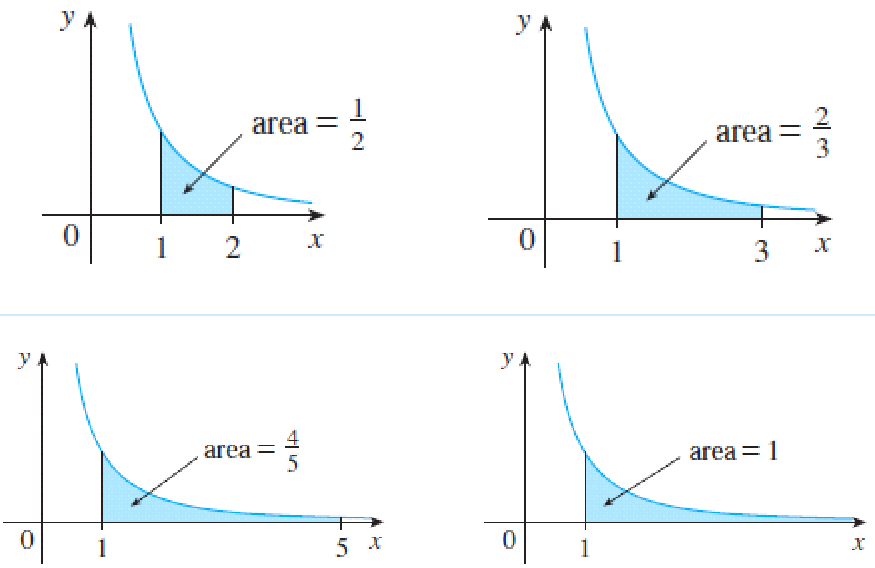


FIGURE 1

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$$



1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 74).

V EXAMPLE 1 Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

SOLUTION According to part (a) of Definition 1, we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty}(\ln t - \ln 1) = \lim_{t\to\infty}\ln t = \infty$$

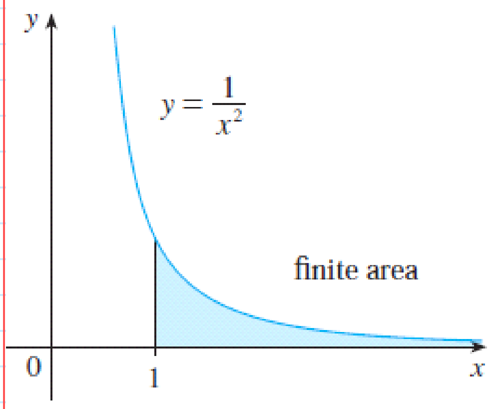


FIGURE 4 $\int_1^\infty (1/x^2)\,dx$ converges

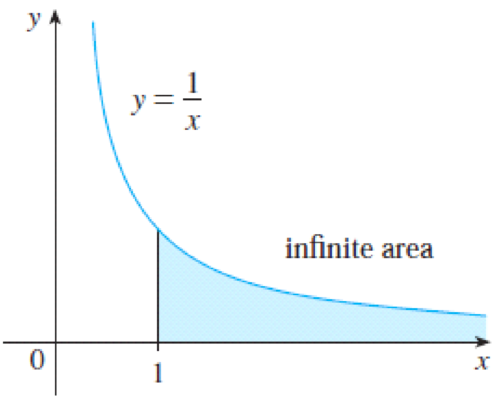


FIGURE 5 $\int_1^\infty (1/x)\,dx$ diverges

EXAMPLE 2 Evaluate $\int_{-\infty}^{0} xe^x\,dx$.

SOLUTION Using part (b) of Definition 1, we have

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$$

We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$:

$$\int_t^0 xe^x\,dx = xe^x\Big]_t^0 - \int_t^0 e^x\,dx$$

$$= -te^t - 1 + e^t$$

We know that $e^t \to 0$ as $t \to -\infty$, and by l'Hospital's Rule we have

$$\begin{aligned}\lim_{t\to-\infty} te^t &= \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{-e^{-t}} \\ &= \lim_{t\to-\infty} (-e^t) = 0\end{aligned}$$

Therefore

$$\begin{aligned}\int_{-\infty}^{0} xe^x\,dx &= \lim_{t\to-\infty} (-te^t - 1 + e^t) \\ &= -0 - 1 + 0 = -1\end{aligned}$$

EXAMPLE 3 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

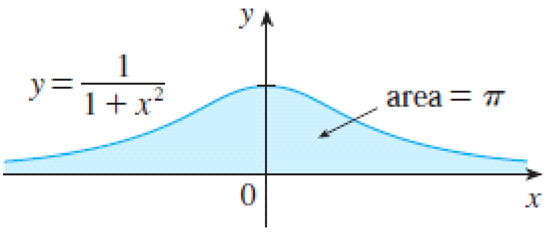


FIGURE 6

SOLUTION It's convenient to choose $a = 0$ in Definition 1(c):

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

We must now evaluate the integrals on the right side separately:

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{t\to\infty} \int_{0}^{t} \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1}x\Big]_0^t \\ &= \lim_{t\to\infty} (\tan^{-1}t - \tan^{-1}0) = \lim_{t\to\infty} \tan^{-1}t = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{1+x^2} = \lim_{t\to-\infty} \tan^{-1}x\Big]_t^0 \\ &= \lim_{t\to-\infty} (\tan^{-1}0 - \tan^{-1}t) \\ &= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Since $1/(1+x^2) > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = 1/(1+x^2)$ and above the x-axis (see Figure 6).

EXAMPLE 4 For what values of p is the integral

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

convergent?

2 $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Type 2: Discontinuous Integrands

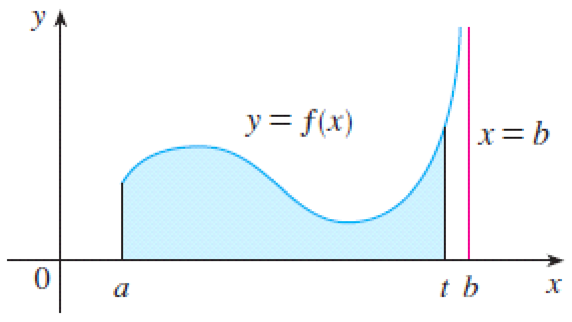


FIGURE 7

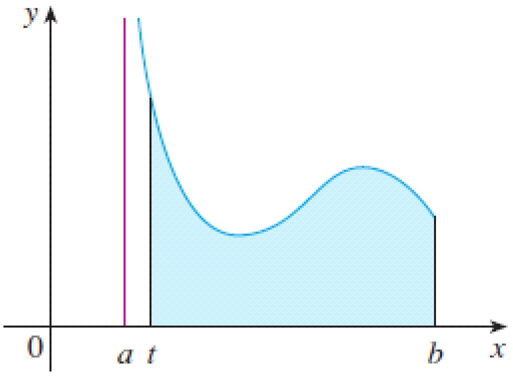


FIGURE 8

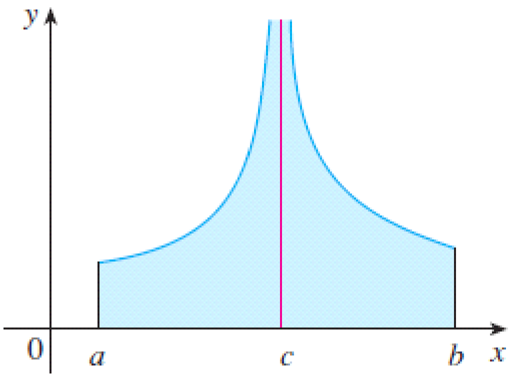


FIGURE 9

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

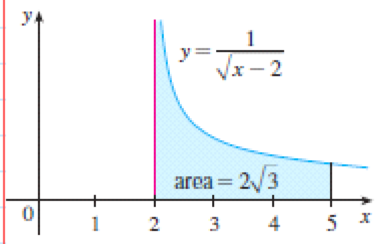
if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

EXAMPLE 5 Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.



SOLUTION We note first that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$. Since the infinite discontinuity occurs at the left endpoint of $[2, 5]$, we use part (b) of Definition 3:

$$\begin{aligned}\int_2^5 \frac{dx}{\sqrt{x-2}} &= \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}}\\ &= \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5\\ &= \lim_{t\to 2^+} 2\big(\sqrt{3} - \sqrt{t-2}\big)\\ &= 2\sqrt{3}\end{aligned}$$

EXAMPLE 7 Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\,\Big]_0^3 = \ln 2 - \ln 1 = \ln 2$$

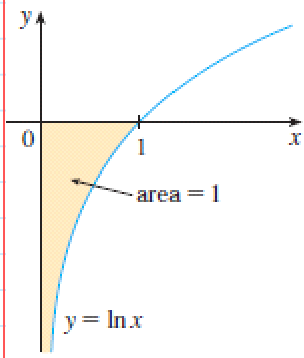
SOLUTION Observe that the line $x = 1$ is a vertical asymptote of the integrand. Since it occurs in the middle of the interval $[0, 3]$, we must use part (c) of Definition 3 with $c = 1$:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where

$$\begin{aligned}\int_0^1 \frac{dx}{x-1} &= \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \ln|x-1|\,\Big]_0^t\\ &= \lim_{t\to 1^-} \big(\ln|t-1| - \ln|-1|\big)\\ &= \lim_{t\to 1^-} \ln(1-t) = -\infty\end{aligned}$$

EXAMPLE 8 Evaluate $\int_0^1 \ln x\,dx$.



SOLUTION We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x\to 0^+} \ln x = -\infty$. Thus the given integral is improper and we have

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx$$

Now we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\begin{aligned}\int_t^1 \ln x\,dx &= x\ln x\Big]_t^1 - \int_t^1 dx \\ &= 1\ln 1 - t\ln t - (1 - t) \\ &= -t\ln t - 1 + t\end{aligned}$$

To find the limit of the first term we use l'Hospital's Rule:

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} = \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} (-t) = 0$$

Therefore $$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} (-t\ln t - 1 + t) = -0 - 1 + 0 = -1$$

Figure 11 shows the geometric interpretation of this result. The area of the shaded region above $y = \ln x$ and below the x-axis is 1.