Partial Fraction

Saturday, 14 September 2024 7:10 pm

Integration by Substitution

$$\int \frac{2x-1}{x^2-x-6}\,dx = \int \frac{1}{u}\,du \quad \text{using } u = x^2-x-6 \text{ and } du = (2x-1)\,dx$$
$$= \ln\left|x^2-x-6\right| + c$$

But what about this one...

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

It breaks!!!

$$\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} - \frac{1}{x+2}$$

Let's figure it out

Partial fractions are the fractions used for the decomposition of a rational expression. When an algebraic expression is split into a sum of two or more rational expressions, then each part is called a partial fraction. Hence, basically, it is the reverse of the addition of rational expressions. Similar to fractions, a partial fraction will have a numerator and denominator, where the denominator represents the decomposed part of a rational function.

$$\frac{3X+5}{2X^2-5x-3} = \frac{2}{x-3} - \frac{1}{2x+1}$$

Rational Expression

Partial Fractions

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k},\ k=1,2,3,\ldots$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k},\ k=1,2,3,\ldots$

Example: Integrate the function

$$\frac{1}{(x-3)(x+1)}$$

with respect to x.

$$\frac{1}{(x-3)(x+1)}=\frac{A}{(x-3)}+\frac{B}{(x+1)}$$

1= A(x+1)+B(x-3)

⇒1=x(A+B)+A-3B

A + B = 0

A − 3B = 1

A =1/4 and B = -1/4.

$$\frac{1}{(x-3)(x+1)}=\frac{1}{4(x-3)}+\frac{-1}{4(x+1)}$$

Example 1: Write the partial fraction decomposition

$(20x + 35)/(x + 4)^2$

Solution:

$(20x + 35)/(x + 4)^2$

$(20x + 35)/(x + 4)^2 = [A/(x + 4)] + [B/(x + 4)^2]$

$(20x + 35)/(x + 4)^2 = [A(x + 4) + B]/ (x + 4)^2$

Now, equating the numerators,

$20x + 35 = A(x + 4) + B$

$20x + 35 = Ax + 4A + B$

$20x + 35 = Ax + (4A + B)$

By equating the coefficients,

$A = 20$

$4A + B = 35$

$4(20) + B = 35$

$B = 35 - 80 = -45$

Therefore, $(20x + 35)/(x + 4)^2 = [20/(x + 4)] - [45/(x + 4)^2]$

7.4 Integration of Rational Functions by Partial Fractions

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$
$$= 2\ln|x-1| - \ln|x+2| + C$$

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Example: Integrate the function

$$\frac{1}{(x-3)(x+1)}$$

with respect to x.

$$\frac{1}{(x-3)(x+1)} = \frac{1}{4(x-3)} + \frac{-1}{4(x+1)}$$

$$\int \frac{1}{(x-3)(x+1)} = \int \frac{1}{4(x-3)} + \int \frac{-1}{4(x+1)}$$

$$= \frac{1}{4}\int \frac{1}{(x-3)} - \frac{1}{4}\int \frac{1}{(x+1)}$$

$$= \frac{1}{4}\ln|x-3| - \frac{1}{4}\ln|x+1|$$

$$= \frac{1}{4}\ln\left|\frac{x-3}{x+1}\right|$$

$$f(x) = \frac{P(x)}{Q(x)}$$

f is *improper*, that is, $\deg(P) \geqslant \deg(Q)$,

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

V EXAMPLE 1 Find $\int \frac{x^3 + x}{x - 1}\, dx$.

$$\begin{array}{r|l} & x^2 + x + 2 \\ \hline x - 1 & x^3 \qquad + x \\ & \underline{x^3 - x^2} \\ & \quad x^2 + x \\ & \quad \underline{x^2 - x} \\ & \qquad 2x \\ & \qquad \underline{2x - 2} \\ & \qquad\quad 2 \end{array}$$

$$\int \frac{x^3 + x}{x - 1}\, dx = \int \left(x^2 + x + 2 + \frac{2}{x - 1} \right) dx$$
$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln|x - 1| + C$$

CASE I The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k}$$

V EXAMPLE 2 Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$.

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

Just Integrate without computing A,B, and C.

EXAMPLE 3 Find $\displaystyle\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$A(x+a) + B(x-a) = 1$$

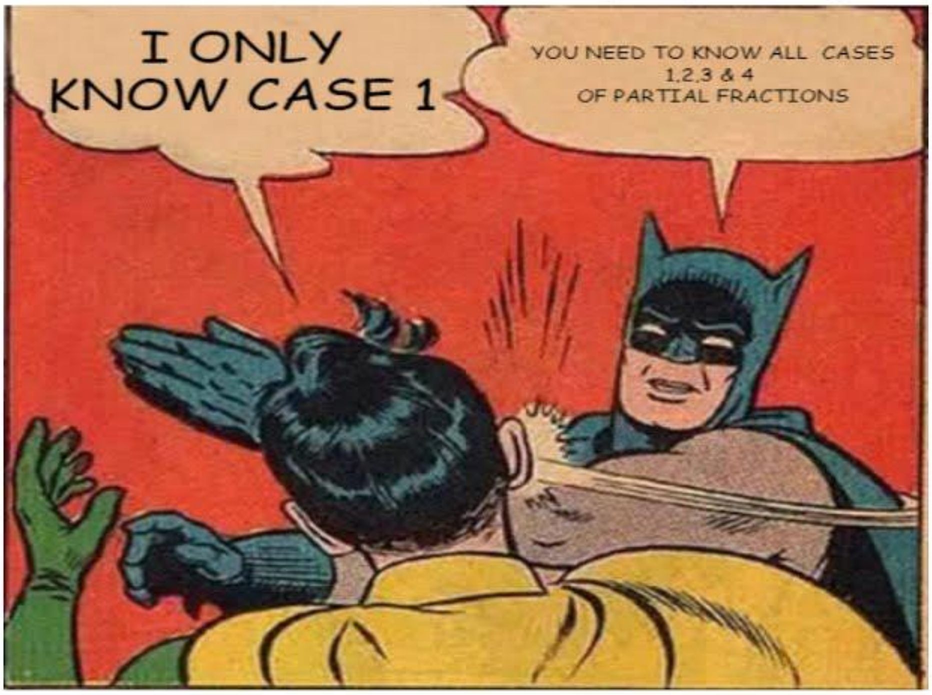
Using the method of the preceding note, we put $x = a$ in this equation and get $A(2a) = 1$, so $A = 1/(2a)$. If we put $x = -a$, we get $B(-2a) = 1$, so $B = -1/(2a)$. Thus

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$$

$$= \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + C$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

6 $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$



CASE II $Q(x)$ **is a product of linear factors, some of which are repeated.**
Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we

would use

7 $$\frac{A_1}{\quad} + \frac{A_2}{\quad} + \cdots + \frac{A_r}{\quad}$$

would use

$$\boxed{7} \qquad \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

but we prefer to work out in detail a simpler example.

$$f(x) = \frac{1}{(x+3)^2(x-2)(x+5)^3}$$

$$= \boxed{?} + \boxed{?} + \boxed{?}$$

CASE III $Q(x)$ **contains irreducible quadratic factors, none of which is repeated.**

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\boxed{9} \qquad \frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. For instance, the function given by $f(x) = x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}$$

The term given in $\boxed{9}$ can be integrated by completing the square (if necessary) and using the formula

$$\boxed{10} \qquad \boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

V EXAMPLE 5 Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

SOLUTION Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by $x(x^2 + 4)$, we have

$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 4A \end{aligned}$$

Equating coefficients, we obtain

$$A + B = 2 \qquad C = -1 \qquad 4A = 4$$

Thus $A = 1$, $B = 1$, and $C = -1$ and so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4} \right) dx$$

In order to integrate the second term we split it into two parts:

$$\int \frac{x - 1}{x^2 + 4}\,dx = \int \frac{x}{x^2 + 4}\,dx - \int \frac{1}{x^2 + 4}\,dx$$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2x\,dx$. We evaluate the second integral by means of Formula 10 with $a = 2$:

$$\begin{aligned} \int \frac{2x^2 - x + 4}{x(x^2 + 4)}\,dx &= \int \frac{1}{x}\,dx + \int \frac{x}{x^2 + 4}\,dx - \int \frac{1}{x^2 + 4}\,dx \\ &= \ln|x| + \tfrac{1}{2}\ln(x^2 + 4) - \tfrac{1}{2}\tan^{-1}(x/2) + K \end{aligned}$$

CASE IV ***$Q(x)$ contains a repeated irreducible quadratic factor.***
If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction $\boxed{9}$, the sum

$$\boxed{11} \qquad \frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$

EXAMPLE 7 Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x - 1)(x^2 + x + 1)(x^2 + 1)^3}$$

SOLUTION

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$