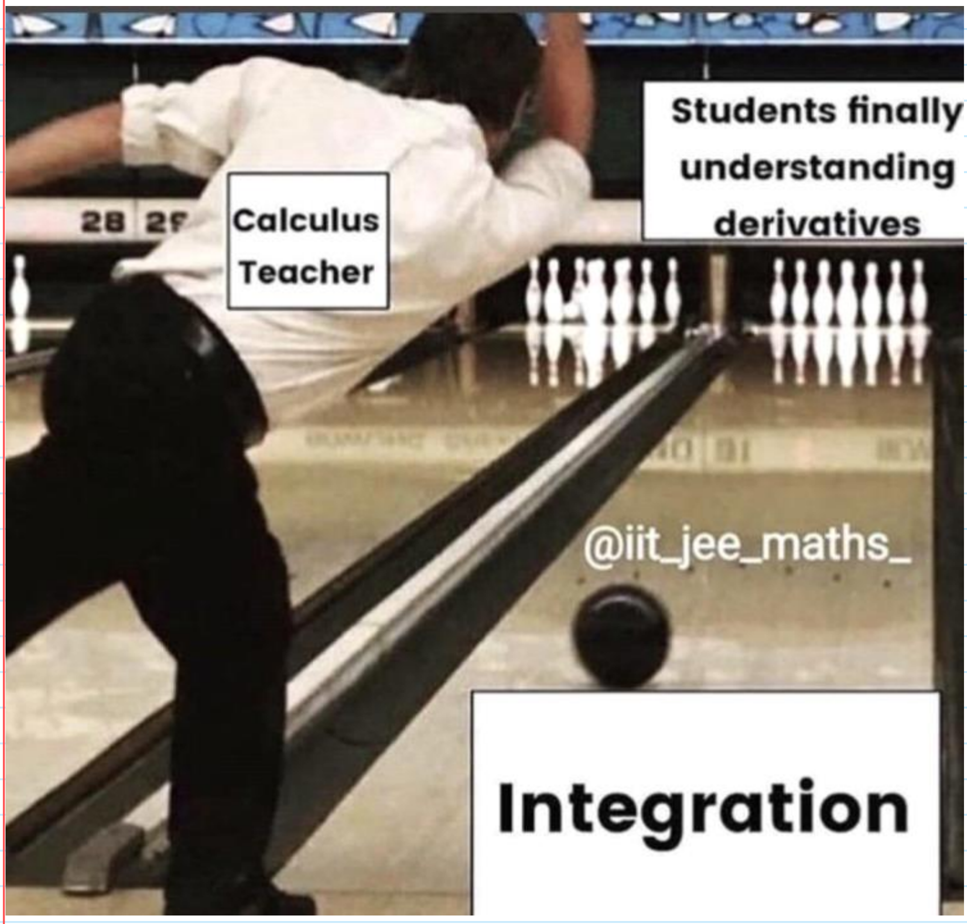
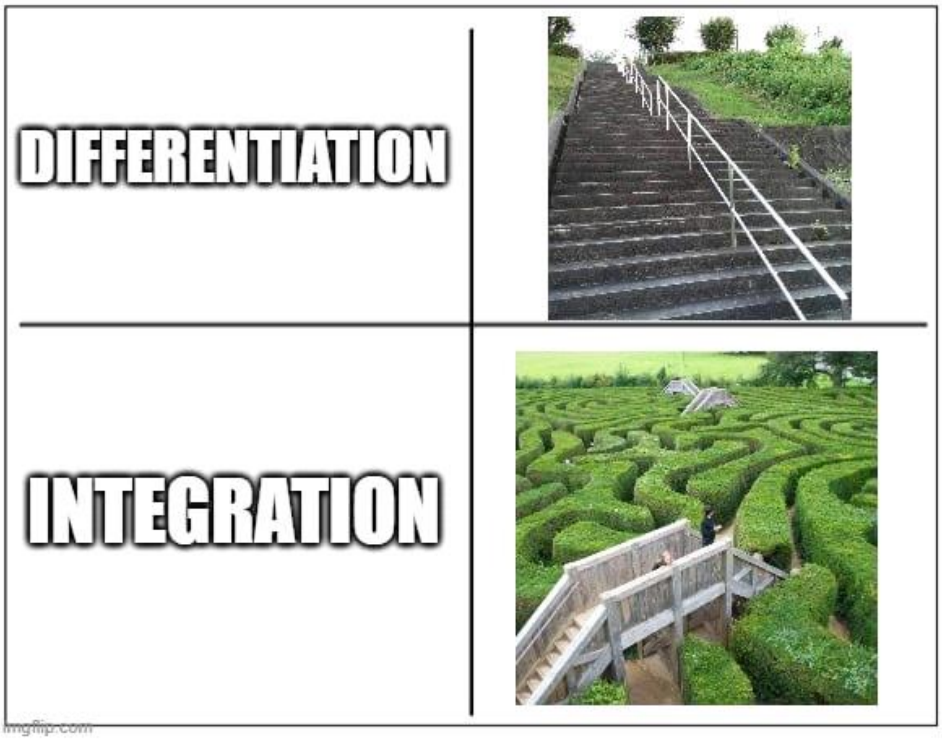
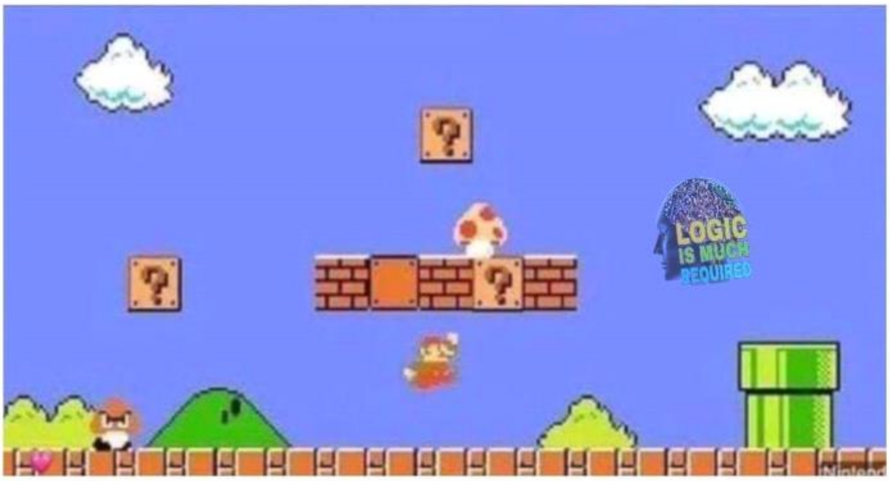
Substitution Method

Wednesday, 14 August 2024 11:11 pm





Chapter: Differentiaton



Chapter: Integration

Differentiation

Simple integration

Integration by substitution

Integration by parts

Literally every type of integration in existence

The Substitution Rule

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

1 $$\int 2x\sqrt{1 + x^2}\, dx$$

2 $$\int 2x\sqrt{1 + x^2}\, dx = \int \sqrt{1 + x^2}\, 2x\, dx = \int \sqrt{u}\, du$$
$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(x^2 + 1)^{3/2} + C$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))\, g'(x)\, dx$. Observe that if $F' = f$, then

3 $$\int F'(g(x))\, g'(x)\, dx = F(g(x)) + C$$

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = F'(g(x))\,g'(x)$$

If we make the "change of variable" or "substitution" $u = g(x)$, then from Equation 3 we have

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C = F(u) + C = \int F'(u)\,du$$

or, writing $F' = f$, we get

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Thus we have proved the following rule.

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

$$\int 18x^2\sqrt[4]{6x^3+5}\,dx$$

$$u = 6x^3 + 5$$

$$du = 18x^2 dx$$

$$\int 18x^2\sqrt[4]{6x^3+5}\,dx = \int (6x^3+5)^{\frac{1}{4}}\,(18x^2 dx)$$
$$= \int u^{\frac{1}{4}}\,du$$

$$\int 18x^2\sqrt[4]{6x^3+5}\,dx = \int u^{\frac{1}{4}}\,du = \frac{4}{5}u^{\frac{5}{4}} + c = \frac{4}{5}(6x^3+5)^{\frac{5}{4}} + c$$

EXAMPLE 1 Find $\int x^3 \cos(x^4 + 2)\,dx$.

$$\int x^3\cos(x^4+2)\,dx = \int \cos u \cdot \tfrac{1}{4}\,du = \tfrac{1}{4}\int \cos u\,du$$
$$= \tfrac{1}{4}\sin u + C$$
$$= \tfrac{1}{4}\sin(x^4+2) + C$$

EXAMPLE 2 Evaluate $\int \sqrt{2x+1}\,dx$.

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u} \cdot \tfrac{1}{2}\,du = \tfrac{1}{2}\int u^{1/2}\,du$$
$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}u^{3/2} + C$$
$$= \tfrac{1}{3}(2x+1)^{3/2} + C$$

V **EXAMPLE 3** Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x\,dx$, so $x\,dx = -\frac{1}{8}\,du$ and

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\tfrac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\tfrac{1}{8}\int u^{-1/2}\,du$$
$$= -\tfrac{1}{8}\left(2\sqrt{u}\right) + C = -\tfrac{1}{4}\sqrt{1-4x^2} + C$$

EXAMPLE 4 Calculate $\int e^{5x}\,dx$.

SOLUTION If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$. Therefore

$$\int e^{5x}\,dx = \tfrac{1}{5}\int e^u\,du = \tfrac{1}{5}e^u + C = \tfrac{1}{5}e^{5x} + C$$

V **EXAMPLE 6** Calculate $\int \tan x\,dx$.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

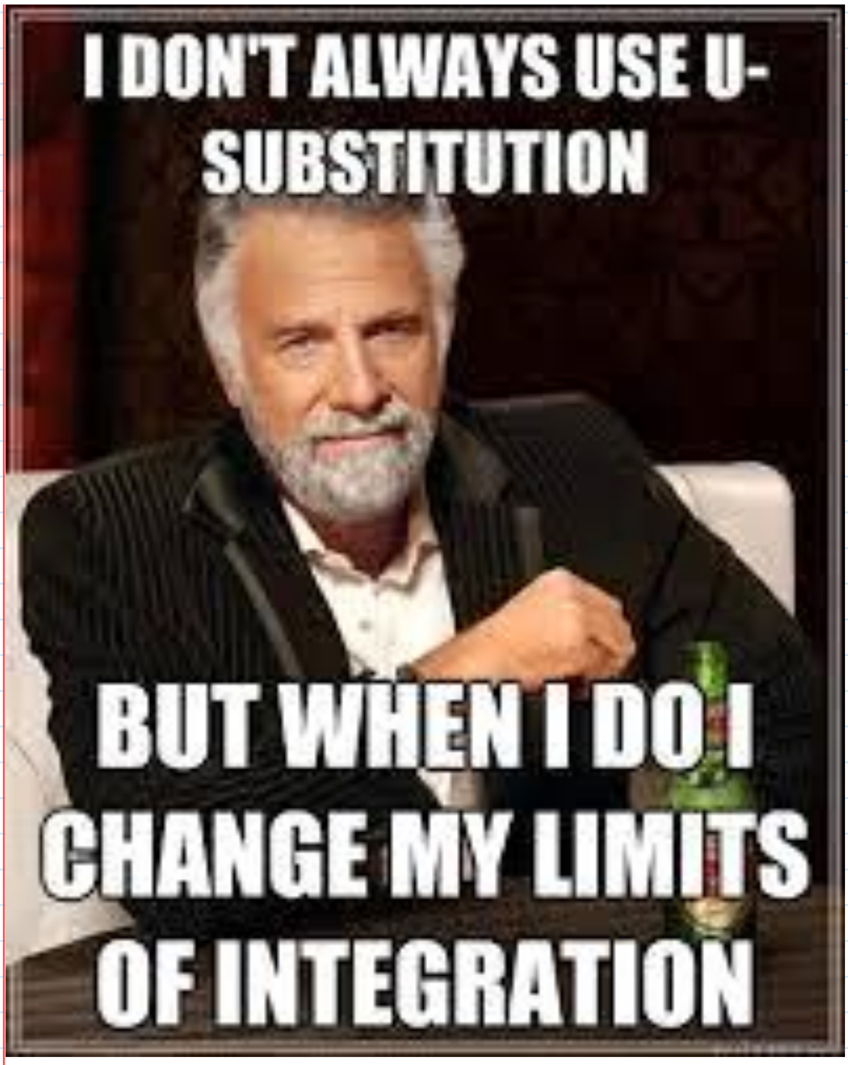
This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\,dx$ and so $\sin x\,dx = -du$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du$$
$$= -\ln|u| + C = -\ln|\cos x| + C$$

Definite Integrals

6 **The Substitution Rule for Definite Integrals** If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$



EXAMPLE 7 Evaluate $\int_0^4 \sqrt{2x+1}\,dx$ using $\boxed{6}$.

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = \frac{1}{2}\,du$. To find the new limits of integration we note that

$$\text{when } x = 0,\ u = 2(0) + 1 = 1 \qquad \text{and} \qquad \text{when } x = 4,\ u = 2(4) + 1 = 9$$

Therefore

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \tfrac{1}{2}\sqrt{u}\,du$$

$$= \tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2}\Big]_1^9$$

$$= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3}$$

EXAMPLE 8 Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

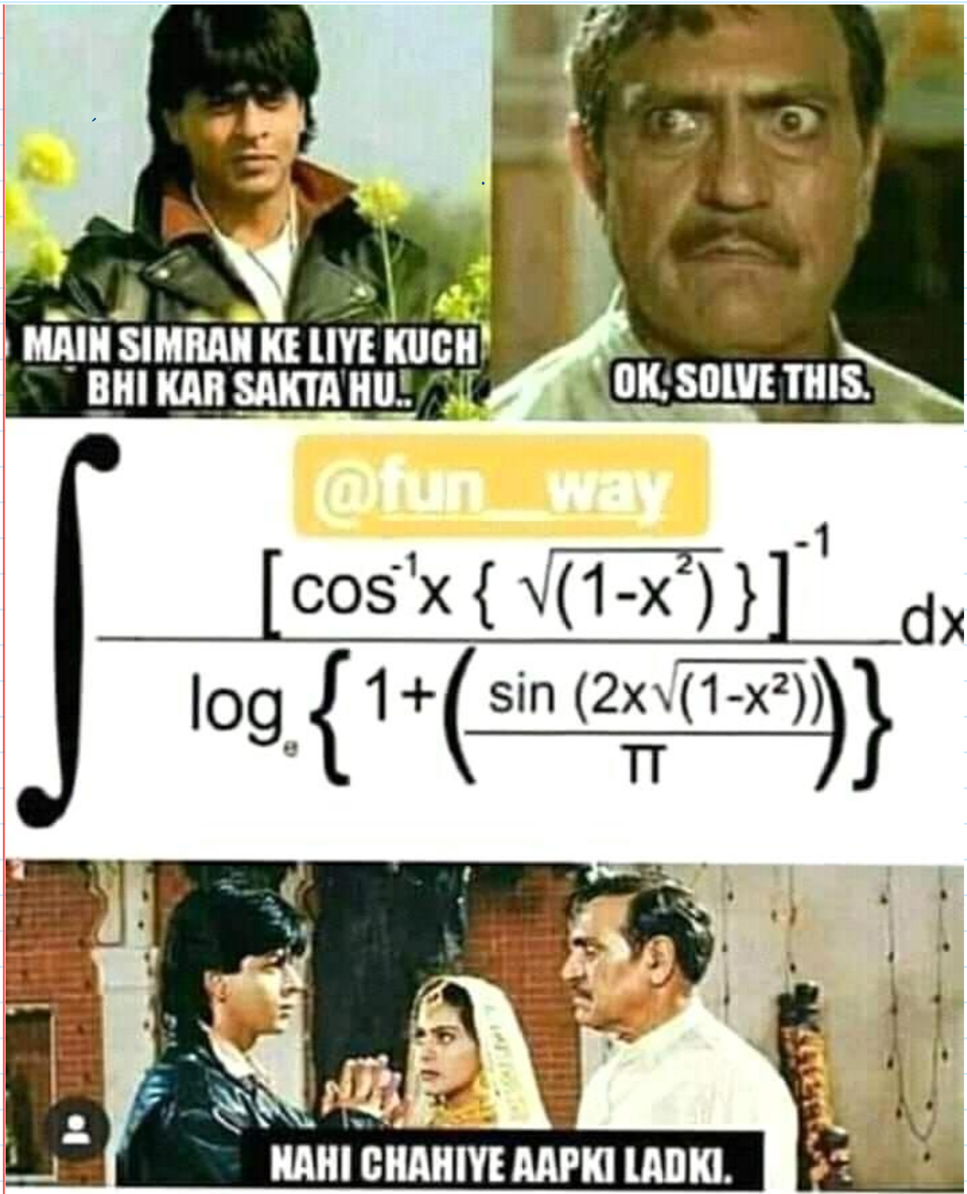
SOLUTION Let $u = 3 - 5x$. Then $du = -5\,dx$, so $dx = -\frac{1}{5}\,du$. When $x = 1$, $u = -2$ and when $x = 2$, $u = -7$. Thus

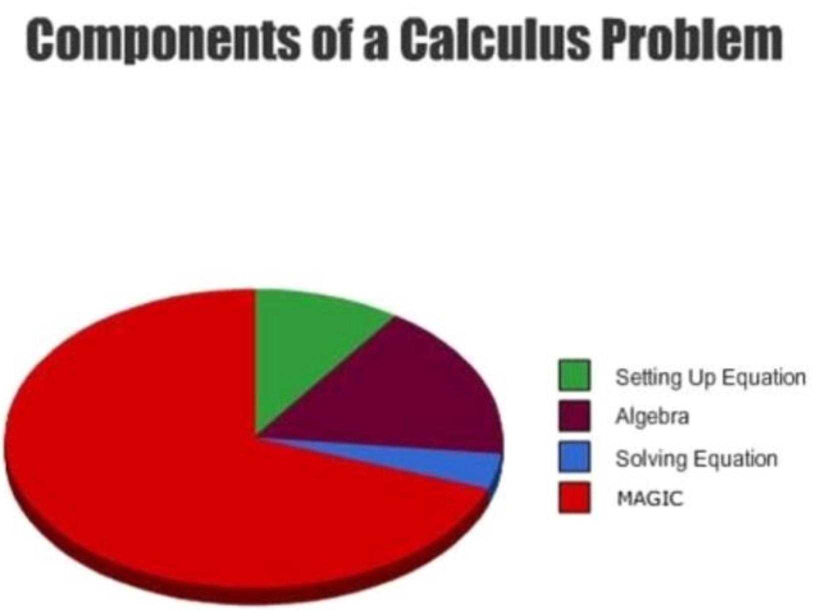
$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2}$$

$$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Bigg]_{-2}^{-7}$$

$$= \frac{1}{5}\left(-\frac{1}{7} + \frac{1}{2}\right) = \frac{1}{14}$$

V **EXAMPLE 9** Calculate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.


MAIN SIMRAN KE LIYE KUCH BHI KAR SAKTA HU..
OK, SOLVE THIS.
@fun_way
$\int \frac{\left[\cos^{-1}x \{ \sqrt{(1-x^2)} \}\right]^{-1}}{\log_e \left\{ 1+\left(\frac{\sin (2x\sqrt{(1-x^2)})}{\pi} \right) \right\}} dx$
NAHI CHAHIYE AAPKI LADKI.


Components of a Calculus Problem
Setting Up Equation
Algebra
Solving Equation
MAGIC