Indeterminate Forms & L'Hospital Rule

Sunday, 10 November 2024 12:48 am

Determinate-Indeterminate Forms Table	
Indeterminate Forms	Determinate Forms
$0/0$	$\infty + \infty = \infty$
$\pm\infty / \pm\infty$	$-\infty - \infty = -\infty$
$\infty - \infty$	$0^{\infty} = 0$
$0(\infty)$	$0^{-\infty} = \infty$
0^0	$(\infty) \cdot (\infty) = \infty$
1^{∞}	
∞^0	
Use L'Hôpital's Rule	Do *Not* Use L'Hôpital's Rule

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

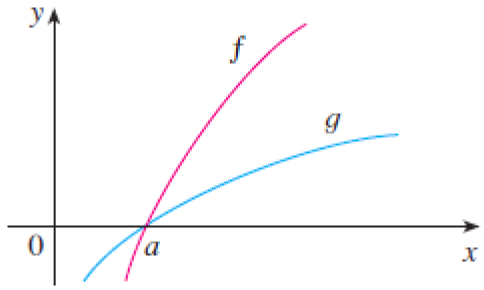
or that

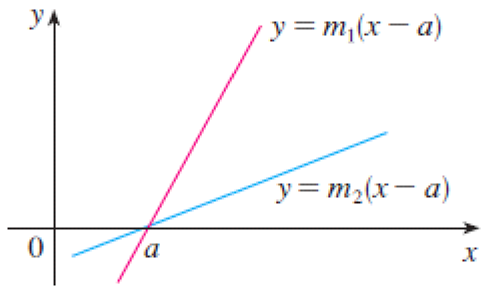
$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).





V EXAMPLE 1 Find $\lim_{x \to 1} \dfrac{\ln x}{x - 1}$.

SOLUTION Since

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \qquad \text{and} \qquad \lim_{x \to 1} (x - 1) = 0$$

we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1/x}{1}$$

$$= \lim_{x \to 1} \frac{1}{x} = 1$$

V EXAMPLE 2 Calculate $\lim_{x \to \infty} \dfrac{e^x}{x^2}$.

SOLUTION We have $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, so l'Hospital's Rule gives

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{\dfrac{d}{dx}(e^x)}{\dfrac{d}{dx}(x^2)} = \lim_{x\to\infty} \frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

V **EXAMPLE 3** Calculate $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt[3]{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0$$

V **EXAMPLE 5** Find $\displaystyle\lim_{x\to\pi^-} \frac{\sin x}{1 - \cos x}$.

Indeterminate Products

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

indeterminate form of type $\frac{0}{0}$ or ∞/∞

V **EXAMPLE 6** Evaluate $\displaystyle\lim_{x\to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

NOTE In solving Example 6 another possible option would have been to write

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{x}{1/\ln x}$$

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

EXAMPLE 7 Compute $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$.

SOLUTION First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x\to(\pi/2)^-} (\sec x - \tan x) = \lim_{x\to(\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to(\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x\to(\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$

$$\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$$

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$$

$$\lim_{x\to\infty} (x - \ln x)$$

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

V EXAMPLE 9 Find $\lim_{x \to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed as in Example 8 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore $$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$

$$\lim_{x \to 1^+} x^{1/(1-x)}$$

$y = x^{1/(1-x)} \Rightarrow \ln y = \frac{1}{1-x} \ln x$, so $\lim_{x \to 1^+} \ln y = \lim_{x \to 1^+} \frac{1}{1-x} \ln x = \lim_{x \to 1^+} \frac{\ln x}{1-x} \overset{\text{H}}{=} \lim_{x \to 1^+} \frac{1/x}{-1} = -1 \Rightarrow$

$\lim_{x \to 1^+} x^{1/(1-x)} = \lim_{x \to 1^+} e^{\ln y} = e^{-1} = \frac{1}{e}$.

$$\lim_{x \to \infty} (e^x + x)^{1/x}$$

$y = (e^x + x)^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln(e^x + x)$,

so $\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln(e^x + x)}{x} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{e^x + 1}{e^x + x} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{e^x}{e^x + 1} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{e^x}{e^x} = 1 \Rightarrow$

$\lim_{x \to \infty} (e^x + x)^{1/x} = \lim_{x \to \infty} e^{\ln y} = e^1 = e$.

7–66 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x}$$

$$\lim_{x \to 1} \frac{x^3 - 2x^2 + 1}{x^3 - 1}$$

$$\lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x}$$

$$\lim_{t \to 0} \frac{e^{2t} - 1}{\sin t}$$

$$\lim_{x \to 0^+} \frac{\ln x}{x}$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

$$\lim_{u \to \infty} \frac{e^{u/10}}{u^3}$$