Integration By Parts

Saturday, 24 August 2024 11:54 pm

Techniques of Integration

7.1 Integration by Parts

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

Derive

Integration by Parts

u and v are Functions x

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

Performing Integration By Parts

Integration By Parts

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int g(x)\,f'(x)\,dx$$

$$f(x) = u \qquad g(x) = v$$

$$f'(x)dx = du \qquad g'(x)dx = dv$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

I - inverse trig (arc functions)

L - logarithmic functions

A - algebraic (polynomials)

T - trigonometric functions

E - exponential functions

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Optional order but not standard

How to Choose u and v'

- Whatever you let v' be, you need to be able to find v.
- It helps if u' is simpler than u (or at least no more complicated than u).
- It helps if v is simpler than v' (or at least no more complicated than v').

EXAMPLE 1 Find $\int x \sin x \, dx$.

$$\int x \sin x \, dx = f(x)g(x) - \int g(x) f'(x) \, dx$$

$$= x(-\cos x) - \int (-\cos x) \, dx$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

V EXAMPLE 2 Evaluate $\int \ln x \, dx$.

SOLUTION Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then

$$du = \frac{1}{x} dx \qquad v = x$$

Integrating by parts, we get

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x}$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

V EXAMPLE 3 Find $\int t^2 e^t \, dt$.

$$u = t^2 \qquad dv = e^t \, dt$$

$$du = 2t \, dt \qquad v = e^t$$

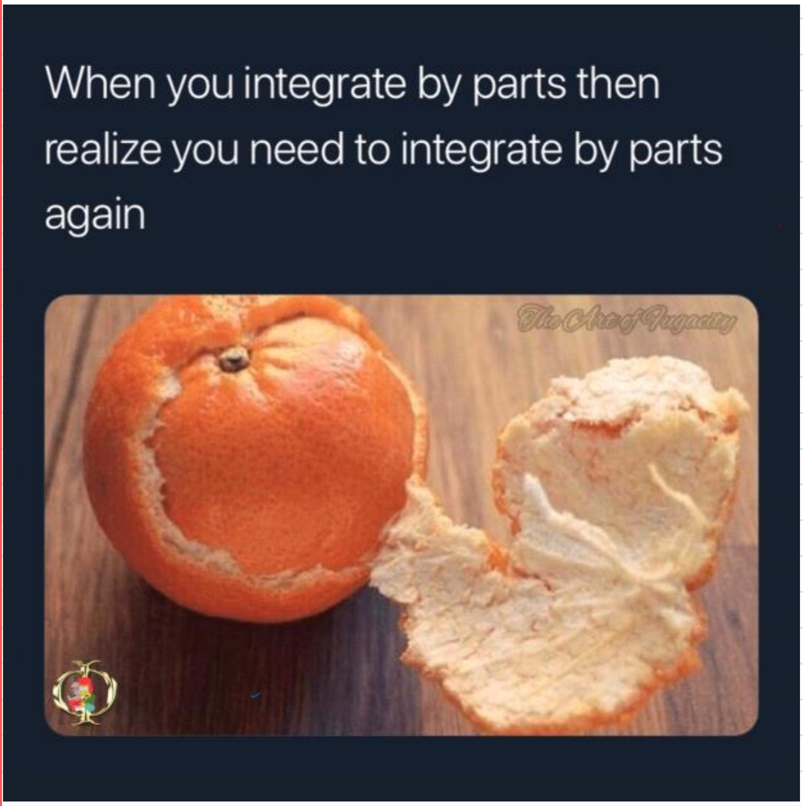
$$\int t^2 e^t \, dt = t^2 e^t - 2 \int t e^t \, dt$$

$$\int t e^t \, dt = t e^t - \int e^t \, dt$$

$$= t e^t - e^t + C$$

$$\int t^2 e^t\,dt = t^2 e^t - 2\int te^t\,dt$$
$$= t^2 e^t - 2(te^t - e^t + C)$$
$$= t^2 e^t - 2te^t + 2e^t + C_1 \qquad \text{where } C_1 = -2C$$

V EXAMPLE 4 Evaluate $\int e^x \sin x\,dx$.



$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx}$$

EXAMPLE 5 Calculate $\int_0^1 \tan^{-1}x\,dx$.

SOLUTION Let

$$u = \tan^{-1}x \qquad dv = dx$$

Then

$$du = \frac{dx}{1+x^2} \qquad v = x$$

So Formula 6 gives

$$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= 1\cdot\tan^{-1}1 - 0\cdot\tan^{-1}0 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx$$

To evaluate this integral we use the substitution $t = 1 + x^2$ (since u has another meaning in this example). Then $dt = 2x\,dx$, so $x\,dx = \frac{1}{2}\,dt$. When $x = 0$, $t = 1$; when $x = 1$, $t = 2$; so

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}\ln|t|\Big]_1^2$$

$$= \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Therefore

$$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

Evaluate the integral.

$$\int te^{-3t}\,dt$$

$$\int p^5 \ln p\,dp$$

$$\int_0^{1/2} x\cos \pi x\,dx$$

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy$$

(a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4 x \, dx$.