Limit at Infinity

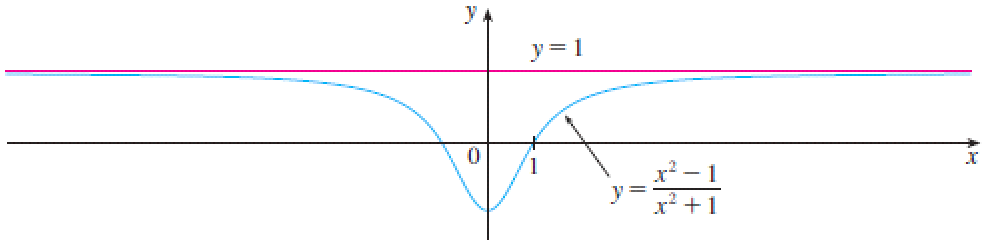
Monday, 4 August 2025 9:47 am

Limits at Infinity; Horizontal Asymptotes

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

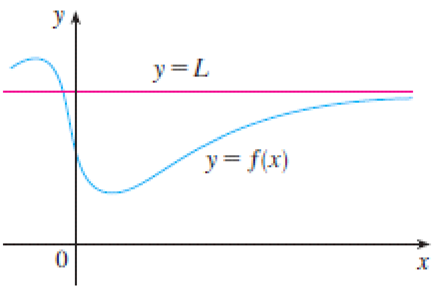
as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

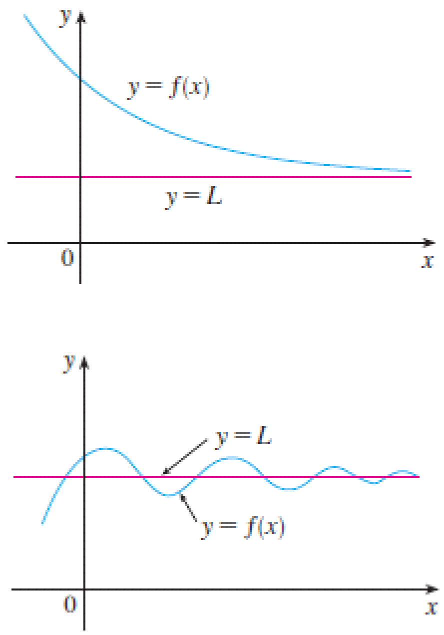


x	$f(x)$
0	−1
±1	0
±2	0.600000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

$$\lim_{x \to \infty} f(x) = L$$





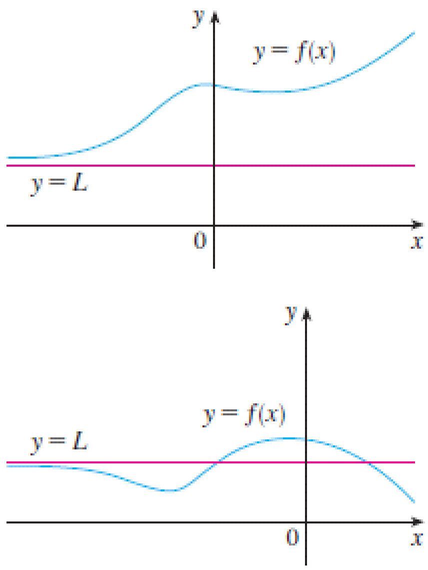


FIGURE 3

Examples illustrating $\lim_{x \to -\infty} f(x) = L$

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

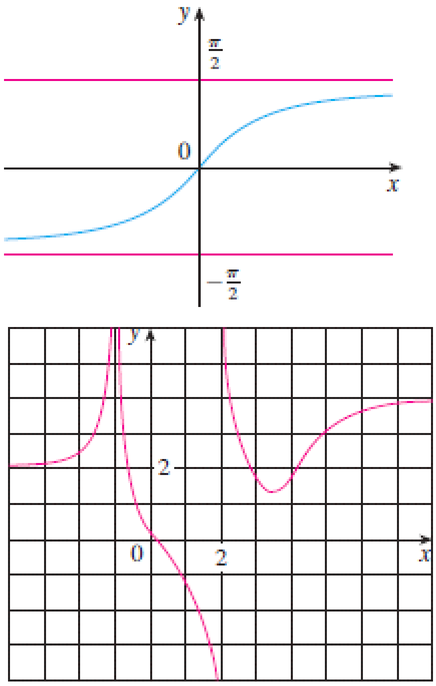


FIGURE 5

$$\lim_{x \to \infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

EXAMPLE 2 Find $\displaystyle\lim_{x \to \infty} \frac{1}{x}$ and $\displaystyle\lim_{x \to -\infty} \frac{1}{x}$.

SOLUTION Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

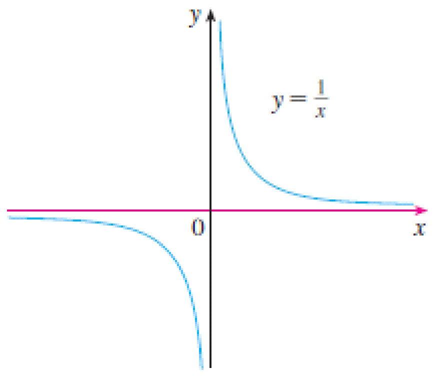


FIGURE 6

$\lim_{x\to\infty}\frac{1}{x}=0, \quad \lim_{x\to-\infty}\frac{1}{x}=0$

V EXAMPLE 3 Evaluate

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}$$

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}=\lim_{x\to\infty}\frac{\dfrac{3x^2-x-2}{x^2}}{\dfrac{5x^2+4x+1}{x^2}}=\lim_{x\to\infty}\frac{3-\dfrac{1}{x}-\dfrac{2}{x^2}}{5+\dfrac{4}{x}+\dfrac{1}{x^2}}$$

$$=\frac{\lim_{x\to\infty}\left(3-\dfrac{1}{x}-\dfrac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5+\dfrac{4}{x}+\dfrac{1}{x^2}\right)} \qquad \text{(by Limit Law 5)}$$

$$=\frac{\lim_{x\to\infty}3-\lim_{x\to\infty}\dfrac{1}{x}-2\lim_{x\to\infty}\dfrac{1}{x^2}}{\lim_{x\to\infty}5+4\lim_{x\to\infty}\dfrac{1}{x}+\lim_{x\to\infty}\dfrac{1}{x^2}} \qquad \text{(by 1, 2, and 3)}$$

$$=\frac{3-0-0}{5+0+0} \qquad \text{(by 7 and Theorem 5)}$$

$$=\frac{3}{5}$$

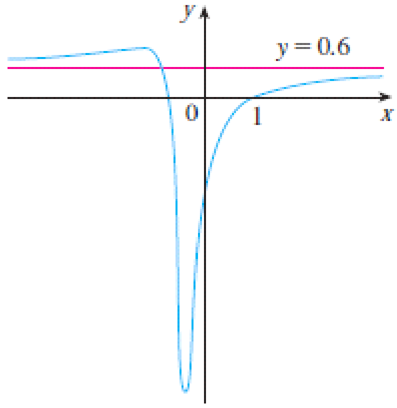


FIGURE 7
$y = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

EXAMPLE 4 Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

SOLUTION Dividing both numerator and denominator by x and using the properties of limits, we have

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to \infty} \frac{\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}} \qquad \text{(since } \sqrt{x^2} = x \text{ for } x > 0\text{)}$$

$$= \frac{\lim_{x \to \infty} \sqrt{2 + \dfrac{1}{x^2}}}{\lim_{x \to \infty} \left(3 - \dfrac{5}{x}\right)} = \frac{\sqrt{\lim_{x \to \infty} 2 + \lim_{x \to \infty} \dfrac{1}{x^2}}}{\lim_{x \to \infty} 3 - 5 \lim_{x \to \infty} \dfrac{1}{x}} = \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0} = \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$. So when we divide the numerator by x, for $x < 0$ we get

$$\frac{1}{x}\sqrt{2x^2 + 1} = -\frac{1}{\sqrt{x^2}}\sqrt{2x^2 + 1} = -\sqrt{2 + \frac{1}{x^2}}$$

Therefore

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to -\infty} \frac{-\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}} = \frac{-\sqrt{2 + \lim_{x \to -\infty} \dfrac{1}{x^2}}}{3 - 5 \lim_{x \to -\infty} \dfrac{1}{x}} = -\frac{\sqrt{2}}{3}$$

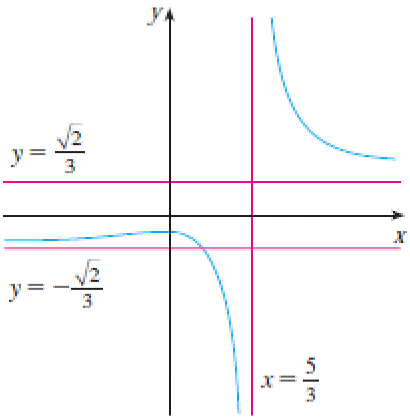


FIGURE 8

$y = \dfrac{\sqrt{2x^2+1}}{3x-5}$

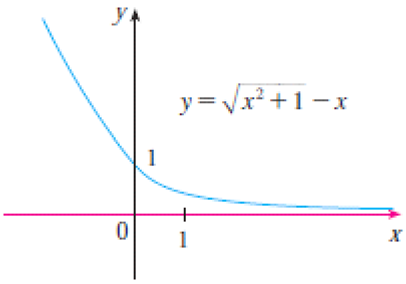
EXAMPLE 5 Compute $\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right)$.

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}$$

$$= \lim_{x \to \infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x}$$

Notice that the denominator of this last expression $\left(\sqrt{x^2+1} + x\right)$ becomes large as $x \to \infty$ (it's bigger than x). So

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x} = 0$$



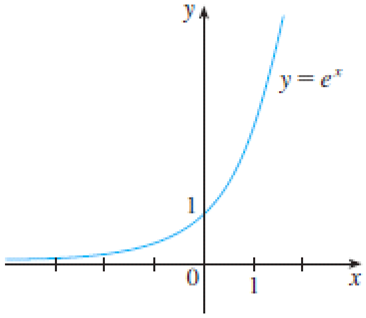
EXAMPLE 6 Evaluate $\lim_{x \to 2^+} \arctan\left(\dfrac{1}{x-2}\right)$.

SOLUTION If we let $t = 1/(x-2)$, we know that $t \to \infty$ as $x \to 2^+$. Therefore, by the second equation in $\boxed{4}$, we have

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t \to \infty} \arctan t = \frac{\pi}{2}$$

6 $$\boxed{\lim_{x \to -\infty} e^x = 0}$$

Notice that the values of e^x approach 0 very rapidly.



x	e^x
0	1.00000
-1	0.36788
-2	0.13534
-3	0.04979
-5	0.00674
-8	0.00034
-10	0.00005

V EXAMPLE 7 Evaluate $\lim_{x \to 0^-} e^{1/x}$.

SOLUTION If we let $t = 1/x$, we know that $t \to -\infty$ as $x \to 0^-$. Therefore, by $\boxed{6}$,

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

(See Exercise 75.)

EXAMPLE 8 Evaluate $\lim_{x \to \infty} \sin x$.

SOLUTION As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often and so they don't approach any definite number. Thus $\lim_{x \to \infty} \sin x$ does not exist.