

Derivatives, Rates of Change, Tangent line & Rule

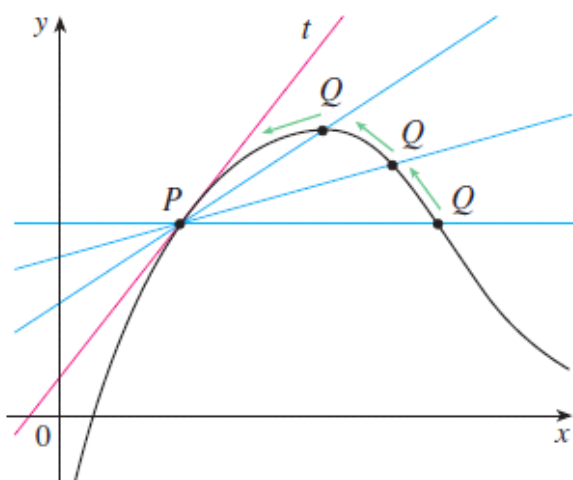
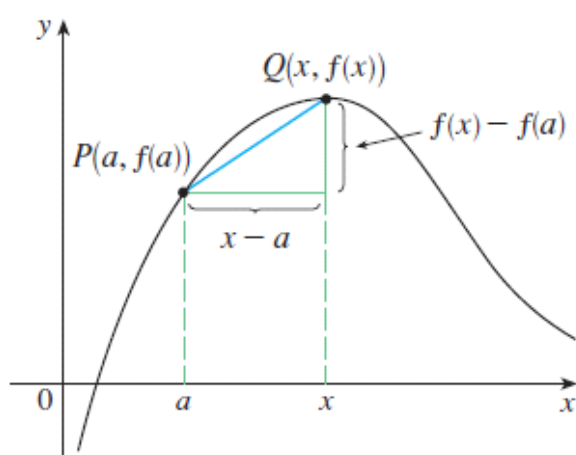
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Derivatives and Rates of Change

Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ :

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$



1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

V EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$\begin{aligned} m &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 1) = 1 + 1 = 2 \end{aligned}$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$$

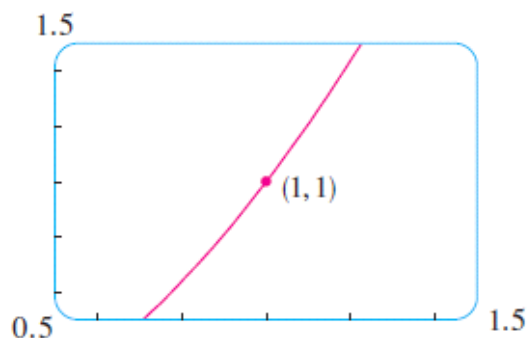
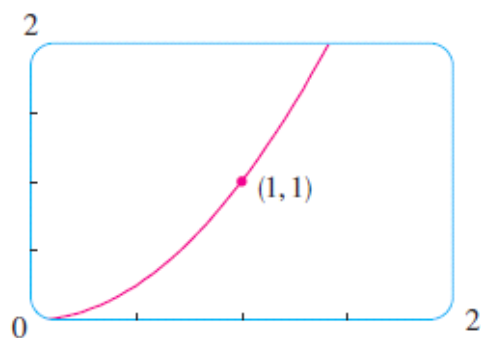
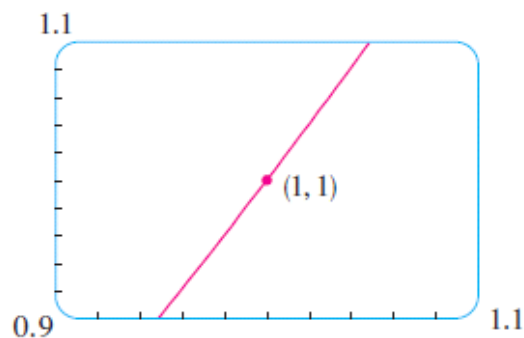


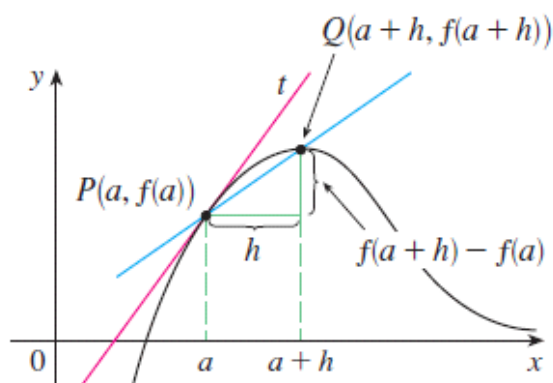
FIGURE 2 Zooming in toward the point $(1, 1)$ on the parabola $y = x^2$



There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

$$m = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$



EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

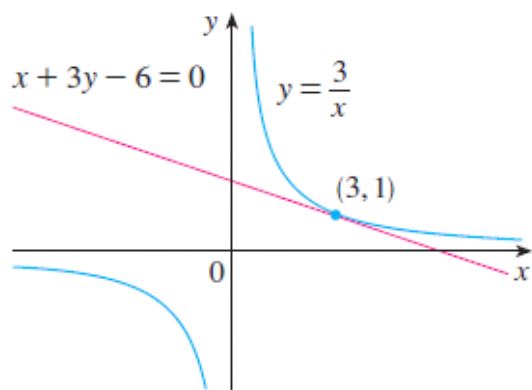
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SOLUTION Let $f(x) = 3/x$. Then the slope of the tangent at $(3, 1)$ is

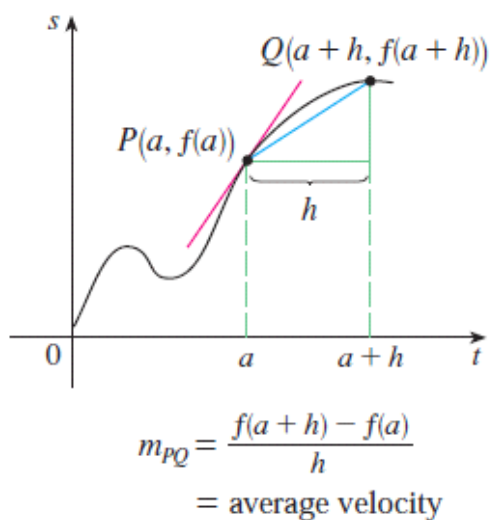
$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3}{3 + h} - 1}{h} = \lim_{h \rightarrow 0} \frac{\frac{3 - (3 + h)}{3 + h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h(3 + h)} = \lim_{h \rightarrow 0} -\frac{1}{3 + h} = -\frac{1}{3} \end{aligned}$$

Therefore an equation of the tangent at the point $(3, 1)$ is

$$y - 1 = -\frac{1}{3}(x - 3)$$



Velocities



$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Derivatives

4 Definition The derivative of a function f at a number a , denoted by $f'(a)$, is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

V EXAMPLE 4 Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a .

SOLUTION From Definition 4 we have

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \rightarrow 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \rightarrow 0} (2a + h - 8) \\ &= 2a - 8 \end{aligned}$$

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a .

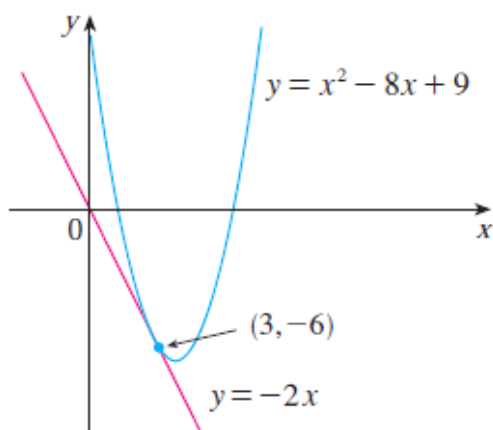
If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

V EXAMPLE 5 Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

SOLUTION From Example 4 we know that the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$. Therefore the slope of the tangent line at $(3, -6)$ is $f'(3) = 2(3) - 8 = -2$. Thus an equation of the tangent line, shown in Figure 7, is

$$y - (-6) = (-2)(x - 3) \quad \text{or} \quad y = -2x$$



Rates of Change

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

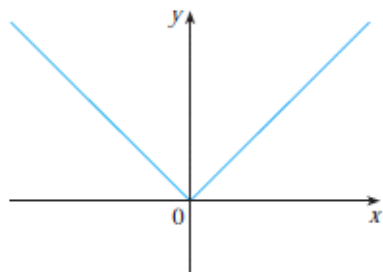
$$\text{instantaneous rate of change} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

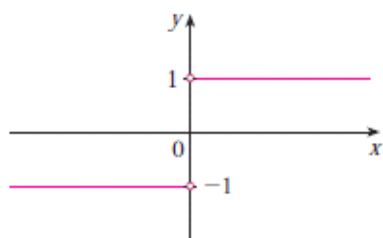
V EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable?

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} = \lim_{h \rightarrow 0} \frac{(x+h) - x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} = \lim_{h \rightarrow 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} = \lim_{h \rightarrow 0} (-1) = -1 \end{aligned}$$

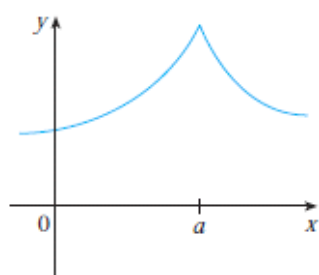


(a) $y = f(x) = |x|$

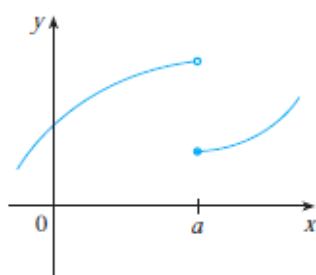


(b) $y = f'(x)$

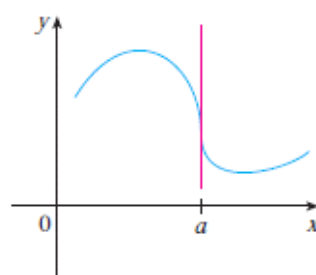
4 Theorem If f is differentiable at a , then f is continuous at a .



(a) A corner



(b) A discontinuity



(c) A vertical tangent

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf'$$

$$(f + g)' = f' + g'$$

$$(f - g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

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Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$