Midterm Exam – Solutions

Calculus & Analytical Geometry (BSCS / BSSE)

Question 1 [10 Marks]

(a) Explain the domain of the following functions. (5 Marks)

i.

$$f(x) = \sqrt{x^2 - 9}$$

Solution:
The expression inside the square root must be non–negative:

$$x^2 - 9 \geq 0$$

$$x^2 \geq 9$$

$$|x| \geq 3$$

Thus the domain is:

$$\boxed{(-\infty, -3] \cup [3, \infty)}$$

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ii.

$$g(x) = \frac{\ln(2x - 5)}{x^2 - 4x + 3}$$

Domain restrictions:
1. Log requires:

$$2x - 5 > 0 \Rightarrow x > \frac{5}{2}$$

2. Denominator cannot be zero:

$$x^2 - 4x + 3 = (x - 1)(x - 3)$$

$$x \neq 1, \quad x \neq 3$$

Since $x > 2.5$, only $x = 3$ is excluded.

$$\boxed{\text{Domain: } \left(\frac{5}{2}, 3\right) \cup (3, \infty)}$$

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(b) Compute the limit at infinity. (2 Marks)

$$\lim_{x\to\infty} \frac{2x^2+3x}{4x^2-x+1}$$

Divide numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{2+3/x}{4-1/x+1/x^2} = \frac{2}{4}$$

$$\boxed{\frac{1}{2}}$$

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(c) Discuss limit existence and continuity at $x = -2, 2, 4$. (3 Marks)

The function is:

$$f(x) = \begin{cases} x^2-4x+3, & x<2, \\ 0, & x=2, \\ \cos\left(\frac{x-2}{5}\right)-2, & x>2. \end{cases}$$

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(i) At $x = -2$

Since $-2 < 2$, we use the first piece:

$$f(x) = x^2 - 4x + 3$$

This is a polynomial $\rightarrow$ continuous everywhere.

Thus:

$$\boxed{\text{Limit exists and the function is continuous at } x = -2.}$$

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(ii) At $x = 2$

Left-hand limit:

$$\lim_{x\to 2^-} (x^2-4x+3) = 4-8+3 = -1$$

Right-hand limit:

$$\lim_{x\to 2^+} \left(\cos\left(\frac{x-2}{5}\right)-2\right) = \cos(0)-2 = 1-2 = -1$$

Both limits are -1, so limit exists.
But:

$$f(2) = 0 \neq -1$$

$$\boxed{\text{Not continuous at } x = 2.}$$

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(iii) At $x = 4$
Since $4 > 2$:

$$f(x) = \cos\left(\frac{x-2}{5}\right) - 2$$

Cosine is continuous everywhere $\implies$ limit exists and equals the function value.

$$\boxed{\text{Continuous at } x = 4.}$$

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Question 2 [15 Marks]

(a) Using first principle, find tangent and normal for $f(x) = 3x^2 - 8$ at $x = 2$. (4 Marks)

First principle:

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - 8 - (3x^2 - 8)}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h} = 6x$$

At $x = 2$:

$$f'(2) = 6(2) = 12$$

Tangent line slope $m_t = 12$. Point:

$$f(2) = 3(4) - 8 = 4$$

Tangent equation:

$$y - 4 = 12(x - 2)$$

Normal slope:

$$m_n = -\frac{1}{12}$$

Normal equation:

$$y - 4 = -\frac{1}{12}(x - 2)$$

$$\boxed{\text{Tangent: } y - 4 = 12(x - 2)}$$

$$\boxed{\text{Normal: } y - 4 = -\frac{1}{12}(x - 2)}$$

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(b) Use implicit differentiation to find $\frac{dy}{dx}$. (4 Marks)

$$3y^4 - \frac{(2x^2 + 1)^3}{e^x} + \sin(xy) + 1 = 0$$

Differentiate:

$$12y^3\frac{dy}{dx} - \frac{d}{dx}\left(\frac{(2x^2 + 1)^3}{e^x}\right) + \cos(xy)(y + x\frac{dy}{dx}) = 0$$

Use quotient rule:

$$\frac{d}{dx}\left(\frac{u}{e^x}\right) = \frac{u'e^x - ue^x}{e^{2x}} = \frac{u' - u}{e^x}$$

where

$$u = (2x^2 + 1)^3, \quad u' = 3(2x^2 + 1)^2(4x)$$

Now collect $\frac{dy}{dx}$:

$$12y^3\frac{dy}{dx} + x\cos(xy)\frac{dy}{dx} = \frac{u' - u}{e^x} - y\cos(xy)$$

Factor:

$$\frac{dy}{dx}\left(12y^3 + x\cos(xy)\right) = \frac{u' - u}{e^x} - y\cos(xy)$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{u'-u}{e^x} - y\cos(xy)}{12y^3 + x\cos(xy)}}$$

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(c) Solve using L'Hôpital's Rule. (3 Marks)

i.

$$\lim_{n\to 0} \frac{\sin(3n)}{5n}$$

$$= \lim_{n\to 0} \frac{3\cos(3n)}{5} = \frac{3}{5}$$

$$\boxed{\frac{3}{5}}$$

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ii.

$$\lim_{t\to\infty} \frac{2t^2}{e^t}$$

As $t \to \infty$, exponential dominates any polynomial.

$$\boxed{0}$$

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(d) Related Rates – Balloon. (4 Marks)

Volume of sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Given:

$$\frac{dr}{dt} = 2 \text{ m/s}, \quad r = 5 \text{ cm} = 0.05 \text{ m}$$

$$\frac{dV}{dt} = 4\pi(0.05)^2(2) = 4\pi(0.0025)(2) = 0.02\pi$$

$$\boxed{\frac{dV}{dt} = 0.02\pi \text{ m}^3/\text{s}}$$

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