Numerical Integration

Monday, 23 December 2024 1:07 pm

Numerical Integration

CHAPTER OBJECTIVES

The primary objective of this chapter is to introduce you to numerical integration. Specific objectives and topics covered are

- Recognizing that Newton-Cotes integration formulas are based on the strategy of replacing a complicated function or tabulated data with a polynomial that is easy to integrate.
- Knowing how to implement the following single application Newton-Cotes formulas:
  - Trapezoidal rule
  - Simpson's 1/3 rule
  - Simpson's 3/8 rule

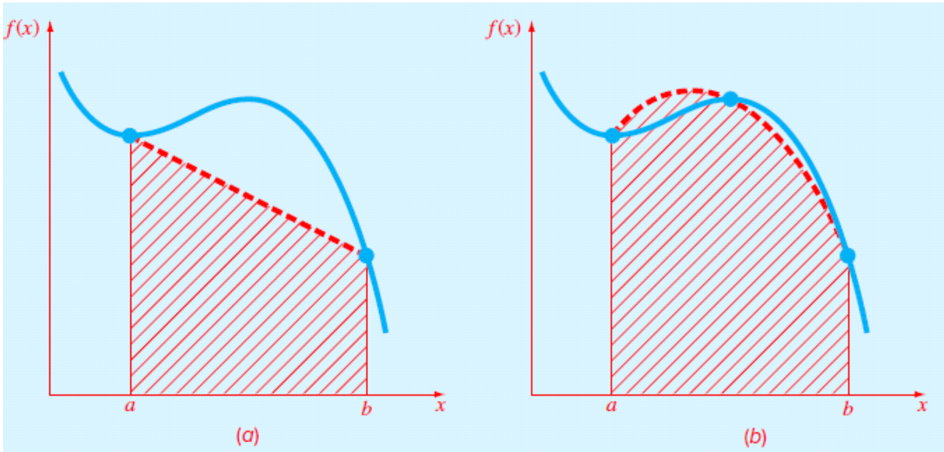
NEWTON-COTES FORMULAS

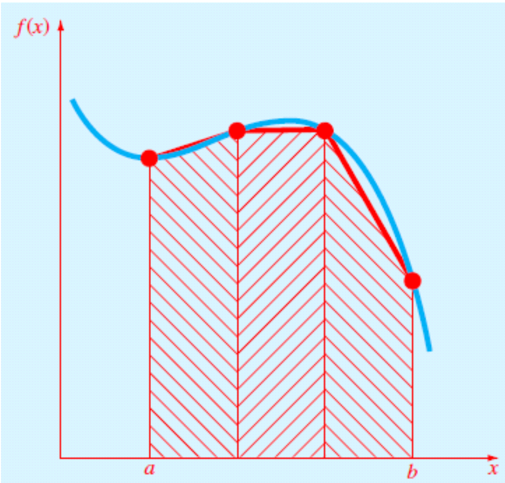
The *Newton-Cotes formulas* are the most common numerical integration schemes. They are based on the strategy of replacing a complicated function or tabulated data with a polynomial that is easy to integrate:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx \tag{19.8}$$

where $f_n(x)$ = a polynomial of the form

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$





THE TRAPEZOIDAL RULE

The *trapezoidal rule* is the first of the Newton-Cotes closed integration formulas. It corresponds to the case where the polynomial in Eq. (19.8) is first-order:

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx \tag{19.10}$$

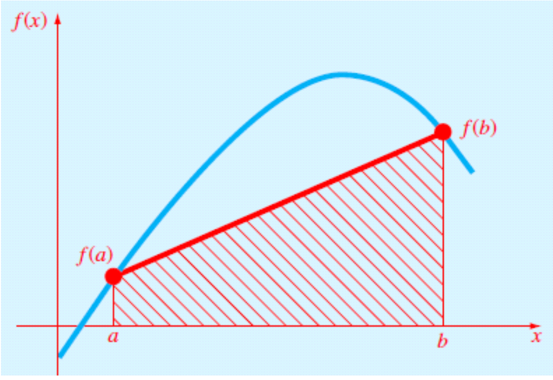
The result of the integration is

$$I = (b - a)\frac{f(a) + f(b)}{2} \tag{19.11}$$

which is called the *trapezoidal rule*.

Geometrically, the trapezoidal rule is equivalent to approximating the area of the trapezoid under the straight line connecting $f(a)$ and $f(b)$ in Fig. 19.7. Recall from geometry that the formula for computing the area of a trapezoid is the height times the average of the bases. In our case, the concept is the same but the trapezoid is on its side. Therefore, the integral estimate can be represented as

$$I = \text{width} \times \text{average height} \tag{19.12}$$



$I = (b - a) \times$ average height

Error of the Trapezoidal Rule

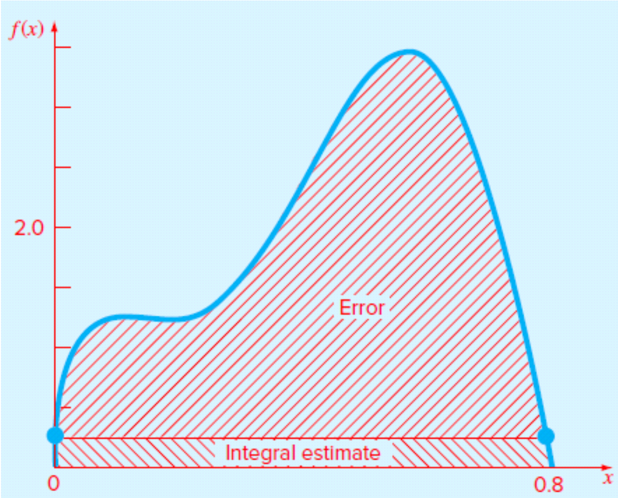
$$E_t = -\frac{1}{12} f''(\xi)(b - a)^3$$

Single Application of the Trapezoidal Rule

Problem Statement. Use Eq. (19.11) to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Note that the exact value of the integral can be determined analytically to be 1.640533.



Solution. The function values $f(0) = 0.2$ and $f(0.8) = 0.232$ can be substituted into Eq. (19.11) to yield

$$I = (0.8 - 0)\frac{0.2 + 0.232}{2} = 0.1728$$

which represents an error of $E_t = 1.640533 - 0.1728 = 1.467733$, which corresponds to a percent relative error of $\varepsilon_t = 89.5\%$. The reason for this large error is evident from the graphical depiction in Fig. 19.8. Notice that the area under the straight line neglects a significant portion of the integral lying above the line.

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$I = \underbrace{(b - a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}}_{\text{Average height}}$$

$$E_t = -\frac{(b - a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$$

$$E_a = -\frac{(b - a)^3}{12n^2}\bar{f}''$$

Composite Application of the Trapezoidal Rule

Problem Statement. Use the two-segment trapezoidal rule to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Employ Eq. (19.21) to estimate the error. Recall that the exact value of the integral is 1.640533.

Solution. For $n = 2$ ($h = 0.4$):

$$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$$

$$I = 0.8\frac{0.2 + 2(2.456) + 0.232}{4} = 1.0688$$

$$E_t = 1.640533 - 1.0688 = 0.57173 \qquad \varepsilon_t = 34.9\%$$

$$E_a = -\frac{0.8^3}{12(2)^2}(-60) = 0.64$$

SIMPSON'S RULES

Simpson's 1/3 Rule

$$I = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$I = (b-a)\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

$$E_t = -\frac{1}{90}h^5 f^{(4)}(\xi)$$

or, because $h = (b-a)/2$:

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

Example 1 Compare the Trapezoidal rule and Simpson's rule approximations to $\int_0^2 f(x)\,dx$ when $f(x)$ is

(a)	x^2	**(b)**	x^4	**(c)**	$(x+1)^{-1}$
(d)	$\sqrt{1+x^2}$	**(e)**	$\sin x$	**(f)**	e^x

T

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi), \quad \text{where} \quad x_0 < \xi < x_1. \qquad (4.25)$$

S

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi), \quad \text{where} \quad x_0 < \xi < x_2. \qquad (4.26)$$

Screen clipping taken: 23/12/2024 4:21 pm

Solution On [0, 2] the Trapezoidal and Simpson's rule have the forms

$$\text{Trapezoid:} \quad \int_0^2 f(x)\,dx \approx f(0) + f(2) \quad \text{and}$$

$$\text{Simpson's:} \quad \int_0^2 f(x)\,dx \approx \frac{1}{3}[f(0) + 4f(1) + f(2)].$$

Screen clipping taken: 23/12/2024 4:21 pm

When $f(x) = x^2$ they give

$$\text{Trapezoid: } \int_0^2 f(x)\,dx \approx 0^2 + 2^2 = 4 \quad \text{and}$$

$$\text{Simpson's: } \int_0^2 f(x)\,dx \approx \frac{1}{3}[(0^2) + 4 \cdot 1^2 + 2^2] = \frac{8}{3}.$$

The approximation from Simpson's rule is exact because its truncation error involves $f^{(4)}$, which is identically 0 when $f(x) = x^2$.

Screen clipping taken: 23/12/2024 4:21 pm

	(a)	(b)	(c)	(d)	(e)	(f)
$f(x)$	x^2	x^4	$(x+1)^{-1}$	$\sqrt{1+x^2}$	$\sin x$	e^x
Exact value	2.667	6.400	1.099	2.958	1.416	6.389
Trapezoidal	4.000	16.000	1.333	3.326	0.909	8.389
Simpson's	2.667	6.667	1.111	2.964	1.425	6.421

Single Application of Simpson's 1/3 Rule

Problem Statement. Use Eq. (19.23) to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Employ Eq. (19.24) to estimate the error. Recall that the exact integral is 1.640533.

Solution. $n = 2(h = 0.4)$:

$$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$$

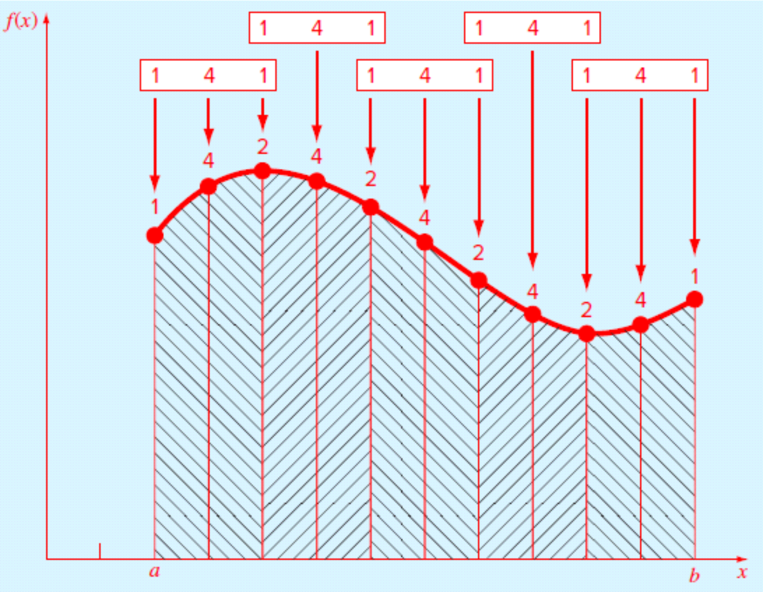
$$I = 0.8\,\frac{0.2 + 4(2.456) + 0.232}{6} = 1.367467$$

$$E_t = 1.640533 - 1.367467 = 0.2730667 \qquad \varepsilon_t = 16.6\%$$

$$E_a = -\frac{0.8^5}{2880}(-2400) = 0.2730667$$

The Composite Simpson's 1/3 Rule

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$



$$I = (b-a)\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$$

$$E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}$$

Composite Simpson's 1/3 Rule

Problem Statement. Use Eq. (19.26) with $n = 4$ to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Employ Eq. (19.27) to estimate the error. Recall that the exact integral is 1.640533.

Solution. $n = 4 (h = 0.2)$:

$$f(0) = 0.2 \qquad f(0.2) = 1.288$$
$$f(0.4) = 2.456 \qquad f(0.6) = 3.464$$
$$f(0.8) = 0.232$$

From Eq. (19.26):

$$I = 0.8\,\frac{0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232}{12} = 1.623467$$

$$E_t = 1.640533 - 1.623467 = 0.017067 \qquad \varepsilon_t = 1.04\%$$

The estimated error [Eq. (19.27)] is

$$E_a = -\frac{(0.8)^5}{180(4)^4}(-2400) = 0.017067$$

which is exact (as was also the case for Example 19.3).

Simpson's 3/8 Rule

$$I = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$I = (b-a)\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}$$

$$E_t = -\frac{3}{80} h^5 f^{(4)}(\xi)$$

or, because $h = (b - a)/3$:

$$E_t = -\frac{(b-a)^5}{6480} f^{(4)}(\xi)$$

Simpson's 3/8 Rule

Problem Statement. **(a)** Use Simpson's 3/8 rule to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. **(b)** Use it in conjunction with Simpson's 1/3 rule to integrate the same function for five segments.

Solution. **(a)** A single application of Simpson's 3/8 rule requires four equally spaced points:

$$f(0) = 0.2 \qquad f(0.2667) = 1.432724$$
$$f(0.5333) = 3.487177 \qquad f(0.8) = 0.232$$

Using Eq. (19.28):

$$I = 0.8 \frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8} = 1.51917$$

(b) The data needed for a five-segment application ($h = 0.16$) are

$$f(0) = 0.2 \qquad f(0.16) = 1.296919$$
$$f(0.32) = 1.743393 \qquad f(0.48) = 3.186015$$
$$f(0.64) = 3.181929 \qquad f(0.80) = 0.232$$

The integral for the first two segments is obtained using Simpson's 1/3 rule:

$$I = 0.32 \frac{0.2 + 4(1.296919) + 1.743393}{6} = 0.3803237$$

For the last three segments, the 3/8 rule can be used to obtain

$$I = 0.48 \frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8} = 1.264754$$

The total integral is computed by summing the two results:

$$I = 0.3803237 + 1.264754 = 1.645077$$