ODE Solution

Monday, 30 December 2024 9:45 pm

Ordinary Differential Equations

Equations which are composed of an unknown function and its derivatives are called *differential equations*.

Ordinary Differential Equations (ODEs) involve one or more ordinary derivatives of unknown functions with respect to one independent variable

Examples:

$$\frac{dv(t)}{dt} - v(t) = e^t$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

x(t): unknown function

t: independent variable

Examples:

$$\frac{dx(t)}{dt} - x(t) = e^t$$ First order ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$ Second order ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + 2x^4(t) = 1$$ Second order ODE

Examples:

$$\frac{dx(t)}{dt} - x(t) = e^{t}$$

Linear ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2t^2x(t) = \cos(t)$$

Linear ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + \sqrt{x(t)} = 1$$

Non-linear ODE

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

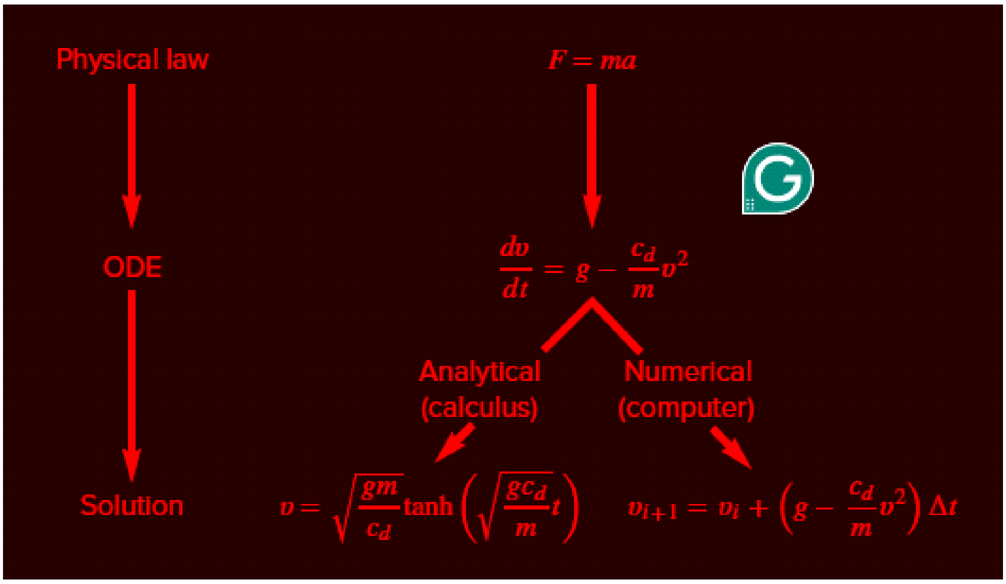
$$x(0) = 1,\quad \dot{x}(0) = 2.5$$

Boundary-Value Problems

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1,\quad x(2) = 1.5$$



$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Initial-Value Problems

EULER'S METHOD

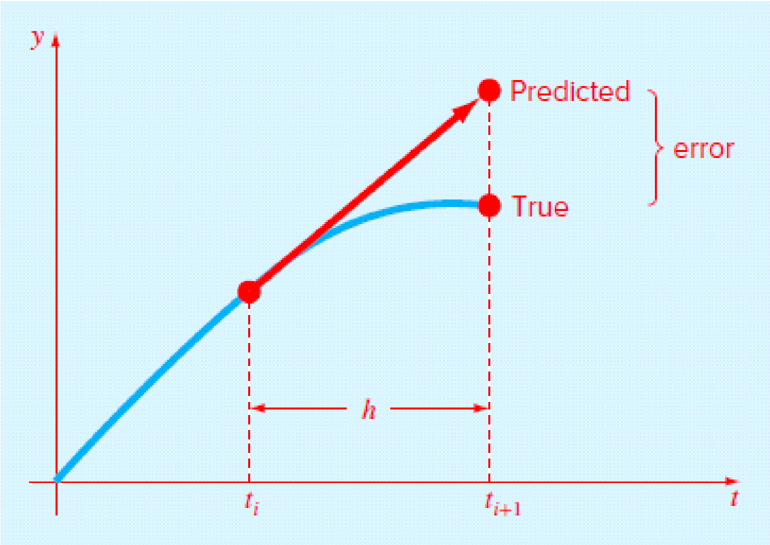
The first derivative provides a direct estimate of the slope at t_i (Fig. 22.1):

$$\phi = f(t_i, y_i)$$

where $f(t_i, y_i)$ is the differential equation evaluated at t_i and y_i. This estimate can be substituted into Eq. (22.1):

$$y_{i+1} = y_i + f(t_i, y_i)h \tag{22.5}$$

This formula is referred to as *Euler's method* (or the Euler-Cauchy or point-slope method). A new value of y is predicted using the slope (equal to the first derivative at the original value of t) to extrapolate linearly over the step size h (Fig. 22.1).



Euler's Method

Problem Statement. Use Euler's method to integrate $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 4 with a step size of 1. The initial condition at $t = 0$ is $y = 2$. Note that the exact solution can be determined analytically as

$$y = \frac{4}{1.3}(e^{0.8t} - e^{-0.5t}) + 2e^{-0.5t}$$

Solution. Equation (22.5) can be used to implement Euler's method:

$$y(1) = y(0) + f(0, 2)(1)$$

where $y(0) = 2$ and the slope estimate at $t = 0$ is

$$f(0, 2) = 4e^0 - 0.5(2) = 3$$

Therefore,

$$y(1) = 2 + 3(1) = 5$$

The true solution at $t = 1$ is

$$y = \frac{4}{1.3}(e^{0.8(1)} - e^{-0.5(1)}) + 2e^{-0.5(1)} = 6.19463$$

Thus, the percent relative error is

$$\varepsilon_t = \left|\frac{6.19463 - 5}{6.19463}\right| \times 100\% = 19.28\%$$

For the second step:

$$y(2) = y(1) + f(1, 5)(1)$$
$$= 5 + [4e^{0.8(1)} - 0.5(5)]\,(1) = 11.40216$$

TABLE 22.1 Comparison of true and numerical values of the integral of $y' = 4e^{0.8t} - 0.5y$, with the initial condition that $y = 2$ at $t = 0$. The numerical values were computed using Euler's method with a step size of 1.

t	y_{true}	y_{Euler}	$\lvert\varepsilon_t\rvert$ (%)
0	2.00000	2.00000	
1	6.19463	5.00000	19.28
2	14.84392	11.40216	23.19
3	33.67717	25.51321	24.24
4	75.33896	56.84931	24.54

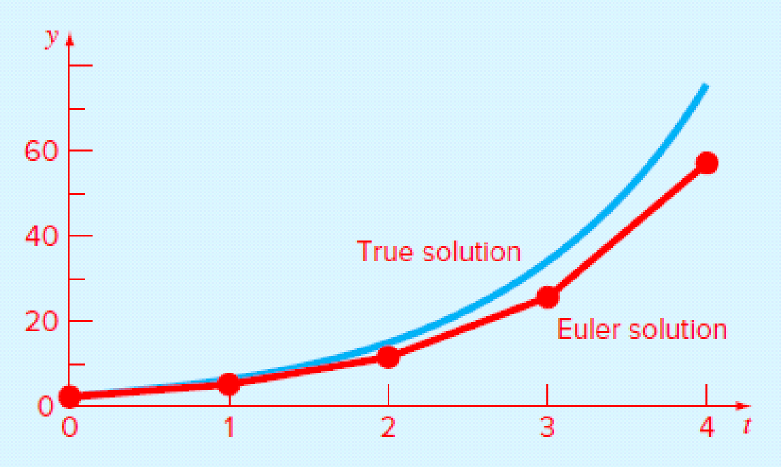


FIGURE 22.2
Comparison of the true solution with a numerical solution using Euler's method for the integral of $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 4 with a step size of 1.0. The initial condition at $t = 0$ is $y = 2$.

The true solution at $t = 2.0$ is 14.84392 and, therefore, the true percent relative error is 23.19%. The computation is repeated, and the results compiled in Table 22.1 and Fig. 22.2. Note that although the computation captures the general trend of the true solution, the error is considerable. As discussed in the next section, this error can be reduced by using a smaller step size.

22.2.1 Error Analysis for Euler's Method

The numerical solution of ODEs involves two types of error (recall Chap. 4):

1. *Truncation*, or discretization, errors caused by the nature of the techniques employed to approximate values of y.
2. *Roundoff* errors caused by the limited numbers of significant digits that can be retained by a computer.

22.2.2 Stability of Euler's Method

$$\frac{dy}{dt} = -ay$$

If $y(0) = y_0$, calculus can be used to determine the solution as

$$y = y_0 e^{-at}$$

Thus, the solution starts at y_0 and asymptotically approaches zero.

Now suppose that we use Euler's method to solve the same problem numerically:

$$y_{i+1} = y_i + \frac{dy_i}{dt} h$$

Substituting Eq. (22.12) gives

$$y_{i+1} = y_i - ay_i h$$

or

$$y_{i+1} = y_i (1 - ah) \qquad (22.1$$

Use Euler's method to solve the I.V.P

$$y' = \frac{t-y}{2} \quad \text{on } [0,3] \text{ with } y(0) = 1.$$

Compare solutions for $h = 1$, $1/2$, $1/4$, and $1/8$.

solution:

The step size $h = 0.25$, the calculations are

$$y_{k+1} = y_k + h\, f(t_k, y_k)$$

$$y_1 = 1.0 + 0.25\left(\frac{0.0 - 1.0}{2}\right) = 0.875$$

$$y_2 = 0.875 + 0.25\left(\frac{0.25 - 0.875}{2}\right) = 0.796875$$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9375	0.943239
0.25			0.875	0.886719	0.897491
0.375				0.846924	0.862087
0.50		0.75	0.796875	0.817429	0.836402
0.75			0.759766	0.786802	0.811868
1.00	0.5	0.6875	0.758545	0.790158	0.819592
1.50		0.765625	0.846386	0.882855	0.917100
2.00	0.75	0.949219	1.030827	1.068222	1.103638
2.50		1.211914	1.289227	1.325176	1.359514
3.00	1.375	1.533936	1.604252	1.637429	1.669390

Use Euler method to solve the ODE:

$$\frac{dy}{dx} = 1 + x^2, \qquad y(1) = -4$$

to determine $y(1.01)$, $y(1.02)$ and $y(1.03)$.

solution:

$$f(x,y)=1+x^2\,, \quad x_0=1,\ \ y_0=-4,\ \ h=0.01$$

Euler Method

$$y_{k+1}=y_k+h\,f(x_k,y_k)$$

$$Step1: \quad y_1=y_0+h\,f(x_0,y_0)=-4+0.01(1+(1)^2)=-3.98$$

$$Step2: \quad y_2=y_1+h\,f(x_1,y_1)=-3.98+0.01\left(1+(1.01)^2\right)=-3.9598$$

$$Step3: \quad y_3=y_2+h\,f(x_2,y_2)=-3.9598+0.01\left(1+(1.02)^2\right)=-3.9394$$

$$f(x,y)=1+x^2\,, \quad x_0=1,\ \ y_0=-4,\ \ h=0.01$$

Summary of the result:

k	x_k	y_k
0	1.00	-4.00
1	1.01	-3.98
2	1.02	-3.9595
3	1.03	-3.9394

Let $h = 0.2$ and do two steps by hand calculation. Then let $h = 0.1$ and do four steps by hand calculation.

$$y' = t^2 - y \quad \text{with} \quad y(0) = 1$$

Euler Method

$$y_{k+1} = y_k + h\, f(t_k, y_k)$$

$$y_1 = y_0 + h(t_0^{\,2} - y_0)$$

$$y_2 = y_1 + h(t_1^{\,2} - y_1)$$

$$y_1 = 1 + 0.2(0^2 - 1)$$

$$y_2 = 0.8 + 0.2(0.2^2 - 0.8)$$

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RUNGE-KUTTA METHODS

Runge-Kutta (RK) methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives. Many variations exist but all can be cast in the generalized form of Eq. (22.4):

$$y_{i+1} = y_i + \phi h \tag{22.33}$$

where ϕ is called an *increment function,* which can be interpreted as a representative slope over the interval. The increment function can be written in general form as

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n \tag{22.34}$$

where the a's are constants and the k's are

$$k_1 = f(t_i, y_i) \tag{22.34a}$$

$$k_2 = f(t_i + p_1 h, y_i + q_{11} k_1 h) \tag{22.34b}$$

$$k_3 = f(t_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h) \tag{22.34c}$$

$$\vdots$$

$$k_n = f(t_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h) \tag{22.34d}$$

22.4.2 Classical Fourth-Order Runge-Kutta Method

The most popular RK methods are fourth order. As with the second-order approaches, there are an infinite number of versions. The following is the most commonly used form, and we therefore call it the *classical fourth-order RK method:*

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \tag{22.44}$$

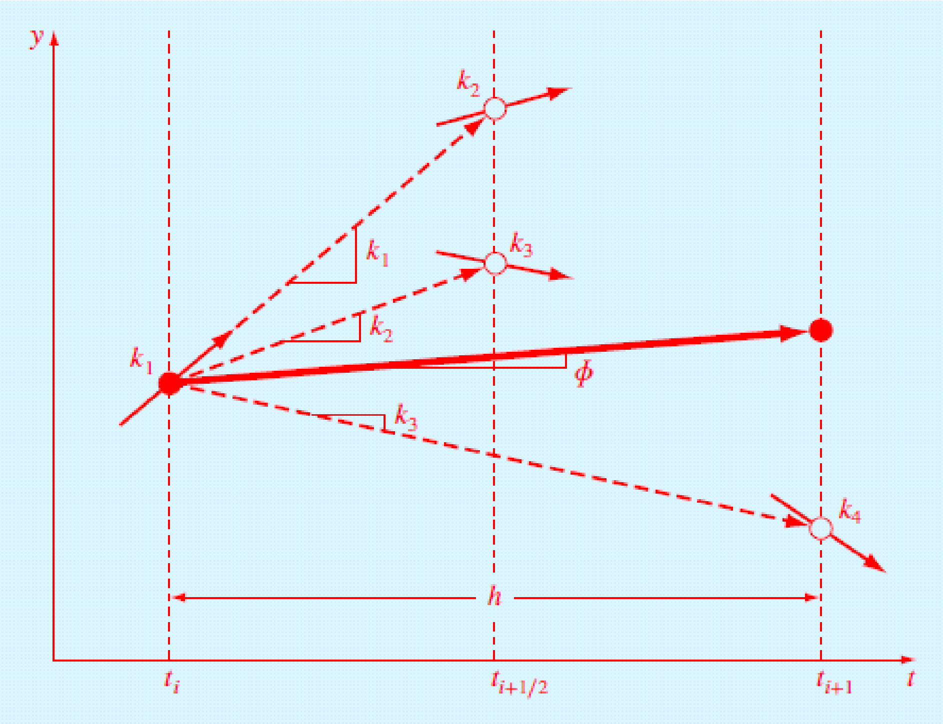

FIGURE 22.7
Graphical depiction of the slope estimates comprising the fourth-order RK method.

where

$$k_1 = f(t_i, y_i)$$

$$k_2 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right)$$

$$k_3 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right)$$

$$k_4 = f(t_i + h, y_i + k_3h)$$

Classical Fourth-Order RK Method

Problem Statement. Employ the classical fourth-order RK method to integrate $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 1 using a step size of 1 with $y(0) = 2$.

Solution. For this case, the slope at the beginning of the interval is computed as

$$k_1 = f(0, 2) = 4e^{0.8(0)} - 0.5(2) = 3$$

This value is used to compute a value of y and a slope at the midpoint:

$$y(0.5) = 2 + 3(0.5) = 3.5$$
$$k_2 = f(0.5, 3.5) = 4e^{0.8(0.5)} - 0.5(3.5) = 4.217299$$

This slope in turn is used to compute another value of y and another slope at the midpoint:

$$y(0.5) = 2 + 4.217299(0.5) = 4.108649$$
$$k_3 = f(0.5, 4.108649) = 4e^{0.8(0.5)} - 0.5(4.108649) = 3.912974$$

Next, this slope is used to compute a value of y and a slope at the end of the interval:

$$y(1.0) = 2 + 3.912974(1.0) = 5.912974$$
$$k_4 = f(1.0, 5.912974) = 4e^{0.8(1.0)} - 0.5(5.912974) = 5.945677$$

Finally, the four slope estimates are combined to yield an average slope. This average slope is then used to make the final prediction at the end of the interval.

$$\phi = \frac{1}{6}[3 + 2(4.217299) + 2(3.912974) + 5.945677] = 4.201037$$

$$y(1.0) = 2 + 4.201037(1.0) = 6.201037$$

which compares favorably with the true solution of 6.194631 ($\varepsilon_t = 0.103\%$).

Use the RK4 method to solve $y' = (t - y)/2$ on [0, 3] with $y(0) = 1$. Compare solutions for h = 1, 1/2, 1/4, and 1/8.

Solution:

For the step size $h = 0.25$, a sample calculation is

$$y_{k+1} = y_k + \frac{h(f_1 + 2f_2 + 2f_3 + f_4)}{6}$$

$$f_1 = f(t_k, y_k), \qquad f_1 = \frac{0.0 - 1.0}{2} = -0.5,$$

$$f_2 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_1\right),$$

$$f_2 = \frac{0.125 - (1 + 0.25(0.5)(-0.5))}{2} = -0.40625,$$

$$f_3 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_2\right),$$

$$f_3 = \frac{0.125 - (1 + 0.25(0.5)(-0.40625))}{2} = -0.4121094,$$

$$f_4 = f(t_k + h, y_k + h f_3).$$

$$f_4 = \frac{0.25 - (1 + 0.25(-0.4121094))}{2} = -0.3234863,$$

$$y_{k+1} = y_k + \frac{h(f_1 + 2f_2 + 2f_3 + f_4)}{6}$$

$$y_1 = 1.0 + 0.25\left(\frac{-0.5 + 2(-0.40625) + 2(-0.4121094) - 0.3234863}{6}\right)$$

$$= 0.8974915.$$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9432392	0.9432392
0.25			0.8974915	0.8974908	0.8974917
0.375				0.8620874	0.8620874
0.50		0.8364258	0.8364037	0.8364024	0.8364023
0.75			0.8118696	0.8118679	0.8118678
1.00	0.8203125	0.8196285	0.8195940	0.8195921	0.8195920
1.50		0.9171423	0.9171021	0.9170998	0.9170997
2.00	1.1045125	1.1036826	1.1036408	1.1036385	1.1036383
2.50		1.3595575	1.3595168	1.3595145	1.3595144
3.00	1.6701860	1.6694308	1.6693928	1.6693906	1.6693905

$$\frac{dy}{dt} = 1 + y + t^2$$

$$y(0) = 0.5$$

$$h = 0.2$$

$$Use\ RK4\ to\ compute\ \ y(0.2)\ and\ \ y(0.4)$$

Problem:

$$\frac{dy}{dt} = 1 + y + t^2, \qquad y(0) = 0.5$$

$$Use\ RK4\ to\ find\ \ y(0.2), y(0.4)$$

$$h = 0.2$$

$$f(t, y) = 1 + y + t^2$$

$$t_0 = 0, \quad y_0 = 0.5$$

Step 1

$$f_1 = f(t_0, y_0) = (1 + y_0 + t_0^{\,2}) = 1.5$$

$$f_2 = f(t_0 + \frac{1}{2}h, y_0 + \frac{1}{2}f_1 h) = 1 + (y_0 + 0.15) + (t_0 + 0.1)^2 = 1.64$$

$$f_3 = f(t_0 + \frac{1}{2}h, y_0 + \frac{1}{2}f_2 h) = 1 + (y_0 + 0.164) + (t_0 + 0.1)^2 = 1.654$$

$$f_4 = f(t_0 + h, y_0 + f_3 h) = 1 + (y_0 + 0.16545) + (t_0 + 0.2)^2 = 1.7908$$

$$y_1 = y_0 + \frac{h}{6}(f_1 + 2f_2 + 2f_3 + f_4) = 0.8293$$

Problem:

$$\frac{dy}{dt} = 1 + y + t^2, \qquad y(0) = 0.5$$

$$Use\ RK4\ to\ find\ \ y(0.2), y(0.4)$$

$$h = 0.2$$

$$f(t, y) = 1 + y + t^2$$

$$t_1 = 0.2, \quad y_1 = 0.8293$$

Step 2

$$f_1 = f(t_1, y_1) = 1.7893$$

$$f_2 = f(t_1 + \frac{1}{2}h, y_1 + \frac{1}{2}f_1 h) = 1.9182$$

$$f_3 = f(t_1 + \frac{1}{2}h, y_1 + \frac{1}{2}f_2 h) = 1.9311$$

$$f_4 = f(t_1 + h, y_1 + f_3 h) = 2.0555$$

$$y_2 = y_1 + \frac{0.2}{6}(f_1 + 2f_2 + 2f_3 + f_4) = 1.2141$$

Problem:

$$\frac{dy}{dt} = 1 + y + t^2, \qquad y(0) = 0.5$$

$Use\ RK4\ to\ find\ y(0.2), y(0.4)$

Summary of the solution

x_i	y_i
0.0	0.5
0.2	0.8293
0.4	1.2141

Use the RK4 method to solve $y' = t^2 - y$ on [0,0.2] with $y(0) = 1$. Compare solutions for $h = 0.05$, 0.1, 0.2.

SYSTEMS OF EQUATIONS

Many practical problems in engineering and science require the solution of a system of simultaneous ordinary differential equations rather than a single equation. Such systems may be represented generally as

$$
\begin{aligned}
\frac{dy_1}{dt} &= f_1(t, y_1, y_2, \ldots, y_n) \\
\frac{dy_2}{dt} &= f_2(t, y_1, y_2, \ldots, y_n) \\
&\vdots \\
\frac{dy_n}{dt} &= f_n(t, y_1, y_2, \ldots, y_n)
\end{aligned}
\tag{22.46}
$$

The solution of such a system requires that n initial conditions be known at the starting value of t.

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = g - \frac{c_d}{m}v^2$$

22.5.1 Euler's Method

Solving Systems of ODEs with Euler's Method

Problem Statement. Solve for the velocity and position of the free-falling bungee jumper using Euler's method. Assuming that at $t = 0$, $x = v = 0$, and integrate to $t = 10$ s with a step size of 2 s. As was done previously in Examples 1.1 and 1.2, the gravitational acceleration is 9.81 m/s^2, and the jumper has a mass of 68.1 kg with a drag coefficient of 0.25 kg/m.

Recall that the analytical solution for velocity is [Eq. (1.9)]:

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}}\, t\right)$$

This result can be substituted into Eq. (22.47) which can be integrated to determine an analytical solution for distance as

$$x(t) = \frac{m}{c_d} \ln\left[\cosh\left(\sqrt{\frac{gc_d}{m}}\, t\right)\right]$$

Use these analytical solutions to compute the true relative errors of the results.

Solution. The ODEs can be used to compute the slopes at $t = 0$ as

$$\frac{dx}{dt} = 0$$

$$\frac{dv}{dt} = 9.81 - \frac{0.25}{68.1}(0)^2 = 9.81$$

Euler's method is then used to compute the values at $t = 2$ s,

$$x = 0 + 0(2) = 0$$

$$v = 0 + 9.81(2) = 19.62$$

The analytical solutions can be computed as $x(2) = 19.16629$ and $v(2) = 18.72919$. Thus, the percent relative errors are 100% and 4.756%, respectively.

The process can be repeated to compute the results at $t = 4$ as

$$x = 0 + 19.62(2) = 39.24$$

$$v = 19.62 + \left(9.81 - \frac{0.25}{68.1}(19.62)^2\right) 2 = 36.41368$$

Proceeding in a like manner gives the results displayed in Table 22.3.

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TABLE 22.3 Distance and velocity of a free-falling bungee jumper as computed numerically with Euler's method.

t	x_{true}	v_{true}	x_{Euler}	v_{Euler}	$\varepsilon_t(x)$	$\varepsilon_t(v)$
0	0	0	0	0		
2	19.1663	18.7292	0	19.6200	100.00%	4.76%
4	71.9304	33.1118	39.2400	36.4137	45.45%	9.97%
6	147.9462	42.0762	112.0674	46.2983	24.25%	10.03%
8	237.5104	46.9575	204.6640	50.1802	13.83%	6.86%
10	334.1782	49.4214	305.0244	51.3123	8.72%	3.83%

22.5.2 Runge-Kutta Methods

Solving Systems of ODEs with the Fourth-Order RK Method

Problem Statement. Use the fourth-order RK method to solve for the same problem we addressed in Example 22.4.

Solution. First, it is convenient to express the ODEs in the functional format of Eq. (22.46) as

$$\frac{dx}{dt} = f_1(t, x, v) = v$$

$$\frac{dv}{dt} = f_2(t, x, v) = g - \frac{c_d}{m}v^2$$

The first step in obtaining the solution is to solve for all the slopes at the beginning of the interval:

$$k_{1,1} = f_1(0, 0, 0) = 0$$

$$k_{1,2} = f_2(0, 0, 0) = 9.81 - \frac{0.25}{68.1}(0)^2 = 9.81$$

where $k_{i,j}$ is the ith value of k for the jth dependent variable. Next, we must calculate the first values of x and v at the midpoint of the first step:

$$x(1) = x(0) + k_{1,1}\frac{h}{2} = 0 + 0\frac{2}{2} = 0$$

$$v(1) = v(0) + k_{1,2}\frac{h}{2} = 0 + 9.81\frac{2}{2} = 9.81$$

which can be used to compute the first set of midpoint slopes:

$$k_{2,1} = f_1(1, 0, 9.81) = 9.8100$$

$$k_{2,2} = f_2(1, 0, 9.81) = 9.4567$$

These are used to determine the second set of midpoint predictions:

$$x(1) = x(0) + k_{2,1}\frac{h}{2} = 0 + 9.8100\frac{2}{2} = 9.8100$$

$$v(1) = v(0) + k_{2,2}\frac{h}{2} = 0 + 9.4567\frac{2}{2} = 9.4567$$

which can be used to compute the second set of midpoint slopes:

$$k_{3,1} = f_1(1, 9.8100, 9.4567) = 9.4567$$

$$k_{3,2} = f_2(1, 9.8100, 9.4567) = 9.4817$$

These are used to determine the predictions at the end of the interval:

$$x(2) = x(0) + k_{3,1}h = 0 + 9.4567(2) = 18.9134$$

$$v(2) = v(0) + k_{3,2}h = 0 + 9.4817(2) = 18.9634$$

which can be used to compute the endpoint slopes:

$$k_{4,1} = f_1(2, 18.9134, 18.9634) = 18.9634$$

$$k_{4,2} = f_2(2, 18.9134, 18.9634) = 8.4898$$

The values of k can then be used to compute [Eq. (22.44)]:

$$x(2) = 0 + \frac{1}{6}[0 + 2(9.8100 + 9.4567) + 18.9634]\,2 = 19.1656$$

$$v(2) = 0 + \frac{1}{6}[9.8100 + 2(9.4567 + 9.4817) + 8.4898]\,2 = 18.7256$$

TABLE 22.4 Distance and velocity of a free-falling bungee jumper as computed numerically with the fourth-order RK method.

t	x_{true}	v_{true}	x_{RK4}	v_{RK4}	$\varepsilon_t(x)$	$\varepsilon_t(v)$
0	0	0	0	0		
2	19.1663	18.7292	19.1656	18.7256	0.004%	0.019%
4	71.9304	33.1118	71.9311	33.0995	0.001%	0.037%
6	147.9462	42.0762	147.9521	42.0547	0.004%	0.051%
8	237.5104	46.9575	237.5104	46.9345	0.000%	0.049%
10	334.1782	49.4214	334.1626	49.4027	0.005%	0.038%