Cal1 Improper Integral

Wednesday, 12 June 2024 10:51 pm

Indefinite Integral	Proper Definite Integral	Improper Definite Integral
$\int e^{-t}\,dt$	$\int_0^1 e^{-t}\,dt$	$\int_0^\infty e^{-t}\,dt$

7.8 Improper Integrals

Definition:

The definite integral $\int_a^b f(x)dx$ is called an improper integral if

(a) At least one of the limits of integration is infinite, or

(b) The integrand *f*(*x*) has one or more points of discontinuity on the interval [*a*, *b*].

WHAT IS AN IMPROPER INTEGRAL?

- AN INTEGRAL WITH UPPER AND LOWER LIMITS THAT GO TO INFINITY IN ONE DIRECTION OR BOTH

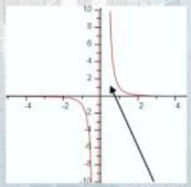
$$\int_{-2}^{\infty} \sin x\,dx$$

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

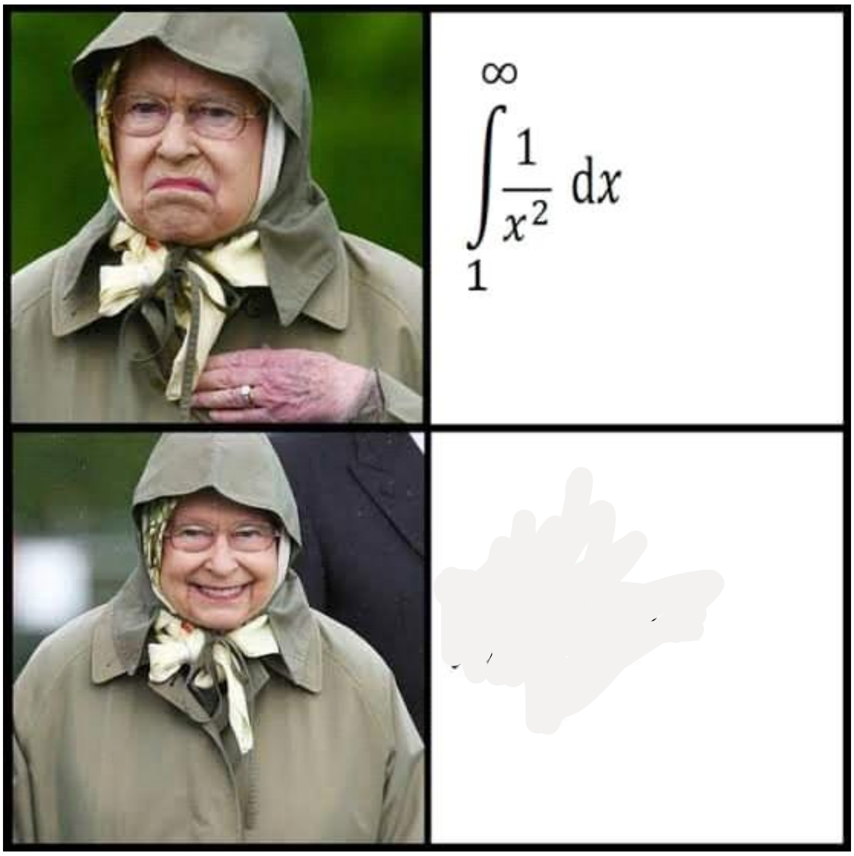
$$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx$$

- TYPE II: LOOKS NORMAL BUT CANNOT BE EVALUATED WITH FTC II BECAUSE OF A DISCONTINUITY.

$$\int_{-2}^{3} \frac{1}{x^3}\,dx$$



VERTICAL ASYMPTOTE


$\int_1^\infty \frac{1}{x^2} \, \mathrm{d}x$

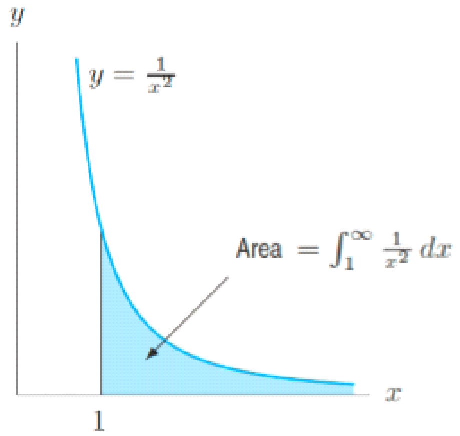
Improper Integrals

Our original discussion of the definite integral $\int_a^b f(x)dx$ assumed that the interval $a \le x \le b$ was of finite length and that f was continuous. Integrals that arise in applications do not necessarily have these nice properties.

Consider the integral

$$\int_1^\infty \frac{1}{x^2}\,dx.$$

We have not come across limits of integration like these before and this cannot be a simple computation since infinity is not a number we can plug in to get a value. Graphically, we can observe what the integral above suggests.



The graph goes on forever without touching the x-axis and this would make an area calculation seem impossible. However, with limits, we have a way forward!

First compute the definite integral $\int_1^b \left(\frac{1}{x^2}\right)\cdot dx$ and get an expression in terms of b.

Now take the limit as $b \to \infty$, i.e.

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx =$$

We can now see that as b gets larger and larger, the value of the definite integral *converges* to 1.

The integral $\int_1^\infty \left(\frac{1}{x^2}\right)\cdot dx$ is an example of an **improper integral.**

Of course, in another example, we might not get a finite limit as b gets larger and larger. In that case we say the improper integral *diverges*.

Suppose $f(x)$ is positive for $x \geq a$.
If $\lim_{b\to\infty} \int_a^b f(x)\,dx$ is a finite number, we say that $\int_a^\infty f(x)\,dx$ **converges** and define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

Otherwise, we say that $\int_a^\infty f(x)\,dx$ **diverges**. We define $\int_{-\infty}^b f(x)\,dx$ similarly.

Next, consider the improper integral $\int_1^\infty \left(1/\sqrt{x}\right)\cdot dx$. Evaluate this integral using the technique used before and determine whether this integral *converges* or *diverges*.

Once you're done, look up as I point something out to you…

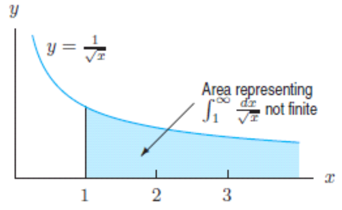


Figure 7.17: $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges

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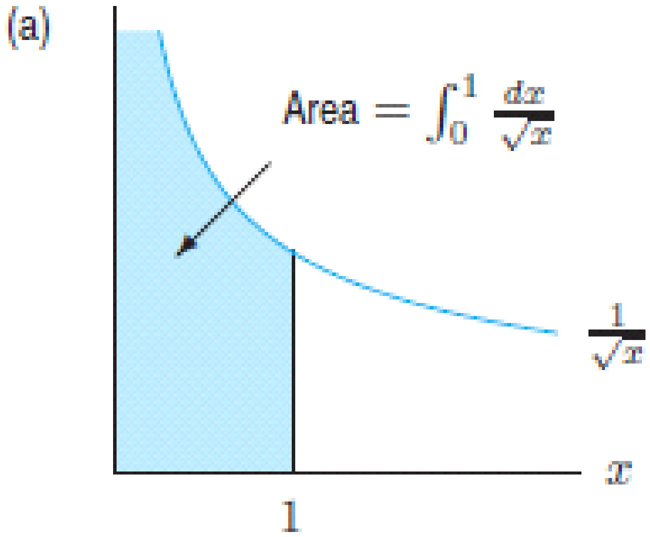
Example 2 Find $\int_0^{\infty} e^{-5x}\, dx$.

Another Type of Improper Integral: When the Integrand Becomes Infinite



Example 5 Investigate the convergence of $\int_0^2 \frac{1}{(x-2)^2}\, dx$.