Integration Application Questions & Answers

Wednesday, 19 June 2024 8:28 pm

61. Water is pumped into a cylindrical tank, standing vertically, at a decreasing rate given at time t minutes by

$$r(t) = 120 - 6t \text{ ft}^3/\text{min} \quad \text{for } 0 \le t \le 10.$$

The tank has radius 5 ft and is empty when $t = 0$. Find the depth of water in the tank at $t = 4$.

The rate at which water is entering the tank (in volume per unit time) is

$$\frac{dV}{dt} = 120 - 6t \text{ ft}^3/\text{min}.$$

Thus, the total quantity of water in the tank at time $t = 4$, in ft^3, is

$$V = \int_0^4 (120 - 6t)\, dt.$$

Since an antiderivative to $120 - 6t$ is

$$120t - 3t^2,$$

we have

$$\begin{aligned} V &= \int_0^4 (120 - 6t)\, dt = (120t - 3t^2)\Big|_0^4 \\ &= (120 \cdot 4 - 3 \cdot 4^2) - (120 \cdot 0 - 3 \cdot 0^2) \\ &= 432 \text{ ft}^3. \end{aligned}$$

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The radius is 5 feet, so if the height is h ft, the volume is $V = \pi 5^2 h = 25\pi h$. Thus, at time $t = 4$, we have $V = 432$, so

$$\begin{aligned} 432 &= 25\pi h \\ h &= \frac{432}{25\pi} = 5.500 \text{ ft.} \end{aligned}$$

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62. A car moves along a straight line with velocity, in feet/second, given by

$$v(t) = 6 - 2t \quad \text{for } t \ge 0.$$

(a) Describe the car's motion in words. (When is it moving forward, backward, and so on?)

(b) The car's position is measured from its starting point. When is it farthest forward? Backward?

(c) Find s, the car's position measured from its starting point, as a function of time.

(a) The formula $v = 6 - 2t$ implies that $v > 0$ (the car is moving forward) if $0 \le t < 3$ and that $v < 0$ (the car is moving backward) if $t > 3$. When $t = 3$, $v = 0$, so the car is not moving at the instant $t = 3$. The car is decelerating when $|v|$ is decreasing; since v decreases (from 6 to 0) on the interval $0 \le t < 3$, the car decelerates on that interval. The car accelerates when $|v|$ is increasing, which occurs on the domain $t > 3$.

(b) The car moves forward on the interval $0 \le t < 3$, so it is furthest to the right at $t = 3$. For all $t > 3$, the car is decelerating. There is no upper bound on the car's distance behind its starting point since it is decelerating for all $t > 3$.

(c) Let $s(t)$ be the position of the car at time t. Then

$$v(t) = s'(t),$$

so $s(t)$ is an antiderivative of $v(t)$. Thus,

$$s(t) = \int v(t)\,dt = \int (6 - 2t)\,dt = 6t - t^2 + C.$$

Since the car's position is measured from its starting point, we have $s(0) = 0$, so $C = 0$. Thus, $s(t) = 6t - t^2$.

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64. In drilling an oil well, the total cost, C, consists of fixed costs (independent of the depth of the well) and marginal costs, which depend on depth; drilling becomes more expensive, per meter, deeper into the earth. Suppose the fixed costs are 1,000,000 riyals (the riyal is the unit of currency of Saudi Arabia), and the marginal costs are

$$C'(x) = 4000 + 10x \text{ riyals/meter},$$

where x is the depth in meters. Find the total cost of drilling a well x meters deep.

Since $C'(x) = 4000 + 10x$ we want to evaluate the indefinite integral

$$\int (4000 + 10x)\,dx = 4000x + 5x^2 + K$$

where K is a constant. Thus $C(x) = 5x^2 + 4000x + K$, and the fixed cost of 1,000,000 riyal means that $C(0) = 1{,}000{,}000 = K$. Therefore, the total cost is

$$C(x) = 5x^2 + 4000x + 1{,}000{,}000.$$

Since $C(x)$ depends on x^2, the square of the depth drilled, costs will increase dramatically when x grows large.

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65. Use the Fundamental Theorem to find the area under $f(x) = x^2$ between $x = 0$ and $x = 3$.

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$$\int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = 9 - 0 = 9.$$

67. Calculate the exact area above the graph of $y = \frac{1}{2}\left(\frac{3}{\pi}x\right)^2$ and below the graph of $y = \cos x$. The curves intersect at $x = \pm\pi/3$.

The area we want (the shaded area in Figure 6.28) is symmetric about the y-axis and so is given by

$$\begin{aligned}\text{Area } &= 2\int_0^{\pi/3}\left(\cos x - \frac{1}{2}\left(\frac{3}{\pi}x\right)^2\right)dx \\ &= 2\int_0^{\pi/3}\cos x\,dx - \int_0^{\pi/3}\frac{9}{\pi^2}x^2\,dx \\ &= 2\sin x\Big|_0^{\pi/3} - \frac{9}{\pi^2}\cdot\frac{x^3}{3}\Big|_0^{\pi/3} \\ &= 2\cdot\frac{\sqrt{3}}{2} - \frac{3}{\pi^2}\cdot\frac{\pi^3}{3^3} = \sqrt{3} - \frac{\pi}{9}.\end{aligned}$$

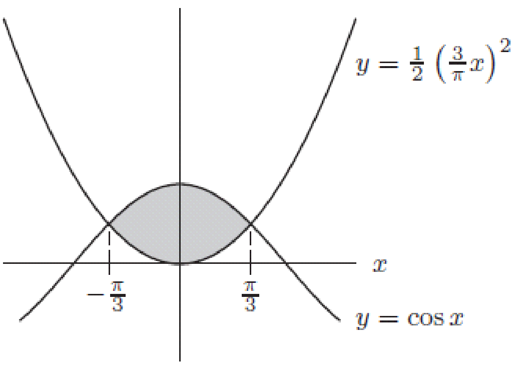

Figure 6.28

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68. Find the exact area of the shaded region in Figure 6.23 between $y = 3x^2 - 3$ and the x-axis.

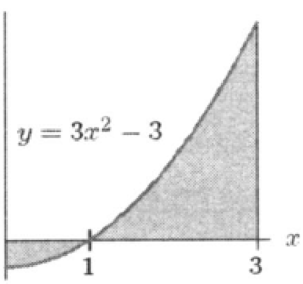

Figure 6.23

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Since $y < 0$ from $x = 0$ to $x = 1$ and $y > 0$ from $x = 1$ to $x = 3$, we have

$$\begin{aligned}
\text{Area} &= -\int_0^1 \left(3x^2 - 3\right)\, dx + \int_1^3 \left(3x^2 - 3\right)\, dx \\
&= -\left(x^3 - 3x\right)\Big|_0^1 + \left(x^3 - 3x\right)\Big|_1^3 \\
&= -(-2 - 0) + (18 - (-2)) = 2 + 20 = 22.
\end{aligned}$$

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69. **(a)** Find the exact area between $f(x) = x^3 - 7x^2 + 10x$, the x-axis, $x = 0$, and $x = 5$.

(b) Find $\int_0^5 (x^3 - 7x^2 + 10x)\, dx$ exactly and interpret this integral in terms of areas.

(a) See Figure 6.29. Since $f(x) > 0$ for $0 < x < 2$ and $f(x) < 0$ for $2 < x < 5$, we have

$$\begin{aligned}
\text{Area} &= \int_0^2 f(x)\, dx - \int_2^5 f(x)\, dx \\
&= \int_0^2 (x^3 - 7x^2 + 10x)\, dx - \int_2^5 (x^3 - 7x^2 + 10x)\, dx \\
&= \left(\frac{x^4}{4} - \frac{7x^3}{3} + 5x^2\right)\Big|_0^2 - \left(\frac{x^4}{4} - \frac{7x^3}{3} + 5x^2\right)\Big|_2^5 \\
&= \left[\left(4 - \frac{56}{3} + 20\right) - (0 - 0 + 0)\right] - \left[\left(\frac{625}{4} - \frac{875}{3} + 125\right) - \left(4 - \frac{56}{3} + 20\right)\right] \\
&= \frac{253}{12}.
\end{aligned}$$

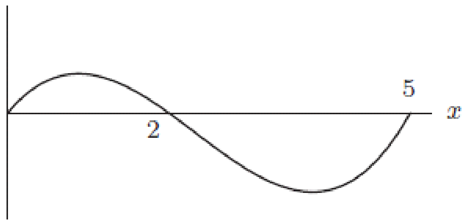

Figure 6.29: Graph of $f(x) = x^3 - 7x^2 + 10x$

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(b) Calculating $\int_0^5 f(x)\,dx$ gives

$$\begin{aligned}\int_0^5 f(x)\,dx &= \int_0^5 (x^3 - 7x^2 + 10x)\,dx \\ &= \left(\frac{x^4}{4} - \frac{7x^3}{3} + 5x^2\right)\Big|_0^5 \\ &= \left(\frac{625}{4} - \frac{875}{3} + 125\right) - (0 - 0 + 0) \\ &= -\frac{125}{12}.\end{aligned}$$

This integral measures the difference between the area above the x-axis and the area below the x-axis. Since the definite integral is negative, the graph of $f(x)$ lies more below the x-axis than above it. Since the function crosses the axis at $x = 2$,

$$\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx = \frac{16}{3} - \frac{63}{4} = \frac{-125}{12},$$

whereas

$$\text{Area} = \int_0^2 f(x)\,dx - \int_2^5 f(x)\,dx = \frac{16}{3} + \frac{64}{4} = \frac{253}{12}.$$

70. Find the exact area between the curve $y = e^x - 2$ and the x-axis for $0 \le x \le 2$.

The graph of $y = e^x - 2$ is below the x-axis at $x = 0$ and above the x-axis at $x = 2$. The graph crosses the axis where

$$\begin{aligned}e^x - 2 &= 0 \\ x &= \ln 2.\end{aligned}$$

See Figure 6.30. Thus we find the area by dividing the region at $x = \ln 2$:

$$\begin{aligned}\text{Area} &= -\int_0^{\ln 2} (e^x - 2)\,dx + \int_{\ln 2}^2 (e^x - 2)\,dx \\ &= (-e^x + 2x)\Big|_0^{\ln 2} + (e^x - 2x)\Big|_{\ln 2}^2 \\ &= -e^{\ln 2} + 2\ln 2 + e^0 + e^2 - 4 - (e^{\ln 2} - 2\ln 2) \\ &= -2e^{\ln 2} + 4\ln 2 - 3 + e^2 \\ &= -2\cdot 2 + 4\ln 2 - 3 + e^2 = e^2 + 4\ln 2 - 7.\end{aligned}$$

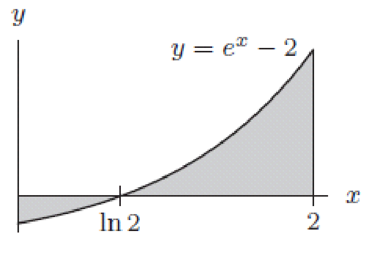

Figure 6.30

74. The area under $1/\sqrt{x}$ on the interval $1 \le x \le b$ is equal to 6. Find the value of b using the Fundamental Theorem.

Since the area under the curve is 6, we have

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = 2x^{1/2}\Big|_1^b = 2b^{1/2} - 2(1) = 6.$$

Thus $b^{1/2} = 4$ and $b = 16$.