Partial Derivative III: Chain Rule

Sunday, 4 August 2024 4:39 pm

The Chain Rule for $z = f(x,y), x = g(t), y = h(t)$

Since $z = f(g(t), h(t))$ is a function of t, we can consider the derivative dz/dt. The chain rule gives dz/dt in terms of the derivatives of f, g, and h. Since dz/dt represents the rate of change of z with t, we look at the change Δz generated by a small change, Δt.

We substitute the local linearizations

$$\Delta x \approx \frac{dx}{dt}\Delta t \quad \text{and} \quad \Delta y \approx \frac{dy}{dt}\Delta t$$

into the local linearization

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y,$$

yielding

$$\begin{aligned}\Delta z &\approx \frac{\partial z}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial z}{\partial y}\frac{dy}{dt}\Delta t \\ &= \left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\Delta t.\end{aligned}$$

Thus,

$$\frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Taking the limit as $\Delta t \to 0$, we get the following result.

> If f, g, and h are differentiable and if $z = f(x,y)$, and $x = g(t)$, and $y = h(t)$, then
>
> $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 2 Suppose that $z = f(x,y) = x \sin y$, where $x = t^2$ and $y = 2t + 1$. Let $z = g(t)$. Compute $g'(t)$ directly and using the chain rule.

Visualizing the Chain Rule with a Diagram

The diagram in Figure 14.50 provides a way of remembering the chain rule. It shows the chain of dependence: z depends on x and y, which in turn depend on t. Each line in the diagram is labeled with a derivative relating the variables at its ends.

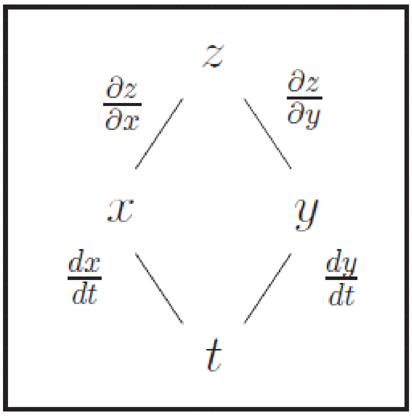

Figure 14.50: Diagram for $z = f(x, y)$, $x = g(t)$, $y = h(t)$. Lines represent dependence of z on x and y, and of x and y on t

Solution Since $z = g(t) = f(t^2, 2t+1) = t^2 \sin(2t+1)$, it is possible to compute $g'(t)$ directly by one-variable methods:

$$g'(t) = t^2 \frac{d}{dt}(\sin(2t+1)) + \left(\frac{d}{dt}(t^2)\right)\sin(2t+1) = 2t^2\cos(2t+1) + 2t\sin(2t+1).$$

The chain rule provides an alternative route to the same answer. We have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (\sin y)(2t) + (x\cos y)(2) = 2t\sin(2t+1) + 2t^2\cos(2t+1).$$

To find the rate of change of one variable with respect to another in a chain of composed differentiable functions:

- Draw a diagram expressing the relationship between the variables, and label each link in the diagram with the derivative relating the variables at its ends.
- For each path between the two variables, multiply together the derivatives from each step along the path.
- Add the contributions from each path.

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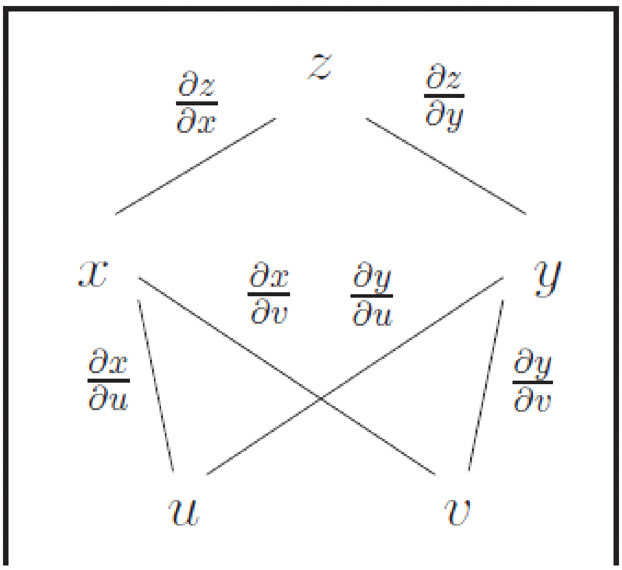

Figure 14.51: Diagram for $z = f(x, y)$, $x = g(u, v)$, $y = h(u, v)$. Lines represent dependence of z on x and y, and of x and y on u and v

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If f, g, h are differentiable and if $z = f(x, y)$, with $x = g(u, v)$ and $y = h(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u},$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

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Example 4 Let $w = x^2 e^y$, $x = 4u$, and $y = 3u^2 - 2v$. Compute $\partial w/\partial u$ and $\partial w/\partial v$ using the chain rule.

Solution Using the previous result, we have

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = 2xe^y(4) + x^2e^y(6u) = (8x + 6x^2u)e^y$$
$$= (32u + 96u^3)e^{3u^2-2v}.$$

Similarly,

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = 2xe^y(0) + x^2e^y(-2) = -2x^2e^y$$
$$= -32u^2e^{3u^2-2v}.$$

Example 3 The capacity, C, of a communication channel, such as a telephone line, to carry information depends on the ratio of the signal strength, S, to the noise, N. For some positive constant k,

$$C = k\ln\left(1 + \frac{S}{N}\right).$$

Suppose that the signal and noise are given as a function of time, t in seconds, by

$$S(t) = 4 + \cos(4\pi t) \qquad N(t) = 2 + \sin(2\pi t).$$

What is dC/dt one second after transmission started? Is the capacity increasing or decreasing at that instant?

EXAMPLE: If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, Find $\frac{dz}{dt}$ when $t = 0$.

EXAMPLE: The pressure P, volume V, and temperature T of a mole of an ideal gas are related by the equation $PV = 8.31\,T$. Find the rate at which Pressure 'P' is changing when the temperature is 300K and increasing at a rate of 0.1 K/s and volume is 100L and increasing at a rate of 0.2 L/s.

EXAMPLE: $z = e^x \sin y$ where $x = st^2$ and $y = s^2t$. Find dz/ds and dz/dt

Q $f(x,y) = e^{xy}$

when $x(u,v) = 3u \sin v$

and $y(u,v) = 4v^2u$

Find $f_u(x,y)$ and $f_v(x,y)$

EXAMPLE: If $z = f(x,y)$ and

$x = r^2 + s^2$; $y = 2rs$

Find a) $\partial z/\partial r$; (b) $\partial^2 z/\partial r^2$