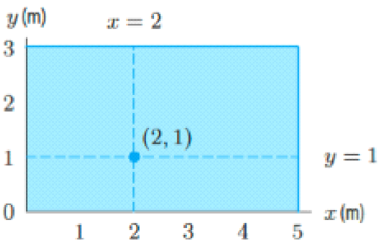
Partial Derivatives I: Application & Introduction

Sunday, 4 February 2024 8:17 pm

Say we have an unevenly heated metal rod.

x (m)	0	1	2	3	4	5
$u(x)$ (°C)	125	128	135	160	175	160

y (m)	3	85	90	**110**	135	155	180
	2	100	110	**120**	145	190	170
	1	**125**	**128**	**135**	**160**	**175**	**160**
	0	120	135	**155**	160	160	150
		0	1	2	3	4	5
				x (m)			



Now imagine an unevenly heated metal plate.

Partial Derivatives of f With Respect to x and y

For all points at which the limits exist, we define the **partial derivatives at the point** (a, b) by

$$f_x(a,b) = \begin{array}{c}\text{Rate of change of } f \text{ with respect to } x \\ \text{at the point } (a,b)\end{array} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h},$$

$$f_y(a,b) = \begin{array}{c}\text{Rate of change of } f \text{ with respect to } y \\ \text{at the point } (a,b)\end{array} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}.$$

If we let a and b vary, we have the **partial derivative functions** $f_x(x, y)$ and $f_y(x, y)$.

Just as with ordinary derivatives, there is an alternative notation:

Alternative Notation for Partial Derivatives

If $z = f(x, y)$, we can write

$$f_x(x,y) = \frac{\partial z}{\partial x} \quad \text{and} \quad f_y(x,y) = \frac{\partial z}{\partial y},$$

$$f_x(a,b) = \left.\frac{\partial z}{\partial x}\right|_{(a,b)} \quad \text{and} \quad f_y(a,b) = \left.\frac{\partial z}{\partial y}\right|_{(a,b)}.$$

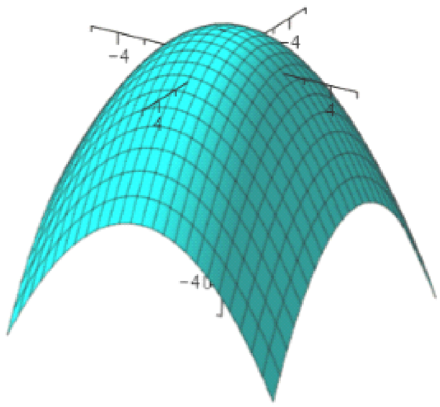


Figure 14.6 shows the contour diagram for the temperature $H(x,t)$ (in °C) in a room as a function of distance x (in meters) from a heater and time t (in minutes) after the heater has been turned on. What are the signs of $H_x(10,20)$ and $H_t(10,20)$? Estimate these partial derivatives and explain the answers in practical terms.

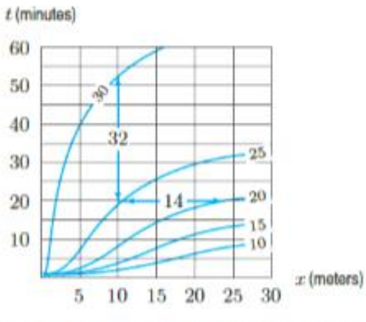
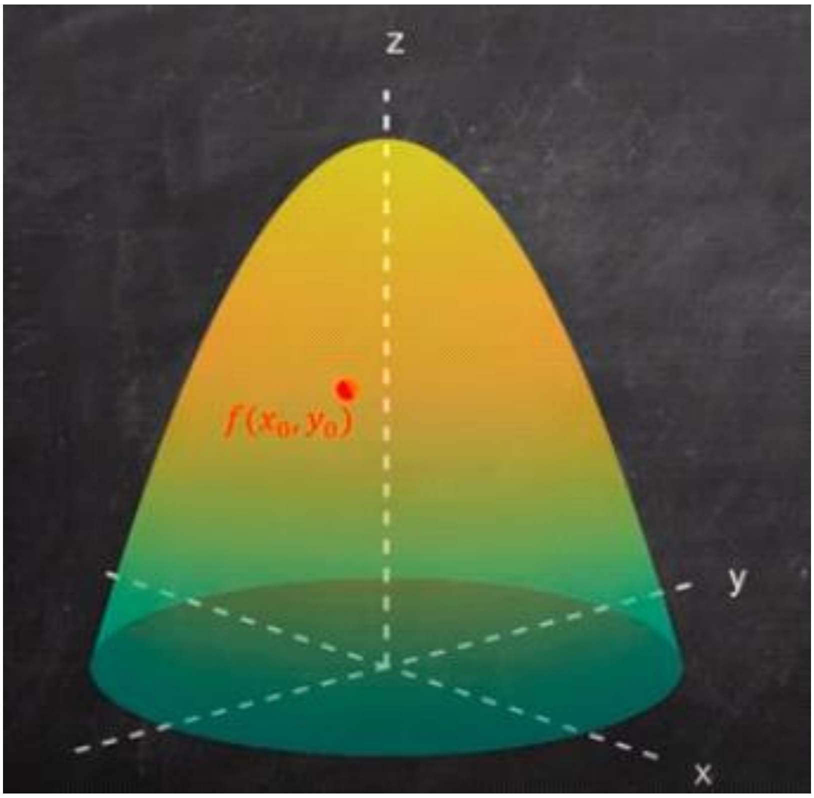


Figure 14.6: Temperature in a heated room: Heater at $x = 0$ is turned on at $t = 0$

Graphical Representation:



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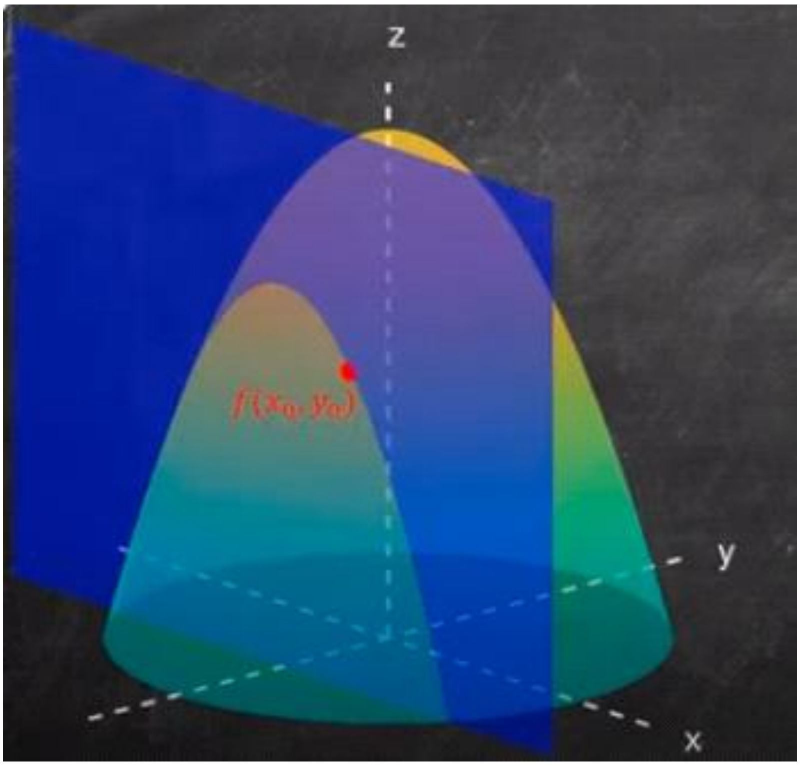

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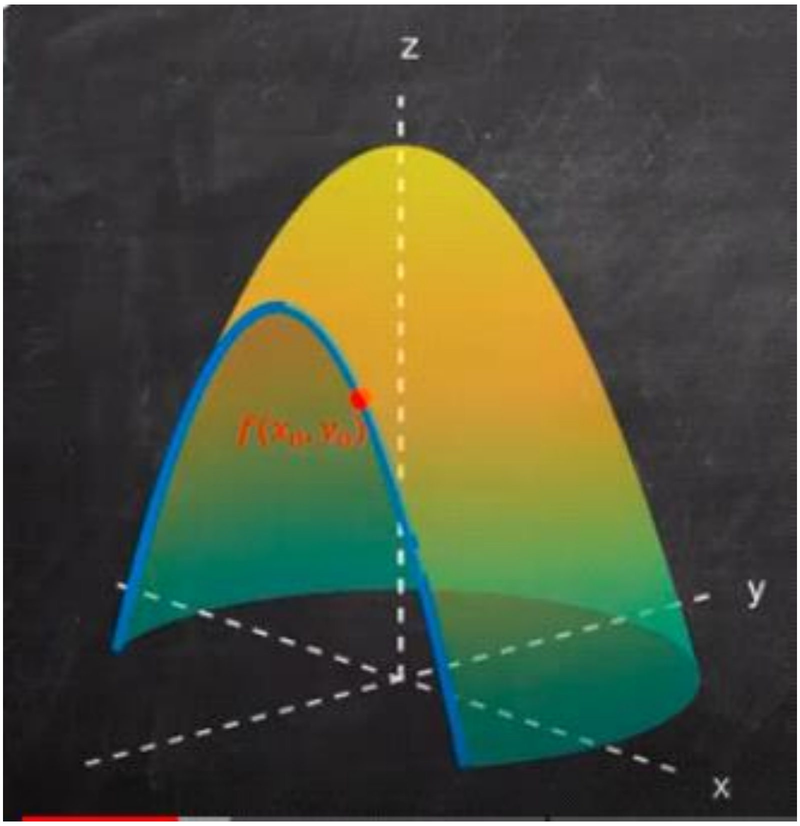

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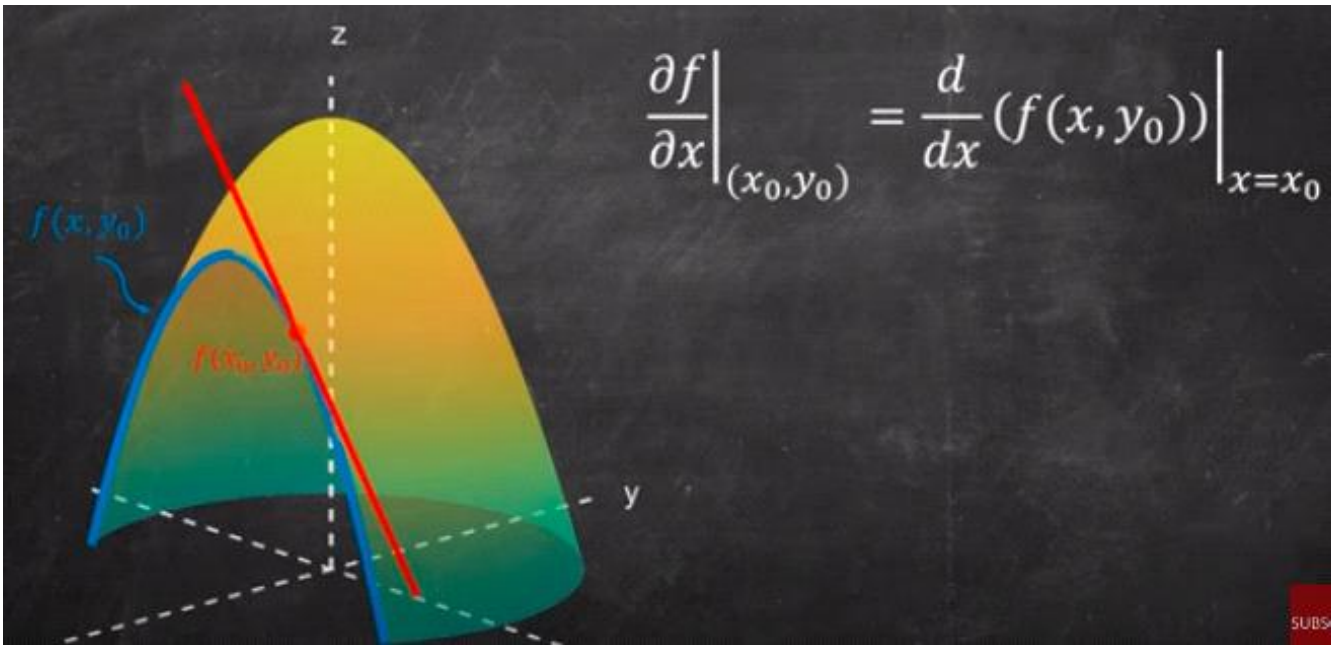

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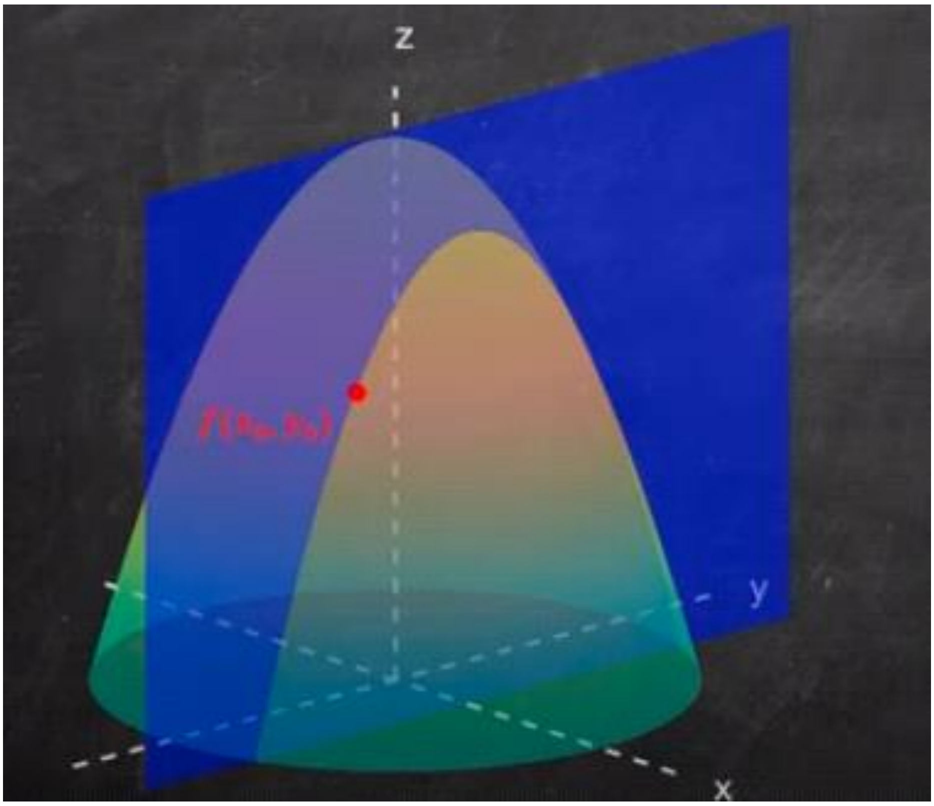

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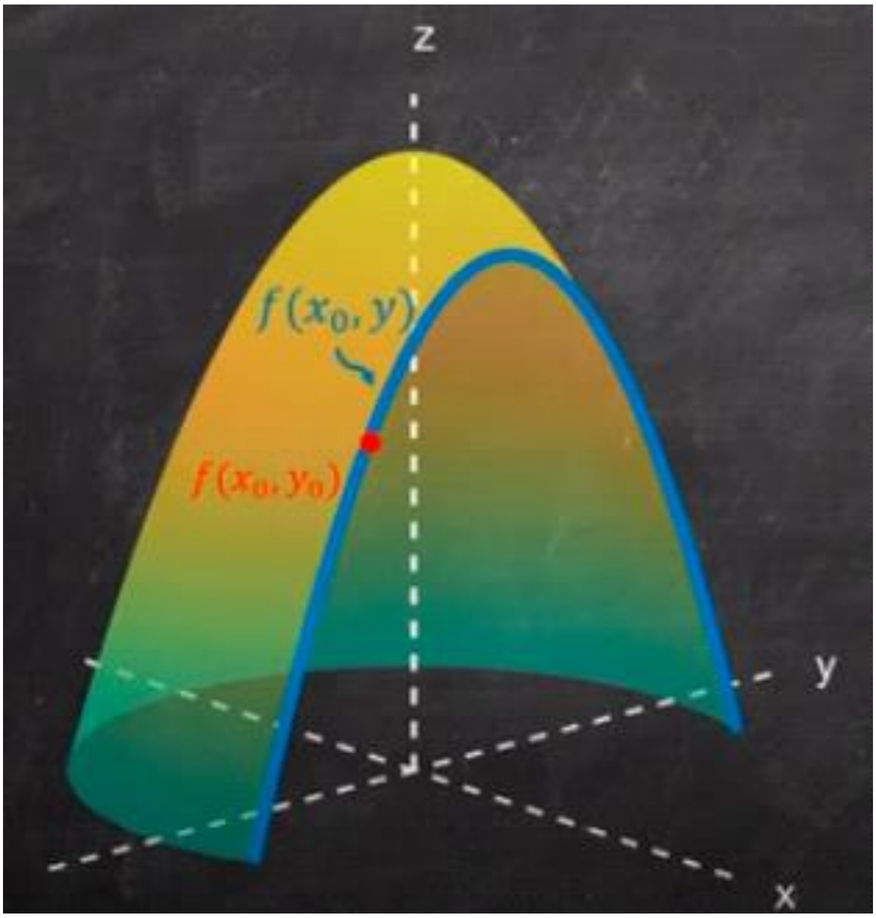

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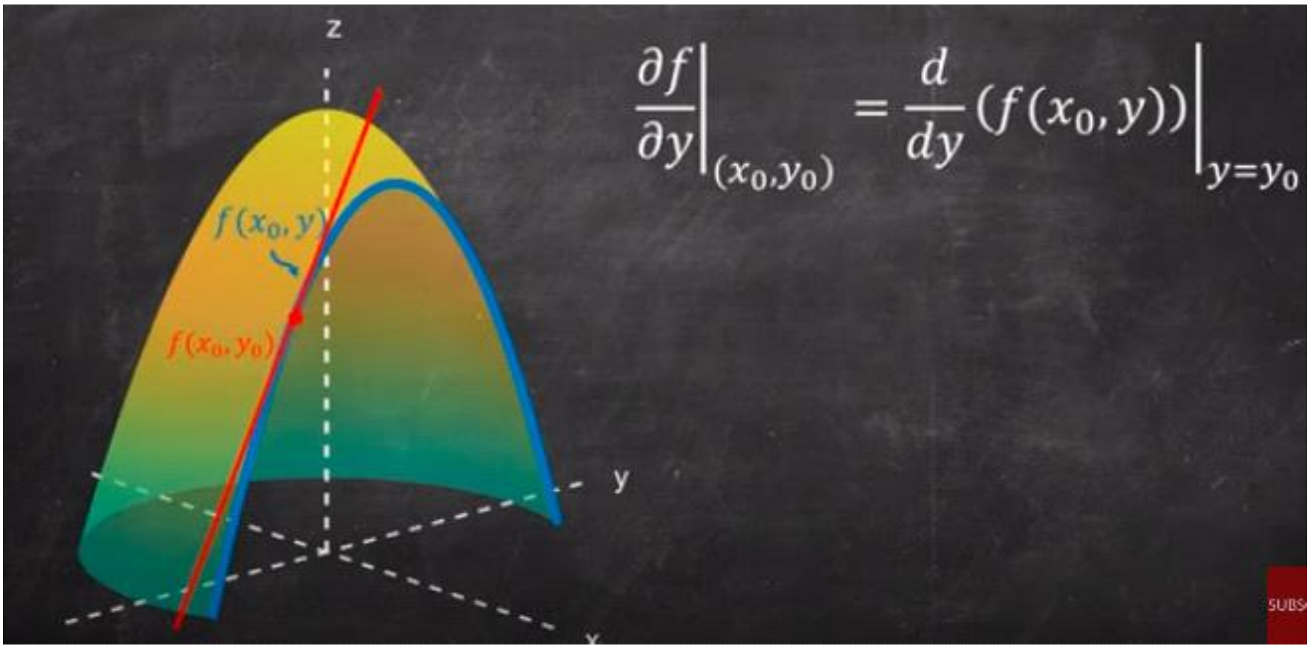

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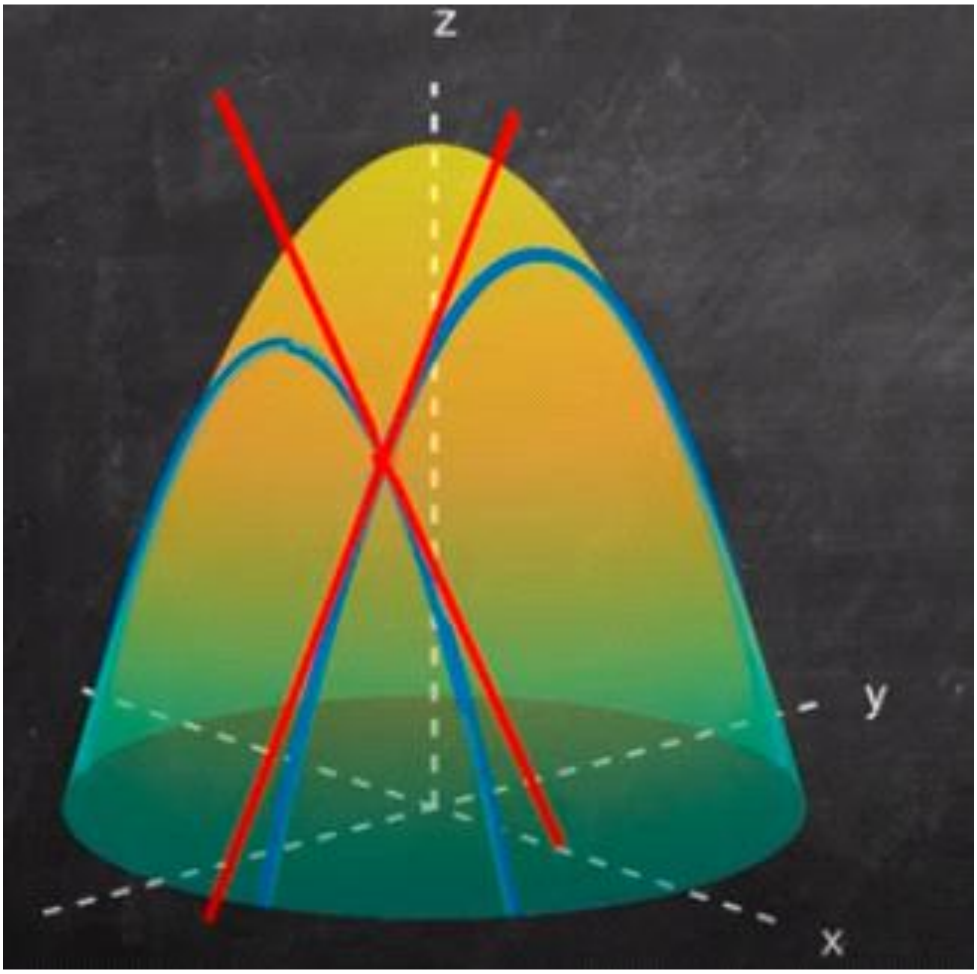

Example 1 Let $f(x,y) = \dfrac{x^2}{y+1}$. Find $f_x(3,2)$ algebraically.

Example 2 Compute the partial derivatives with respect to x and with respect to y for the following functions.
(a) $f(x,y) = y^2 e^{3x}$ (b) $z = (3xy + 2x)^5$ (c) $g(x,y) = e^{x+3y}\sin(xy)$

Example 3 Find all the partial derivatives of $f(x, y, z) = \dfrac{x^2y^3}{z}$.

Money in a bank account earns interest at a continuous rate, r. The amount of money, $\$B$, in the account depends on the amount deposited, $\$P$, and the time, t, it has been in the bank according to the formula

$$B = Pe^{rt}.$$

Find $\partial B/\partial t$ and $\partial B/\partial P$ and interpret each in financial terms.

The acceleration g due to gravity, at a distance r from the center of a planet of mass m, is given by

$$g = \frac{Gm}{r^2},$$

where G is the universal gravitational constant.

(a) Find $\partial g/\partial m$ and $\partial g/\partial r$.

(b) Interpret each of the partial derivatives you found in part (a) as the slope of a graph in the plane and sketch the graph.

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The Dubois formula relates a person's surface area, s, in m^2, to weight, w, in kg, and height, h, in cm, by

$$s = f(w, h) = 0.01w^{0.25}h^{0.75}.$$

Find $f(65, 160)$, $f_w(65, 160)$, and $f_h(65, 160)$. Interpret your answers in terms of surface area, height, and weight.

The energy, E, of a body of mass m moving with speed v is given by the formula

$$E = mc^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right).$$

The speed, v, is nonnegative and less than the speed of light, c, which is a constant.

(a) Find $\partial E/\partial m$. What would you expect the sign of $\partial E/\partial m$ to be? Explain.

(b) Find $\partial E/\partial v$. Explain what you would expect the sign of $\partial E/\partial v$ to be and why.