Differential & Its Application

Friday, 9 August 2024 4:13 pm

The Differential

We are often interested in the change in the value of the function as we move from the point (a, b) to a nearby point (x, y). Then we use the notation

$$\Delta f = f(x, y) - f(a, b) \quad \text{and} \quad \Delta x = x - a \quad \text{and} \quad \Delta y = y - b$$

to rewrite the tangent plane approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

in the form

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y.$$

For fixed a and b, the right side of this is a linear function of Δx and Δy that can be used to estimate Δf. We call this linear function the *differential*. To define the differential in general, we introduce new variables dx and dy to represent changes in x and y.

The Differential of a Function $z = f(x, y)$

The **differential**, df (or dz), at a point (a, b) is the linear function of dx and dy given by the formula

$$df = f_x(a, b)\, dx + f_y(a, b)\, dy.$$

The differential at a general point is often written $df = f_x\, dx + f_y\, dy$.

Example 4 Compute the differentials of the following functions.

(a) $f(x, y) = x^2 e^{5y}$ (b) $z = x \sin(xy)$ (c) $f(x, y) = x \cos(2x)$

Solution (a) Since $f_x(x, y) = 2xe^{5y}$ and $f_y(x, y) = 5x^2 e^{5y}$, we have

$$df = 2xe^{5y}\, dx + 5x^2 e^{5y}\, dy.$$

Example 5 The density ρ (in g/cm^3) of carbon dioxide gas CO_2 depends upon its temperature T (in °C) and pressure P (in atmospheres). The ideal gas model for CO_2 gives what is called the state equation:

$$\rho = \frac{0.5363P}{T + 273.15}.$$

Compute the differential $d\rho$. Explain the signs of the coefficients of dT and dP.

EXAMPLE: $z = x^2 + 3xy - y^2$

(i) Find dz

(ii) Find dz and Δz when x changes from 2 to 2.05 and y changes from 3 to 2.96

Q Find the total differential of

(i) $f(x,y) = ye^x + \sin x$

(iv) $R(\alpha, \beta, \gamma) = \alpha\beta^2 \cos\gamma$

(v) The Length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error of at most 0.1 cm in each.
Use differentials to estimate max. error in area of rectangle.

PROBLEM: Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter. If the metal on top and the bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

Application:

LOCAL LINEARITY AND THE DIFFERENTIAL

Zooming In to See Local Linearity

For a function of one variable, local linearity means that as we zoom in on the graph, it looks like a straight line. As we zoom in on the graph of a two-variable function, the graph usually looks like a plane, which is the graph of a linear function of two variables. (See Figure 14.22.)

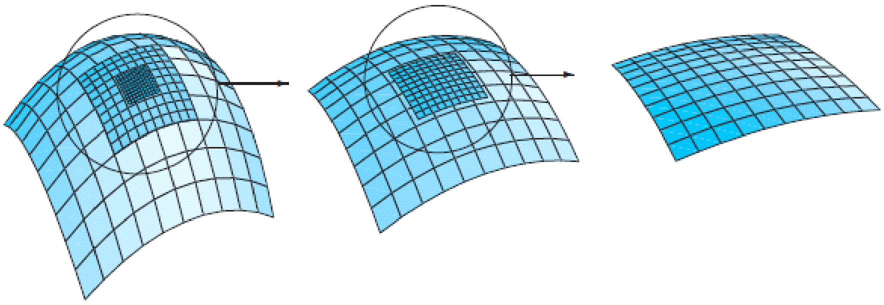


Figure 14.22: Zooming in on the graph of a function of two variables until the graph looks like a plane

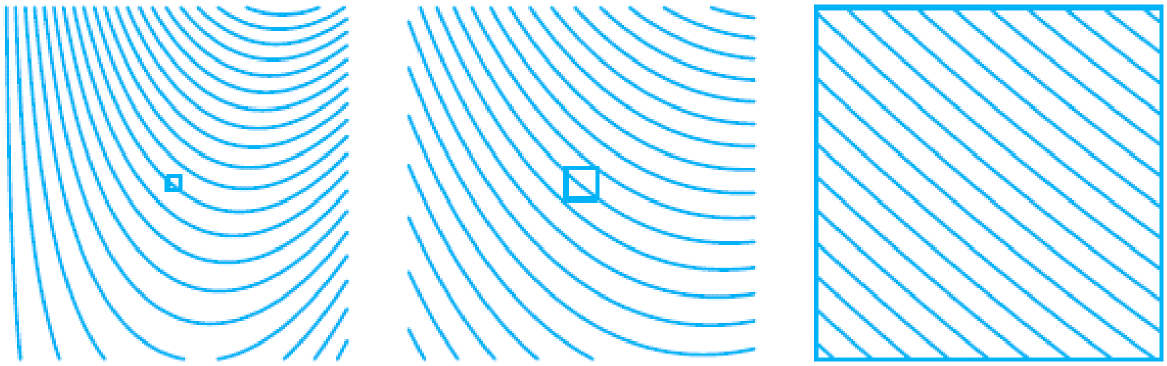


Figure 14.23: Zooming in on a contour diagram until the lines look parallel and equally spaced

Table 14.4 *Zooming in on values of* $f(x,y) = x^2 + y^3$ *near* $(2,1)$ *until the table looks linear*

$x \backslash y$	0	1	2
1	1	2	9
2	4	5	12
3	9	10	17

$x \backslash y$	0.9	1.0	1.1
1.9	4.34	4.61	4.94
2.0	4.73	5.00	5.33
2.1	5.14	5.41	5.74

$x \backslash y$	0.99	1.00	1.01
1.99	4.93	4.96	4.99
2.00	4.97	5.00	5.03
2.01	5.01	5.04	5.07

The Tangent Plane

Tangent Plane to the Surface $z = f(x, y)$ at the Point (a, b)

Assuming f is differentiable at (a, b), the equation of the tangent plane is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Here we are thinking of a and b as fixed, so $f(a, b)$, and $f_x(a, b)$, and $f_y(a, b)$ are constants. Thus, the right side of the equation is a linear function of x and y.

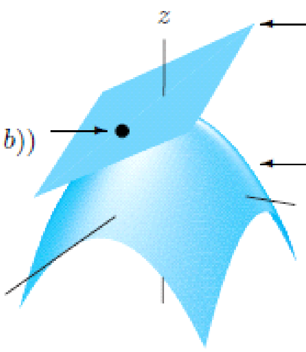


Figure 14.24: The tangent plane to the surface $z = f(x, y)$ at the point (a, b)

Example 1 Find the equation for the tangent plane to the surface $z = x^2 + y^2$ at the point $(3, 4)$.

Solution We have $f_x(x, y) = 2x$, so $f_x(3, 4) = 6$, and $f_y(x, y) = 2y$, so $f_y(3, 4) = 8$. Also, $f(3, 4) = 3^2 + 4^2 = 25$. Thus, the equation for the tangent plane at $(3, 4)$ is

$$z = 25 + 6(x - 3) + 8(y - 4).$$

Tangent Plane Approximation to $f(x, y)$ for (x, y) Near the Point (a, b)

Provided f is differentiable at (a, b), we can approximate $f(x, y)$:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We are thinking of a and b as fixed, so the expression on the right side is linear in x and y. The right side of this approximation gives the **local linearization** of f near $x = a$, $y = b$.

Example 2 Find the local linearization of $f(x, y) = x^2 + y^2$ at the point $(3, 4)$. Estimate $f(2.9, 4.2)$ and $f(2, 2)$ using the linearization and compare your answers to the true values.

Solution Let $z = f(x, y) = x^2 + y^2$. In Example 1 on page 772, we found the equation of the tangent plane at $(3, 4)$ to be

$$z = 25 + 6(x - 3) + 8(y - 4).$$

Therefore, for (x, y) near $(3, 4)$, we have the local linearization

$$f(x, y) \approx 25 + 6(x - 3) + 8(y - 4).$$

Substituting $x = 2.9, y = 4.2$ gives

$$f(2.9, 4.2) \approx 25 + 6(-0.1) + 8(0.2) = 26.$$

This compares favorably with the true value $f(2.9, 4.2) = (2.9)^2 + (4.2)^2 = 26.05$.

However, the local linearization does not give a good approximation at points far away from $(3, 4)$. For example, if $x = 2$, $y = 2$, the local linearization gives

$$f(2, 2) \approx 25 + 6(-1) + 8(-2) = 3,$$

whereas the true value of the function is $f(2, 2) = 2^2 + 2^2 = 8$.

18. A student was asked to find the equation of the tangent plane to the surface $z = x^3 - y^2$ at the point $(x, y) = (2, 3)$. The student's answer was

$$z = 3x^2(x - 2) - 2y(y - 3) - 1.$$

(a) At a glance, how do you know this is wrong?
(b) What mistake did the student make?
(c) Answer the question correctly.

Find the Equation of Tangent.

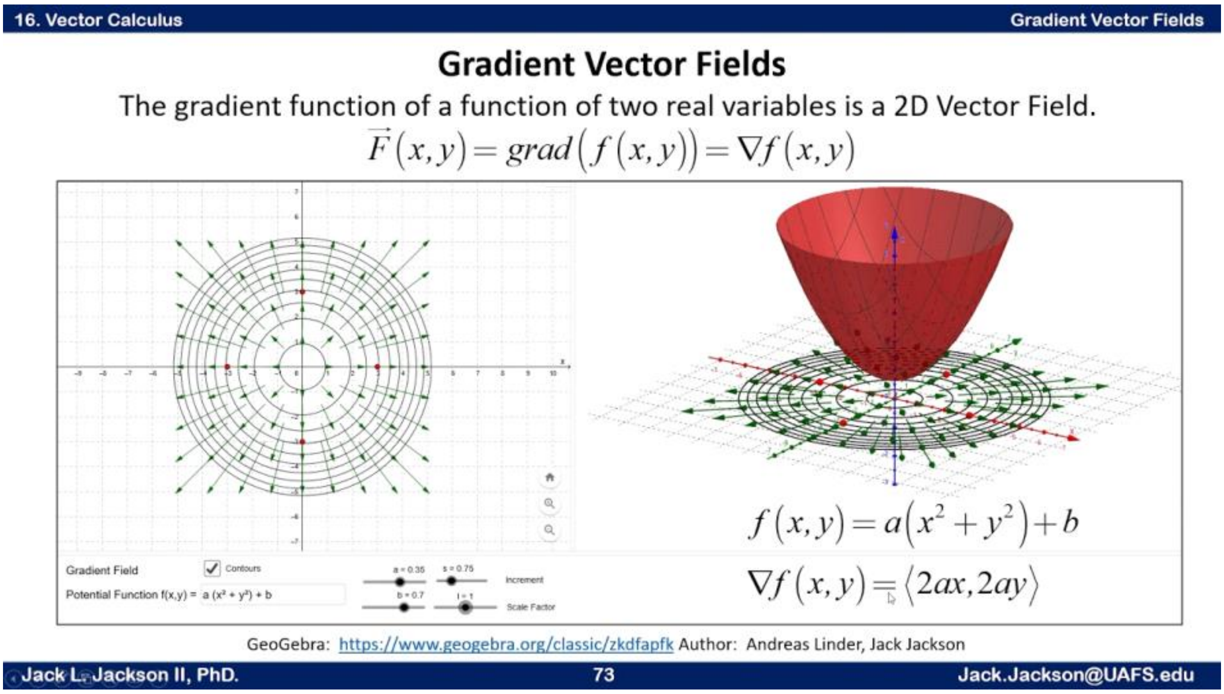
7. $x^2y^2 + z - 40 = 0$ at $x = 2, y = 3$

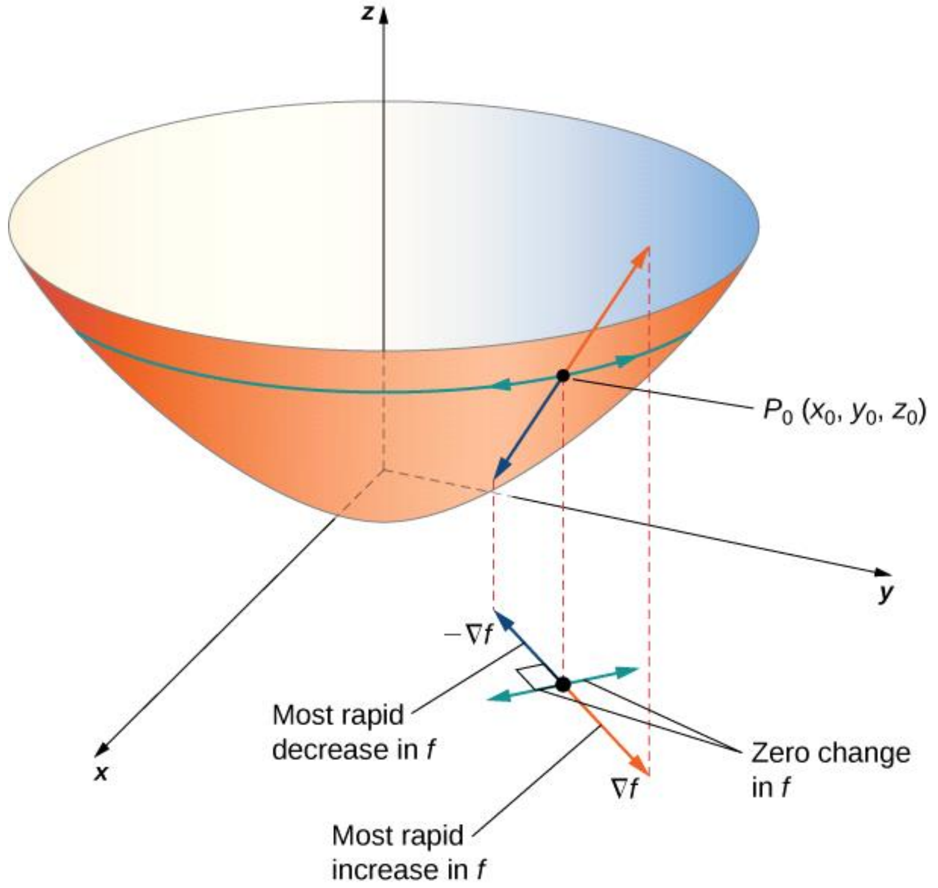
8. $x^2y + \ln(xy) + z = 6$ at the point $(4, 0.25, 2)$

22. (a) Find the equation of the plane tangent to the graph of $f(x, y) = x^2 e^{xy}$ at $(1, 0)$.

(b) Find the linear approximation of $f(x, y)$ for (x, y) near $(1, 0)$.

(c) Find the differential of f at the point $(1, 0)$.





The Gradient Vector of a differentiable function f at the point (a, b) is

$$\text{grad}\, f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$$

Example 5 Find the gradient vector of $f(x, y) = x + e^y$ at the point $(1, 1)$.

Solution Using the definition, we have

$$\text{grad}\ f = f_x\vec{i} + f_y\vec{j} = \vec{i} + e^y\vec{j},$$

so at the point $(1, 1)$

$$\text{grad}\ f(1,1) = \vec{i} + e\vec{j}.$$

Alternative Notation for the Gradient

You can think of $\dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j}$ as the result of applying the vector operator (pronounced "del")

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$$

to the function f. Thus, we get the alternative notation

$$\text{grad}\, f = \nabla f.$$

If $z = f(x, y)$, we can write grad z or ∇z for grad f or for ∇f.

The gradient is a fancy word for derivative, or the rate of change of a function. It's a vector (a direction to move) that. Points in the direction of greatest increase of a function (intuition on why) Is zero at a local maximum or local minimum (because there is no single direction of increase)

Screen clipping taken: 21/02/2024 8:57 pm