Two Variable Function: Intro, Sketching & Domain

Tuesday, 30 July 2024 1:29 pm

Example 1 Graph $x = 3$ in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

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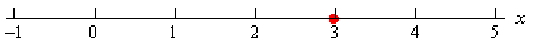

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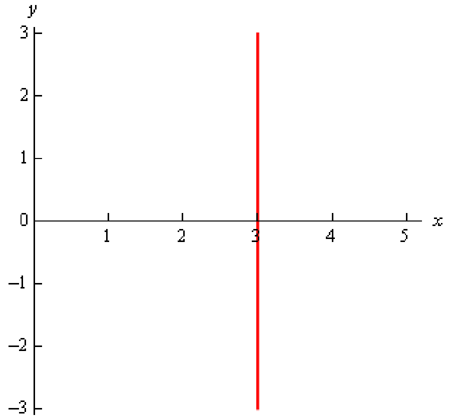
Here is the graph of $x = 3$ in $\mathbb{R}^2$.



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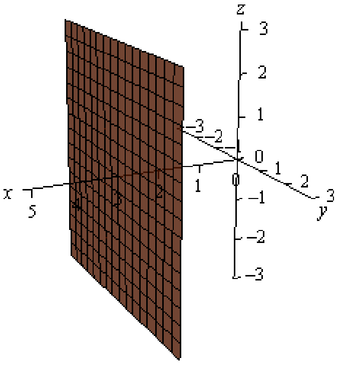

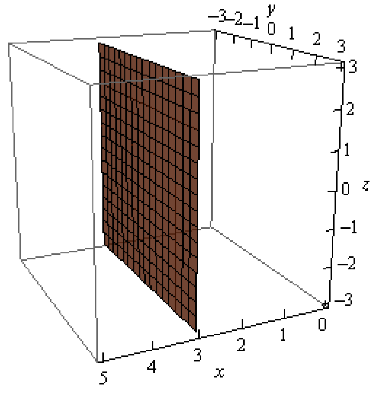

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Numerical Example: Beef Consumption

Suppose you are a beef producer and you want to know how much beef people will buy. This depends on how much money people have and on the price of beef. The consumption of beef, C (in pounds per week per household) is a function of household income, I (in thousands of dollars per year), and the price of beef, p (in dollars per pound). In function notation, we write:

$$C = f(I, p).$$

Table 12.1 contains values of this function. Values of p are shown across the top, values of I are down the left side, and corresponding values of $f(I, p)$ are given in the table.[1] For example, to find the value of $f(40, 3.50)$, we look in the row corresponding to $I = 40$ under $p = 3.50$, where we find the number 4.05. Thus,

$$f(40, 3.50) = 4.05.$$

This means that, on average, if a household's income is $40,000 a year and the price of beef is $3.50/lb, the family will buy 4.05 lbs of beef per week.

Table 12.1 *Quantity of beef bought (pounds/household/week)*

Household income per year, I ($1000)	Price of beef ($/lb): 3.00	3.50	4.00	4.50
20	2.65	2.59	2.51	2.43
40	4.14	4.05	3.94	3.88
60	5.11	5.00	4.97	4.84
80	5.35	5.29	5.19	5.07
100	5.79	5.77	5.60	5.53

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Example 3 A cylinder with closed ends has radius r and height h. If its volume is V and its surface area is A, find formulas for the functions $V = f(r, h)$ and $A = g(r, h)$.

Solution Since the area of the circular base is πr^2, we have

$$V = f(r, h) = \text{Area of base} \cdot \text{Height} = \pi r^2 h.$$

The surface area of the side is the circumference of the bottom, $2\pi r$, times the height h, giving $2\pi rh$. Thus,

$$A = g(r, h) = 2 \cdot \text{Area of base} + \text{Area of side} = 2\pi r^2 + 2\pi rh.$$

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A Tour of 3-Space

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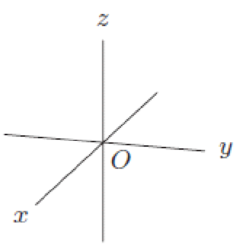

Figure 12.2: Coordinate axes in three-dimensional space

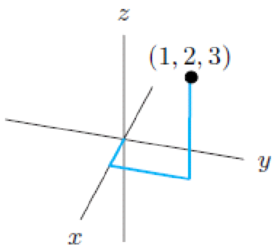

Figure 12.3: The point $(1, 2, 3)$ in 3-space

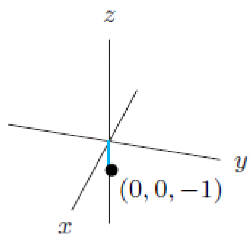

Figure 12.4: The point $(0, 0, -1)$ in 3-space

Graphing Equations in 3-Space

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Example 6 What do the graphs of the equations $z = 0$, $z = 3$, and $z = -1$ look like?

Solution To graph $z = 0$, we visualize the set of points whose z-coordinate is zero. If the z-coordinate is 0, then we must be at the same vertical level as the origin, that is, we are in the horizontal plane containing the origin. So the graph of $z = 0$ is the middle plane in Figure 12.6. The graph of $z = 3$ is a plane parallel to the graph of $z = 0$, but three units above it. The graph of $z = -1$ is a plane parallel to the graph of $z = 0$, but one unit below it.

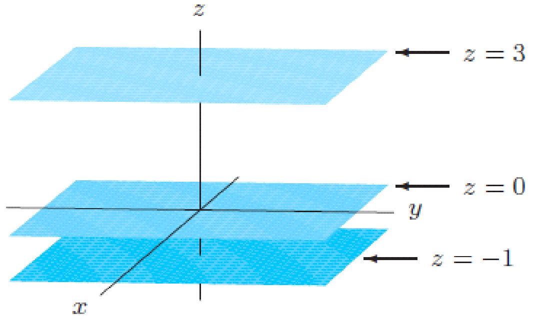

Figure 12.6: The planes $z = -1$, $z = 0$, and $z = 3$

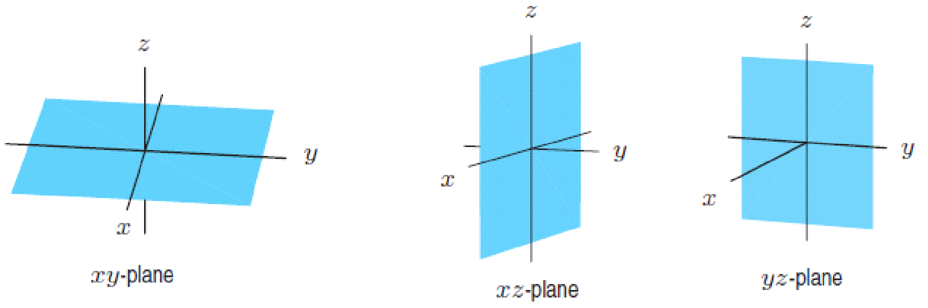

Figure 12.7: The three coordinate planes

Visualizing a Function of Two Variables Using a Graph

For a function of one variable, $y = f(x)$, the graph of f is the set of all points (x, y) in 2-space such that $y = f(x)$. In general, these points lie on a curve in the plane. When a computer or calculator graphs f, it approximates by plotting points in the xy-plane and joining consecutive points by line segments. The more points, the better the approximation.

Now consider a function of two variables.

> The **graph** of a function of two variables, f, is the set of all points (x, y, z) such that $z = f(x, y)$. In general, the graph of a function of two variables is a surface in 3-space.

Plotting the Graph of the Function $f(x, y) = x^2 + y^2$

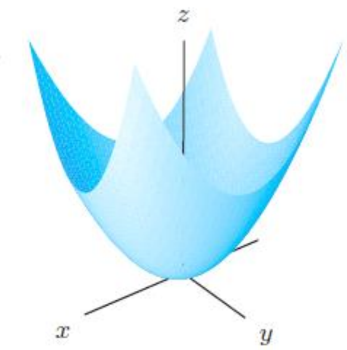

Figure 12.13: Graph of $f(x, y) = x^2 + y^2$ for $-3 \le x \le 3, -3 \le y \le 3$

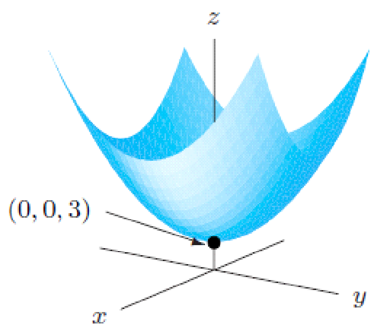

Figure 12.14: Graph of $g(x, y) = x^2 + y^2 + 3$

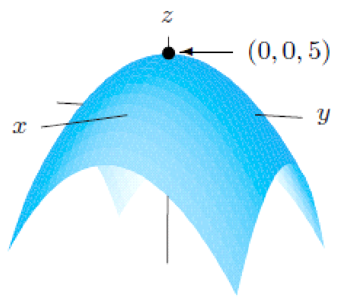

Figure 12.15: Graph of $h(x, y) = 5 - x^2 - y^2$

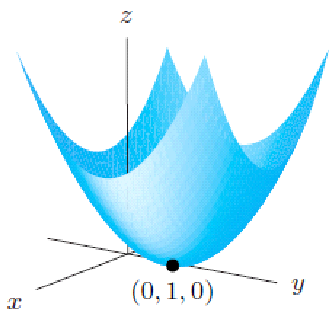

Figure 12.16: Graph of $k(x, y) = x^2 + (y - 1)^2$

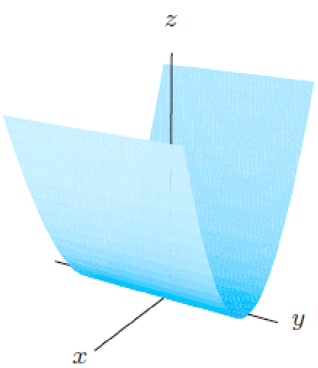

Figure 12.25: A parabolic cylinder $z = x^2$

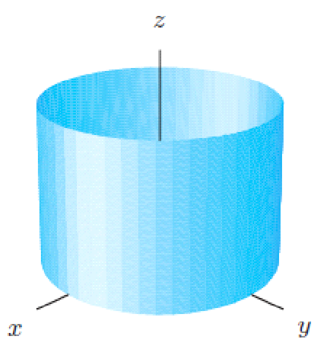

Figure 12.26: Circular cylinder $x^2 + y^2 = 1$

Recall that the **equation of a plane** is given by

$$ax + by + cz = d$$

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$$f(x,y) = 12 - 3x - 4y$$

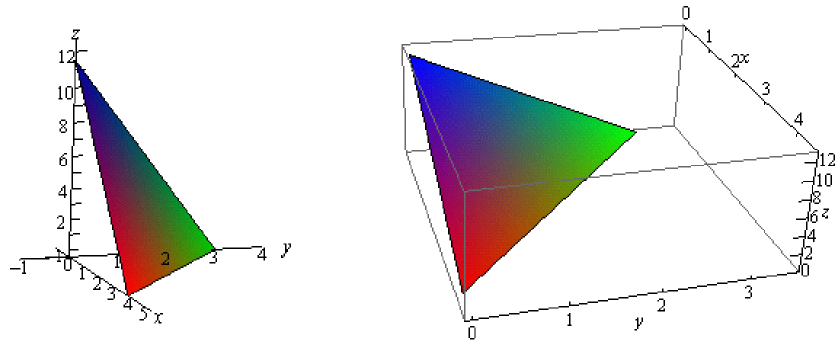
For purposes of graphing this it would probably be easier to write this as,

$$z = 12 - 3x - 4y \qquad \Rightarrow \qquad 3x + 4y + z = 12$$

Now, each of the intersection points with the three main coordinate axes is defined by the fact that two of the coordinates are zero. For instance, the intersection with the z-axis is defined by $x = y = 0$. So, the three intersection points are,

$$\begin{aligned} x - \text{axis}: &\ (4,0,0) \\ y - \text{axis}: &\ (0,3,0) \\ z - \text{axis}: &\ (0,0,12) \end{aligned}$$

Here is the graph of the plane.



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Example 1 Determine the domain of each of the following.

(a) $f(x,y) = \sqrt{x+y}$

(b) $f(x,y) = \sqrt{x} + \sqrt{y}$

(c) $f(x,y) = \ln\left(9 - x^2 - 9y^2\right)$

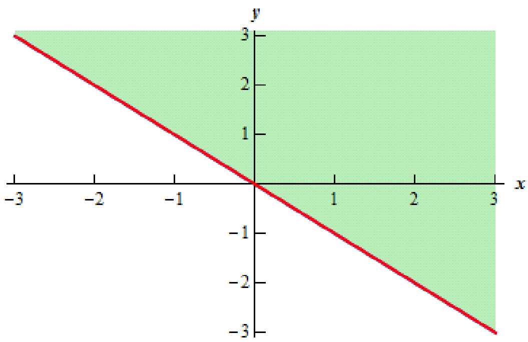
[Show All Solutions] [Hide All Solutions]

(a) $f(x,y) = \sqrt{x+y}$ ***Hide Solution*** ▾

In this case we know that we can't take the square root of a negative number so this means that we must require,

$$x + y \geq 0$$

Here is a sketch of the graph of this region.



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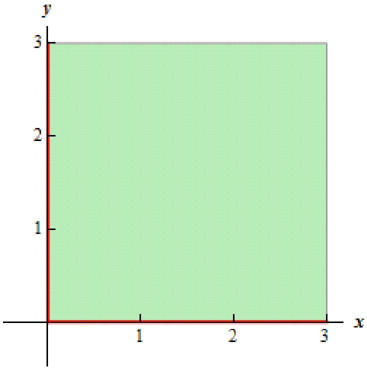
(b) $f(x,y) = \sqrt{x} + \sqrt{y}$ *Hide Solution* ▾

This function is different from the function in the previous part. Here we must require that,

$$x \geq 0 \quad \text{and} \quad y \geq 0$$

and they really do need to be separate inequalities. There is one for each square root in the function. Here is the sketch of this region.



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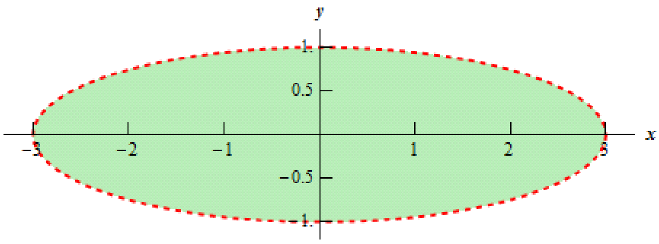
(c) $f(x,y) = \ln(9 - x^2 - 9y^2)$ *Hide Solution* ▾

In this final part we know that we can't take the logarithm of a negative number or zero. Therefore, we need to require th

$$9 - x^2 - 9y^2 > 0 \quad \Rightarrow \quad \frac{x^2}{9} + y^2 < 1$$

and upon rearranging we see that we need to stay interior to an ellipse for this function. Here is a sketch of this region.



Example 2 Determine the domain of the following function,

$$f(x,y,z) = \frac{1}{\sqrt{x^2 + y^2 + z^2 - 16}}$$

Hide Solution ▾

In this case we have to deal with the square root and division by zero issues. These will require,

$$x^2 + y^2 + z^2 - 16 > 0 \quad \Rightarrow \quad x^2 + y^2 + z^2 > 16$$

So, the domain for this function is the set of points that lies completely outside a sphere of radius 4 centered at the origin.

expression ...

$\boxed{\text{EXAMPLE 1}}$ ~~For~~ for the function $f(x,y)$, evaluate $f(3,2)$ and find and sketch the domain.

(a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$

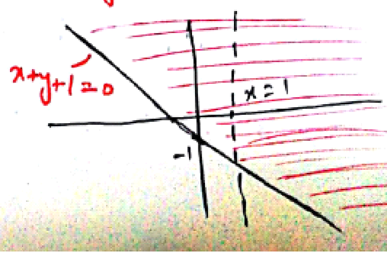
$f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$

Domain: $\sqrt{x+y+1} \geq 0$ (should not be complex)

and $x \neq 1$ (otherwise infinite)

$D = \{(x,y) \mid (x+y+1) \geq 0,\ x \neq 1\}$

Graph: $x+y+1 \geq 0 \Rightarrow y \geq -x-1$ and $x = 1$



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(b) $f(x,y) = x\ln(y^2-x)$

$f(3,2) = 3\ln(4-3) = 3\ln(1)$

$f(3,2) = 0$

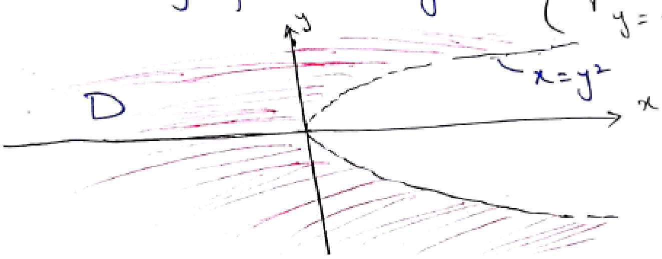
Domain: Since Ln can not have negative values and zero as an input therefore

$y^2 - x > 0$

or $y^2 > x$ or $x < y^2$

$D = \{(x,y) \mid x < y^2\}$

Graph: Plotting for $x = y^2$ $\left\{ \begin{array}{l} \text{2 parts of this equation} \\ y = \sqrt{x} \ \& \ y = -\sqrt{x} \end{array} \right.$



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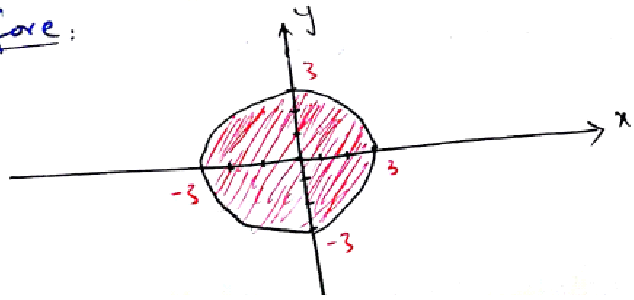
Example 2 Find the domain and Range for
$$g(x,y) = \sqrt{9 - x^2 - y^2}$$

Domain: For $g(x,y)$ to be real valued
$$9 - x^2 - y^2 \geq 0 \quad \text{or} \quad x^2 + y^2 \leq 9$$
$$D = \{(x,y) \mid x^2 + y^2 \leq 9\}$$

Graph of Domain: we know equation of circle
$$x^2 + y^2 = R^2 \quad (R \to \text{radius of circle})$$

Therefore:



Range: Since $z = g(x,y) = \sqrt{9 - x^2 - y^2}$
$$R = \{z \mid z = \sqrt{9 - x^2 - y^2},\ (x,y) \in D\}$$

Since Domain shows that z should be positive ($z \geq 0$), So the max. value that z can achieve can be found using
$$\sqrt{9 - x^2 - y^2} \geq 0$$
$$\sqrt{9 - (x^2 + y^2)} \geq 0$$
if $(x^2 + y^2) = 0$
$$\sqrt{9 - 0} = \sqrt{9} = 3 \to \text{max. value } z \text{ can have}$$
therefore
$$\sqrt{9 - x^2 - y^2} \leq 3$$
So, the resulting range would be
$$\{z \mid 0 \leq z \leq 3\} = [0, 3]$$

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EXAMPLE 3 Sketch the graph of the function of previous Question.

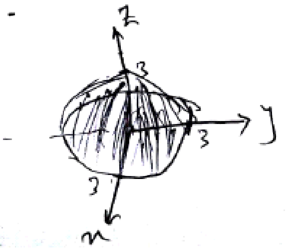
SOLUTION: $z = \sqrt{9 - x^2 - y^2}$

Squaring b.t.s.

$$z^2 = 9 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 9 \quad \text{—— (1)}$$

Since eq (1) is an equation of a sphere with the center at origin and a radius of '3'. But since we concluded $z \geq 0$, therefore the graph will contain only the ~~upper Hemisphere~~ upper half of the sphere.



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Example 4 Sketch the graph for

$$f(x,y) = 6 - 3x - 2y$$

SOLUTION: The graph can be drawn using

$$z = 6 - 3x - 2y$$

or

$$3x + 2y + z = 6$$

which is a linear equation which will always represent a plane.

To plot a plane we find intercepts

for x-intercept put $y = z = 0$

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$$3x + 0 + 0 = 6$$

for y-intercept put $\boxed{x = 2}$

$x = z = 0$

$$0 + 2y + 0 = 6$$

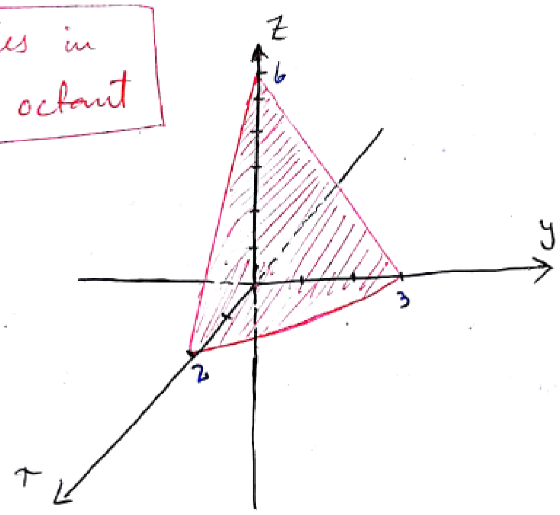
$$\boxed{y = 3}$$

for z-intercept

$$0 + 0 + z = 6$$

$$\boxed{z = 6}$$

raph lies in
first octant



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