Arclength & Parametric Equation

Monday, 2 December 2024 5:37 pm

2 The Arc Length Formula If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

EXAMPLE 1 Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$. (See Figure 5.)

V EXAMPLE 4 Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1, 1)$ as the starting point.

SOLUTION If $f(x) = x^2 - \frac{1}{8}\ln x$, then

$$f'(x) = 2x - \frac{1}{8x}$$

$$1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x}$$

Thus the arc length function is given by

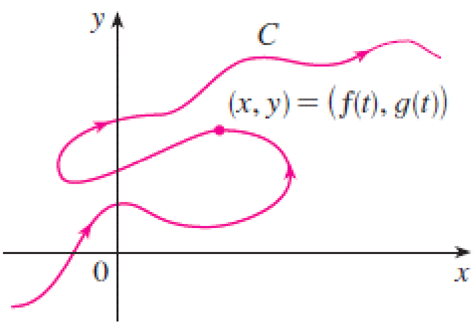
$$\begin{aligned} s(x) &= \int_1^x \sqrt{1 + [f'(t)]^2}\, dt \\ &= \int_1^x \left(2t + \frac{1}{8t}\right) dt = t^2 + \tfrac{1}{8}\ln t\Big]_1^x \\ &= x^2 + \tfrac{1}{8}\ln x - 1 \end{aligned}$$

For instance, the arc length along the curve from (1, 1) to (3, $f(3)$) is

For instance, the arc length along the curve from (1, 1) to (3, $f(3)$) is

$$s(3) = 3^2 + \tfrac{1}{8}\ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373$$

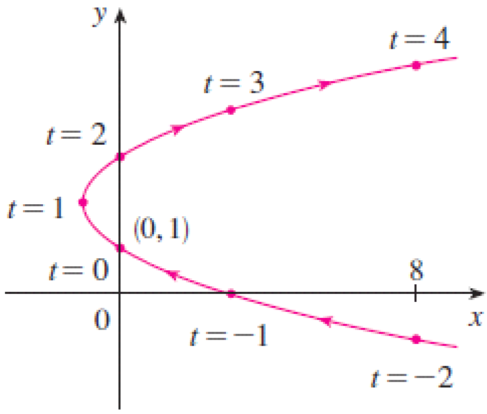
Curves Defined by Parametric Equations



EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



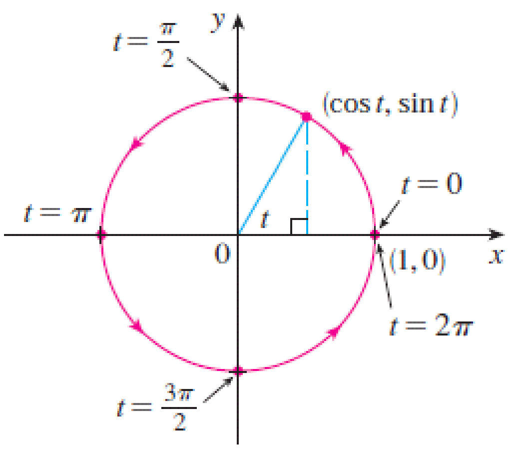
$x = y^2 - 4y + 3.$

$$x = f(t) \qquad y = g(t) \qquad a \leqslant t \leqslant b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

V EXAMPLE 2 What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

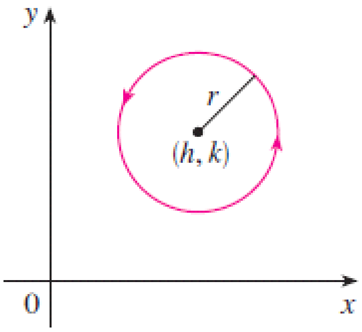


EXAMPLE 3 What curve is represented by the given parametric equations?

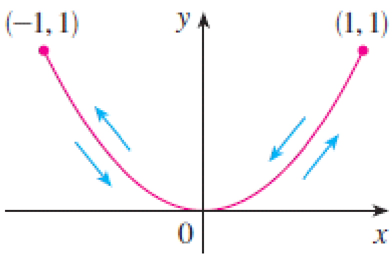
$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

EXAMPLE 4 Find parametric equations for the circle with center (h, k) and radius r.

$$x = h + r\cos t \qquad y = k + r\sin t \qquad 0 \leqslant t \leqslant 2\pi$$



V **EXAMPLE 5** Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.



EXAMPLE 6 Use a graphing device to graph the curve $x = y^4 - 3y^2$.

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

EXAMPLE 4 If we use the representation of the unit circle given in Example 2 in Section 10.1,

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

V **EXAMPLE 5** Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.