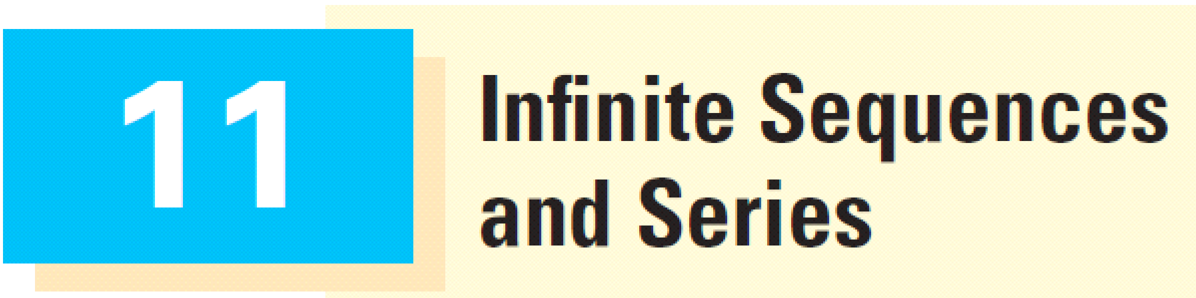
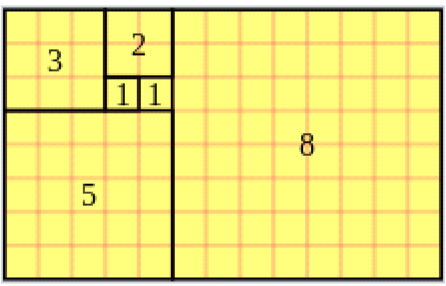
Sequence

Saturday, 2 November 2024 5:34 pm



The prime numbers are the natural numbers greater than 1 that have no divisors but 1 and themselves. Taking these in their natural order gives the sequence (2, 3, 5, 7, 11, 13, 17, ...). The prime numbers are widely used in mathematics, particularly in number theory where many results related to them exist.



The Fibonacci numbers comprise the integer sequence whose elements are the sum of the previous two elements. The first two elements are either 0 and 1 or 1 and 1 so that the sequence is (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...).[1]

Other examples of sequences include those made up of rational numbers, real numbers and complex numbers. The sequence (.9, .99, .999, .9999, ...), for instance, approaches the number 1. In fact, every real number can be written as the limit of a sequence of rational numbers (e.g. via its decimal expansion). As another example, π is the

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\qquad a_n = \dfrac{n}{n+1}$ $\qquad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$ $\qquad a_n = \dfrac{(-1)^n(n+1)}{3^n}$ $\qquad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \ldots, \dfrac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

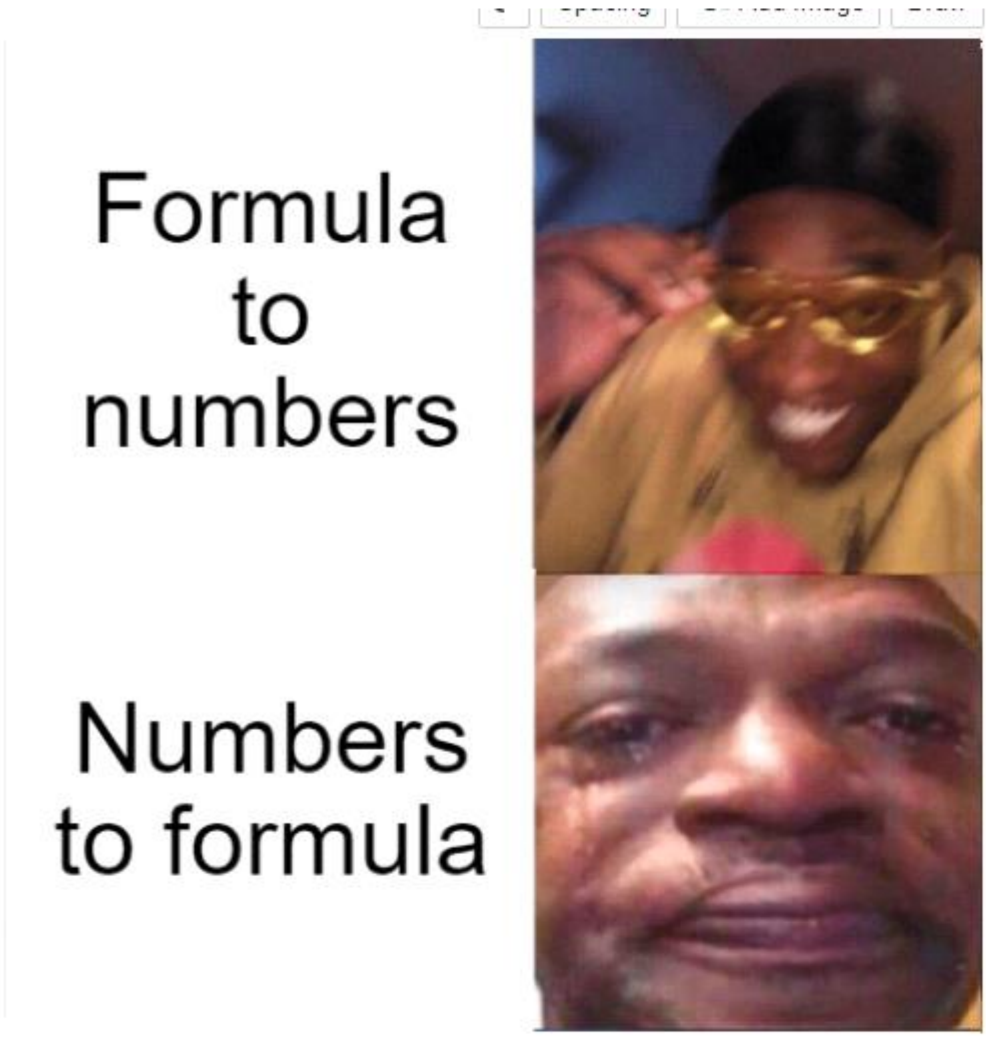
(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\qquad a_n = \sqrt{n-3},\ n \geqslant 3$ $\qquad \left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

(d) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\qquad a_n = \cos\dfrac{n\pi}{6},\ n \geqslant 0$ $\qquad \left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \ldots, \cos\dfrac{n\pi}{6}, \ldots\right\}$

V EXAMPLE 2 Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.



$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

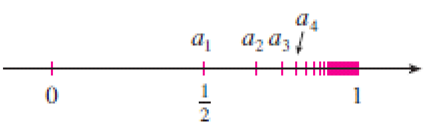
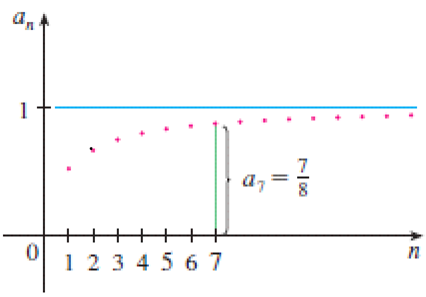


FIGURE 1



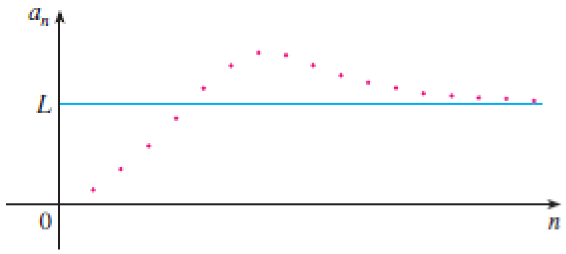
$$\lim_{n\to\infty}\frac{n}{n+1} = 1$$

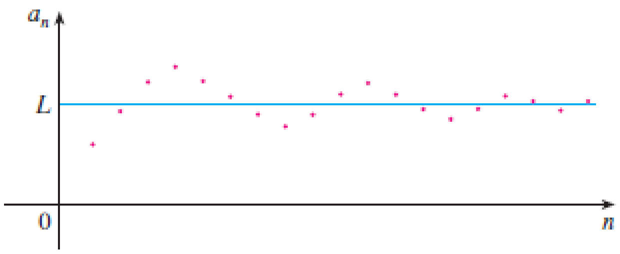
$$\lim_{n\to\infty} a_n = L$$

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

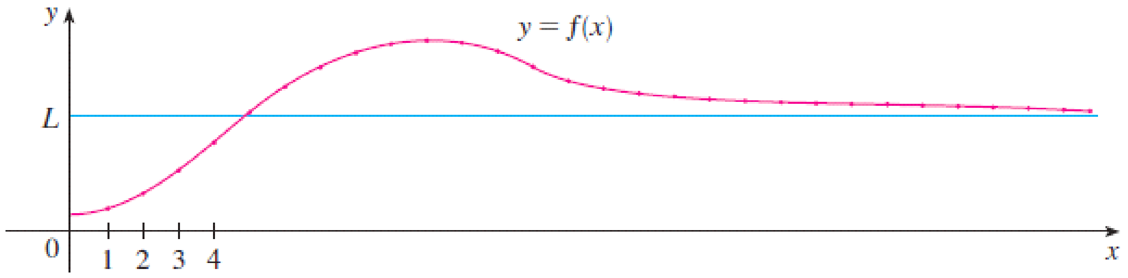
$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).





3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.



In particular, since we know that $\lim_{x\to\infty} (1/x^r) = 0$ when $r > 0$ (Theorem 2.6.5), we have

4 $$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } \ p > 0 \text{ and } a_n > 0$$

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

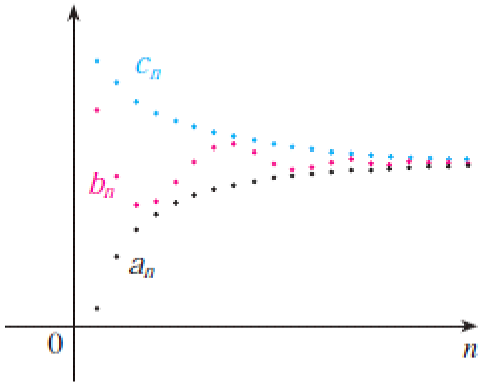


FIGURE 7
The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

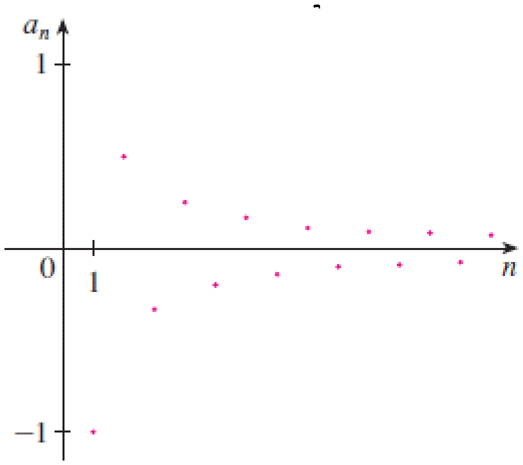
6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 4 Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

EXAMPLE 6 Calculate $\lim_{n \to \infty} \frac{\ln n}{n}$.

EXAMPLE 7 Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

EXAMPLE 8 Evaluate $\lim_{n \to \infty} \frac{(-1)^n}{n}$ if it exists.



$\boxed{7}$ **Theorem** If $\lim_{n \to \infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n \to \infty} f(a_n) = f(L)$$

EXAMPLE 9 Find $\lim_{n \to \infty} \sin(\pi/n)$.

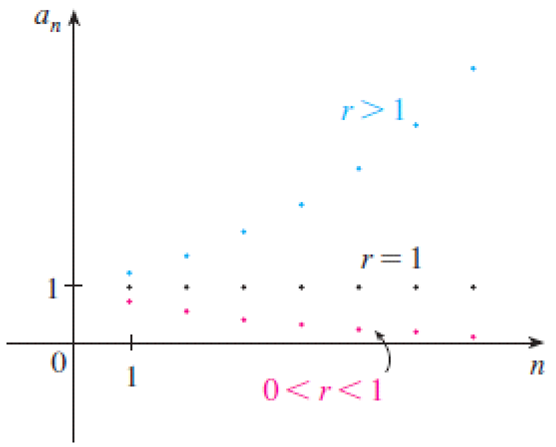
V **EXAMPLE 11** For what values of r is the sequence $\{r^n\}$ convergent?

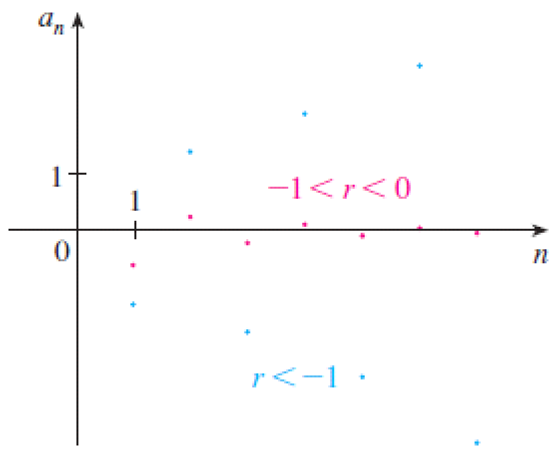
$$\lim_{n \to \infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

$$\lim_{n\to\infty} 1^n = 1 \qquad \text{and} \qquad \lim_{n\to\infty} 0^n = 0$$

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$





10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

EXAMPLE 12 The sequence $\left\{\dfrac{3}{n+5}\right\}$ is decreasing because

EXAMPLE 13 Show that the sequence $a_n = \dfrac{n}{n^2 + 1}$ is decreasing.

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

EXAMPLE 14 Investigate the sequence $\{a_n\}$ defined by the *recurrence relation*

$$a_1 = 2 \qquad a_{n+1} = \tfrac{1}{2}(a_n + 6) \qquad \text{for } n = 1, 2, 3, \ldots$$

SOLUTION We begin by computing the first several terms:

$$a_1 = 2 \qquad a_2 = \tfrac{1}{2}(2 + 6) = 4 \qquad a_3 = \tfrac{1}{2}(4 + 6) = 5$$

$$a_4 = \tfrac{1}{2}(5 + 6) = 5.5 \qquad a_5 = 5.75 \qquad a_6 = 5.875$$

$$a_7 = 5.9375 \qquad a_8 = 5.96875 \qquad a_9 = 5.984375$$

Example 3 Give the first six terms of the recursively defined sequences.

(a) $s_n = s_{n-1} + 3$ for $n > 1$ and $s_1 = 4$
(b) $s_n = -3s_{n-1}$ for $n > 1$ and $s_1 = 2$
(c) $s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$ for $n > 2$ and $s_1 = 0$, $s_2 = 1$
(d) $s_n = ns_{n-1}$ for $n > 1$ and $s_1 = 1$

Example 1 Give the first six terms of the following sequences:

(a) $s_n = \dfrac{n(n+1)}{2}$ (b) $s_n = \dfrac{n+(-1)^n}{n}$

Solution (a) Substituting $n = 1, 2, 3, 4, 5, 6$ into the formula for the general term, we get

$$\frac{1\cdot 2}{2}, \frac{2\cdot 3}{2}, \frac{3\cdot 4}{2}, \frac{4\cdot 5}{2}, \frac{5\cdot 6}{2}, \frac{6\cdot 7}{2} = 1,\ 3,\ 6,\ 10,\ 15,\ 21.$$

(b) Substituting $n = 1, 2, 3, 4, 5, 6$ into the formula for the general term, we get

$$\frac{1-1}{1}, \frac{2+1}{2}, \frac{3-1}{3}, \frac{4+1}{4}, \frac{5-1}{5}, \frac{6+1}{6} = 0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}.$$

Example 2 Give a general term for the following sequences:

(a) $1,\ 2,\ 4,\ 8,\ 16,\ 32, \ldots$ (b) $\dfrac{7}{2}, \dfrac{7}{5}, \dfrac{7}{8}, \dfrac{7}{11}, \dfrac{1}{2}, \dfrac{7}{17}, \ldots$

$s_n = 2^{n-1}$.

$s_n = \dfrac{7}{3n-1}$.

Example 5 Do the following sequences converge or diverge? If a sequence converges, find its limit.

(a) $s_n = (0.8)^n$ (b) $s_n = \dfrac{1-e^{-n}}{1+e^{-n}}$ (c) $s_n = \dfrac{n^2+1}{n}$ (d) $s_n = 1 + (-1)^n$

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