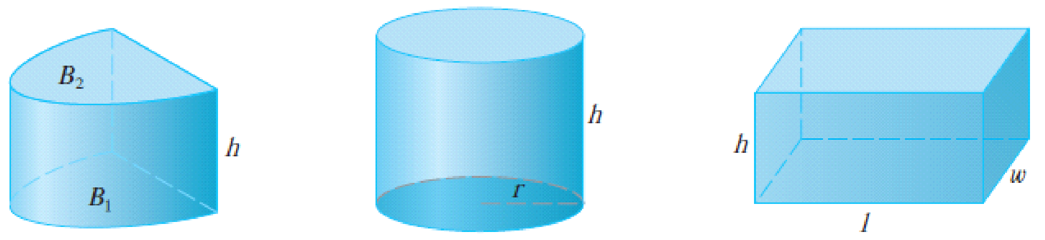
Volume of Revolution: Shell, Washer and Cylinderical

Tuesday, 24 September 2024 3:46 pm

6.2 Volumes



(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

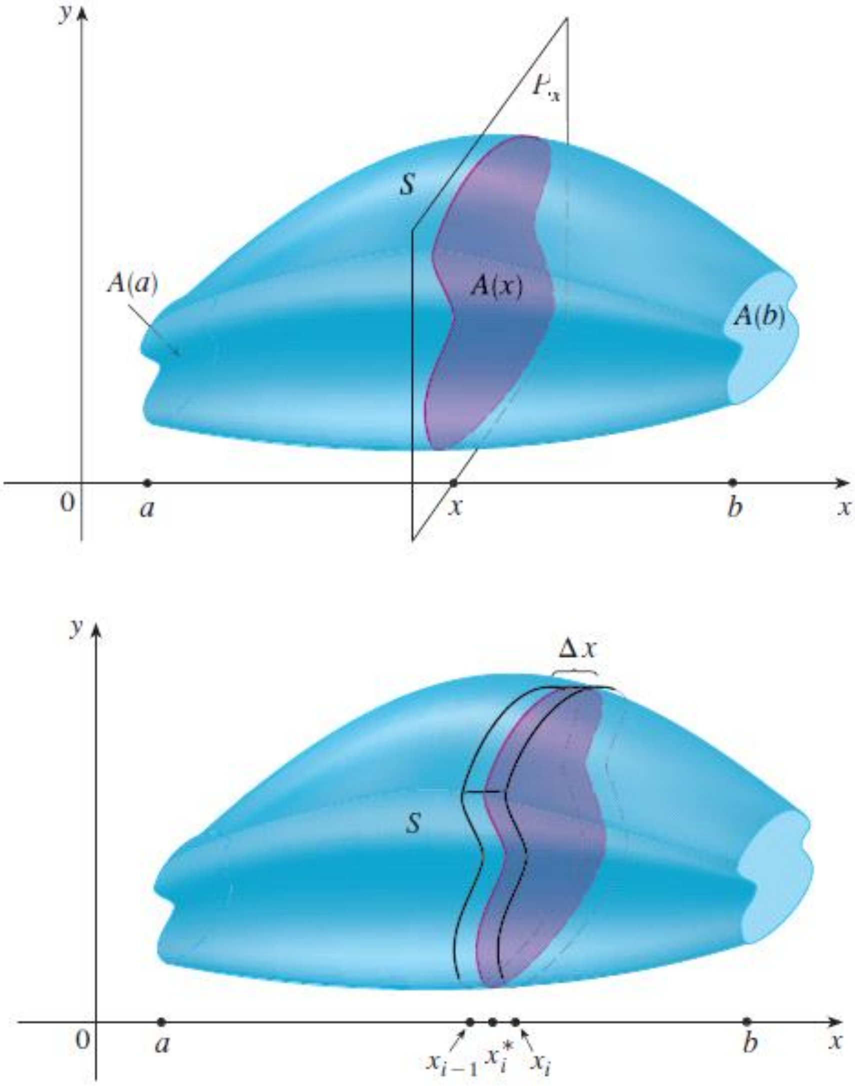
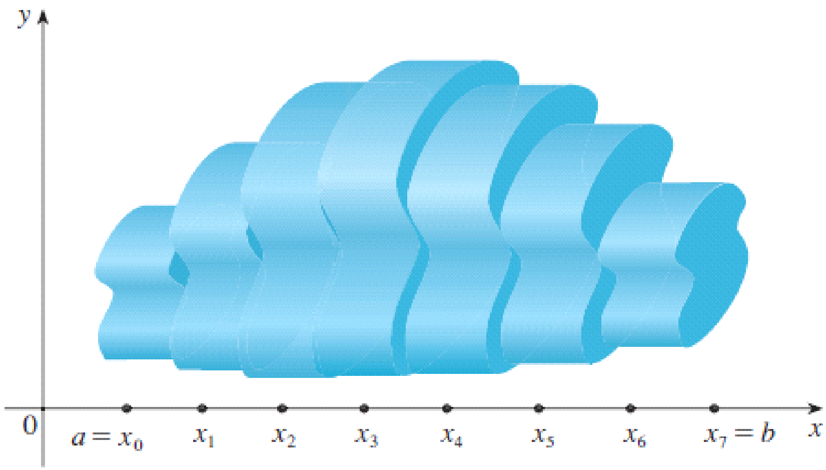


FIGURE 3



Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

EXAMPLE 1 Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

is $y = \sqrt{r^2 - x^2}$. So the cross-sectional area is

$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Using the definition of volume with $a = -r$ and $b = r$, we have

$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$= 2\pi \int_0^r (r^2 - x^2)\,dx \qquad \text{(The integrand is even.)}$$

$$= 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right)$$

$$= \tfrac{4}{3}\pi r^3$$

of Figure 5. sums become closer to the true volume.

(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

FIGURE 5 Approximating the volume of a sphere with radius 1

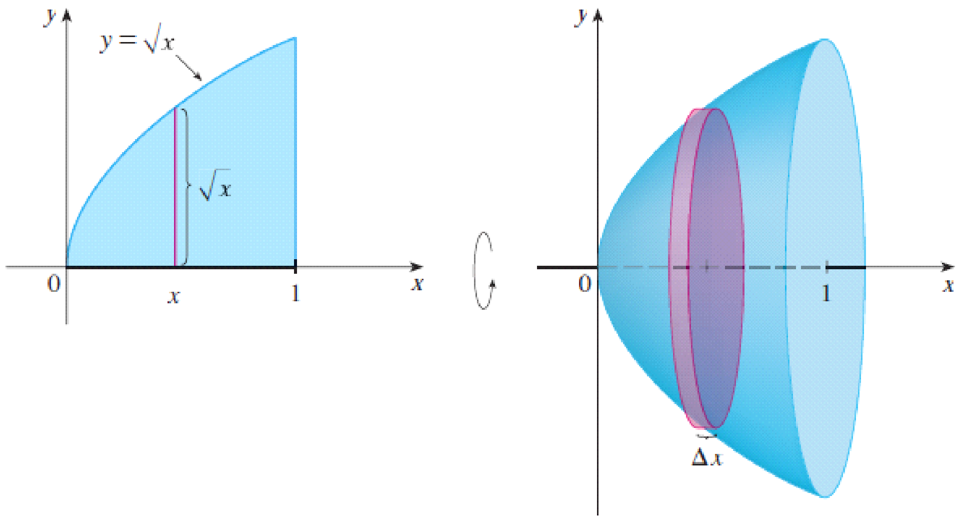
V **EXAMPLE 2** Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

$$A(x) = \pi\left(\sqrt{x}\right)^2 = \pi x$$

$$A(x)\,\Delta x = \pi x\,\Delta x$$

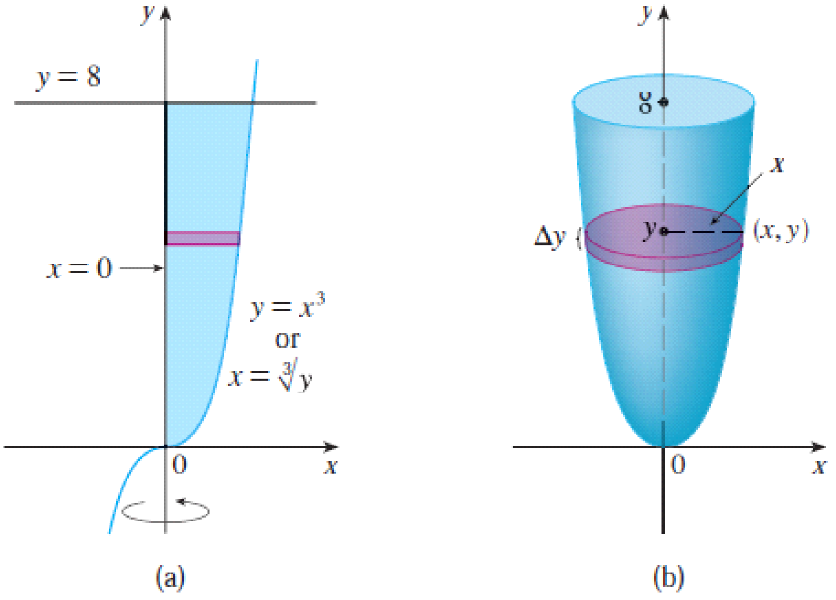
The solid lies between $x = 0$ and $x = 1$, so its volume is

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi \frac{x^2}{2}\Big]_0^1 = \frac{\pi}{2}$$



(a) (b)

V **EXAMPLE 3** Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.



$$A(y) = \pi x^2 = \pi\left(\sqrt[3]{y}\right)^2 = \pi y^{2/3}$$

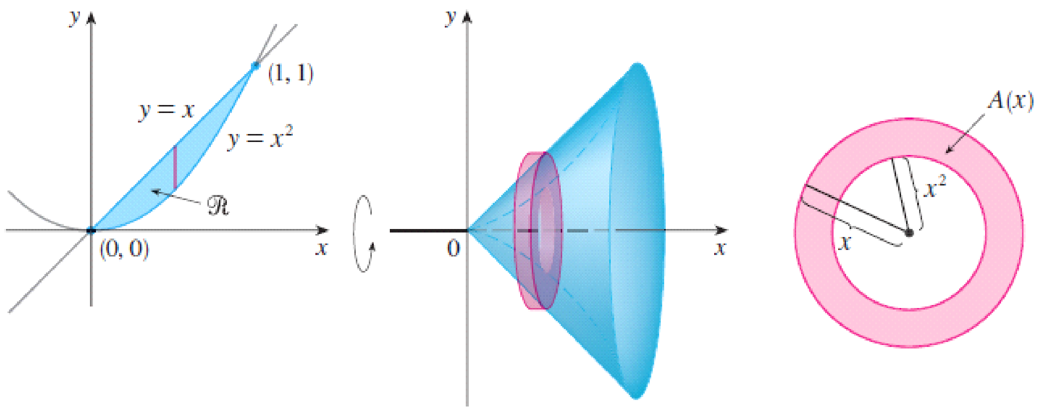
and the volume of the approximating cylinder pictured in Figure 7(b) is

$$A(y)\,\Delta y = \pi y^{2/3}\,\Delta y$$

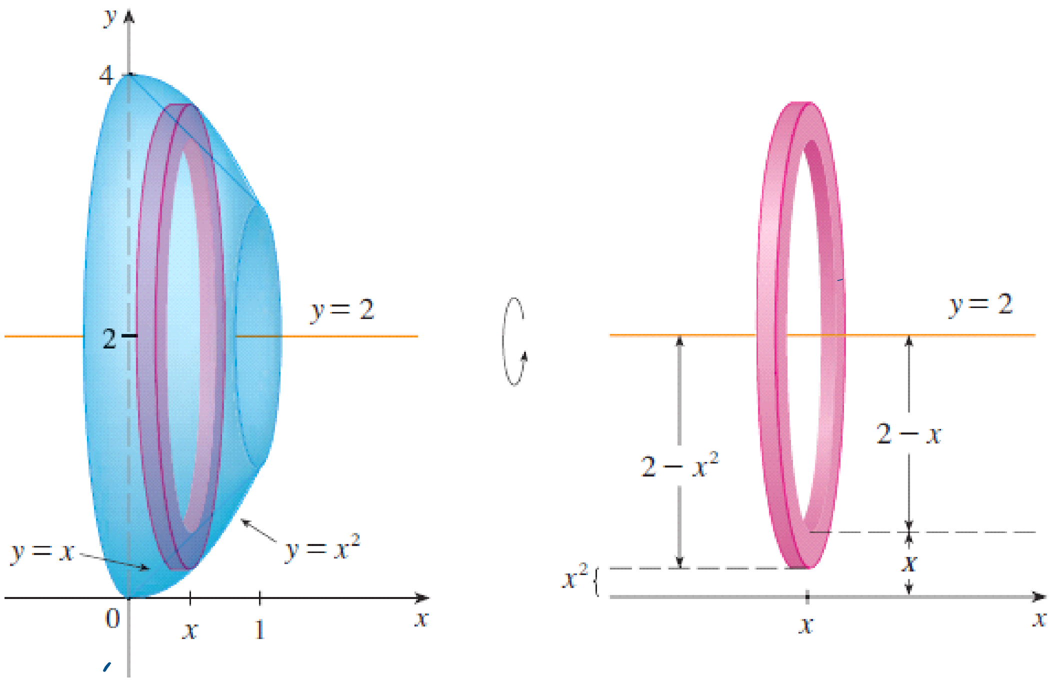
Since the solid lies between $y = 0$ and $y = 8$, its volume is

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi\left[\tfrac{3}{5}y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$

EXAMPLE 4 The region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.



EXAMPLE 5 Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.



- If the cross-section is a disk (as in Examples 1–3), we find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2$$

- If the cross-section is a washer (as in Examples 4 and 5), we find the inner radius r_{in} and outer radius r_{out} from a sketch (as in Figures 8, 9, and 10) and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

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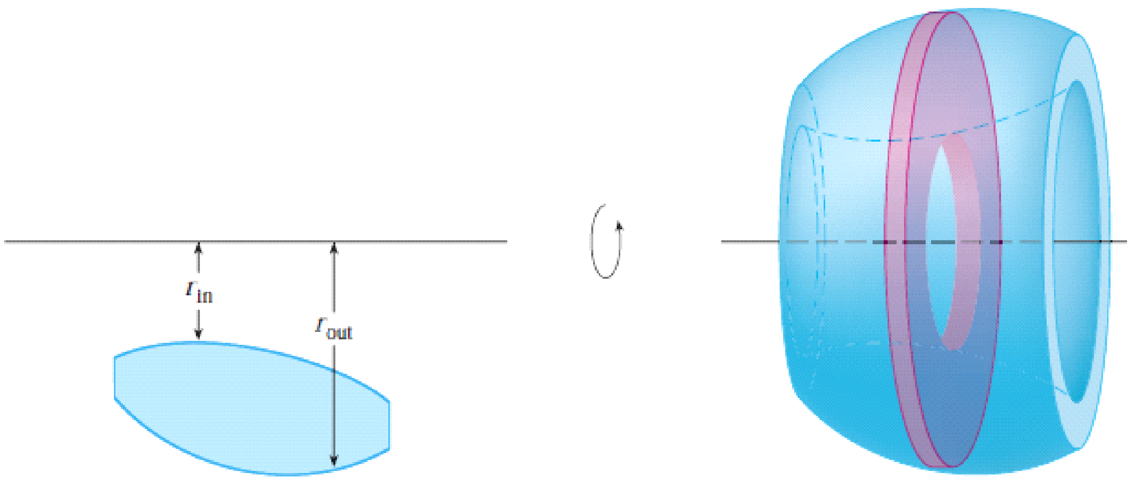
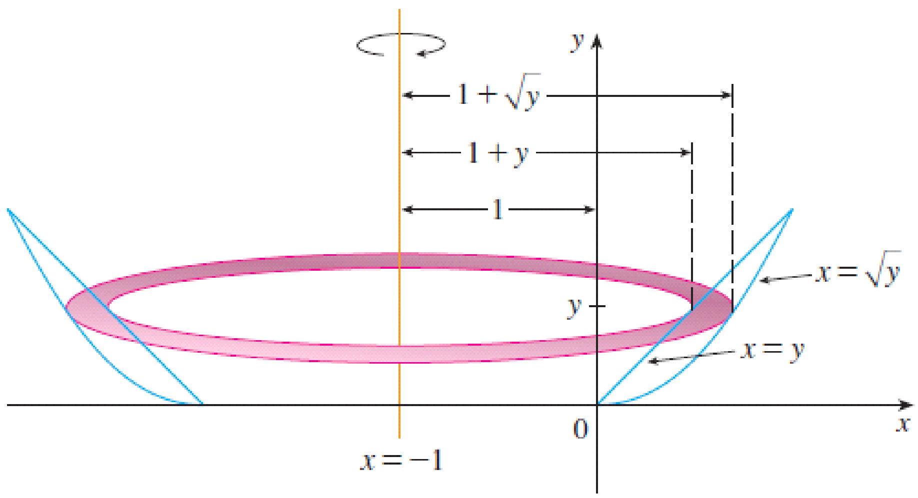
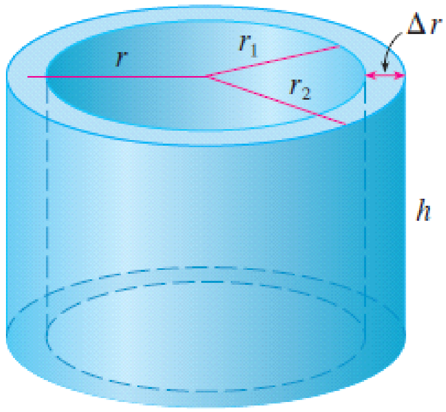


EXAMPLE 6 Find the volume of the solid obtained by rotating the region in Example 4 about the line $x = -1$.



Volumes by Cylindrical Shells

$$
\begin{aligned}
V &= V_2 - V_1 \\
&= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h \\
&= \pi(r_2 + r_1)(r_2 - r_1)h \\
&= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1)
\end{aligned}
$$

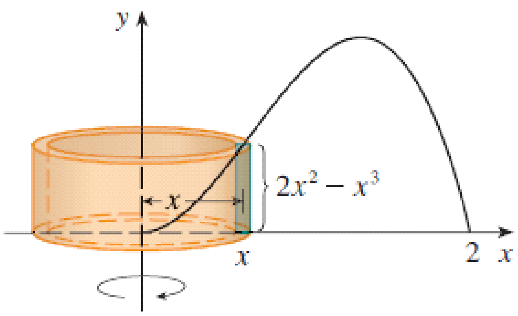


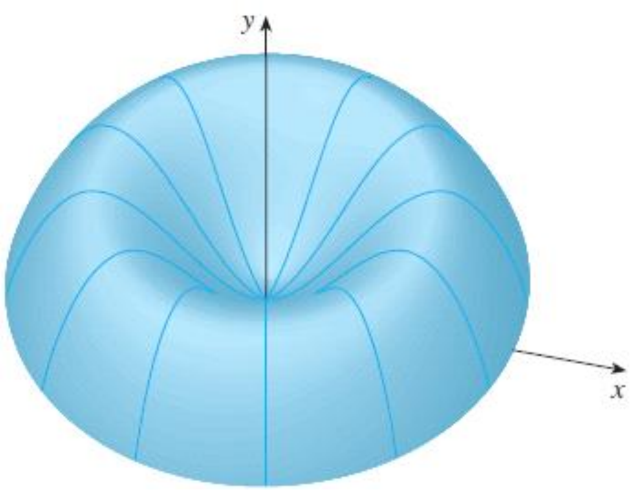
$$\boxed{V = 2\pi r h\,\Delta r}$$

$V =$ [circumference] [height] [thickness]

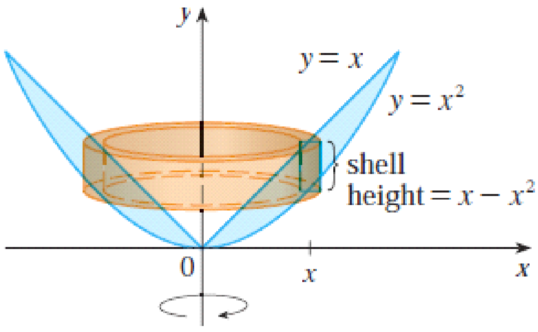
$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \; \underbrace{[\,f(x)]}_{\text{height}} \; \underbrace{dx}_{\text{thickness}}$$

EXAMPLE 1 Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.





V EXAMPLE 2 Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.



V EXAMPLE 4 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.