Partial Fraction

Saturday, 14 September 2024 7:10 pm

7.4 Integration of Rational Functions by Partial Fractions

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$

V EXAMPLE 1 Find $\int \frac{x^3+x}{x-1}\,dx$.

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

CASE I The denominator $Q(x)$ is a product of distinct linear factors.
This means that we can write

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

V EXAMPLE 2 Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

EXAMPLE 3 Find $\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

CASE II $Q(x)$ **is a product of linear factors, some of which are repeated.**

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we

would use

$$\boxed{7} \qquad \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x - 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} + \frac{E}{(x - 1)^3}$$

but we prefer to work out in detail a simpler example.

EXAMPLE 4 Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$.

CASE III $Q(x)$ **contains irreducible quadratic factors, none of which is repeated.**
If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \tag{9}$$

where A and B are constants to be determined. For instance, the function given by $f(x) = x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

The term given in 9 can be integrated by completing the square (if necessary) and using the formula

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C} \tag{10}$$

V EXAMPLE 5 Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

EXAMPLE 6 Evaluate $\displaystyle\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx$.

CASE IV $Q(x)$ **contains a repeated irreducible quadratic factor.**
If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction $\boxed{9}$, the sum

$$\boxed{11} \qquad \frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$

EXAMPLE 7 Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$

EXAMPLE 8 Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\,dx.$

Rationalizing Substitutions

EXAMPLE 9 Evaluate $\displaystyle\int \frac{\sqrt{x + 4}}{x}\,dx.$