

Power Series

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- **Power Series**
- **Taylor's Series**
- **Maclaurin's Series.**

A **power series** is a series of the form

$$\boxed{1} \quad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

V EXAMPLE 1 For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

See solution from the book.

V EXAMPLE 2 For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

See solution from the book.

EXAMPLE 3 Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

SOLUTION Let $a_n = (-1)^n x^{2n} / [2^{2n}(n!)^2]$. Then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} [(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\ &= \frac{x^{2n+2}}{2^{2n+2} (n+1)^2 (n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \\ &= \frac{x^2}{4(n+1)^2} \rightarrow 0 < 1 \quad \text{for all } x \end{aligned}$$

Thus, by the Ratio Test, the given series converges for all values of x . In other words, the domain of the Bessel function J_0 is $(-\infty, \infty) = \mathbb{R}$. ■

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

- (i) The series converges only when $x = a$.
- (ii) The series converges for all x .
- (iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n! x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

EXAMPLE 4 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

See solution from the book.

V EXAMPLE 5 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

See solution from the book.