# Area b/w Curves

Tuesday, 24 September 2024 3:09 pm

## 6.1 Areas Between Curves

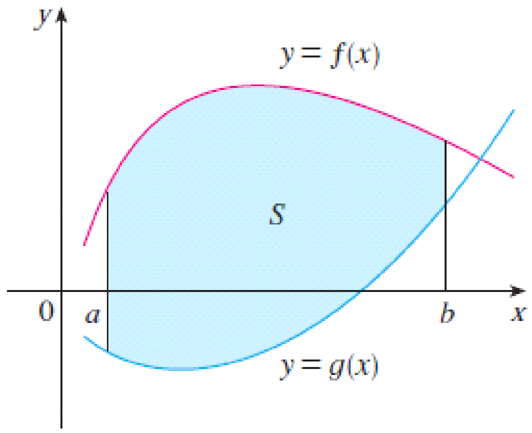
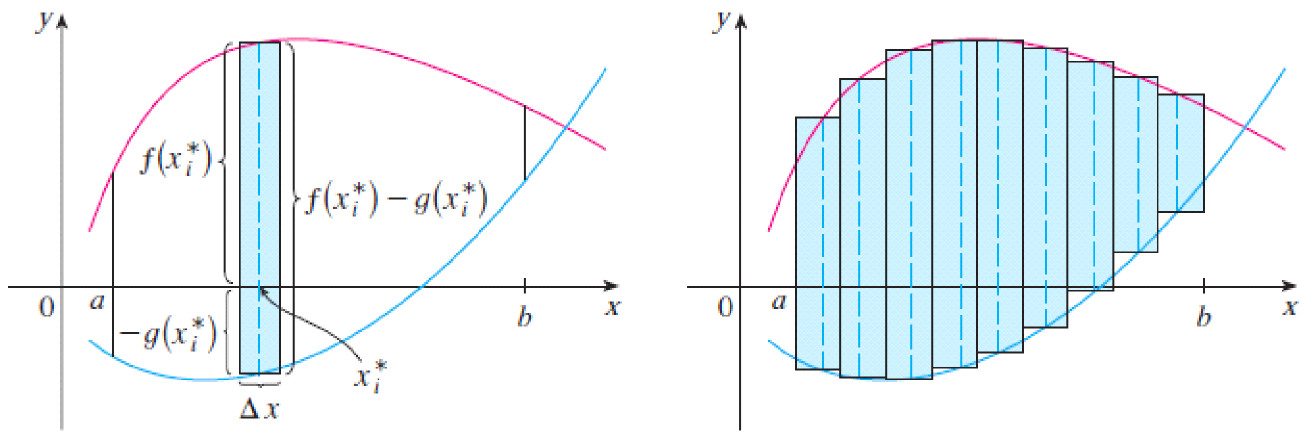



**FIGURE 1**
$S = \{(x, y) \mid a \leqslant x \leqslant b,\ g(x) \leqslant y \leqslant f(x)\}$

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$




(a) Typical rectangle (b) Approximating rectangles

$$\boxed{A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x}$$

**2** The area $A$ of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where $f$ and $g$ are continuous and $f(x) \geqslant g(x)$ for all $x$ in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

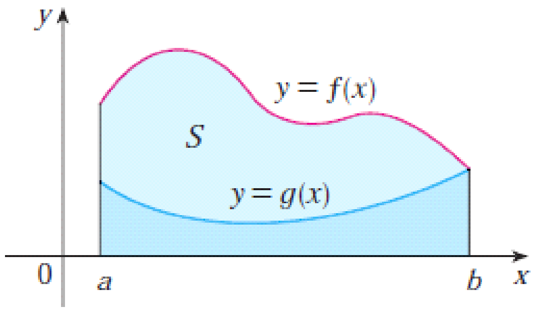



**FIGURE 3**

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$A = [\text{area under } y = f(x)] - [\text{area under } y = g(x)]$$

$$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx$$

**EXAMPLE 1** Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

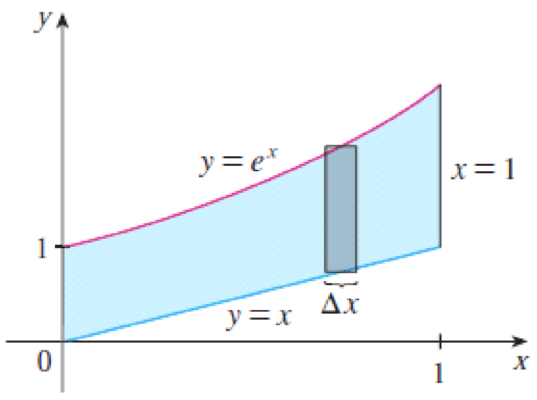



**FIGURE 4**

SOLUTION The region is shown in Figure 4. The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$. So we use the area formula $\boxed{2}$ with $f(x) = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 (e^x - x)\,dx = e^x - \tfrac{1}{2}x^2\Big]_0^1$$

$$= e - \tfrac{1}{2} - 1 = e - 1.5$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (y_T - y_B)\,\Delta x = \int_a^b (y_T - y_B)\,dx$$

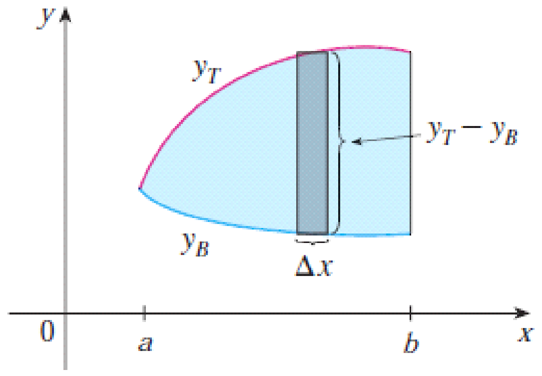



FIGURE 5

**V EXAMPLE 2** Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

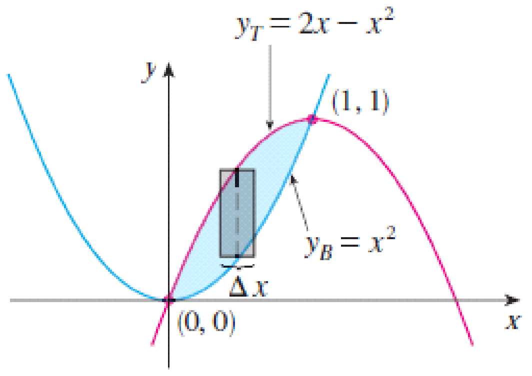



FIGURE 6

SOLUTION We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$, so $x = 0$ or 1. The points of intersection are (0, 0) and (1, 1).

We see from Figure 6 that the top and bottom boundaries are

$$y_T = 2x - x^2 \qquad \text{and} \qquad y_B = x^2$$

The area of a typical rectangle is

$$(y_T - y_B)\,\Delta x = (2x - x^2 - x^2)\,\Delta x$$

and the region lies between $x = 0$ and $x = 1$. So the total area is

$$A = \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

**EXAMPLE 3** Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2 + 1}$ and $y = x^4 - x$.

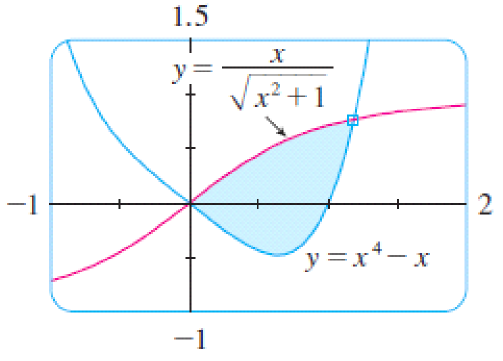



**FIGURE 7**

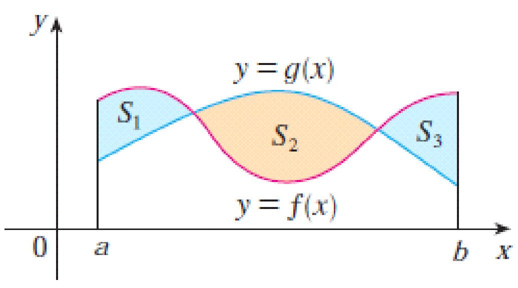



**FIGURE 9**

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geqslant g(x) \\ g(x) - f(x) & \text{when } g(x) \geqslant f(x) \end{cases}$$

**3** The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b | f(x) - g(x) | \, dx$$

**V EXAMPLE 5** Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

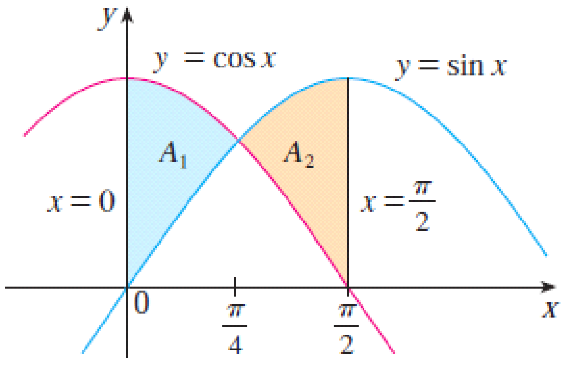



**FIGURE 10**

SOLUTION The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \leqslant x \leqslant \pi/2$). The region is sketched in Figure 10. Observe that $\cos x \geqslant \sin x$ when $0 \leqslant x \leqslant \pi/4$ but $\sin x \geqslant \cos x$ when $\pi/4 \leqslant x \leqslant \pi/2$. Therefore the required area is

$$\begin{aligned}
A &= \int_0^{\pi/2} |\cos x - \sin x| \, dx = A_1 + A_2 \\
&= \int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx \\
&= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\
&= 2\sqrt{2} - 2
\end{aligned}$$

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2 \int_0^{\pi/4} (\cos x - \sin x) \, dx$$

$$A = \int_c^d [\, f(y) - g(y)] \, dy$$

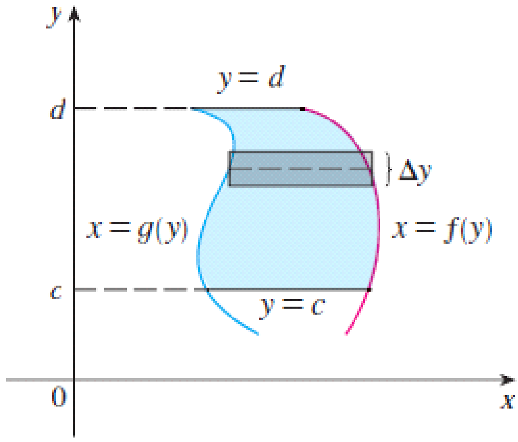



**FIGURE 11**

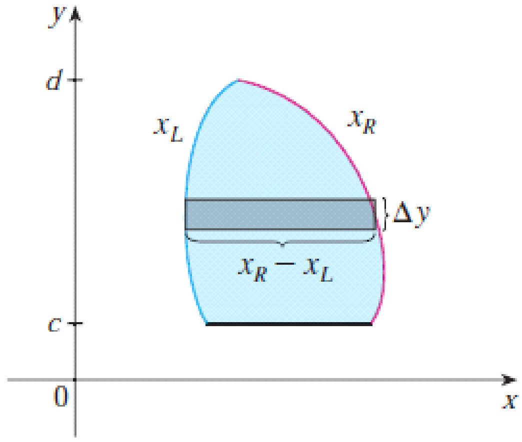



**FIGURE 12**

$$A = \int_c^d (x_R - x_L)\, dy$$

**V EXAMPLE 6** Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

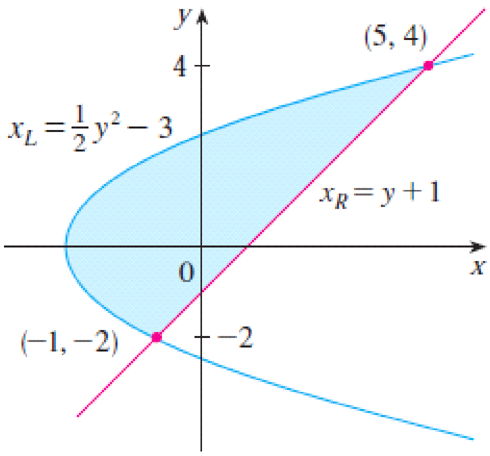



**FIGURE 13**

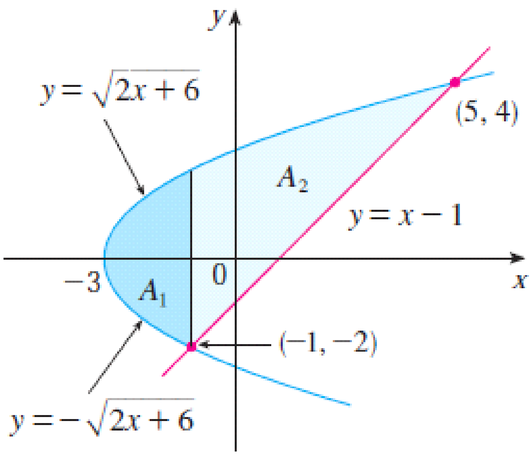



**FIGURE 14**