Improper Integral

Saturday, 14 September 2024 8:40 pm

7.8 Improper Integrals

Type 1: Infinite Intervals

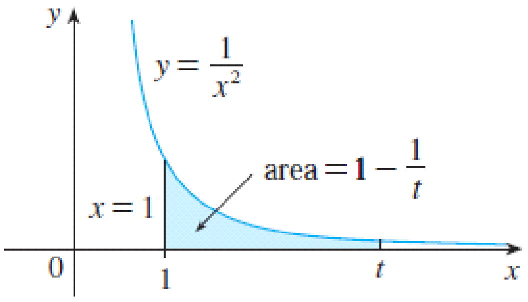
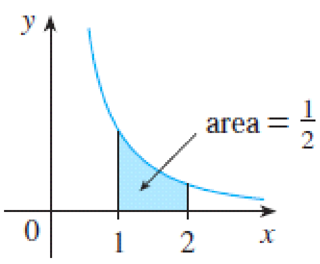


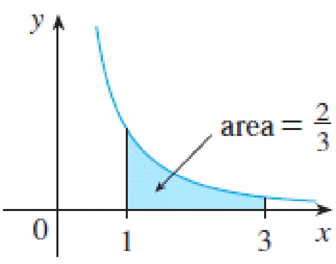
FIGURE 1

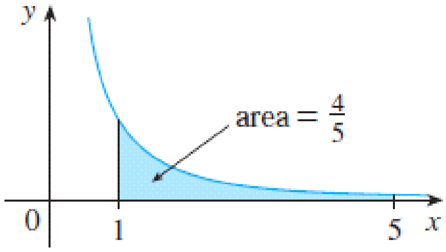
$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$

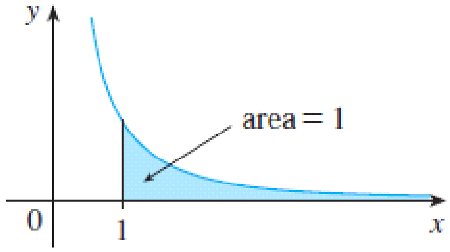
$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$$









1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 74).

V EXAMPLE 1 Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

SOLUTION According to part (a) of Definition 1, we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x}\,dx = \lim_{t \to \infty} \ln|x|\Big]_1^t$$

$$= \lim_{t \to \infty} (\ln t - \ln 1) = \lim_{t \to \infty} \ln t = \infty$$

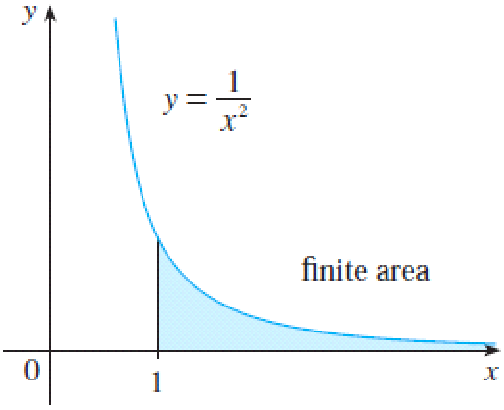


FIGURE 4 $\int_1^\infty (1/x^2)\,dx$ converges

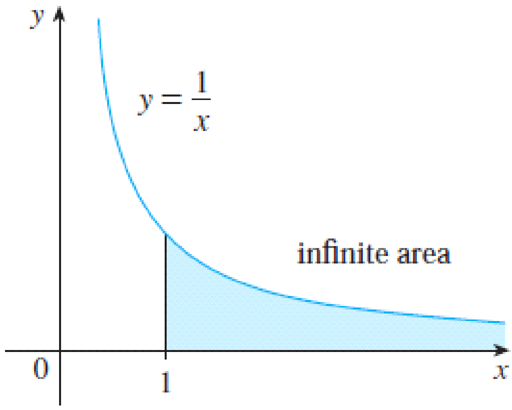


FIGURE 5 $\int_1^\infty (1/x)\,dx$ diverges

EXAMPLE 2 Evaluate $\int_{-\infty}^{0} xe^x\,dx$.

EXAMPLE 3 Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

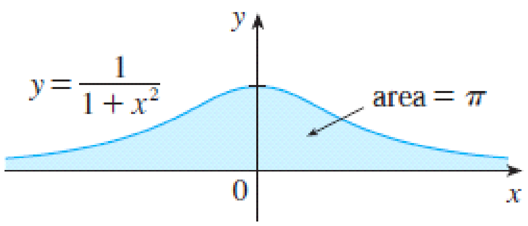


FIGURE 6

EXAMPLE 4 For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

2 $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Type 2: Discontinuous Integrands

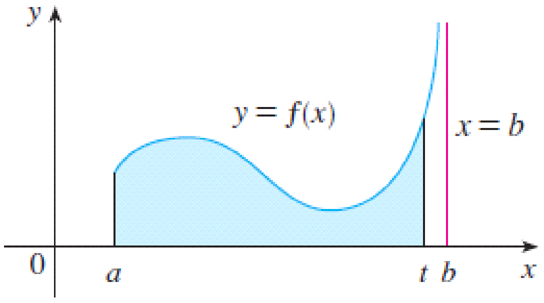


FIGURE 7

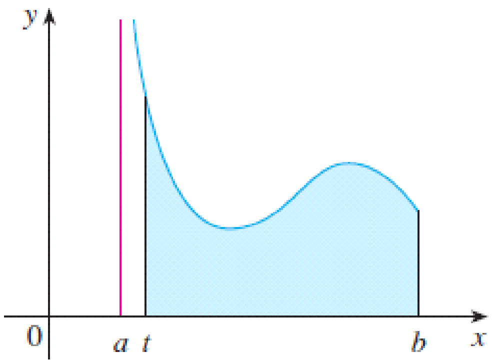


FIGURE 8

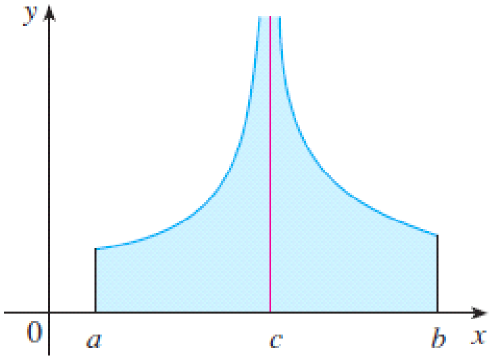


FIGURE 9

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

EXAMPLE 5 Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

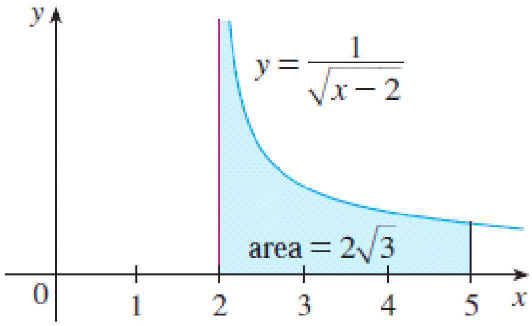


FIGURE 10

V EXAMPLE 6 Determine whether $\int_0^{\pi/2} \sec x\,dx$ converges or diverges.

EXAMPLE 7 Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

EXAMPLE 8 Evaluate $\int_0^1 \ln x \, dx$.

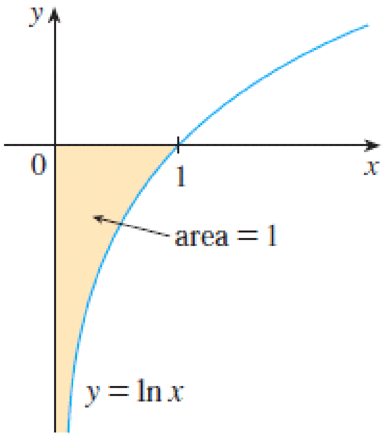


FIGURE 11

A Comparison Test for Improper Integrals

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\, dx$ is convergent, then $\int_a^\infty g(x)\, dx$ is convergent.

(b) If $\int_a^\infty g(x)\, dx$ is divergent, then $\int_a^\infty f(x)\, dx$ is divergent.

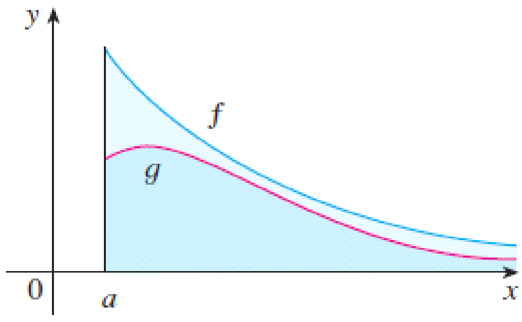


FIGURE 12

V EXAMPLE 9 Show that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

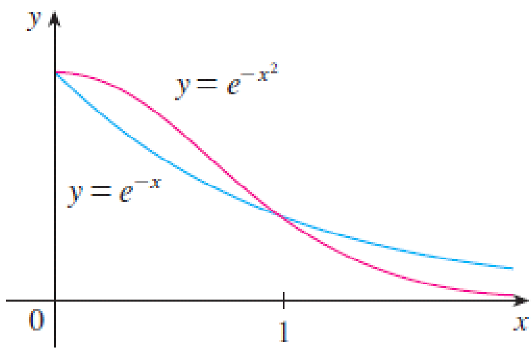


FIGURE 13

EXAMPLE 10 The integral $\int_1^\infty \frac{1 + e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\frac{1 + e^{-x}}{x} > \frac{1}{x}$$

TABLE 2

t	$\int_1^t [(1 + e^{-x})/x]\,dx$
2	0.8636306042
5	1.8276735512
10	2.5219648704
100	4.8245541204
1000	7.1271392134
10000	9.4297243064