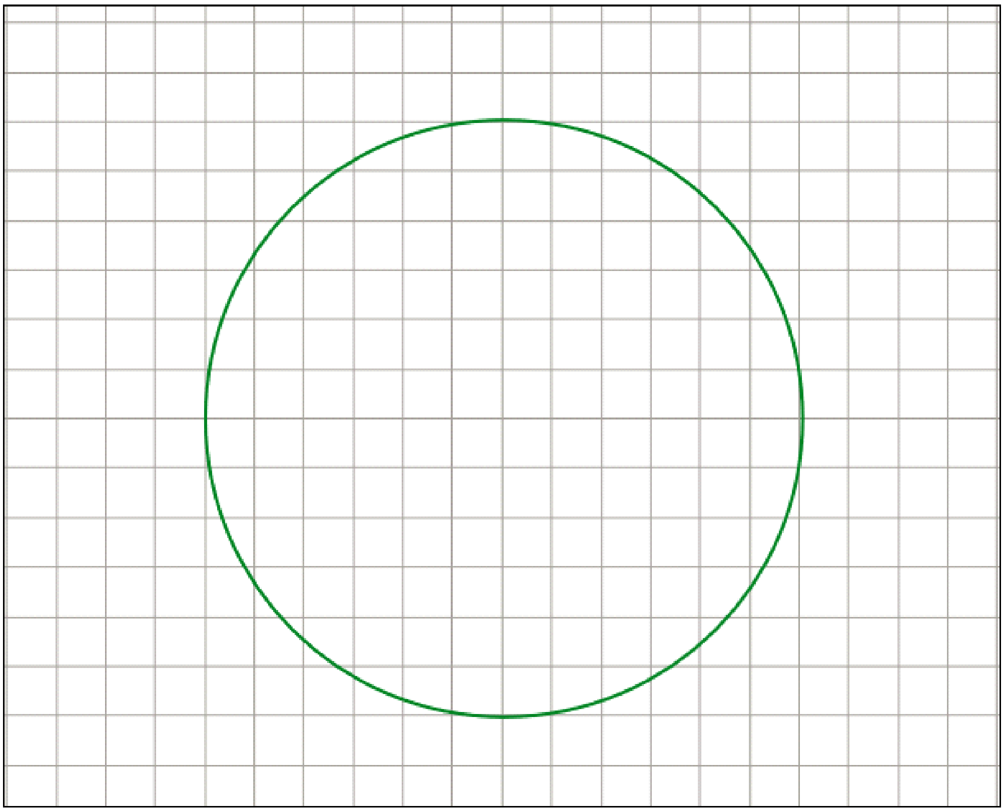
Integration (Fundamental)

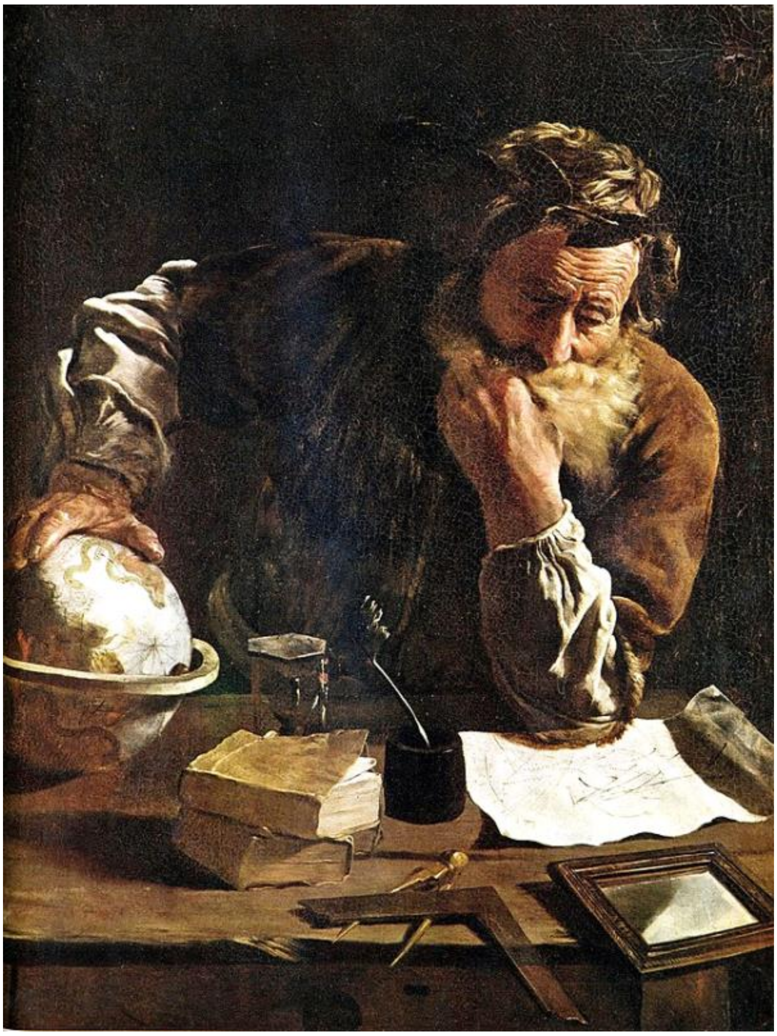
Tuesday, 20 August 2024 11:53 am

Area of a Circle:

A personal Approach

Find a way to approximate the area of the circle below without using any prior knowledge of a formula for the area. You may use your knowledge of areas of rectangles, triangles and other polygons. Document your process clearly and come up with a good approximation. You can assume one of the squares on the grid to be 1 sq. unit.

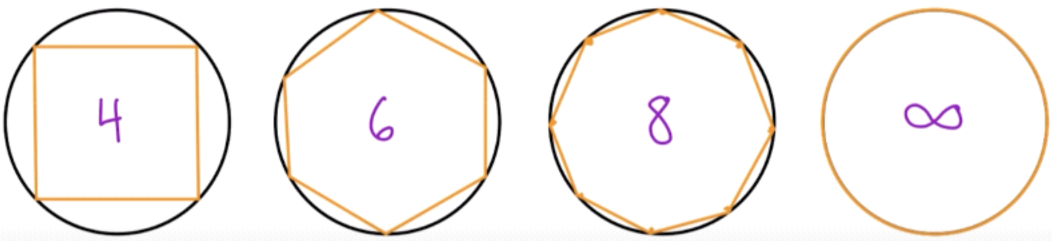




According to the most popular account, Archimedes was contemplating a mathematical diagram when the city was captured. A Roman soldier commanded him to come and meet Marcellus, but he declined, saying that he had to finish working on the problem. This enraged the soldier, who killed Archimedes with his sword.

The Method of Exhaustion

...the **Method of Exhaustion** builds on this to find area of curved shapes:



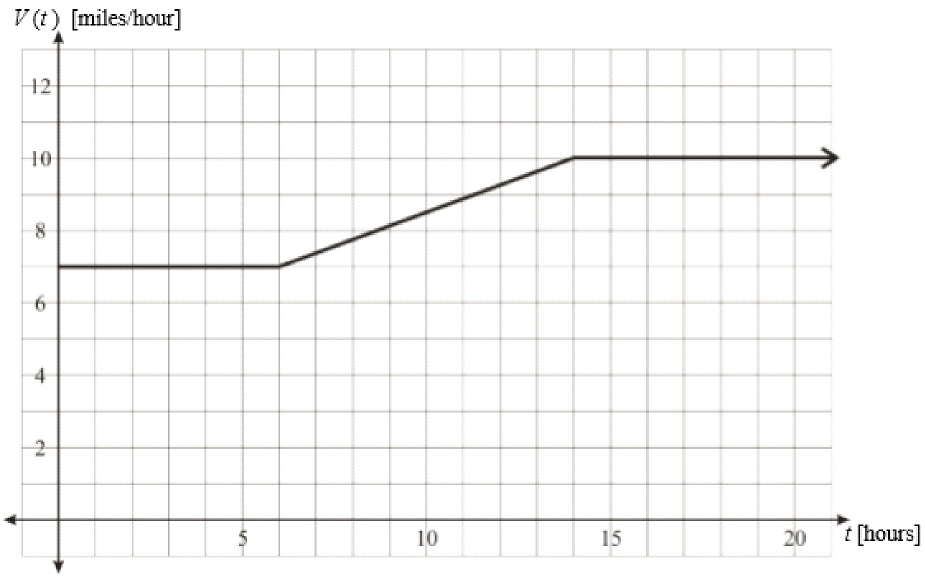
- Integral calculus is directly based on this method.
- Calculus in general relies on the idea of approaching infinity.
  - Leibniz gave field the name "calculus of the infinitesimals"

Calculus 1 *Handout 18*

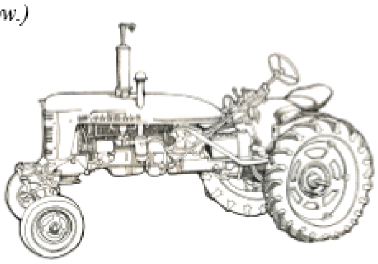
Areas Bounded by Curves

Given a velocity function, we know that finding the distance traveled by the object, over an interval of time, involves finding the area bounded by the function's graph over that same interval.

Below is a hypothetical graph of the velocity of a tractor traveling on a straight road. Find the distance traveled by the tractor between $t = 6$ and $t = 18$.



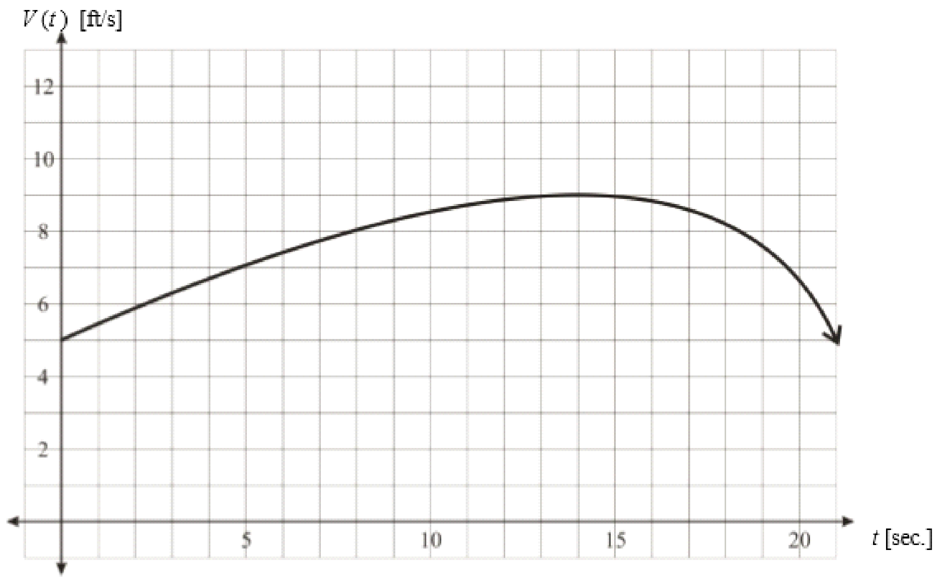
(Document your method of approximating the area in the space below.)



Area over the given interval = _______

Therefore, the distance traveled between the 6th and the 18th hour is ______________ miles.

Next, let's consider the velocity function of a Victorian horse-carriage (graphed below). It is hard to envision being able to get an exact area, judging from the shape of the curve, but you can surely get an approximation. Use your creativity with various shapes and find the area under the curve over the interval [8, 20].



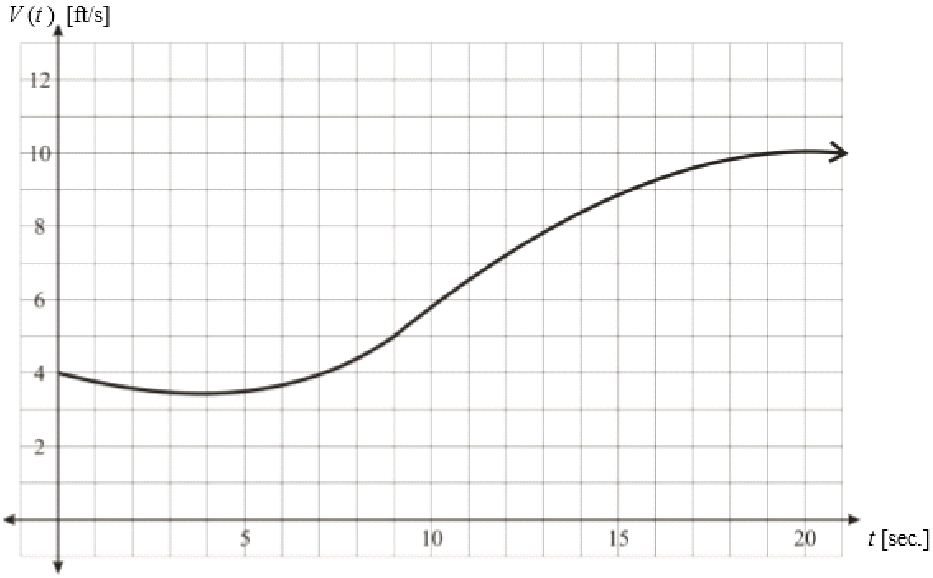
(Document your method of approximating the area in the space below.)

Area = ________

Therefore, the distance traveled between $t = 8$ and $t = 20$ is _________ feet.

Now look up at the board as I point something out to you.....

Graphed below is the velocity of Mr. Kerai running after the unidentified student who stole his special TI 89 calculator! Find an approximation for the distance traveled over the interval [4, 19], but this time restricting yourselves to rectangles of width 1 second as your primary shape.



Area = ___________

Thus, the distance traveled between 4 and 19 seconds is _______________ feet.

Now answer the following questions based on your observations of the work you have done so far.

V EXAMPLE 1 Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

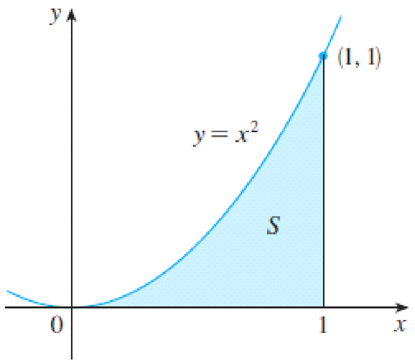
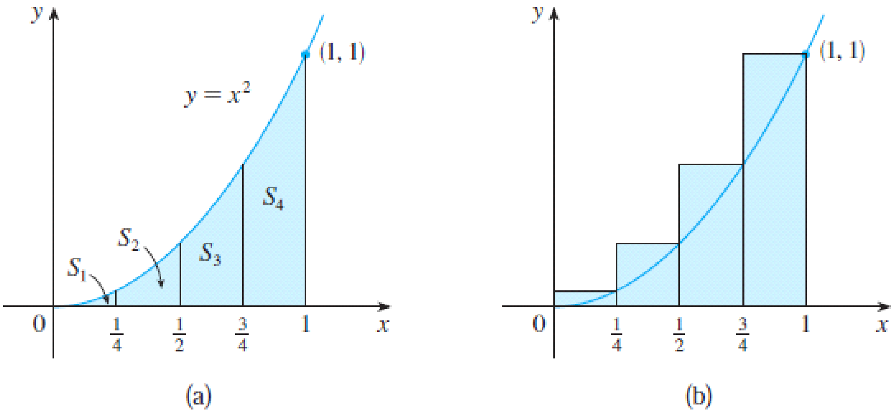


FIGURE 3



$$R_4 = \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4} \cdot 1^2 = \tfrac{15}{32} = 0.46875$$

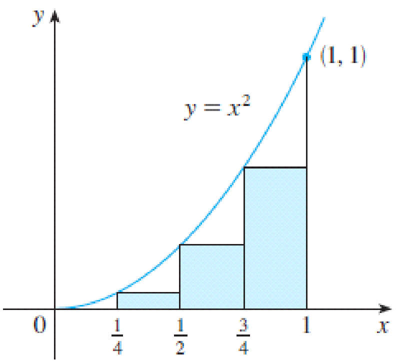
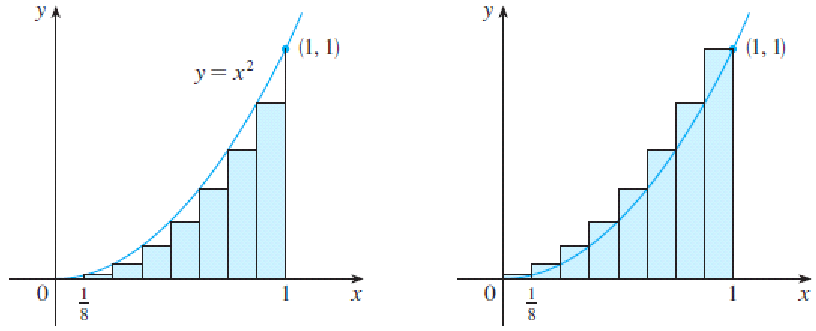


FIGURE 5

$$L_4 = \tfrac{1}{4} \cdot 0^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 = \tfrac{7}{32} = 0.21875$$

$$0.21875 < A < 0.46875$$



(a) Using left endpoints (b) Using right endpoints

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

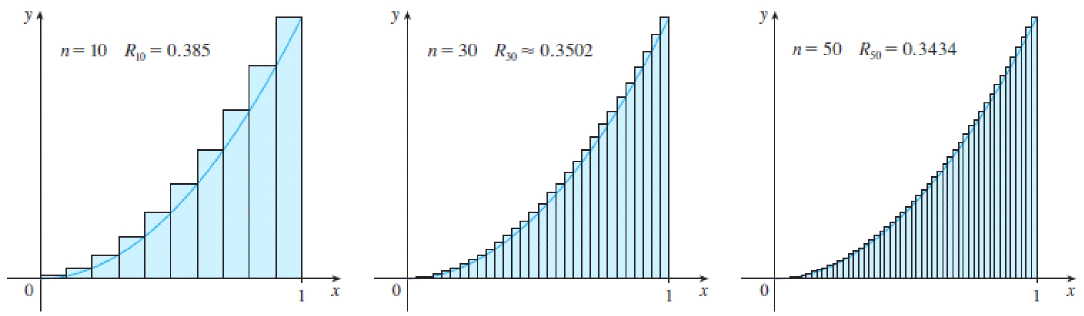


FIGURE 8 Right endpoints produce upper sums because $f(x) = x^2$ is increasing

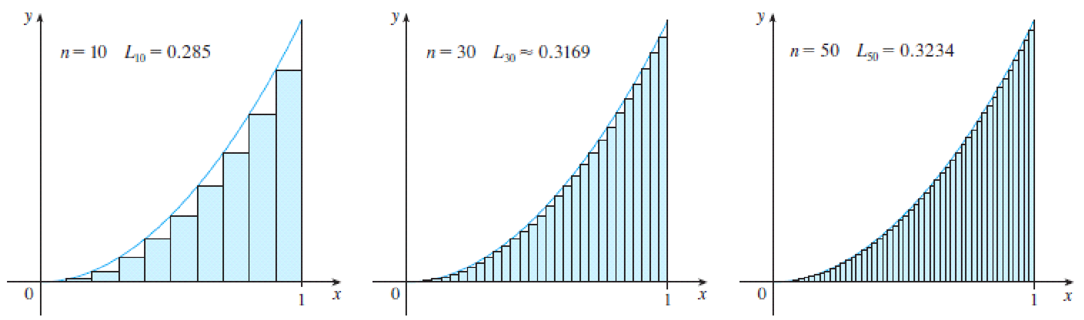


FIGURE 9 Left endpoints produce lower sums because $f(x) = x^2$ is increasing

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f **is integrable** on $[a, b]$.