Substitution Method

Wednesday, 14 August 2024 11:11 pm

The Substitution Rule

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

1 $$\int 2x\sqrt{1 + x^2}\, dx$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))\, g'(x)\, dx$. Observe that if $F' = f$, then

3 $$\int F'(g(x))\, g'(x)\, dx = F(g(x)) + C$$

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = F'(g(x))\, g'(x)$$

If we make the "change of variable" or "substitution" $u = g(x)$, then from Equation 3 we have

$$\int F'(g(x))\, g'(x)\, dx = F(g(x)) + C = F(u) + C = \int F'(u)\, du$$

or, writing $F' = f$, we get

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Thus we have proved the following rule.

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

EXAMPLE 1 Find $\int x^3 \cos(x^4 + 2)\, dx$.

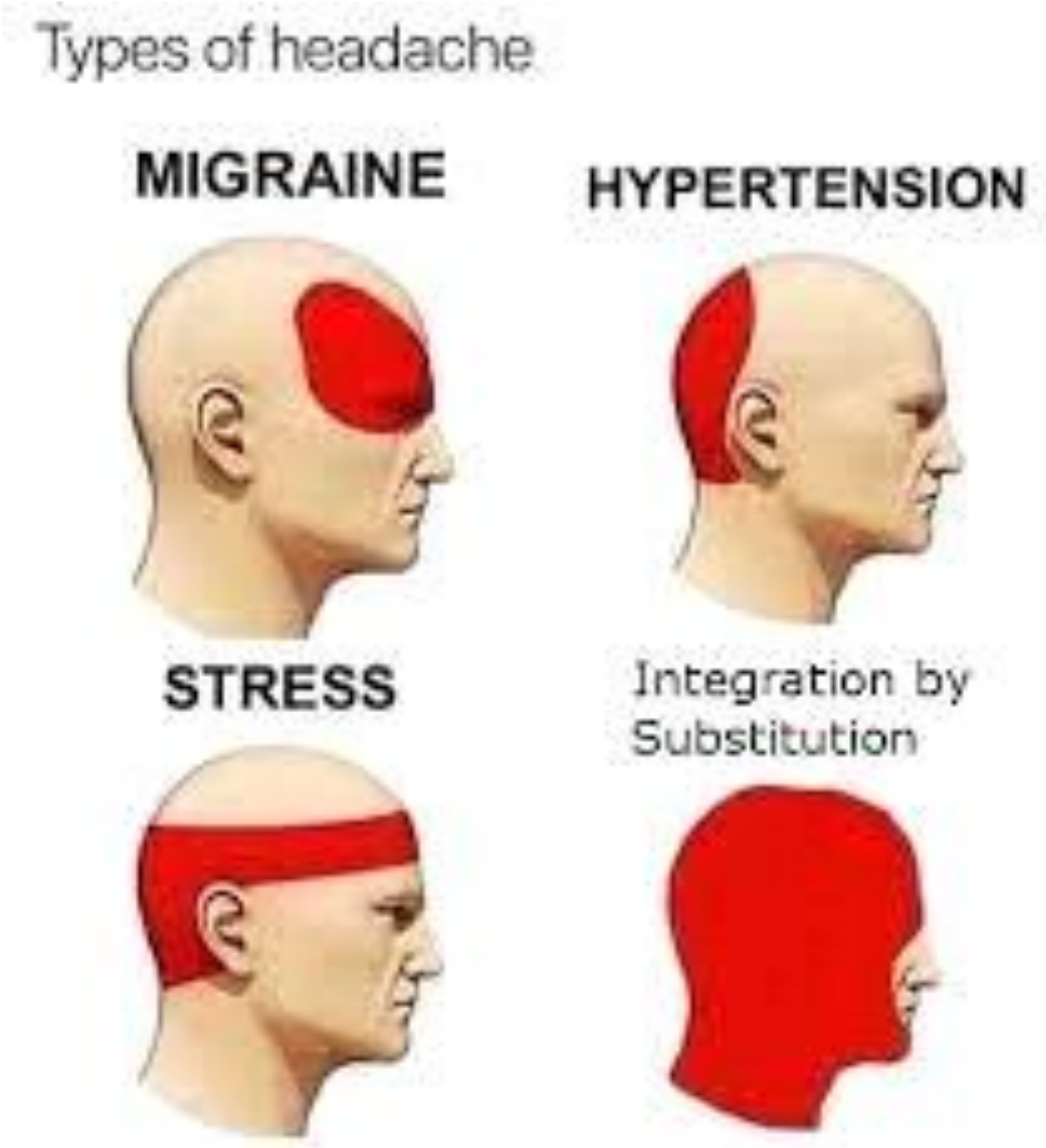
EXAMPLE 2 Evaluate $\int \sqrt{2x + 1}\, dx$.

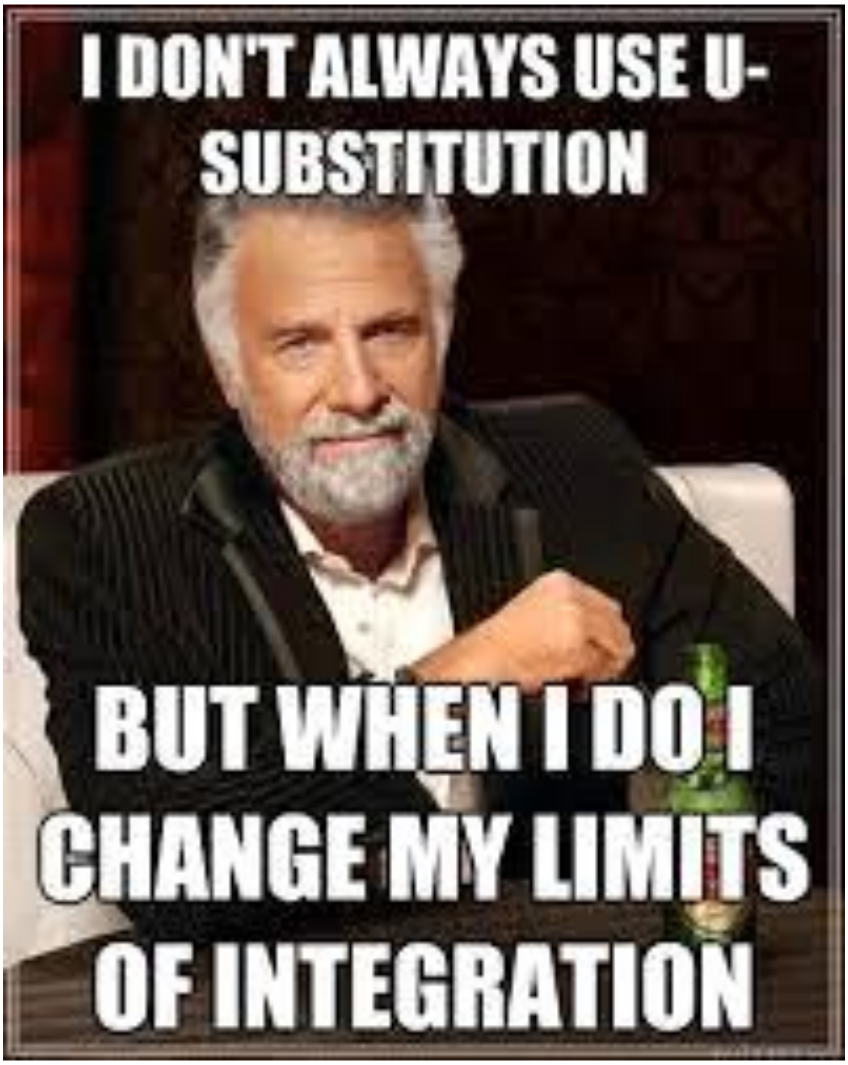
V EXAMPLE 3 Find $\int \frac{x}{\sqrt{1 - 4x^2}}\, dx$.

EXAMPLE 4 Calculate $\int e^{5x}\,dx$.

EXAMPLE 5 Find $\int \sqrt{1+x^2}\,x^5\,dx$.

V **EXAMPLE 6** Calculate $\int \tan x\,dx$.





Definite Integrals

6 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

PROOF Let F be an antiderivative of f. Then, by $\boxed{3}$, $F(g(x))$ is an antiderivative of $f(g(x))\,g'(x)$, so by Part 2 of the Fundamental Theorem, we have

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

But, applying FTC2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

EXAMPLE 7 Evaluate $\int_0^4 \sqrt{2x+1}\,dx$ using $\boxed{6}$.

EXAMPLE 8 Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

V **EXAMPLE 9** Calculate $\int_1^e \frac{\ln x}{x}\,dx$.