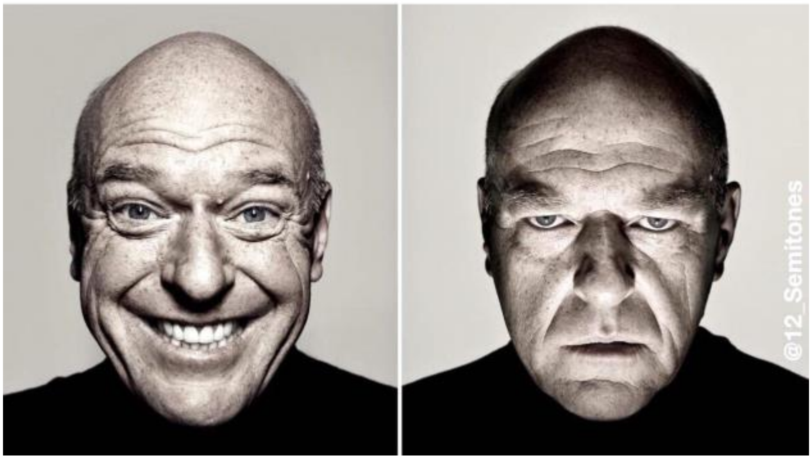
Trig Substitution Part 1

Monday, 9 September 2024 3:21 pm

$$\int \frac{1}{x^2}dx \qquad \int \frac{1}{x^2+1}dx$$



Example 6 Find $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$ using the substitution $x = \sin x$

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

$\sin^2\theta + \cos^2\theta = 1$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta \, dx$$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \theta + C$$

$= \sin^{-1} x + C$

Example 7 Use a trigonometric substitution to find $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx.$

Let $x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$$\Rightarrow \int \frac{1}{\sqrt{4-4\sin^2\theta}} 2\cos\theta \, d\theta$$

$$\int d\theta = \theta + C \Rightarrow$$

To simplify $\sqrt{a^2 - x^2}$, for constant a, try $x = a\sin\theta$, with $-\pi/2 \le \theta \le \pi/2$.

Example 10 Find $\displaystyle\int \frac{1}{x^2+9}\,dx$ using the substitution $x = 3\tan\theta$

$\frac{x}{3} = \tan\theta$

$x = 3\tan\theta \Rightarrow$

$dx = 3\sec^2\theta\, d\theta$

$$\int \frac{1}{(3\tan\theta)^2+9}\, 3\sec^2\theta\, d\theta \Rightarrow \int \frac{3\sec^2\theta\, d\theta}{9\tan^2\theta+9}$$

$$\Rightarrow \frac{\cancel{3}\cancel{\sec^2\theta}}{\cancel{9}\cancel{\sec^2\theta}}$$

$$\Rightarrow \frac{1}{3}\int d\theta = \frac{1}{3}\theta + C$$

$$\Rightarrow \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

To simplify $a^2 + x^2$ or $\sqrt{a^2 + x^2}$, for constant a, try $x = a\tan\theta$, with $-\pi/2 < \theta < \pi/2$.

Completing the Square to Use a Trigonometric Substitution

To make a trigonometric substitution, we may first need to complete the square.

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Example 12 Find $\displaystyle\int \frac{3}{\sqrt{2x - x^2}}\,dx.$ $= 3\displaystyle\int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$

$2x - x^2 = -1 + 1 + 2x - x^2 \quad \ldots^2$

$$2x - x^2 = 1 - 1 + 2x - x^2$$
$$= 1 - (x^2 - 2x + 1) \Rightarrow 1 - (x-1)^2$$

Let $x - 1 = \sin\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{3 \cos\theta}{\sqrt{1 - \sin^2\theta}} \, d\theta = 3 \int d\theta = 3\theta + c$$
$$= 3 \sin^{-1}(x-1) + c$$

Example 8 Find the area of the ellipse $4x^2 + y^2 = 9$.

$$y^2 = 9 - 4x^2$$
$$y = \sqrt{\left(\tfrac{3}{2}\right)^2 - x^2}$$

$a^2 - x^2$

$x = \frac{3}{2}\sin\theta$

$0 \le x \le \frac{3}{2}$

$$= \int_0 \sqrt{\left(\tfrac{3}{2}\right)^2 - x^2} \, dx$$

$$\int \sqrt{\left(\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\sin\theta\right)^2} \; \tfrac{3}{2}\cos\theta \, d\theta$$

$$\tfrac{3}{2} \times \tfrac{3}{2} \sqrt{1 - \sin^2\theta} \; \cos\theta \, d\theta$$

$$\tfrac{9}{4} \cos\theta \; \cos\theta \, d\theta$$

$x =$

$dx =$ $d\theta$

$$\tfrac{9}{4} \cos\theta \, d\theta$$

$$\tfrac{9}{4} \times \tfrac{1}{2}(1 + \cos 2\theta) \, d\theta$$

$$= \int \sqrt{\tfrac{3}{2}^2 - \left(\tfrac{3}{2}\right)^2} \quad d\theta$$

$$= \tfrac{3}{2} \int_0 \quad d\theta$$