Implicit Derivative, Inverse Trig & Logarithmic

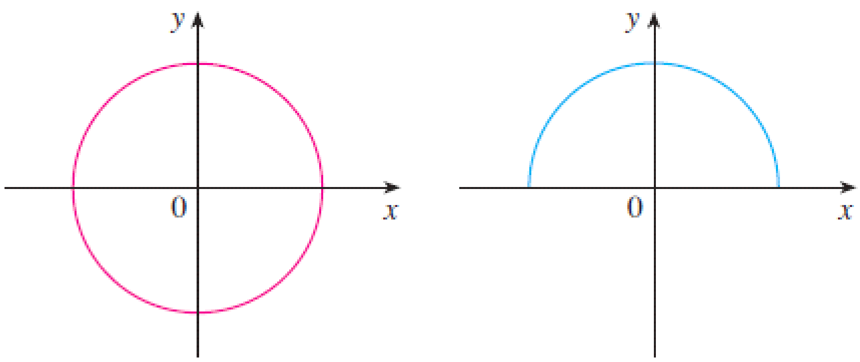
Friday, 8 November 2024 4:49 am

Implicit Differentiation

$y = \sqrt{x^3 + 1}$ or $y = x \sin x$

$x^2 + y^2 = 25$

$x^3 + y^3 = 6xy$



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$

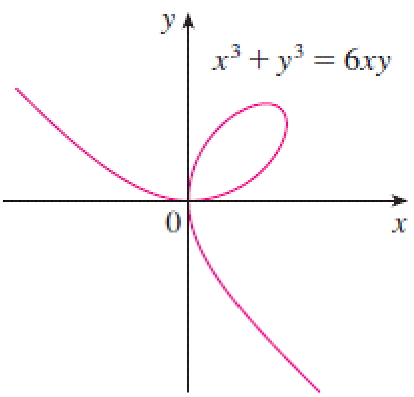


FIGURE 2 The folium of Descartes

V EXAMPLE 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

V **EXAMPLE 2**

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).
(c) At what point in the first quadrant is the tangent line horizontal?

$$3x^2 + 3y^2y' = 6xy' + 6y$$

$$x^2 + y^2y' = 2xy' + 2y$$

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(b) When $x = y = 3$,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

and a glance at Figure 4 confirms that this is a reasonable value for the slope at (3, 3). So an equation of the tangent to the folium at (3, 3) is

$$y - 3 = -1(x - 3) \qquad \text{or} \qquad x + y = 6$$

(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y - x^2 = 0$ (provided that $y^2 - 2x \neq 0$). Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get

$$x^3 + \left(\tfrac{1}{2}x^2\right)^3 = 6x\left(\tfrac{1}{2}x^2\right)$$

which simplifies to $x^6 = 16x^3$. Since $x \neq 0$ in the first quadrant, we have $x^3 = 16$. If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$. Thus the tangent is horizontal at $(2^{4/3}, 2^{5/3})$, which is approximately (2.5198, 3.1748). Looking at Figure 5, we see that our answer is reasonable.

EXAMPLE 3 Find y' if $\sin(x + y) = y^2 \cos x$.

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)(2yy')$$

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

EXAMPLE 4 Find y'' if $x^4 + y^4 = 16$.

$$y' = -\frac{x^3}{y^3}$$

$$y'' = -\frac{3x^2(16)}{y^7} = -48\,\frac{x^2}{y^7}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

V **EXAMPLE 5** Differentiate (a) $y = \frac{1}{\sin^{-1}x}$ and (b) $f(x) = x\arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$
$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$$

(b)
$$f'(x) = x\frac{1}{1+\left(\sqrt{x}\right)^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$
$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$

Derivatives of Logarithmic Functions

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}}$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

V **EXAMPLE 1** Differentiate $y = \ln(x^3 + 1)$.

EXAMPLE 4 Differentiate $f(x) = \log_{10}(2 + \sin x)$.

$$f'(x) = \frac{d}{dx}\log_{10}(2 + \sin x)$$

$$= \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x)$$

$$= \frac{\cos x}{(2 + \sin x)\ln 10}$$

EXAMPLE 5 Find $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right]$$

$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

EXAMPLE 7 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

V EXAMPLE 8 Differentiate $y = x^{\sqrt{x}}$.

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)$$