Integration by Partial Fraction

Sunday, 29 December 2024 7:53 am

7.4 Integration of Rational Functions by Partial Fractions

Partial Fractions

$$\int \frac{1}{x^2-4}\,dx \qquad \int \frac{x-4}{x^2+2x-15}\,dx$$

$$\int \frac{x^2+9}{(x^2-1)(x^2+4)}\,dx \qquad \int \frac{x}{(x-1)(x-2)^2}\,dx$$

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) dx$$

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

$$= 2\ln|x-1| - \ln|x+2| + C$$

V EXAMPLE 1 Find $\int \frac{x^3+x}{x-1}\,dx$.

$$\int \frac{x^3+x}{x-1}\,dx = \int \left(x^2+x+2+\frac{2}{x-1} \right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\ \hline
x - 1 & x^3 \qquad + x \\
 & \underline{x^3 - x^2} \\
 & \qquad x^2 + x \\
 & \qquad \underline{x^2 - x} \\
 & \qquad\qquad 2x \\
 & \qquad\qquad \underline{2x - 2} \\
 & \qquad\qquad\qquad 2
\end{array}
$$

$$\frac{A}{(ax + b)^i} \qquad \text{or} \qquad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

CASE I The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

V EXAMPLE 2 Evaluate $\displaystyle\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx.$

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

$$x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A &= -1 \end{aligned}$$

EXAMPLE 3 Find $\displaystyle\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

$$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{A}{x - a} + \frac{B}{x + a}$$

$$A(x + a) + B(x - a) = 1$$

$$\begin{aligned} \int \frac{dx}{x^2 - a^2} &= \frac{1}{2a} \int \left(\frac{1}{x - a} - \frac{1}{x + a} \right) dx \\ &= \frac{1}{2a} \big(\ln |x - a| - \ln |x + a| \big) + C \end{aligned}$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

$$\boxed{6} \qquad \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

CASE II $Q(x)$ **is a product of linear factors, some of which are repeated.**
Suppose the first linear factor $(a_1 x + b_1)$ is repeated r times; that is, $(a_1 x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1 x + b_1)$ in Equation 2, we

would use

$$\boxed{7} \qquad \frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3-x+1}{x^2(x-1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}$$

but we prefer to work out in detail a simpler example.

CASE III $Q(x)$ **contains irreducible quadratic factors, none of which is repeated.**

If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\boxed{9} \qquad \frac{Ax+B}{ax^2+bx+c}$$

where A and B are constants to be determined. For instance, the function given by $f(x)=x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2+4}$$

The term given in $\boxed{9}$ can be integrated by completing the square (if necessary) and using the formula

$$\boxed{10} \qquad \boxed{\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$$

V EXAMPLE 5 Evaluate $\displaystyle\int\frac{2x^2-x+4}{x^3+4x}\,dx$.

SOLUTION Since $x^3+4x=x(x^2+4)$ can't be factored further, we write

$$\frac{2x^2-x+4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

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$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$
$$= (A + B)x^2 + Cx + 4A$$

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$$A + B = 2 \qquad C = -1 \qquad 4A = 4$$

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Thus $A = 1$, $B = 1$, and $C = -1$ and so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4}\right) dx$$

In order to integrate the second term we split it into two parts:

$$\int \frac{x - 1}{x^2 + 4}\,dx = \int \frac{x}{x^2 + 4}\,dx - \int \frac{1}{x^2 + 4}\,dx$$

$$\int \frac{2x^2 - x + 4}{x(x^2 + 4)}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2 + 4}\,dx - \int \frac{1}{x^2 + 4}\,dx$$
$$= \ln|x| + \tfrac{1}{2}\ln(x^2 + 4) - \tfrac{1}{2}\tan^{-1}(x/2) + K$$

CASE IV ***$Q(x)$ contains a repeated irreducible quadratic factor.***
If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction $\boxed{9}$, the sum

$$\boxed{11} \qquad \frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$

EXAMPLE 7 Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x - 1)(x^2 + x + 1)(x^2 + 1)^3}$$

SOLUTION

$$\frac{x^3 + x^2 + 1}{x(x - 1)(x^2 + x + 1)(x^2 + 1)^3}$$
$$= \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + x + 1} + \frac{Ex + F}{x^2 + 1} + \frac{Gx + H}{(x^2 + 1)^2} + \frac{Ix + J}{(x^2 + 1)^3}$$

Partial fraction

Summary

Denominator containing...	Expression	Form of Partial Fractions
a. Linear factor	$\dfrac{f(x)}{(x+a)(x+b)}$	$\dfrac{A}{x+a}+\dfrac{B}{x+b}$
b. Repeated linear factors	$\dfrac{f(x)}{(x+a)^3}$	$\dfrac{A}{x+a}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+a)^3}$
c. Quadratic term (which cannot be factored)	$\dfrac{f(x)}{(ax^2+bx+c)(gx+h)}$	$\dfrac{Ax+B}{ax^2+bx+c}+\dfrac{C}{gx+h}$

Rationalizing Substitutions

EXAMPLE 9 Evaluate $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$.

SOLUTION Let $u = \sqrt{x+4}$. Then $u^2 = x+4$, so $x = u^2 - 4$ and $dx = 2u\,du$. Therefore

$$\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\,2u\,du = 2\int \frac{u^2}{u^2-4}\,du$$

$$= 2\int \left(1 + \frac{4}{u^2-4}\right) du$$