# Continuity

Friday, 25 October 2024 3:00 pm

## Continuity

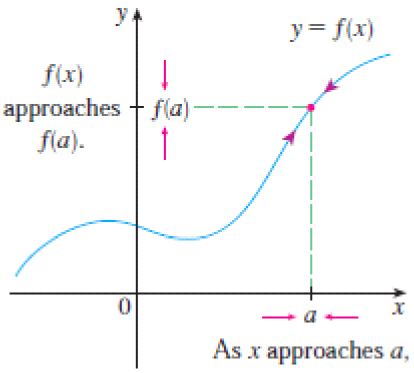
**1 Definition** A function $f$ is **continuous at a number** $a$ if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if $f$ is continuous at $a$:

1. $f(a)$ is defined (that is, $a$ is in the domain of $f$)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

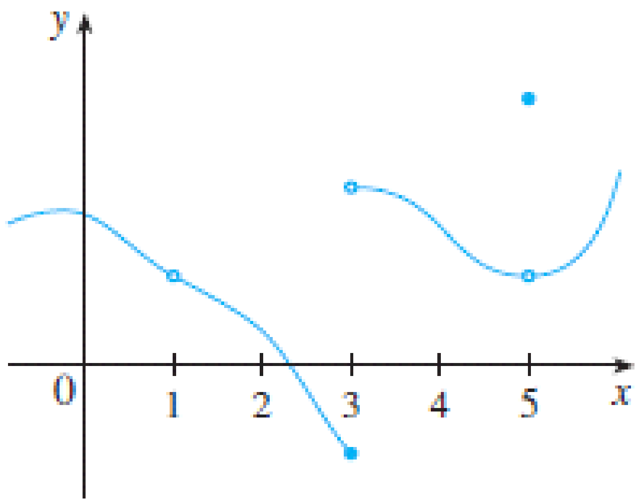
As illustrated in Figure 1, if $f$ is continuous, then the points $(x, f(x))$ on the graph of $f$ approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.




GeoGebra Link: [Limits & Continuity]()

Again, all this means is that there are no **holes**, **breaks**, or **jumps** in the graph. Otherwise, the function is considered discontinuous.

**EXAMPLE 1** Figure 2 shows the graph of a function $f$. At which numbers is $f$ discontinuous? Why?




**V EXAMPLE 2** Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

SOLUTION

(a) Notice that $f(2)$ is not defined, so $f$ is discontinuous at 2. Later we'll see why $f$ is continuous at all other numbers.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. (See Example 8 in Section 2.2.) So $f$ is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

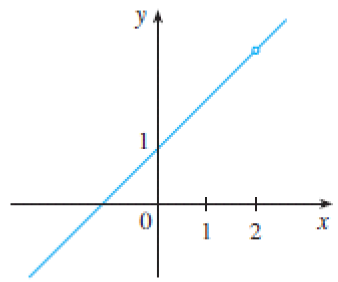
$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3$$

exists. But

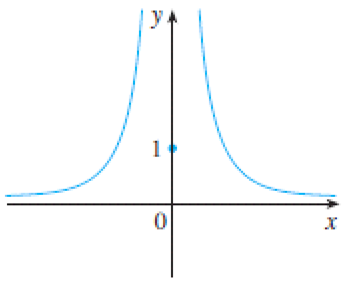
$$\lim_{x \to 2} f(x) \neq f(2)$$

so $f$ is not continuous at 2.

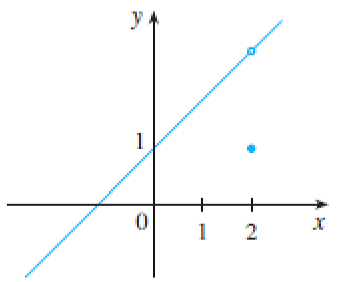
(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x \to n} [\![x]\!]$ does not exist if $n$ is an integer. (See Example 10 and Exercise 51 in Section 2.3.)



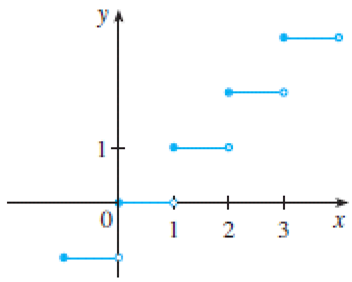

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$




(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



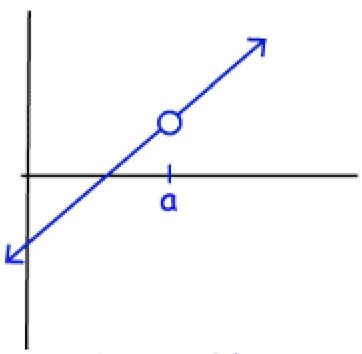

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



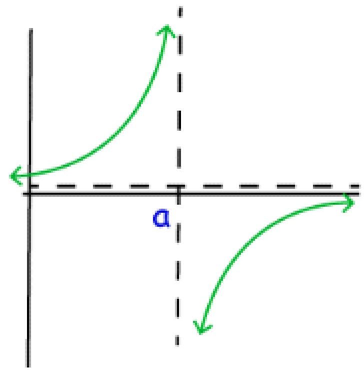

(d) $f(x) = [\![ x ]\!]$

Recall that there are four types of discontinuity:

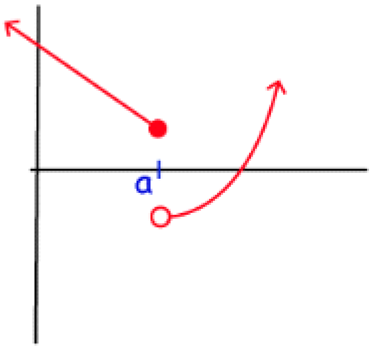
1. Removable
2. Infinite
3. Jump
4. Oscillating

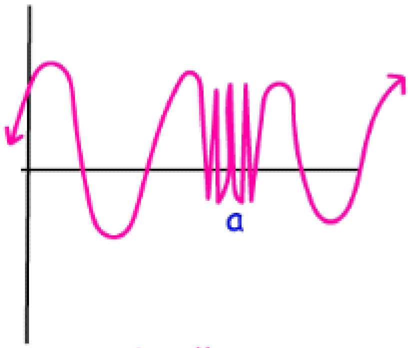



Removable Discontinuity




Infinite Discontinuity




**2 Definition** A function $f$ is **continuous from the right at a number $a$** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and $f$ is **continuous from the left at $a$** if

$$\lim_{x \to a^-} f(x) = f(a)$$

**EXAMPLE 3** At each integer $n$, the function $f(x) = [\![x]\!]$ [see Figure 3(d)] is continuous from the right but discontinuous from the left because

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [\![x]\!] = n = f(n)$$

but
$$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [\![x]\!] = n - 1 \neq f(n)$$

**3 Definition** A function $f$ is **continuous on an interval** if it is continuous at every number in the interval. (If $f$ is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*)

**4 Theorem** If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:

1. $f + g$
2. $f - g$
3. $cf$
4. $fg$
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

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**5 Theorem**

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

**7 Theorem** The following types of functions are continuous at every number in their domains:

| | | |
|---|---|---|
| polynomials | rational functions | root functions |
| trigonometric functions | inverse trigonometric functions | |
| exponential functions | logarithmic functions | |

**EXAMPLE 7** Evaluate $\displaystyle\lim_{x\to\pi} \frac{\sin x}{2 + \cos x}$.

**8 Theorem** If $f$ is continuous at $b$ and $\displaystyle\lim_{x\to a} g(x) = b$, then $\displaystyle\lim_{x\to a} f\big(g(x)\big) = f(b)$. In other words,

$$\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big)$$

**EXAMPLE 8** Evaluate $\displaystyle\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$.

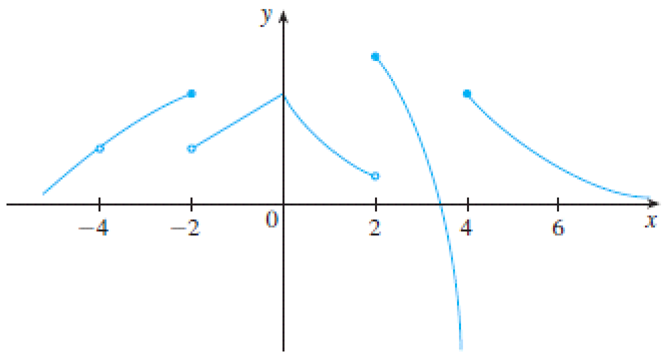
SOLUTION Because arcsin is a continuous function, we can apply Theorem 8:

$$\begin{aligned}
\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) &= \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right) \\
&= \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}\right) \\
&= \arcsin\left(\lim_{x\to 1} \frac{1}{1+\sqrt{x}}\right) \\
&= \arcsin\frac{1}{2} = \frac{\pi}{6}
\end{aligned}$$

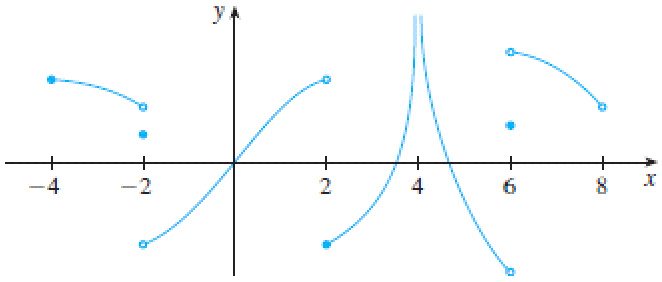
**10 The Intermediate Value Theorem** Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c$ in $(a, b)$ such that $f(c) = N$.

(a) From the graph of $f$, state the numbers at which $f$ is discontinuous and explain why.

(b) For each of the numbers stated in part (a), determine whether $f$ is continuous from the right, or from the left, or neither.




From the graph of $g$, state the intervals on which $g$ is continuous.




$$f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \qquad a = 0$$

## Applications of Continuity

Continuity (or concept of continuous function) is used in optimization problems for finding maximum and minimum values of the function to experience a smooth change of state. Signal processing has a wide variety of applications which require continuous functions such as analysing and manipulating signals in audio processing and image processing.

Application link: Real Life Applications of Continuity - GeeksforGeeks