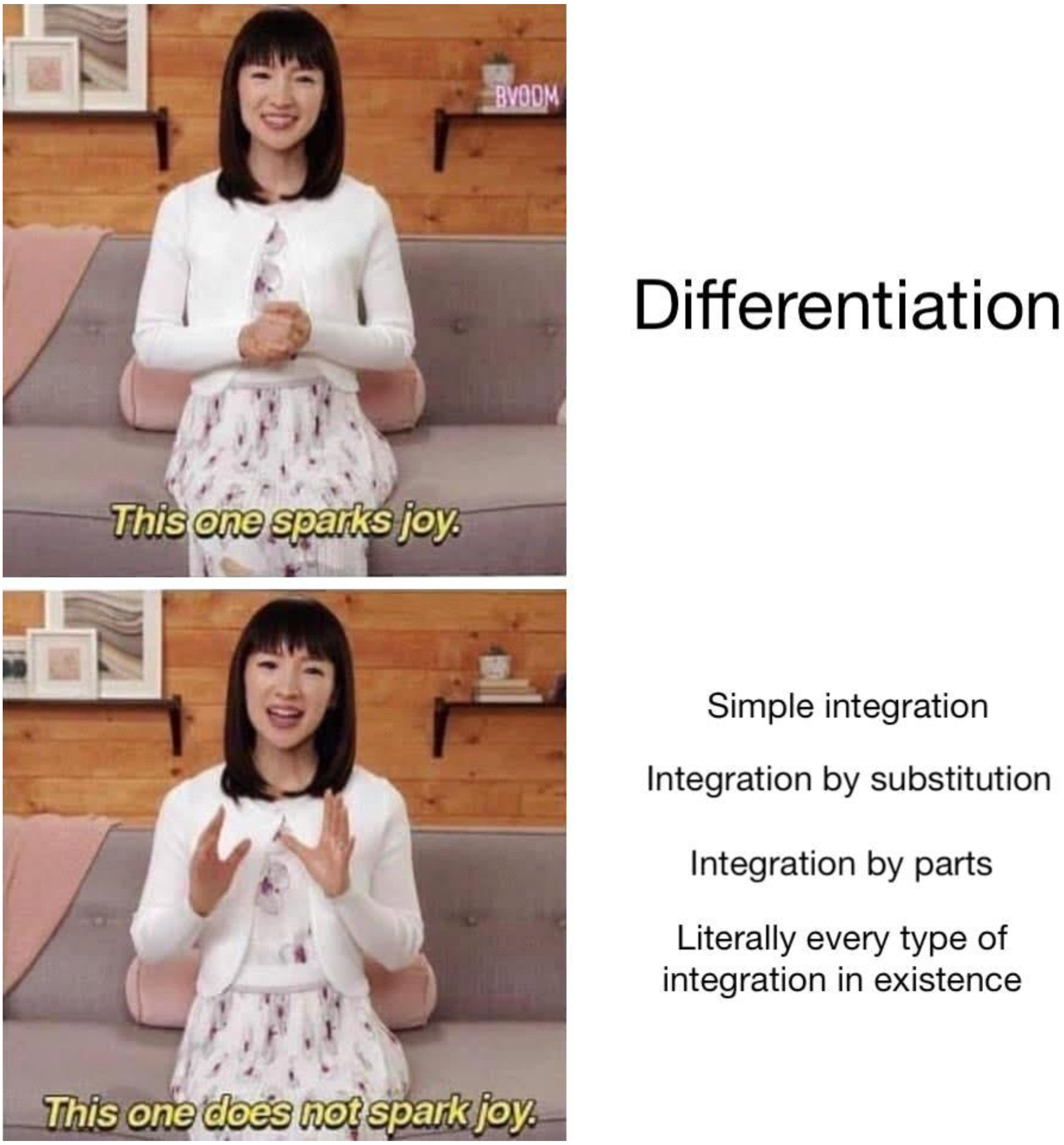
Techniques of Integration(Substitution)

Sunday, 15 December 2024 1:37 pm



The Substitution Rule

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

1 $$\int 2x\sqrt{1 + x^2}\, dx$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))\,g'(x)\, dx$. Observe that if $F' = f$, then

3 $$\int F'(g(x))\,g'(x)\, dx = F(g(x)) + C$$

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = F'(g(x))\,g'(x)$$

If we make the "change of variable" or "substitution" $u = g(x)$, then from Equation 3 we have

$$\int F'(g(x))\,g'(x)\, dx = F(g(x)) + C = F(u) + C = \int F'(u)\, du$$

or, writing $F' = f$, we get

$$\int f(g(x))\,g'(x)\, dx = \int f(u)\, du$$

Thus we have proved the following rule.

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\, dx = \int f(u)\, du$$

EXAMPLE 1 Find $\int x^3 \cos(x^4 + 2)\, dx$.

SOLUTION We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\, dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using $x^3\, dx = \frac{1}{4}\, du$ and the Substitution Rule, we have

$$\begin{aligned}\int x^3 \cos(x^4 + 2)\, dx &= \int \cos u \cdot \tfrac{1}{4}\, du = \tfrac{1}{4}\int \cos u\, du \\ &= \tfrac{1}{4}\sin u + C \\ &= \tfrac{1}{4}\sin(x^4 + 2) + C\end{aligned}$$

EXAMPLE 2 Evaluate $\int \sqrt{2x+1}\, dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\, dx$, so $dx = \frac{1}{2}\, du$. Thus the Substitution Rule gives

$$\begin{aligned}\int \sqrt{2x+1}\, dx &= \int \sqrt{u} \cdot \tfrac{1}{2}\, du = \tfrac{1}{2} \int u^{1/2}\, du \\ &= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C = \tfrac{1}{3} u^{3/2} + C \\ &= \tfrac{1}{3}(2x+1)^{3/2} + C\end{aligned}$$

V **EXAMPLE 3** Find $\int \frac{x}{\sqrt{1-4x^2}}\, dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x\, dx$, so $x\, dx = -\frac{1}{8}\, du$ and

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\, dx &= -\tfrac{1}{8} \int \frac{1}{\sqrt{u}}\, du = -\tfrac{1}{8} \int u^{-1/2}\, du \\ &= -\tfrac{1}{8}\left(2\sqrt{u}\right) + C = -\tfrac{1}{4}\sqrt{1-4x^2} + C\end{aligned}$$

EXAMPLE 4 Calculate $\int e^{5x}\, dx$.

SOLUTION If we let $u = 5x$, then $du = 5\, dx$, so $dx = \frac{1}{5}\, du$. Therefore

$$\int e^{5x}\, dx = \tfrac{1}{5} \int e^{u}\, du = \tfrac{1}{5} e^{u} + C = \tfrac{1}{5} e^{5x} + C$$

EXAMPLE 5 Find $\int \sqrt{1+x^2}\, x^5\, dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$. Also $x^2 = u - 1$, so $x^4 = (u - 1)^2$:

$$\begin{aligned}
\int \sqrt{1 + x^2}\, x^5\, dx &= \int \sqrt{1 + x^2}\, x^4 \cdot x\, dx \\
&= \int \sqrt{u}\,(u - 1)^2 \cdot \tfrac{1}{2}\, du = \tfrac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\, du \\
&= \tfrac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du \\
&= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2 \cdot \tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C \\
&= \tfrac{1}{7}(1 + x^2)^{7/2} - \tfrac{2}{5}(1 + x^2)^{5/2} + \tfrac{1}{3}(1 + x^2)^{3/2} + C
\end{aligned}$$

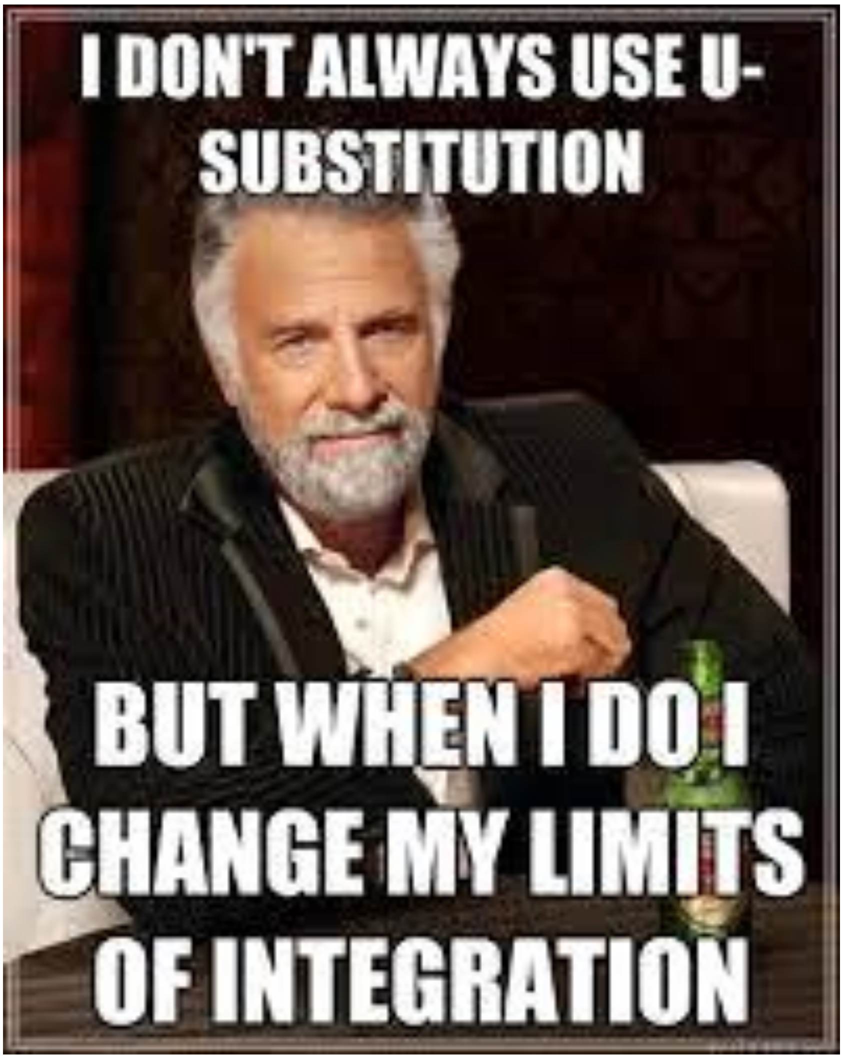
V EXAMPLE 6 Calculate $\int \tan x\, dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\, dx$ and so $\sin x\, dx = -du$:

$$\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du \\
&= -\ln|u| + C = -\ln|\cos x| + C
\end{aligned}$$



Definite Integrals

6 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

PROOF Let F be an antiderivative of f. Then, by $\boxed{3}$, $F(g(x))$ is an antiderivative of $f(g(x))\, g'(x)$, so by Part 2 of the Fundamental Theorem, we have

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

But, applying FTC2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\, du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

EXAMPLE 7 Evaluate $\int_0^4 \sqrt{2x+1}\,dx$ using $\boxed{6}$.

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = \frac{1}{2}\,du$. To find the new limits of integration we note that

$$\text{when } x = 0,\ u = 2(0) + 1 = 1 \qquad \text{and} \qquad \text{when } x = 4,\ u = 2(4) + 1 = 9$$

Therefore

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \tfrac{1}{2}\sqrt{u}\,du$$
$$= \tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2}\Big]_1^9$$
$$= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3}$$

Observe that when using $\boxed{6}$ we do *not* return to the variable x after integrating. We simply evaluate the expression in u between the appropriate values of u. ■

EXAMPLE 8 Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

SOLUTION Let $u = 3 - 5x$. Then $du = -5\,dx$, so $dx = -\frac{1}{5}\,du$. When $x = 1$, $u = -2$ and when $x = 2$, $u = -7$. Thus

$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2}$$
$$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \left[\frac{1}{5u}\right]_{-2}^{-7}$$
$$= \frac{1}{5}\left(-\frac{1}{7} + \frac{1}{2}\right) = \frac{1}{14}$$ ■

V EXAMPLE 9 Calculate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

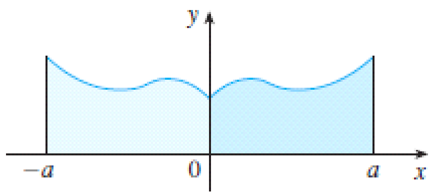
$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\bigg]_0^1 = \frac{1}{2}$$

Symmetry

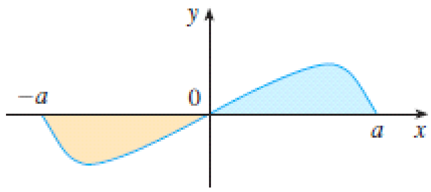
7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

V EXAMPLE 10 Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_0^2 (x^6 + 1)\,dx$$

$$= 2\Big[\tfrac{1}{7}x^7 + x\Big]_0^2 = 2\left(\tfrac{128}{7} + 2\right) = \tfrac{284}{7}$$

EXAMPLE 11 Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\, dx = 0$$