Derivative III(Trigonometric & Chain Rule)

Thursday, 7 November 2024 7:02 pm

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

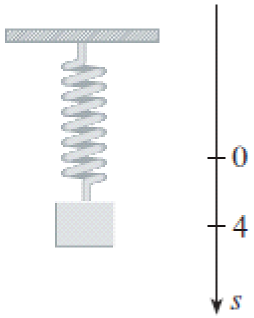
$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

$$\boxed{\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0}$$

V EXAMPLE 3 An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. (See Figure 5 and note that the downward direction is positive.) Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.



EXAMPLE 4 Find the 27th derivative of $\cos x$.

The Chain Rule

$$F(x) = \sqrt{x^2 + 1}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{d}{dx} \underbrace{f}_{\substack{\text{outer}\\ \text{function}}} \underbrace{(g(x))}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} = \underbrace{f'}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}} \underbrace{(g(x))}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}}$$

EXAMPLE 1 Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

SOLUTION 1 (using Equation 2): At the beginning of this section we expressed F as $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x$$

we have

$$\begin{aligned} F'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

SOLUTION 2 (using Equation 3): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}$$

V EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\underbrace{\sin}_{\substack{\text{outer}\\ \text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} = \underbrace{\cos}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{2x}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}} \\ &= 2x\cos(x^2) \end{aligned}$$

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{(\sin x)^2}_{\substack{\text{inner}\\ \text{function}}} = \underbrace{2}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}} \cdot \underbrace{(\sin x)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{\cos x}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}}$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

EXAMPLE 3 Differentiate $y = (x^3 - 1)^{100}$.

Taking $u = g(x) = x^3 - 1$ and $n = 100$ in $\boxed{4}$, we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100} = 100(x^3 - 1)^{99}\frac{d}{dx}(x^3 - 1)$$
$$= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}$$

EXAMPLE 5 Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$
$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$

EXAMPLE 6 Differentiate $y = (2x+1)^5(x^3 - x + 1)^4$.

$$\frac{dy}{dx} = (2x+1)^5 \frac{d}{dx}(x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx}(2x+1)^5$$
$$= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx}(x^3 - x + 1)$$
$$+ (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \frac{d}{dx}(2x+1)$$
$$= 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1) + 5(x^3 - x + 1)^4(2x+1)^4 \cdot 2$$

EXAMPLE 7 Differentiate $y = e^{\sin x}$.

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin x}) = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cos x$$

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

$$\frac{d}{dx}(2^x) =$$

V EXAMPLE 8 If $f(x) = \sin(\cos(\tan x))$, then

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$$f'(x) = \cos(\cos(\tan x))\frac{d}{dx}\cos(\tan x)$$

$$= \cos(\cos(\tan x))[-\sin(\tan x)]\frac{d}{dx}(\tan x)$$

$$= -\cos(\cos(\tan x))\sin(\tan x)\sec^2 x$$

EXAMPLE 9 Differentiate $y = e^{\sec 3\theta}$.

$$\frac{dy}{d\theta} = e^{\sec 3\theta}\frac{d}{d\theta}(\sec 3\theta)$$

$$= e^{\sec 3\theta}\sec 3\theta\tan 3\theta\frac{d}{d\theta}(3\theta)$$

$$= 3e^{\sec 3\theta}\sec 3\theta\tan 3\theta$$