Factorial, Permutation, Combination & Pigeonhole Principle

Wednesday, 17 December 2025 8:41 pm

FACTORIAL
K-SAMPLE
K-PERMUTATION

FACTORIAL OF A POSITIVE INTEGER:

For each positive integer n, its factorial is defined to be the product of all the integers from 1 to n and is denoted n!. Thus $n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$

In addition, we define

$$0! = 1$$

REMARK:

n! can be recursively defined as

Base: $0! = 1$

Recursion $n! = n(n-1)!$ for each positive integer n.

EXERCISE:

Compute each of the following

(i) $\dfrac{7!}{5!}$ (ii) $(-2)!$

(iii) $\dfrac{(n+1)!}{n!}$ (iv) $\dfrac{(n-1)!}{(n+1)!}$

SOLUTION:

(i) $\dfrac{7!}{5!} = \dfrac{7\cdot 6\cdot 5!}{5!} = 7\cdot 6 = 42$

(ii) $(-2)!$ is not defined

(iii) $\dfrac{(n+1)!}{n!} = \dfrac{(n+1)n!}{n!} = n+1$

(iv) $\dfrac{(n-1)!}{(n+1)!} = \dfrac{(n-1)!}{(n+1)\cdot n\cdot(n-1)!} = \dfrac{1}{(n+1)n} = \dfrac{1}{n^2+n}$

EXERCISE:

Write in terms of factorials.

(i) $25\cdot 24\cdot 23\cdot 22$ (ii) $n(n-1)(n-2)\ldots(n-r+1)$

(iii) $\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)\cdot r}$

SOLUTION:

(i) $$25\cdot 24\cdot 23\cdot 22 = \frac{25\cdot 24\cdot 23\cdot 22\cdot 21!}{21!} = \frac{25!}{21!}$$

(ii) $$n(n-1)(n-2)\cdots(n-r+1) = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

(iii) $$\frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)\cdot r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{r!(n-r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

FORMULA FOR K-SAMPLE:

Suppose there are n distinct elements and we draw a k-sample from it. The first element of the k-sample can be drawn in n ways. Since, repetition of elements is allowed, so the second element can also be drawn in n ways.

Similarly each of third, fourth, …, k-th element can be drawn in n ways.

Hence, by product rule, the total number of ways in which a k-sample can be drawn from n distinct elements is

$$n \cdot n \cdot n \cdot \ldots \cdot n \qquad \text{(k-times)}$$
$$= n^k$$

EXERCISE:

How many possible outcomes are there when a fair coin is tossed three times.

SOLUTION:

Each time a coin is tossed it's outcome is either a head (H) or a tail (T). Hence in successive tosses, H and T are repeated. Also the order in which they appear is important. Accordingly, the problem is of 3-samples from a set of two elements H and T. [k = 3, n = 2]

Hence number of samples $= n^k$

$= 2^3 = 8$

These 8-samples may be listed as:

HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

EXERCISE:

Suppose repetition of digits is permitted.

(a) How many three-digit numbers can be formed from the six digits 2, 3, 4, 5, 7 and 9

SOLUTION:

Given distinct elements = n = 6

Digits to be chosen = k = 3

While forming numbers, order of digits is important. Also digits may be repeated. Hence, this is the case of 3-sample from 6 elements.

Number of 3-digit numbers $= n^k = 6^3 = 216$

(b) How many of these numbers are less than 400?

SOLUTION:

From the given six digits 2, 3, 4, 5, 7 and 9, a three-digit number would be less than 400 if and only if its first digit is either 2 or 3.

The next two digits positions may be filled with any one of the six digits.

Hence, by product rule, there are

$$2 \cdot 6 \cdot 6 = 72$$

three-digit numbers less than 400.

(c) How many are even?

SOLUTION:

A number is even if its right most digit is even. Thus, a 3-digit number formed by the digits 2, 3, 4, 5, 7 and 9 is even if it last digit is 2 or 4. Thus the last digit position may be filled in 2 ways only while each of the first two positions may be filled in 6 ways.

Hence, there are

$$6 \cdot 6 \cdot 2 = 72$$

3-digit even numbers.

(d) How many are odd?

EXERCISE:

How many 2-permutation are there of {W, X, Y, Z}? Write them all.

SOLUTION:

Number of 2-permutation of 4 elements is

$$P(4,2) = {}^{4}_{2}p = \frac{4!}{(4-2)!}$$
$$= \frac{4\cdot 3\cdot 2!}{2!}$$
$$= 4\cdot 3 = 12$$

These 12 permutations are:

WX, WY, WZ,
XW, XY, XZ,
YW, YX, YZ,
ZW, ZX, ZY.

EXERCISE:

Find	(a)	P(8, 3)	(b)	P(8,8)
	(c)	P(8,1)	(d)	P(6,8)

EXERCISE:

(a) How many ways can five of the letters of the word ALGORITHM be selected and written in a row?

(b) How many ways can five of the letters of the word ALGORITHM be selected and written in a row if the first two letters must be TH?

SOLUTION:

(a) The answer equals the number of 5-permutation of a set of 9 elements and

$$P(9,5) = \frac{9!}{(9-5)!} = 9\cdot 8\cdot 7\cdot 6\cdot 5 = 15120$$

(b) Since the first two letters must be TH hence we need to choose the remaining three letters out of the left 9 - 2 = 7 alphabets.

Hence, the answer is the number of 3-permutations of a set of seven elements which is

$$P(7,3) = \frac{7!}{(7-3)!} = 7\cdot 6\cdot 5 = 210$$

EXERCISE:

Find the number of ways that a party of seven persons can arrange themselves in a row of seven chairs.

EXERCISE:

A debating team consists of three boys and two girls. Find the number n of ways they can sit in a row if the boys and girls are each to sit together.

SOLUTION:

There are two ways to distribute them according to sex: BBBGG or GGBBB.
In each case
the boys can sit in a row in P(3,3) = 3! = 6 ways, and
the girls can sit in

P(2,2) = 2! = 2 ways and
Every row consist of boy and girl which is = 2!=2
Thus

The total number of ways=n = 2 · 3! · 2!
= 2 ·6 ·2 = 24

EXERCISE:

Find the number n of ways that five large books, four medium sized book, and three small books can be placed on a shelf so that all books of the same size are together.

NOTE:

k-combinations are also written ${}^{n}C_{k}$ as or $\binom{n}{k}$

REMARK:

With k-combinations of a set of n elements, repetition of elements is not allowed, therefore, k must be less than or equal to n, i.e., $k \leq n$.

EXAMPLE:

Let $X = \{a, b, c\}$. Then 2-combinations of the 3 elements of the set X are: $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$. Hence $C(3,2) = 3$.

EXERCISE:

Let $X = \{a, b, c, d, e\}$.
List all 3-combinations of the 5 elements of the set X, and hence find the value of $C(5,3)$.

SOLUTION:

Then 3-combinations of the 5 elements of the set X are:
$\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, e\}$, $\{a, c, d\}$, $\{a, c, e\}$,
$\{a, d, e\}$, $\{b, c, d\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{c, d, e\}$
Hence $C(5, 3) = 10$

EXERCISE:

A student is to answer eight out of ten questions on an exam.

(a) Find the number m of ways that the student can choose the eight questions

(b) Find the number m of ways that the student can choose the eight questions, if the first three questions are compulsory.

SOLUTION:

(a) The eight questions can be answered in $m = C(10, 8) = 45$ ways.

(b) The eight questions can be answered in $m = C(7, 5) = 21$ ways.

EXERCISE:

An examination paper consists of 5 questions in section A and 5 questions in section B. A total of 8 questions must be answered. In how many ways can a student select the questions if he is to answer at least 4 questions from section A.

SOLUTION:

There are two possibilities:

(a) The student answers 4 questions from section A and 4 questions from section B.The number of ways 8 questions can be answered taking 4 questions from section A and 4 questions from section B are

$$C(5, 4) \cdot C(5, 4) = 5 \cdot 5 = 25.$$

(b) The student answers 5 questions from section A and 3 questions from section B.The number of ways 8 questions can be answered taking 5 questions from section A and 3 questions from section B are

$$C(5, 5) \cdot C(5, 3) = 1 \cdot 10 = 10.$$

Thus there will be a total of $25 + 10 = 35$ choices.

EXERCISE:

A computer programming team has 14 members.

(a) How many ways can a group of seven be chosen to work on a project?

(b) Suppose eight team members are women and six are men

(i) How many groups of seven can be chosen that contain four women and three men

(ii) How many groups of seven can be chosen that contain at least one man?

(iii)How many groups of seven can be chosen that contain at most three women?

(c) Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project?

(d) Suppose two team members insist on either working together or not at all on projects. How many groups of seven can be chosen to work on a project?

(e) How many ways a group of 7 be chosen to work on a project?

EXERCISE:

(a) How many 16-bit strings contain exactly 9 1's?

(b)How many 16-bit strings contain at least one 1?

SOLUTION:

(a) 16-bit strings that contain exactly 9 1's= $C(16,9) = \dfrac{16!}{(16-9)!9!} = 11440$

(b) Total no. of 16-bit strings = 2^{16}

Hence number of 16-bit strings that contain at least one 1

$$2^{16} - 1 = 65536 - 1$$
$$= 65535$$

PIGEONHOLE PRINCIPLE

A function from a set of $k + 1$ or more elements to a set of k elements must have at least two elements in the domain that have the same image in the co-domain.

If $k + 1$ or more pigeons fly into k pigeonholes then at least one pigeonhole must contain two or more pigeons.

EXAMPLES:

1. Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

2. In any set of 27 English words, there must be at least two that begin with the same letter, since there are 26 letters in the English alphabet.

EXERCISE:

What is the minimum number of students in a class to be sure that two of them are born in the same month?

SOLUTION:

There are 12 ($= n$) months in a year. The pigeonhole principle shows that among any 13 ($= n + 1$) or more students there must be at least two students who are born in the same month.

EXERCISE:

Given any set of seven integers, must there be two that have the same remainder when divided by 6?

SOLUTION:

The set of possible remainders that can be obtained when an integer is divided by six is $\{0, 1, 2, 3, 4, 5\}$. This set has 6 elements. Thus by the pigeonhole principle if $7 = 6 + 1$ integers are each divided by six, then at least two of them must have the same remainder.

EXERCISE:

How many integers from 1 through 100 must you pick in order to be sure of getting one that is divisible by 5?

SOLUTION:

There are 20 integers from 1 through 100 that are divisible by 5. Hence there are eighty integers from 1 through 100 that are not divisible by 5. Thus by the pigeonhole principle $81 = 80 + 1$ integers from 1 though 100 must be picked in order to be sure of getting one that is divisible by 5.

EXERCISE:

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Suppose six integers are chosen from A. Must there be two integers whose sum is 11.

SOLUTION:

The set A can be partitioned into five subsets:
$\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}$, and $\{5, 6\}$
each consisting of two integers whose sum is 11.
These 5 subsets can be considered as 5 pigeonholes.
If $6 = (5 + 1)$ integers are selected from A, then by the pigeonhole principle at least two must be from one of the five subsets. But then the sum of these two integers is 11.

EXERCISE:

Suppose a laundry bag contains many red, white, and blue socks. Find the minimum number of socks that one needs to choose in order to get two pairs (four socks) of the same colour.

SOLUTION:

Here there are $n = 3$ colours (pigeonholes) and $k + 1 = 4$ or $k = 3$. Thus among any $n \cdot k + 1 = 3 \cdot 3 + 1 = 10$ socks (pigeons), at least four have the same colour.

DEFINITION:

1. Given any real number x, **the floor of x**, denoted $\lfloor x \rfloor$, is the largest integer smaller than or equal to x.

2. Given any real number x, **the ceiling of x**, denoted $\lceil x \rceil$, is the smallest integer greater than or equal to x.

EXAMPLE:

Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of x.

a. 25/4 **b.** 0.999 **c.** –2.01

SOLUTION:

a. $\lfloor 25/4 \rfloor = \lfloor 6 + ¼ \rfloor = 6$

$\lceil 25/4 \rceil = \lceil 6 + ¼ \rceil = 6 + 1 = 7$

b. $\lfloor 0.999 \rfloor = \lfloor 0 + 0.999 \rfloor = 0$

$\lceil 0.999 \rceil = \lceil 0 + 0.999 \rceil = 0 + 1 = 1$

c. $\lfloor -2.01 \rfloor = \lfloor -3 + 0.99 \rfloor = -3$

$\lceil -2.01 \rceil = \lceil -3 + 0.999 \rceil = -3 + 1 = -2$

EXERCISE:

What is the smallest integer N such that

a. $\lceil N/7 \rceil = 5$ b. $\lceil N/9 \rceil = 6$

SOLUTION:

a. $N = 7 \cdot (5 - 1) + 1 = 7 \cdot 4 + 1 = 29$

b. $N = 9 \cdot (6 - 1) + 1 = 9 \cdot 5 + 1 = 46$

PIGEONHOLE PRINCIPLE:

If N pigeons fly into k pigeonholes then at least one pigeonhole must contain $\lceil N/k \rceil$ or more pigeons.

EXAMPLE:

Among 100 people there are at least $\lceil 100/12 \rceil = \lceil 8 + 1/3 \rceil = 9$ who were born in the same month.

EXERCISE:

What is the minimum number of students required in a Discrete Mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F.

SOLUTION:

The minimum number of students needed to guarantee that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$.

The smallest such integer is $N = 5(6-1)+1=5 \cdot 5 + 1 = 26$.

Thus 26 is the minimum number of students needed to be sure that at least 6 students will receive the same grades.