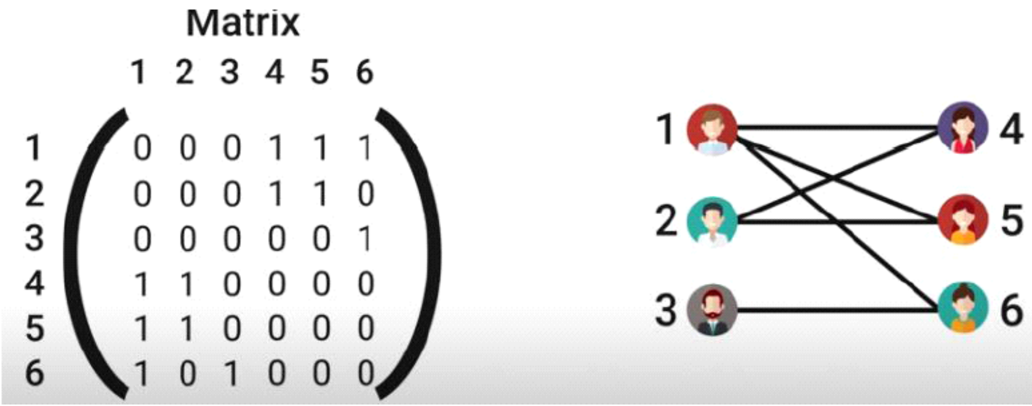
Graph Theory II

Sunday, 21 December 2025 1:23 pm

MATRIX REPRESENTATIONS OF GRAPHS



ADJACENCY MATRIX OF A GRAPH:

Let G be a graph with ordered vertices $v_1, v_2, \dots, v_n$. The adjacency matrix of G is the matrix $A = [a_{ij}]$ over the set of non-negative integers such that

a_{ij} = the number of edges connecting v_i and v_j for all i, j = 1, 2, …, n.

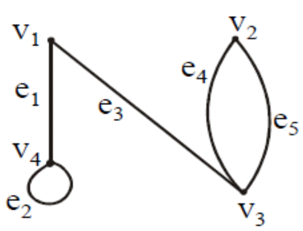
OR

The adjancy matrix say $A = [a_{ij}]$ is also defined as

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

<u>EXAMPLE:</u>

A graph with it's adjacency matrix is shown.



$$A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 0 & 1 & 1 \\ v_2 & 0 & 0 & 2 & 0 \\ v_3 & 1 & 2 & 0 & 0 \\ v_4 & 1 & 0 & 0 & 1 \end{array}$$

<u>EXERCISE:</u>

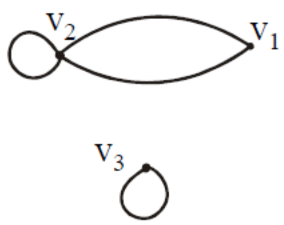
Find a graph that have the following adjacency matrix.

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SOLUTION:

Let the three vertices of the graph be named v_1, v_2 and v_3. We label the adjacency matrix across the top and down the left side with these vertices and draw the graph accordingly(as from v_1 to v_2 there is a value "2",it means that two parallel edges between v_1 and v_2 and same condition occurs between v_2 and v_1 and the value "1" represent the loops of v_2 and v_3).

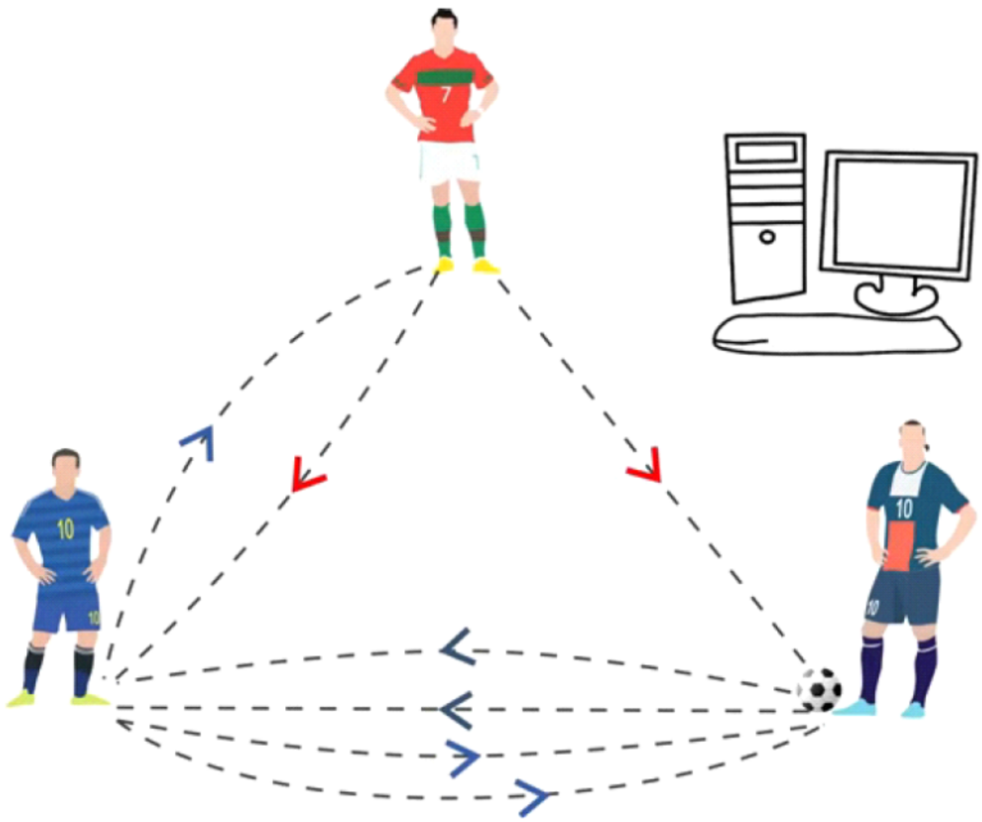
$$\begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



DIRECTED GRAPH:

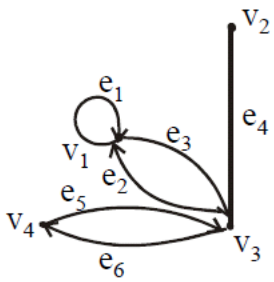
A directed graph or digraph, consists of two finite sets: a set V(G) of vertices and a set D(G) of directed edges, where each edge is associated with an ordered pair of vertices called its end points.

If edge e is associated with the pair (v, w) of vertices, then e is said to be the directed edge from v to w and is represented by drawing an arrow from v to w.



$$
\begin{array}{cc}
 & \begin{array}{ccc} R & M & Z \end{array} \\
\begin{array}{c} R \\ M \\ Z \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}
\end{array}
$$

EXAMPLE OF A DIGRAPH:



ADJACENCY MATRIX OF A DIRECTED GRAPH:

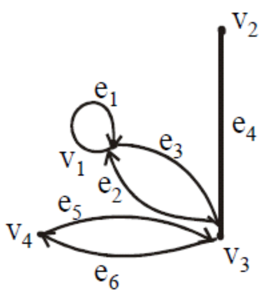
Let G be a graph with ordered vertices $v_1, v_2, \ldots, v_n$.

The adjacency matrix of G is the matrix $A = [a_{ij}]$ over the set of non-negative integers such that

a_{ij} = the number of arrows from v_i to v_j for all $i, j = 1, 2, \ldots, n$.

EXAMPLE:

A directed graph with its adjacency matrix is shown



$$
A = \begin{array}{cc}
 & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{array}
$$

is the adjacency matrix

EXERCISE:

Find directed graph that has the adjacency matrix

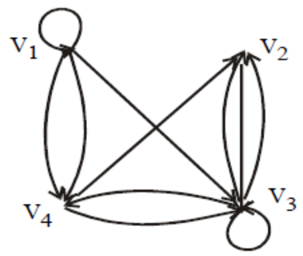
$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

SOLUTION:

The 4×4 adjacency matrix shows that the graph has 4 vertices say v_1, v_2, v_3 and v_4 labeled across the top and down the left side of the matrix.

$$A = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

A corresponding directed graph is



It means that a loop exists from v_1 and v_3 , two arrows go from v_1 to v_4 and two from v_3 and v_2 and one arrow go from $v_1$to v_3 , v_2 to v_3 , v_3 to v_4, v_4 to v_2 and v_3.

THEOREM

If G is a graph with vertices $v_1, v_2, \ldots, v_m$ and A is the adjacency matrix of G, then for each positive integer n,
the ijth entry of A^n = the number of walks of length n from v_i to v_j
for all integers i,j = 1, 2, …, n

PROBLEM:

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

be the adjacency matrix of a graph G with vertices v_1, v_2, and v_3. Find
(a) the number of walks of length 2 from v_2 to v_3
(b) the number of walks of length 3 from v_1 to v_3
Draw graph G and find the walks by visual inspection for (a)

SOLUTION:

(a)

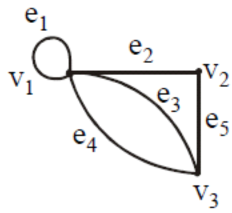
$$A^2 = AA = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix} \longrightarrow$$ it shows the entry (2,3) from v_2 to v_3

Hence, number of walks of length 2(means "multiply matrix A two times") from v_2 to v_3 = the entry at (2,3) of A^2 = 2

(b) $$A^3 = AA^2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 9 & 15 \\ 9 & 5 & 8 \\ 15 & 8 & 8 \end{bmatrix} \longrightarrow$$ it shows the entry (1,3) from v_1 to v_3

Hence, number of walks of length 3 from v_1 to v_3 = the entry at (1,3) of $A^3 = 15$
Walks from v_2 to v_3 by visual inspection of graph is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$



so in part (a)two Walks of length 2 from v_2 to v_3 are
(i) $v_2\ e_2\ v_1\ e_3\ v_3$ (by using the above theorem).
(ii) $v_2\ e_2\ v_1\ e_4\ v_3$

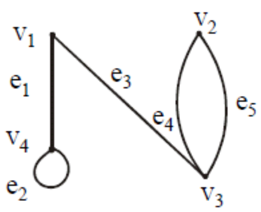
INCIDENCE MATRIX OF A SIMPLE GRAPH:

Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_n$. The incidence matrix of G is the matrix $M = [m_{ij}]$ of size $n \times m$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if the vertex } v_i \text{ is incident on the edge } e_j \\ 0 & \text{otherwise} \end{cases}$$

EXAMPLE:

A graph with its incidence matrix is shown.



$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

REMARK:

In the incidence matrix

1. Multiple edges are represented by columns with identical entries (in this matrix e_4 & e_5 are multiple edges).
2. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop and other zeros (here e_2 is only a loop).