Equivalence Relation

Saturday, 24 August 2024 8:34 pm

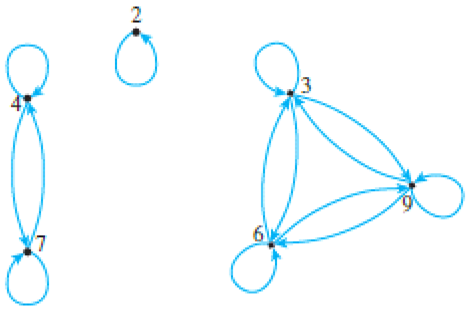
8.2 Reflexivity, Symmetry, and Transitivity

Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field. —P. A. M. Dirac, 1902–1984

Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation R on A as follows: For every $x, y \in A$,

$$x \, R \, y \quad \Leftrightarrow \quad 3 \mid (x - y).$$

Then $2 \, R \, 2$ because $2 - 2 = 0$, and $3 \mid 0$. Similarly, $3 \, R \, 3$, $4 \, R \, 4$, $6 \, R \, 6$, $7 \, R \, 7$, and $9 \, R \, 9$. Also $6 \, R \, 3$ because $6 - 3 = 3$, and $3 \mid 3$. And $3 \, R \, 6$ because $3 - 6 = -(6 - 3) = -3$, and $3 \mid (-3)$. Similarly, $3 \, R \, 9$, $9 \, R \, 3$, $6 \, R \, 9$, $9 \, R \, 6$, $4 \, R \, 7$, and $7 \, R \, 4$. Thus the directed graph for R has the appearance shown below.



This graph has three important properties:

1. Each point of the graph has an arrow looping around from it and going back to it.
2. In each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.
3. In each case where there is an arrow going from one point to a second and from the second point to a third, there is an arrow going from the first point to the third. That is, there are no "incomplete directed triangles" in the graph.

Caution! The definition of symmetric does not say that x is related to y by R; rather, it states only that **if it happens** that x is related to y, then y must be related to x.

Definition

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for every $x \in A$, $x \, R \, x$.
2. R is **symmetric** if, and only if, for every $x, y \in A$, if $x \, R \, y$ then $y \, R \, x$.
3. R is **transitive** if, and only if, for every $x, y, z \in A$, if $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$.

Because of the equivalence of the expressions $x \, R \, y$ and $(x, y) \in R$ for every x and y in A, the reflexive, symmetric, and transitive properties can also be written as follows:

1. R is reflexive	$\Leftrightarrow$	for every x in A, $(x, x) \in R$.
2. R is symmetric	$\Leftrightarrow$	for every x and y in A, *if* $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive	$\Leftrightarrow$	for every x, y, and z in A, *if* $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

In informal terms, properties (1)–(3) say the following:

1. **Reflexive:** Each element is related to itself.
2. **Symmetric:** If any one element is related to any other element, then the second element is related to the first.
3. **Transitive:** If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

Note that the definitions of reflexivity, symmetry, and transitivity are universal statements. This means that to prove a relation has one of the properties, you use either the method of exhaustion or the method of generalizing from the generic particular.

Now consider what it means for a relation *not* to have one of the properties defined previously. Recall that the negation of a universal statement is existential. Hence if R is a relation on a set A, then

1. **R is not reflexive** $\Leftrightarrow$ there is an element x in A such that $x \not R x$ *[that is, such that $(x, x) \notin R$].*
2. **R is not symmetric** $\Leftrightarrow$ there are elements x and y in A such that $x R y$ but $y \not R x$ *[that is, such that $(x, y) \in R$ but $(y, x) \notin R$].*
3. **R is not transitive** $\Leftrightarrow$ there are elements x, y, and z in A such that $x R y$ and $y R z$ but $x \not R z$ *[that is, such that $(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$].*

It follows that you can show that a relation does *not* have one of the properties by finding a counterexample.

Example 8.2.1 Properties of Relations on Finite Sets

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\},$$
$$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\},$$
$$T = \{(0, 1), (2, 3)\}.$$

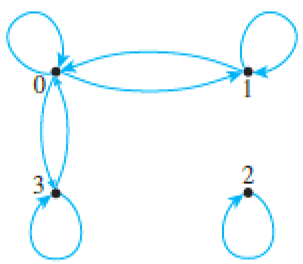
a. Is R reflexive? symmetric? transitive?

b. Is S reflexive? symmetric? transitive?

c. Is T reflexive? symmetric? transitive?

Solution

a. The directed graph of R has the appearance shown below.

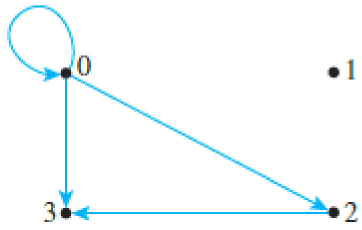


***R is reflexive*:** There is a loop at each point of the directed graph. This means that each element of A is related to itself, so R is reflexive.

***R is symmetric*:** In each case where there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first. This means that whenever one element of A is related by R to a second, then the second is related to the first. Hence R is symmetric.

***R is not transitive*:** There is an arrow going from 1 to 0 and an arrow going from 0 to 3, but there is no arrow going from 1 to 3. This means that there are elements of A—0, 1, and 3—such that $1\,R\,0$ and $0\,R\,3$ but $1 \not R\, 3$. Hence R is not transitive.

b. The directed graph of S has the appearance shown below.

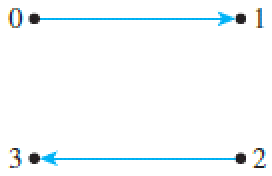


***S is not reflexive*:** There is no loop at 1, for example. Thus $(1, 1) \notin S$, and so S is not reflexive.

***S is not symmetric*:** There is an arrow from 0 to 2 but not from 2 to 0. Hence $(0, 2) \in S$ but $(2, 0) \notin S$, and so S is not symmetric.

***S is transitive*:** There are three cases for which there is an arrow going from one point of the graph to a second and from the second point to a third. In particular, there are arrows going from 0 to 2 and from 2 to 3; there are arrows going from 0 to 0 and from 0 to 2; and there are arrows going from 0 to 0 and from 0 to 3. In each case there is an arrow going from the first point to the third. (Note again that the "first," "second," and "third" points need not be distinct.) This means that whenever $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$, for every $x, y, z \in \{0, 1, 2, 3\}$, and so S is transitive.

c. The directed graph of T has the appearance shown below.



T is not reflexive: There is no loop at 0, for example. Thus $(0, 0) \notin T$, so T is not reflexive.

T is not symmetric: There is an arrow from 0 to 1 but not from 1 to 0. Thus $(0, 1) \in T$ but $(1, 0) \notin T$, and so T is not symmetric.

T is transitive: The transitivity condition is vacuously true for T. To see this, observe that the transitivity condition says that

$$\text{For every } x, y, z \in A, \quad \text{if } (x, y) \in T \text{ and } (y, z) \in T \text{ then } (x, z) \in T.$$

The only way for this to be false would be for there to exist elements of A that make the hypothesis true and the conclusion false. That is, there would have to be elements x, y, and z in A such that

$$(x, y) \in T \quad \text{and} \quad (y, z) \in T \quad \text{and} \quad (x, z) \notin T.$$

In other words, there would have to be two ordered pairs in T that have the potential to "link up" by having the *second* element of one pair be the *first* element of the other pair. But the only elements in T are $(0, 1)$ and $(2, 3)$, and these do not have the potential to link up. Hence the hypothesis is never true. It follows that it is impossible for T *not* to be transitive, and thus T is transitive. ■

Properties of Relations on Infinite Sets

Suppose a relation R is defined on an infinite set A. To prove the relation is reflexive, symmetric, or transitive, first write down what is to be proved. For instance, for symmetry you need to prove that

$$\forall x, y \in A, \text{ if } x\, R\, y \text{ then } y\, R\, x.$$

Then use the definitions of A and R to rewrite the statement for the particular case in question. For instance, for the "equality" relation on the set of real numbers, the rewritten statement is

$$\forall x, y \in \mathbf{R}, \text{ if } x = y \text{ then } y = x.$$

Sometimes the truth of the rewritten statement will be immediately obvious (as it is here). At other times you will need to prove it using the method of generalizing from the generic particular. We give examples of both cases in this section. We begin with the relation of equality, one of the simplest and yet most important relations.

Example 8.2.2 **Properties of Equality**

Define a relation R on $\mathbf{R}$ as follows: For all real numbers x and y,

$$\boxed{x\,R\,y \quad \Leftrightarrow \quad x = y.}$$

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

Solution

a. ***R is reflexive:*** R is reflexive if, and only if, the following statement is true:

$$\text{For every } x \in \mathbf{R}, \quad x\,R\,x.$$

Since $x\,R\,x$ just means that $x = x$, this is the same as saying

$$\text{For every } x \in \mathbf{R}, \quad x = x.$$

But this statement is certainly true; every real number is equal to itself.

b. ***R is symmetric:*** R is symmetric if, and only if, the following statement is true:

$$\text{For every } x, y \in \mathbf{R}, \quad \text{if } x\,R\,y \text{ then } y\,R\,x.$$

By definition of R, $x\,R\,y$ means that $x = y$ and $y\,R\,x$ means that $y = x$. Hence R is symmetric if, and only if,

$$\text{For every } x, y \in \mathbf{R}, \quad \text{if } x = y \text{ then } y = x.$$

But this statement is certainly true; if one number is equal to a second, then the second is equal to the first.

c. ***R is transitive:*** R is transitive if, and only if, the following statement is true:

$$\text{For every } x, y, z \in \mathbf{R}, \quad \text{if } x\,R\,y \text{ and } y\,R\,z \text{ then } x\,R\,z.$$

By definition of R, $x\,R\,y$ means that $x = y$, $y\,R\,z$ means that $y = z$, and $x\,R\,z$ means that $x = z$. Hence R is transitive if, and only if, the following statement is true:

$$\text{For every } x, y, z \in \mathbf{R}, \quad \text{if } x = y \text{ and } y = z \text{ then } x = z.$$

But this statement is certainly true: If one real number equals a second and the second equals a third, then the first equals the third. ■

Example 8.2.3 **Properties of "Less Than"**

Define a relation R on $\mathbf{R}$ as follows: For all real numbers x and y,

$$\boxed{x \, R \, y \quad \Leftrightarrow \quad x < y.}$$

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

Solution

a. ***R is not reflexive*:** R is reflexive if, and only if, $\forall x \in \mathbf{R}$, $x \, R \, x$. By definition of R, this means that $\forall x \in \mathbf{R}$, $x < x$. But this is false: $\exists x \in \mathbf{R}$ such that $x \not< x$. As a counterexample, let $x = 0$ and note that $0 \not< 0$. Hence R is not reflexive.

b. ***R is not symmetric*:** R is symmetric if, and only if, $\forall x, y \in \mathbf{R}$, if $x \, R \, y$ then $y \, R \, x$. By definition of R, this means that $\forall x, y \in \mathbf{R}$, if $x < y$ then $y < x$. But this is false: $\exists x, y \in \mathbf{R}$ such that $x < y$ and $y \not< x$. As a counterexample, let $x = 0$ and $y = 1$ and note that $0 < 1$ but $1 \not< 0$. Hence R is not symmetric.

c. ***R is transitive*:** R is transitive if, and only if, $\forall x, y, z \in \mathbf{R}$, if $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$. By definition of R, this means that $\forall x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$, then $x < z$. But this statement is true by the transitive law of order for real numbers (Appendix A, T18). Hence R is transitive. ■

TEST YOURSELF

1. For a relation R on a set A to be reflexive means that ______.
2. For a relation R on a set A to be symmetric means that ______.
3. For a relation R on a set A to be transitive means that ______.
4. To show that a relation R on an infinite set A is reflexive, you suppose that ______ and you show that ______.
5. To show that a relation R on an infinite set A is symmetric, you suppose that ______ and you show that ______.
6. To show that a relation R on an infinite set A is transitive, you suppose that ______ and you show that ______.
7. To show that a relation R on a set A is not reflexive, you ______.
8. To show that a relation R on a set A is not symmetric, you ______.
9. To show that a relation R on a set A is not transitive, you ______.
10. Given a relation R on a set A, the transitive closure of R is the relation R^t on A that satisfies the following three properties: ______, ______, and ______.

ANSWERS FOR TEST YOURSELF

1. for every x in A, $x \, R \, x$ 2. for every x and y in A, if $x \, R \, y$ then $y \, R \, x$ 3. for every x, y, and z in A, if $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$ 4. x is any element of A; $x \, R \, x$ 5. x and y are any elements of A such that $x \, R \, y$; $y \, R \, x$ 6. x, y, and z are any elements of A such that $x \, R \, y$ and $y \, R \, z$; $x \, R \, z$ 7. show that there is an element x in A such that $x \not\!R \, x$ 8. show that there are elements x and y in A such that $x \, R \, y$ but $y \not\!R \, x$ 9. show that there are elements x, y, and z in A such that $x \, R \, y$ and $y \, R \, z$ but $x \not\!R \, z$ 10. R^t is transitive; $R \subseteq R^t$; if S is any other transitive relation that contains R, then $R^t \subseteq S$

In 1–8, a number of relations are defined on the set $A = \{0, 1, 2, 3\}$. For each relation:

a. Draw the directed graph.

b. Determine whether the relation is reflexive.

c. Determine whether the relation is symmetric.

d. Determine whether the relation is transitive.

Give a counterexample in each case in which the relation does not satisfy one of the properties.

1. $R_1 = \{(0, 0), (0, 1), (0, 3), (1, 1), (1, 0), (2, 3), (3, 3)\}$
2. $R_2 = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 3)\}$
3. $R_3 = \{(2, 3), (3, 2)\}$
4. $R_4 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$
5. $R_5 = \{(0, 0), (0, 1), (0, 2), (1, 2)\}$
6. $R_6 = \{(0, 1), (0, 2)\}$
7. $R_7 = \{(0, 3), (2, 3)\}$
8. $R_8 = \{(0, 0), (1, 1)\}$

In 9–33, determine whether the given relation is reflexive, symmetric, transitive, or none of these. Justify your answers.

9. R is the "greater than or equal to" relation on the set of real numbers: For every $x, y \in \mathbf{R}$, $x\, R\, y \Leftrightarrow x \geq y$.
10. C is the circle relation on the set of real numbers: For every $x, y \in \mathbf{R}$, $x\, C\, y \Leftrightarrow x^2 + y^2 = 1$.
11. D is the relation defined on $\mathbf{R}$ as follows: For every $x, y \in \mathbf{R}$, $x\, D\, y \Leftrightarrow xy \geq 0$.