Series

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SERIES:

The sum of the terms of a sequence forms a series. If $a_1, a_2, a_3, \ldots$ represent a sequence of numbers, then the corresponding series is

$$a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k$$

SUMMATION NOTATION:

The capital Greek letter sigma $\sum$ is used to write a sum in a short hand notation. where k varies from 1 to n represents the sum given in expanded form by

$= a_1 + a_2 + a_3 + \ldots + a_n$

More generally if m and n are integers and $m \leq n$, then the summation from k equal m to n of a_k is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

Here **k** is called the index of the summation; **m** the lower limit of the summation and **n** the upper limit of the summation.

COMPUTING SUMMATIONS:

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$. Compute each of the summations:

$$(a) \quad \sum_{i=0}^{4} a_i \qquad (b) \quad \sum_{j=0}^{2} a_{2j} \qquad (c) \quad \sum_{k=1}^{1} a_k$$

SOLUTION:

(a) $\displaystyle\sum_{i=0}^{4} a_i = a_0 + a_1 + a_2 + a_3 + a_4$
$= 2 + 3 + (-2) + 1 + 0 = 4$

(b) $\displaystyle\sum_{j=0}^{2} a_{2j} = a_0 + a_2 + a_4$
$= 2 + (-2) + 0 = 0$

(c) $\displaystyle\sum_{k=1}^{1} a_k = a_1$
$= 3$

EXERCISE:

Compute the summations

1. $\displaystyle\sum_{i=1}^{3} (2i-1) = [2(1)-1]+[2(2)-1]+[2(3)-1]$

$= 1+3+ \quad 5$

$= 9$

2. $\sum_{k=-1}^{1}(k^3+2) = [(-1)^3+2]+[(0)^3+2]+[(1)^3+2]$

$= [-1+2]+[0+2]+[1+2]$

$= 1+ \quad 2+ \quad 3$

$= 6$

SUMMATION NOTATION TO EXPANDED FORM:

Write the summation $\sum_{i=0}^{n}\frac{(-1)^i}{i+1}$ to expanded form.

SOLUTION:

$$\sum_{i=0}^{n}\frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1}+\frac{(-1)^1}{1+1}+\frac{(-1)^2}{2+1}+\frac{(-1)^3}{3+1}+\cdots+\frac{(-1)^n}{n+1}$$

$$= \frac{1}{1}+\frac{(-1)}{2}+\frac{1}{3}+\frac{(-1)}{4}+\cdots+\frac{(-1)^n}{n+1}$$

$$= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^n}{n+1}$$

EXPANDED FORM TO SUMMATION NOTATION:

Write the following using summation notation:

1. $\frac{1}{n}+\frac{2}{n+1}+\frac{3}{n+2}+\cdots+\frac{n+1}{2n}$

SOLUTION:

We find the kth term of the series.

The numerators forms an arithmetic sequence 1, 2, 3,…, n+1, in which

a = first term = 1

& d = common difference = 1

$a_k = a + (k - 1)d$

$= 1 + (k - 1)(1) = 1 + k - 1 = k$

Similarly, the denominators forms an arithmetic sequence

n, n+1, n+2, …, 2n, in which

a = first term = n

d = common difference = 1

∴ $a_k = a + (k - 1)d$

$= n + (k - 1)(1)$

$= k + n - 1$

Hence the kth term of the series is

$$\frac{k}{(n-1)+k}$$

And the expression for the series is given by

$$\therefore \quad \frac{1}{n}+\frac{2}{n+1}+\frac{3}{n+2}+\cdots+\frac{n+1}{2n}=\sum_{k=1}^{n+1}\frac{k}{(n-1)+k}$$

$$=\sum_{k=0}^{n}\frac{k+1}{n+k}$$

TRANSFORMING A SUM BY A CHANGE OF VARIABLE:

Consider $\displaystyle\sum_{k=1}^{3}k^2=1^2+2^2+3^2$

and $\displaystyle\sum_{i=1}^{3}i^2=1^2+2^2+3^2$

Hence $\displaystyle\sum_{k=1}^{3}k^2=\sum_{i=1}^{3}i^2$

The index of a summation can be replaced by any other symbol. The index of a summation is therefore called a dummy variable.

EXERCISE:

Consider

$$\sum_{k=1}^{n+1}\frac{k}{(n-1)+k}$$

Substituting k = j + 1 so that j = k – 1
When k = 1, j = k - 1 = 1 - 1 = 0
When k = n + 1, j = k - 1 = (n + 1) - 1 = n
Hence

$$\sum_{k=1}^{n+1} \frac{k}{(n-1)+k} = \sum_{j=0}^{n} \frac{j+1}{(n-1)+(j+1)}$$

$$= \sum_{j=0}^{n} \frac{j+1}{n+j} = \sum_{k=0}^{n} \frac{k+1}{n+k} \quad \text{(changing variable)}$$

Transform by making the change of variable j = i - 1, in the summation

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2} \qquad **$$

<u>PROPERTIES OF SUMMATIONS:</u>

1. $\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k; \quad a_k, b_k \in R$

2. $\sum_{k=m}^{n} ca_k = c\sum_{k=m}^{n} a_k \qquad c \in R$

3. $\sum_{k=a-i}^{b-i} (k+i) = \sum_{k=a}^{b} k \qquad i \in N$

4. $\sum_{k=a+i}^{b+i}(k-i) = \sum_{k=a}^{b} k \qquad i \in N$

5. $\sum_{k=1}^{n} c = c + c + \cdots + c = nc$

EXERCISE:

Express the following summation more simply:

SOLUTION: $3\sum_{k=1}^{n}(2k-3) + \sum_{k=1}^{n}(4-5k)$

$$3\sum_{k=1}^{n}(2k-3) + \sum_{k=1}^{n}(4-5k)$$

$$= 3\sum_{k=1}^{n}3(2k-3) + \sum_{k=1}^{n}(4-5k)$$

$$= \sum_{k=1}^{n}[3(2k-3) + (4-5k)]$$

$$= \sum_{k=1}^{n}(k-5)$$

$$= \sum_{k=1}^{n}k - \sum_{k=1}^{n}5$$

$$= \sum_{k=1}^{n}k - 5n$$

ARITHMETIC SERIES:

The sum of the terms of an arithmetic sequence forms an arithmetic series (A.S). For example

$$1 + 3 + 5 + 7 + \ldots$$

is an arithmetic series of positive odd integers.

In general, if a is the first term and d the common difference of an arithmetic series, then the series is given as: $a + (a+d) + (a+2d) + \ldots$

SUM OF n TERMS OF AN ARITHMETIC SERIES:

Let a be the first term and d be the common difference of an arithmetic series. Then its nth term is:

$$a_n = a + (n-1)d; \qquad n \geq 1$$

If S_n denotes the sum of first n terms of the A.S, then

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + \ldots + [a + (n-1)d] \\ &= a + (a+d) + (a+2d) + \ldots + a_n \\ &= a + (a+d) + (a+2d) + \ldots + (a_n - d) + a_n \ \ldots\ldots\ldots(1) \end{aligned}$$

where $a_n = a + (n-1)d$

Rewriting the terms in the series in reverse order,

$$Sn = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a + d) + a \ \ldots\ldots\ldots(2)$$

Adding (1) and (2) term by term, gives

$$2\ S_n = (a + a_n) + (a + a_n) + (a + a_n) + \ldots + (a + a_n) \qquad \text{(n terms)}$$
$$2\ S_n = n\ (a + a_n)$$
$$\Rightarrow \quad S_n = n(a + a_n)/2$$
$$S_n = n(a + l)/2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where $\quad l = an = a + (n - 1)d$

Therefore

$$S_n = n/2\ [a + a + (n - 1)\ d]$$
$$S_n\ n/2\ [2\ a + (n - 1)\ d] \ldots\ldots\ldots(4)$$

EXERCISE:

Find the sum of first n natural numbers.

SOLUTION:

Let $S_n = 1 + 2 + 3 + \ldots + n$

Clearly the right hand side forms an arithmetic series with

$a = 1, \quad d = 2 - 1 = 1 \qquad$ and $\ n = n$

$$\therefore \quad S_n = \frac{n}{2}[2a + (n-1)d]$$
$$= \frac{n}{2}[2(1) + (n-1)(1)]$$
$$= \frac{n}{2}[2 + n - 1]$$
$$= \frac{n(n+1)}{2}$$

EXERCISE:

Find the sum of all two digit positive integers which are neither divisible by 5 nor by 2.

SOLUTION:

The series to be summed is:

$11 + 13 + 17 + 19 + 21 + 23 + 27 + 29 + \ldots + 91 + 93 + 97 + 99$

which is not an arithmetic series.

If we make group of four terms we get

$(11 + 13 + 17 + 19) + (21 + 23 + 27 + 29) + (31 + 33 + 37 + 39) + \ldots + (91 + 93 + 97 + 99) = 60 + 100 + 140 + \ldots + 380$

which now forms an arithmetic series in which

$$a = 60;\ \ d = 100 - 60 = 40 \quad \text{and} \quad l = a_n = 380$$

To find n, we use the formula

$$a_n = a + (n - 1)\ d$$
$$\Rightarrow 380 = 60 + (n - 1)\ (40)$$
$$\Rightarrow 380 - 60 = (n - 1)\ (40)$$
$$\Rightarrow 320 = (n - 1)\ (40)$$

$$\frac{320}{40} = n - 1$$

$$8 = n - 1$$

$$\Rightarrow \quad n = 9$$

Now

$$S_n = \frac{n}{2}(a+l)$$

$$\therefore \quad S_9 = \frac{9}{2}(60+380) = 1980$$

GEOMETRIC SERIES:

The sum of the terms of a geometric sequence forms a geometric series (G.S.). For example

$$1 + 2 + 4 + 8 + 16 + \ldots$$

is geometric series.

In general, if **a** is the first term and **r** the common ratio of a geometric series, then the series is given as: $a + ar + ar^2 + ar^3 + \ldots$

SUM OF n TERMS OF A GEOMETRIC SERIES:

Let **a** be the first term and **r** be the common ratio of a geometric series. Then its nth term is:

$$a_n = ar^{n-1}; \quad n \geq 1$$

If Sn denotes the sum of first n terms of the G.S. then

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \ldots\ldots\ldots\ldots\ldots(1)$$

Multiplying both sides by r we get.

$r\,S_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$..................(2)

Subtracting (2) from (1) we get

$$S_n - rS_n = a - ar^n$$

$$\Rightarrow \quad (1 - r)\, S_n = a\,(1 - r^n)$$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

EXERCISE:

Find the sum of the geometric series

$$6 - 2 + \frac{2}{3} - \frac{2}{9} + \cdots + \quad \text{to 10 terms}$$

SOLUTION:

In the given geometric series

$$a = 6, \quad r = \frac{-2}{6} = -\frac{1}{3} \quad \text{and } n = 10$$

$$\therefore \qquad S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{10} = \frac{6\left(1-\left(-\frac{1}{3}\right)^{10}\right)}{1-\left(-\frac{1}{3}\right)} = \frac{6\left(1+\frac{1}{3^{10}}\right)}{\left(\frac{4}{3}\right)}$$

$$= \frac{9\left(1+\frac{1}{3^{10}}\right)}{2}$$

INFINITE GEOMETRIC SERIES:

Consider the infinite geometric series

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$$

then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

If $S_n \to S$ as $n \to \infty$, then the series is convergent and S is its sum.

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

$$\therefore \qquad S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r}$$

If S_n increases indefinitely as n becomes very large then the series is said to be divergent.

EXERCISE:

Find the sum of the infinite geometric series:

$$\frac{9}{4}+\frac{3}{2}+1+\frac{2}{3}+\cdots$$

SOLUTION:

Here we have

$$a=\frac{9}{4}, \quad r=\frac{3/2}{9/4}=\frac{2}{3}$$

Note that |r| < 1 So we can use the above formula.

$$\begin{aligned}
\therefore \quad S &= \frac{a}{1-r} \\
&= \frac{9/4}{1-2/3} \\
&= \frac{9/4}{1/3}=\frac{9}{4}\times\frac{3}{1}=\frac{27}{4}
\end{aligned}$$

EXERCISE:

Find a common fraction for the recurring decimal 0.81

SOLUTION:

$$\begin{aligned}
0.81 &= 0.8181818181\ \ldots \\
&= 0.81 + 0.0081 + 0.000081 + \ldots
\end{aligned}$$

which is an infinite geometric series with

$$a=0.81, \quad r=\frac{0.0081}{0.81}=0.01$$

$$\begin{aligned}
\therefore \quad \text{Sum} &= \frac{a}{1-r} \\
&= \frac{0.81}{1-0.01} = \frac{0.81}{0.99} \\
&= \frac{81}{99} = \frac{9}{11}
\end{aligned}$$

IMPORTANT SUMS:

1. $$1+2+3+\cdots+n=\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

2. $$1^2+2^2+3^2+\cdots+n^2=\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

3. $$1^3+2^3+3^3+\cdots+n^3=\sum_{k=1}^{n}k^3=\frac{n^2(n+1)}{4}=\left[\frac{n(n+1)}{2}\right]^2$$