Logic III

Sunday, 4 August 2024 11:00 pm

2.2 Conditional Statements

. . . hypothetical reasoning implies the subordination of the real to the realm of the possible . . . —Jean Piaget, 1972

When you make a logical inference or deduction, you reason *from* a hypothesis *to* a conclusion. Your aim is to be able to say, "*If* such and such is known, *then* something or other must be the case."

Let p and q be statements. A sentence of the form "If p then q" is denoted symbolically by "$p \to q$"; p is called the *hypothesis* and q is called the *conclusion*. For instance, consider the following statement:

$$\text{If } \underbrace{\text{4,686 is divisible by 6}}_{\text{hypothesis}}, \text{ then } \underbrace{\text{4,686 is divisible by 3}}_{\text{conclusion}}$$

Such a sentence is called *conditional* because the truth of statement q is conditioned on the truth of statement p.

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Truth Table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Definition

If p and q are statement variables, the **conditional** of q by p is "If p then q" or "p implies q" and is denoted $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).

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Example 2.2.1 A Conditional Statement with a False Hypothesis

Consider the statement

$$\text{If } 0 = 1 \text{ then } 1 = 2.$$

As strange as it may seem, since the hypothesis of this statement is false, the statement as a whole is true. ■

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Example 2.2.2 **Truth Table for $p \vee \sim q \to \sim p$**

Construct a truth table for the statement form $p \vee \sim q \to \sim p$.

Solution By the order of operations given above, the following two expressions are equivalent: $p \vee \sim q \to \sim p$ and $(p \vee (\sim q)) \to (\sim p)$, and this order governs the construction of the truth table. First fill in the four possible combinations of truth values for p and q, and then enter the truth values for $\sim p$ and $\sim q$ using the definition of negation. Next fill in the $p \vee \sim q$ column using the definition of $\vee$. Finally, fill in the $p \vee \sim q \to \sim p$ column using the definition of $\to$.

Note The only rows in which the hypothesis $p \vee \sim q$ is true and the conclusion $\sim p$ is false are the first and second rows. So you put F's in those two rows and T's in the other two rows.

		conclusion		hypothesis	
p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$p \vee \sim q \to \sim p$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

■

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Example 2.2.3 **Division into Cases: Showing That $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$**

Use truth tables to show the logical equivalence of the statement forms $p \vee q \to r$ and $(p \to r) \wedge (q \to r)$. Annotate the table with a sentence of explanation.

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p	q	r	$p \vee q$	$p \to r$	$q \to r$	$p \vee q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$p \vee q \to r$ and $(p \to r) \wedge (q \to r)$ always have the same truth values, so they are logically equivalent ■

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Representation of If-Then as Or

In exercise 13(a) at the end of this section you are asked to use truth tables to show that

$$p \to q \equiv \sim p \vee q.$$

The logical equivalence of "if p then q" and "not p or q" is occasionally used in everyday speech. Here is one instance.

Example 2.2.4 **Application of the Equivalence between $\sim p \vee q$ and $p \to q$**

Rewrite the following statement in if-then form.

Either you get to work on time or you are fired.

Solution Let $\sim p$ be

You get to work on time.

and q be

You are fired.

Then the given statement is $\sim p \vee q$. Also p is

You do not get to work on time.

So the equivalent if-then version, $p \to q$, is

If you do not get to work on time, then you are fired. ■

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The Negation of a Conditional Statement

By definition, $p \to q$ is false if, and only if, its hypothesis, p, is true and its conclusion, q, is false. It follows that

> The negation of "if p then q" is logically equivalent to "p and not q."

This can be restated symbolically as follows:

$$\boxed{\sim(p \to q) \equiv p \wedge \sim q}$$

To obtain this result you can also start from the logical equivalence $p \to q \equiv \sim p \vee q$. Take the negation of both sides to obtain

$$\begin{aligned}
\sim(p \to q) &\equiv \sim(\sim p \vee q) \\
&\equiv \sim(\sim p) \wedge (\sim q) && \text{by De Morgan's laws} \\
&\equiv p \wedge \sim q && \text{by the double negative law.}
\end{aligned}$$

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Example 2.2.5 Negations of *If-Then* Statements

Write negations for each of the following statements:

a. If my car is in the repair shop, then I cannot get to class.

b. If Sara lives in Athens, then she lives in Greece.

Solution

a. My car is in the repair shop and I can get to class.

b. Sara lives in Athens and she does not live in Greece. (Sara might live in Athens, Georgia; Athens, Ohio; or Athens, Wisconsin.) ■

Caution! Remember that the negation of an if-then statement does not start with the word *if*.

It is tempting to write the negation of an if-then statement as another if-then statement. Please resist that temptation!

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The Contrapositive of a Conditional Statement

One of the most fundamental laws of logic is the equivalence between a conditional statement and its contrapositive.

Definition

The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p.$$

Symbolically,

$$\text{The contrapositive of } p \to q \text{ is } \sim q \to \sim p.$$

The fact is that

A conditional statement is logically equivalent to its contrapositive.

Example 2.2.6 Writing the Contrapositive

Write each of the following statements in its equivalent contrapositive form:

a. If Howard can swim across the lake, then Howard can swim to the island.

b. If today is Easter, then tomorrow is Monday.

Solution

a. If Howard cannot swim to the island, then Howard cannot swim across the lake.

b. If tomorrow is not Monday, then today is not Easter. ■

The Converse and Inverse of a Conditional Statement

The fact that a conditional statement and its contrapositive are logically equivalent is very important and has wide application. Two other variants of a conditional statement are *not* logically equivalent to the statement.

Definition

Suppose a conditional statement of the form "If p then q" is given.

1. The **converse** is "If q then p."
2. The **inverse** is "If $\sim p$ then $\sim q$."

Symbolically,

$$\text{The converse of } p \to q \text{ is } q \to p,$$

and

$$\text{The inverse of } p \to q \text{ is } \sim p \to \sim q.$$

Example 2.2.7 **Writing the Converse and the Inverse**

Write the converse and inverse of each of the following statements:

a. If Howard can swim across the lake, then Howard can swim to the island.

b. If today is Easter, then tomorrow is Monday.

Caution! Many people believe that if a conditional statement is true, then its converse and inverse must also be true. This is not correct! The converse might be true, but it does not have to be true.

Solution

a. *Converse:* If Howard can swim to the island, then Howard can swim across the lake.

Inverse: If Howard cannot swim across the lake, then Howard cannot swim to the island.

b. *Converse:* If tomorrow is Monday, then today is Easter.

Inverse: If today is not Easter, then tomorrow is not Monday. ■

Note that while the statement "If today is Easter, then tomorrow is Monday" is always true, both its converse and inverse are false on every Sunday except Easter.

1. A conditional statement and its converse are *not* logically equivalent.
2. A conditional statement and its inverse are *not* logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

In exercises 24, 25, and 27 at the end of this section, you are asked to use truth tables to verify the statements in the box above. Note that the truth of statement 3 also follows from the observation that the inverse of a conditional statement is the contrapositive of its converse.

Only If and the Biconditional

To say "p only if q" means that p can take place *only* if q takes place also. That is, if q does not take place, then p cannot take place. Another way to say this is that if p occurs, then q must also occur (by the logical equivalence between a statement and its contrapositive).

Definition

If p and q are statements,

p **only if** q means "if not q then not p,"

or, equivalently,

"if p then q."

Example 2.2.8 **Converting *Only If* to *If-Then***

Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other.

> John will break the world's record for the mile run only if he runs the mile in under four minutes.

Solution *Version 1:* If John does not run the mile in under four minutes, then he will not break the world's record.

Version 2: If John breaks the world's record, then he will have run the mile in under four minutes. ■

Definition

Given statement variables p and q, the **biconditional of** p **and** q is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words *if and only if* are sometimes abbreviated **iff**.

The biconditional has the following truth table:

Truth Table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

In order of operations $\leftrightarrow$ is coequal with $\rightarrow$. As with $\wedge$ and $\vee$, the only way to indicate precedence between them is to use parentheses. The full hierarchy of operations for the five logical operators is shown below.

Order of Operations for Logical Operators

1. $\sim$ Evaluate negations first.
2. $\wedge, \vee$ Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3. $\rightarrow, \leftrightarrow$ Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

According to the separate definitions of *if* and *only if*, saying "p if, and only if, q" should mean the same as saying both "p if q" and "p only if q." The following annotated truth table shows that this is the case:

Truth Table Showing That $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ always have the same truth values, so they are logically equivalent

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Example 2.2.9 *If and Only If*

Rewrite the following statement as a conjunction of two if-then statements:

> This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

Solution If this program is correct, then it produces the correct answers for all possible sets of input data; and if this program produces the correct answers for all possible sets of input data, then it is correct. ■

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Necessary and Sufficient Conditions

The phrases *necessary condition* and *sufficient condition*, as used in formal English, correspond exactly to their definitions in logic.

Definition

If r and s are statements:

r is a **sufficient condition** for s means "if r then s."

r is a **necessary condition** for s means "if not r then not s."

In other words, to say "r is a sufficient condition for s" means that the occurrence of r is *sufficient* to guarantee the occurrence of s. On the other hand, to say "r is a necessary condition for s" means that if r does not occur, then s cannot occur either:

The occurrence of r is *necessary* to obtain the occurrence of s. Note that because of the equivalence between a statement and its contrapositive,

r is a necessary condition for s also means "if s then r."

Consequently,

r is a necessary and sufficient condition for s means "r if, and only if, s."

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Example 2.2.10 **Interpreting Necessary and Sufficient Conditions**

Consider the statement "If John is eligible to vote, then he is at least 18 years old." The truth of the condition "John is eligible to vote" is *sufficient* to ensure the truth of the condition "John is at least 18 years old." In addition, the condition "John is at least 18 years old" is *necessary* for the condition "John is eligible to vote" to be true. If John were younger than 18, then he would not be eligible to vote. ■

Example 2.2.11 **Converting a Sufficient Condition to If-Then Form**

Rewrite the following statement in the form "If A then B":

Pia's birth on U.S. soil is a sufficient condition
for her to be a U.S. citizen.

Solution If Pia was born on U.S. soil, then she is a U.S. citizen. ■

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Example 2.2.12 **Converting a Necessary Condition to If-Then Form**

Use the contrapositive to rewrite the following statement in two ways:

> George's attaining age 35 is a necessary condition for his being president of the United States.

Solution *Version 1:* If George has not attained the age of 35, then he cannot be president of the United States.

Version 2: If George can be president of the United States, then he has attained the age of 35. ■

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Remarks

1. *In logic, a hypothesis and conclusion are not required to have related subject matters.*

   In ordinary speech we never say things like "If computers are machines, then Babe Ruth was a baseball player" or "If $2+2=5$, then Mickey Mouse is president of the United States." We formulate a sentence like "If p then q" only if there is some connection of content between p and q.

   In logic, however, the two parts of a conditional statement need not have related meanings. The reason? If there were such a requirement, who would enforce it? What one person perceives as two unrelated clauses may seem related to someone else. There would have to be a central arbiter to check each conditional sentence before anyone could use it, to be sure its clauses were in proper relation. This is impractical, to say the least!

   Thus a statement like "if computers are machines, then Babe Ruth was a baseball player" is allowed, and it is even called true because both its hypothesis and its conclusion are true. Similarly, the statement "If $2+2=5$, then Mickey Mouse is president of the United States" is allowed and is called true because its hypothesis is false, even though doing so may seem ridiculous.

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2. *In informal language, simple conditionals are often used to mean biconditionals.*

   The formal statement "p if, and only if, q" is seldom used in ordinary language. Frequently, when people intend the biconditional they leave out either the *and only if* or the *if and*. That is, they say either "p if q" or "p only if q" when they really mean "p if, and only if, q." For example, consider the statement "You will get dessert if, and only if, you eat your dinner." Logically, this is equivalent to the conjunction of the following two statements.

   Statement 1: If you eat your dinner, then you will get dessert.
   Statement 2: You will get dessert only if you eat your dinner.
   or
   If you do not eat your dinner, then you will not get dessert.

   Now how many parents in the history of the world have said to their children "You will get dessert if, and only if, you eat your dinner"? Not many! Most say either "If you eat your dinner, you will get dessert" (these take the positive approach—they emphasize the reward) or "You will get dessert only if you eat your dinner" (these take the

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negative approach—they emphasize the punishment). Yet the parents who promise the reward intend to suggest the punishment as well, and those who threaten the punishment will certainly give the reward if it is earned. Both sets of parents expect that their conditional statements will be interpreted as biconditionals.

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TEST YOURSELF

1. An *if-then* statement is false if, and only if, the hypothesis is ________ and the conclusion is ________.
2. The negation of "if *p* then *q*" is ________.
3. The converse of "if *p* then *q*" is ________.
4. The contrapositive of "if *p* then *q*" is ________.
5. The inverse of "if *p* then *q*" is ________.
6. A conditional statement and its contrapositive are ________.
7. A conditional statement and its converse are not ________.
8. "*R* is a sufficient condition for *S*" means "if ________ then ________."
9. "*R* is a necessary condition for *S*" means "if ________ then ________."
10. "*R* only if *S*" means "if ________ then ________."

Sr. No.	Logical Equivalence involving implications
1	$P \to Q \equiv \neg P \vee Q$
2	$P \to Q \equiv \neg Q \to \neg P$
3	$P \vee Q \equiv \neg P \to Q$
4	$P \wedge Q \equiv \neg(P \to \neg Q)$
5	$\neg(P \to Q) \equiv P \wedge \neg Q$
6	$(P \to Q) \wedge (P \to r) \equiv P \to (Q \wedge r)$
7	$(P \to r) \wedge (Q \to r) \equiv (P \vee Q) \to r$
8	$(P \to Q) \vee (P \to r) \equiv P \to (Q \vee r)$
9	$(P \to r) \vee (Q \to r) \equiv (P \wedge Q) r$
10	$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$
11	$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$
12	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$
13	$\neg(P \leftrightarrow Q) \equiv P \leftrightarrow \neg Q$

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