Proof By Induction (Inequalities)

e.g. Prove $2^n > 1+n$ for all $n \geq 2$, $n \in \mathbb{N}$

① Prove for the base case

Let $n = 2 \Rightarrow$ LHS $= 2^2 = 4$

RHS $= 1 + 2 = 3$

LHS > RHS $\therefore$ true for $n = 2$

② Assume true for $n = k$

Let $n = k$ and assume true

i.e. $2^k > 1 + k$

③ Let $n = k + 1$ and prove true using assumption

At this point look at what we need to do to the LHS of our assumption to get the LHS of our target

TARGET (INEQUALITY WITH $n = k+1$)

$2^{k+1} > k + 2$

Assuming $2^k > 1 + k$

$\Rightarrow 2(2^k) > 2(1+k)$ (Multiplying both sides of our assumption by 2 gives another statement we can assume true)

$\Rightarrow \underbrace{2^{k+1} > 2 + 2k}$

Make the RHS of our target appear here

$\Rightarrow 2^{k+1} > (k+2) + k > k+2$ Since $k > 0$ for $\underbrace{k \geq 1}$

always state range of values for which true

$\therefore 2^{k+1} > k + 2$

④ Conclude

If true for $n = k$ then true for $n = k+1$

Since true for $n = 2$ then true for all integers $n \geq 2$

e.g. Prove $2^n > 2n$ for $n \geq 3$, $n \in \mathbb{N}$

① Base case

Let $n = 3 \Rightarrow$ LHS $= 2^3 = 8$

RHS $= 2(3) = 6$

LHS $>$ RHS $\therefore$ true for $n = 3$

② Assumption

Let $n = k$ and assume true

i.e. $2^k > 2k$

③ Inductive Step

TARGET

$2^{k+1} > 2k + 2$

Assuming $2^k > 2k$

$\Rightarrow 2(2^k) > 2(2k)$ ← Multiply the LHS of our assumption by 2 to get the LHS of our target

$\Rightarrow 2^{k+1} > 4k$

$\Rightarrow 2^{k+1} > \underbrace{(2k+2) + (2k-2)}_{\text{Must be equal to above line to be true}} > 2k+2$ since $2k - 2 > 0$ for $k \geq 2$

$\therefore 2^{k+1} > 2k + 2$

④ Conclusion

If true for $n = k$ then true for $n = k + 1$

Since true for $n = 3$ then true for all integers $n \geq 3$

e.g. Prove that $n! > 2^n$ for all $n \geq 4$, $n \in \mathbb{N}$

① Base case

Let $n = 4 \Rightarrow$ LHS $= 4! = 24$

RHS $= 2^4 = 16$

LHS > RHS $\therefore$ true for $n = 4$

② Assumption

Let $n = k$ and assume true

i.e. $k! > 2^k$

③ Inductive step

TARGET

$(k+1)! > 2^{k+1}$

Assuming $k! > 2^k$

$(k+1)k! > (k+1)2^k$ ← Multiply both sides by $k+1$ to get LHS of target

$\Rightarrow (k+1)! > \underbrace{2 \times 2^k}$ since $k+1 > 2$ when $k \geq 2$

We have made RHS smaller so '>' still holds

$\Rightarrow (k+1)! > 2^{k+1}$

④ Conclusion

If true for $n = k$ then true for $n = k+1$

Since true for $n = 4$ then true for all integers $n \geq 4$