Graph

Sunday, 8 September 2024 4:38 pm

Basic Graph Theory

- Graph
  A graph is a mathematical structure consisting of a set of points called **VERTICES** and a set (possibly empty) of lines linking some pair of vertices. It is possible for the edges to oriented; i.e. to be directed edges. The lines are called **EDGES** if they are undirected, and or **ARCS** if they are directed.
- Loop and Multiple edges
  A **loop** is an edge that connects a vertex to itself. If a graph has more than one edge joining some pair of vertices then these edges are called **multiple edges**.
- Simple Graph
  A **simple graph** is a graph that does not have more than one edge between any two vertices and no edge starts and ends at the same vertex. In other words a simple graph is a graph without loops and multiple edges.
- Adjacent Vertices
  Two vertices are said to be **adjacent** if there is an edge (arc) connecting them.
- Adjacent Edges
  Adjacent edges are edges that share a common vertex.
- Degree of a Vertex
  The **degree** of a vertex is the number of edges incident with that vertex.
- Path
  A **path** is a sequence of vertices with the property that each vertex in the sequence is adjacent to the vertex next to it. A path that does not repeat vertices is called a **simple path**.
- Circuit
  A **circuit** is path that begins and ends at the same vertex.
- Cycle
  A circuit that doesn't repeat vertices is called a **cycle**.

- A Connected Graph
  A graph is said to be **connected** if **any two** of its vertices are joined by a path. A graph that is not connected is a **disconnected graph**. A disconnected graph is made up of connected subgraphs that are called **components**.
- Bridge
  A **bridge** is an edge whose deletion from a graph **increases the number of components** in the graph. If a graph was a connected graph then the removal of a bridge-edge disconnects it.
- Euler Path
  An **Euler path** is a path that travels through all edges of a connected graph.
- Euler Circuit
  An **Euler circuit** is a circuit that visits all edges of a connected graph.

The Hand Shaking Lemma

1. The sum of the degrees of all the vertices of a graph is twice the number of edges in the graph.
2. The number of vertices of odd degree is always even.
   An applet on the Hand shaking Lemma

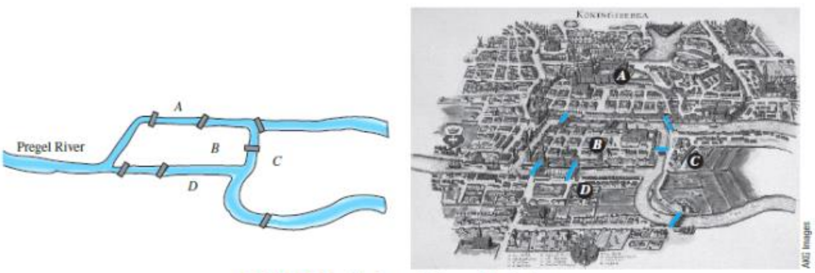


FIGURE 10.1.1 The Seven Bridges of Königsberg

Leonhard Euler
(1707–1783)

To solve this puzzle, Euler translated it into a graph theory problem. He noticed that all points of a given land mass can be identified with each other since a person can travel from any one point to any other point of the same land mass without crossing a bridge. Thus for the purpose of solving the puzzle, the map of Königsberg can be identified with the graph shown in Figure 10.1.2, in which the vertices A, B, C, and D represent land masses and the seven edges represent the seven bridges.

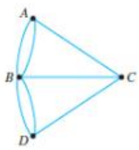


FIGURE 10.1.2 Graph Version of Königsberg Map

In terms of this graph, the question becomes the following:

> Is it possible to find a route through the graph that starts and ends at some vertex, one of A, B, C, or D, and traverses each edge exactly once?

Equivalently:

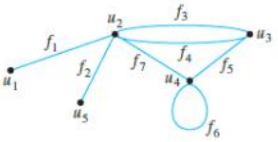
> Is it possible to trace this entire graph, starting and ending at the same point, without either ever lifting your pencil from the paper or crossing an edge more than once?

Take a few minutes to think about the question yourself. Can you find a route that meets the requirements? Try it!

Definitions

Travel in a graph is accomplished by moving from one vertex to another along a sequence of adjacent edges. In the graph below, for instance, you can go from u_1 to u_4 by taking f_1 to u_2 and then f_7 to u_4. This is represented by writing

$$u_1 f_1 u_2 f_7 u_4.$$



Or you could take the roundabout route

$$u_1 f_1 u_2 f_3 u_3 f_4 u_2 f_3 u_3 f_5 u_4 f_6 u_4 f_7 u_2 f_3 u_3 f_5 u_4.$$

Certain types of sequences of adjacent vertices and edges are of special importance in graph theory: those that do not have a repeated edge, those that do not have a repeated vertex, and those that start and end at the same vertex.

Definition

Let G be a graph, and let v and w be vertices in G.

A **walk from v to w** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for each $i = 1, 2, \ldots n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from v to v** consists of the single vertex v.

A **trail from v to w** is a walk from v to w that does not contain a repeated edge.

A **path from v to w** is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

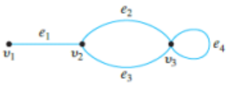
For ease of reference, these definitions are summarized in the following table:

	Repeated Edge?	Repeated Vertex?	Starts and Ends at the Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple circuit	no	first and last only	yes	yes

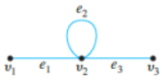
Often a walk can be specified unambiguously by giving either a sequence of edges or a sequence of vertices. The next two examples show how this is done.

Example 10.1.1 Notation for Walks

a. In the graph below, the notation $e_1e_2e_4e_3$ refers unambiguously to the following walk: $v_1e_1v_2e_2v_3e_4v_3e_3v_2$. On the other hand, the notation e_1 is ambiguous if used by itself to refer to a walk. It could mean either $v_1e_1v_2$ or $v_2e_1v_1$.



b. In the graph of part (a), the notation v_2v_3 is ambiguous if used to refer to a walk. It could mean $v_2e_2v_3$ or $v_2e_3v_3$. On the other hand, in the graph below, the notation $v_1v_2v_2v_3$ refers unambiguously to the walk $v_1e_1v_2e_2v_2e_3v_3$.

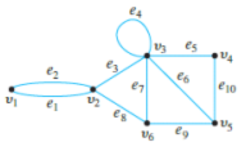


Note that if a graph G does not have any parallel edges, then any walk in G is uniquely determined by its sequence of vertices.

Example 10.1.2 Walks, Trails, Paths, and Circuits

In the graph below, determine which of the following walks are trails, paths, circuits, or simple circuits.

a. $v_1e_1v_2e_3v_3e_4v_3e_5v_4$
b. $e_1e_3e_5e_5e_6$
c. $v_2v_3v_4v_5v_3v_6v_2$
d. $v_2v_3v_4v_5v_6v_2$
e. $v_1e_1v_2e_1v_1$
f. v_1



Solution

a. This walk has a repeated vertex but does not have a repeated edge, so it is a trail from v_1 to v_4 but not a path.

b. This is just a walk from v_1 to v_5. It is not a trail because it has a repeated edge.

c. This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, so it is a circuit. Since the vertex v_3 is repeated in the middle, it is not a simple circuit.

d. This walk starts and ends at v_2, contains at least one edge, does not have a repeated edge, and does not have a repeated vertex. Thus it is a simple circuit.

e. This is just a closed walk starting and ending at v_1. It is not a circuit because edge e_1 is repeated.

f. The first vertex of this walk is the same as its last vertex, but it does not contain an edge, and so it is not a circuit. It is a closed walk from v_1 to v_1. (It is also a trail from v_1 to v_1.)

Subgraphs

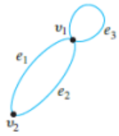
Definition

A graph H is said to be a **subgraph** of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

Example 10.1.3 Subgraphs

List all subgraphs of the graph G with vertex set $\{v_1, v_2\}$ and edge set $\{e_1, e_2, e_3\}$, where the endpoints of e_1 are v_1 and v_2, the endpoints of e_2 are v_1 and v_2, and e_3 is a loop at v_1.

Solution G can be drawn as shown below.



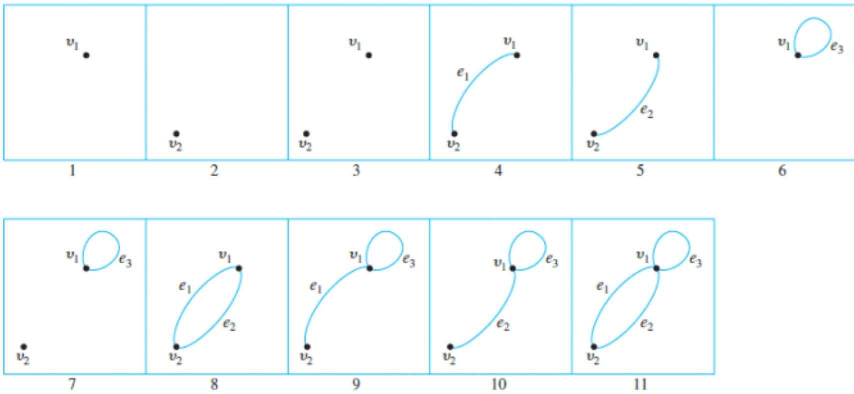


FIGURE 10.1.3

Connectedness

It is easy to understand the concept of connectedness on an intuitive level. Roughly speaking, a graph is connected if it is possible to travel from any vertex to any other vertex along a sequence of adjacent edges of the graph. The formal definition of connectedness is stated in terms of walks.

Definition

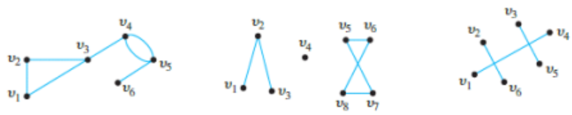
Let G be a graph. Two **vertices v and w of G are connected** if, and only if, there is a walk from v to w. The **graph G is connected** if, and only if, given *any* two vertices v and w in G, there is a walk from v to w. Symbolically:

$$G \text{ is connected} \quad \Leftrightarrow \quad \forall \text{ vertices } v \text{ and } w \text{ in } G, \exists \text{ a walk from } v \text{ to } w.$$

If you take the negation of this definition, you will see that a graph G is *not connected* if, and only if, there exist two vertices of G that are not connected by any walk.

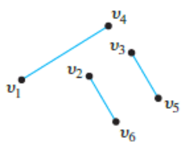
Example 10.1.4 **Connected and Disconnected Graphs**

Which of the following graphs are connected?



(a) (b) (c)

Solution The graph represented in (a) is connected, whereas those of (b) and (c) are not. To understand why (c) is not connected, recall that in a drawing of a graph, two edges may cross at a point that is not a vertex. Thus the graph in (c) can be redrawn as follows:



Euler Circuits

Now we return to consider general problems similar to the puzzle of the Königsberg bridges. The following definition is made in honor of Euler.

Definition

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Theorem 10.1.2

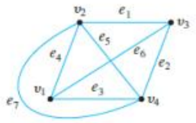
If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Contrapositive Version of Theorem 10.1.2

If some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

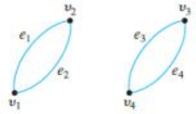
Example 10.1.6 **Showing That a Graph Does Not Have an Euler Circuit**

Show that the graph below does not have an Euler circuit.



Solution Vertices v_1 and v_3 both have degree 3, which is odd. Hence by (the contrapositive form of) Theorem 10.1.2, this graph does not have an Euler circuit. ■

Now consider the converse of Theorem 10.1.2: If every vertex of a graph has even degree, then the graph has an Euler circuit. Is this true? The answer is no. There is a graph G such that every vertex of G has even degree but G does not have an Euler circuit. In fact, there are many such graphs. The illustration below shows one example.



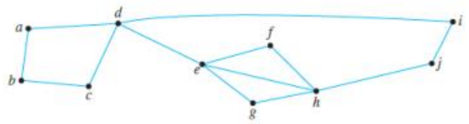
Every vertex has even degree, but the graph does not have an Euler circuit.

Theorem 10.1.3

If a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit.

Example 10.1.7 **Finding an Euler Circuit**

Use Theorem 10.1.3 to check that the graph below has an Euler circuit. Then use the algorithm from the proof of the theorem to find an Euler circuit for the graph.

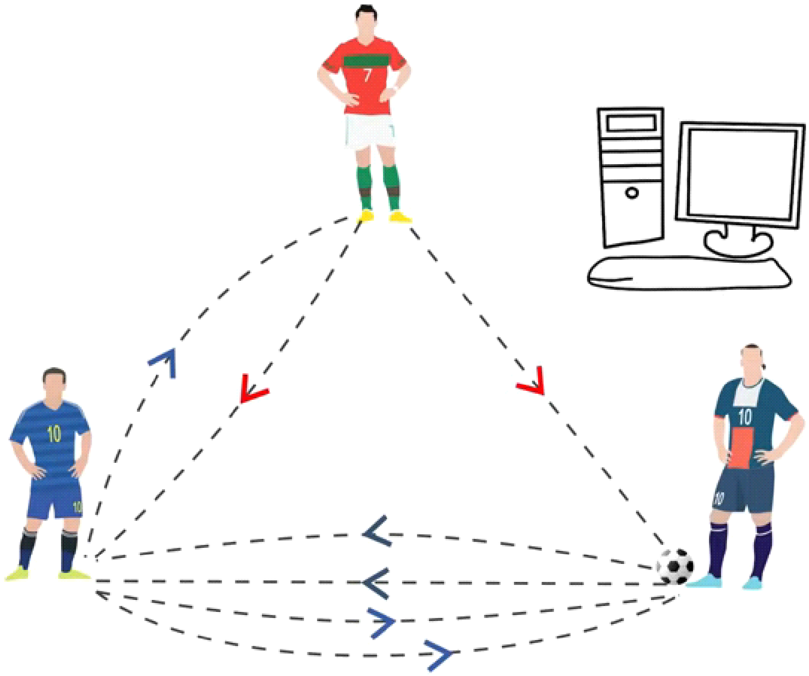


Definition

Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

10.2 Matrix Representations of Graphs

Order and simplification are the first steps toward the mastery of a subject.
—Thomas Mann, *The Magic Mountain*, 1924

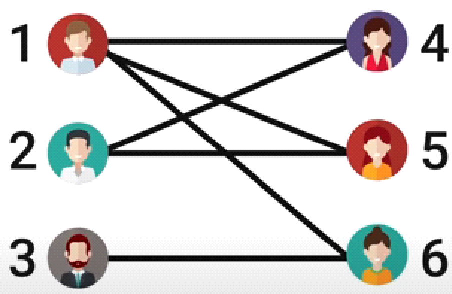


$$\begin{array}{c|c} & \begin{matrix} R & M & Z \end{matrix} \\ \begin{matrix} R \\ M \\ Z \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

Matrix

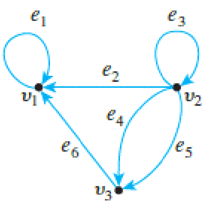
Matrix

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$



Matrices and Directed Graphs

Consider the directed graph shown in Figure 10.2.1. This graph can be represented by the matrix $\mathbf{A} = (a_{ij})$ for which a_{ij} = the number of arrows from v_i to v_j, for every $i = 1, 2, 3$ and $j = 1, 2, 3$. Thus $a_{11} = 1$ because there is one arrow from v_1 to v_1; $a_{12} = 0$ because there is no arrow from v_1 to v_2, $a_{23} = 2$ because there are two arrows from v_2 to v_3, and so forth. **A** is called the *adjacency matrix* of the directed graph. For convenient reference, the rows and columns of **A** are often labeled with the vertices of the graph G.



$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Directed Graph G (a)

Adjacency Matrix of G (b)

FIGURE 10.2.1 A Directed Graph and Its Adjacency Matrix

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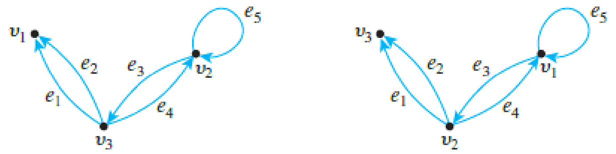
Example 10.2.2 The Adjacency Matrix of a Graph

The two directed graphs shown below differ only in the ordering of their vertices. Find their adjacency matrices.



(a) (b)

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Example 10.2.3 Obtaining a Directed Graph from a Matrix

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

Draw a directed graph that has **A** as its adjacency matrix.

Solution Let G be the graph corresponding to **A**, and let v_1, v_2, v_3, and v_4 be the vertices of G. Label **A** across the top and down the left side with these vertex names, as shown below.

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Then, for instance, the 2 in the fourth row and the first column means that there are two arrows from v_4 to v_1. The 0 in the first row and the fourth column means that there is no arrow from v_1 to v_4. A corresponding directed graph is shown on the next page (without edge labels because the matrix does not determine those).

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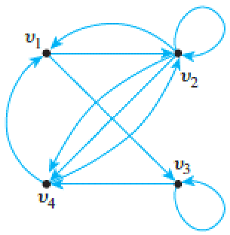

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Matrices and Undirected Graphs

Once you know how to associate a matrix with a directed graph, the definition of the matrix corresponding to an undirected graph should seem natural to you. As before, you must order the vertices of the graph, but in this case you simply set the ijth entry of the adjacency matrix equal to the number of edges connecting the ith and jth vertices of the graph.

Definition

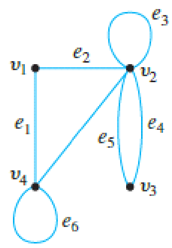
Let G be an undirected graph with ordered vertices $v_1, v_2, \ldots, v_n$. The **adjacency matrix of G** is the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ over the set of nonnegative integers such that

$$a_{ij} = \text{the number of edges connecting } v_i \text{ and } v_j$$

for every i and $j = 1, 2, \ldots, n$.

Example 10.2.4 **Finding the Adjacency Matrix of a Graph**

Find the adjacency matrix for the graph G shown below.



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Solution

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

The entries of $\mathbf{A}$ satisfy the condition, $a_{ij} = a_{ji}$, for every $i, j = 1, 2, \ldots, n$. This implies that the appearance of $\mathbf{A}$ remains the same if the entries of $\mathbf{A}$ are flipped across its main diagonal. A matrix, like $\mathbf{A}$, with this property is said to be *symmetric*.

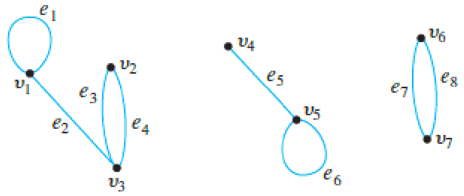
Definition

An $n \times n$ square matrix $\mathbf{A} = (a_{ij})$ is called **symmetric** if, and only if, for every i and $j = 1, 2, \ldots, n$,

$$a_{ij} = a_{ji}.$$

Matrices and Connected Components

Consider a graph G, as shown below, that consists of several connected components.



The adjacency matrix of G is

$$\mathbf{A} = \left[\begin{array}{ccc:cc:cc} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

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$$\mathbf{A} = \left[\begin{array}{ccc|cc|cc} 1 & 0 & 1 & & & & \\ 0 & 0 & 2 & & & & \\ 1 & 2 & 0 & & & & \\ \hline & & & 0 & 1 & & \\ & & & 1 & 1 & & \\ \hline & & & & & 0 & 2 \\ & & & & & 2 & 0 \end{array}\right]$$

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10.3 Isomorphisms of Graphs

Thinking is a momentary dismissal of irrelevancies. —R. Buckminster Fuller, 1969

The two drawings shown in Figure 10.3.1 both represent the same graph: Their vertex and edge sets are identical, and their edge-endpoint functions are the same. Call this graph G.

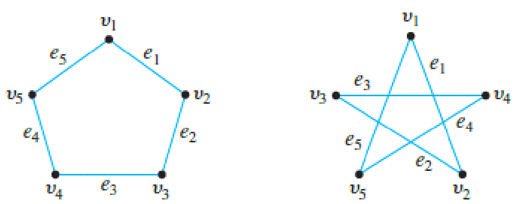

FIGURE 10.3.1

Now consider the graph G' represented in Figure 10.3.2.

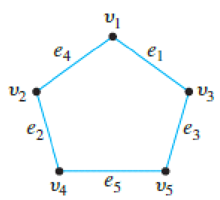

FIGURE 10.3.2

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Observe that G' is a different graph from G (for instance, in G the endpoints of e_1 are v_1 and v_2, whereas in G' the endpoints of e_1 are v_1 and v_3). Yet G' is certainly very similar to G. In fact, if the vertices and edges of G' are relabeled by the functions shown in Figure 10.3.3, then G' becomes the same as G.

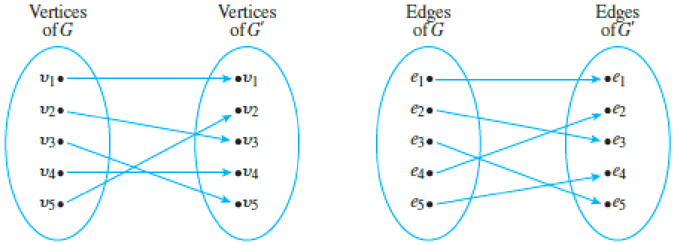


FIGURE 10.3.3

Note that these relabeling functions are one-to-one and onto.

Two graphs that are the same except for the labeling of their vertices and edges are called *isomorphic*. The word *isomorphism* comes from the Greek, meaning "same form." Isomorphic graphs are those that have essentially the same form.

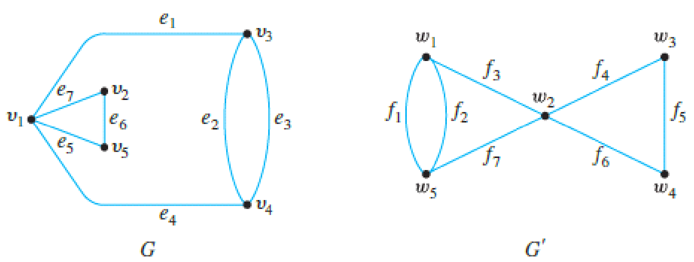
Definition

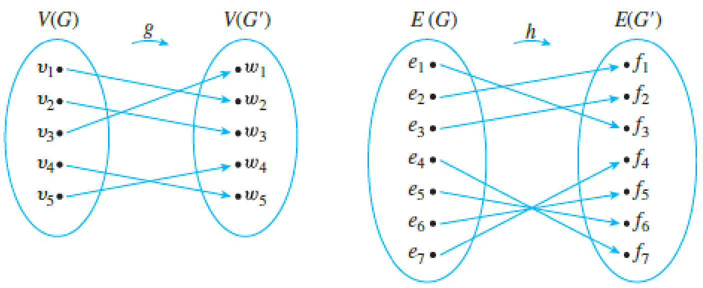
Let G and G' be graphs with vertex sets $V(G)$ and $V(G')$ and edge sets $E(G)$ and $E(G')$, respectively. **G is isomorphic to G'** if, and only if, there exist one-to-one correspondences $g: V(G) \to V(G')$ and $h: E(G) \to E(G')$ that preserve the edge-endpoint functions of G and G' in the sense that for each $v \in V(G)$ and $e \in E(G)$,

$$v \text{ is an endpoint of } e \quad \Leftrightarrow \quad g(v) \text{ is an endpoint of } h(e). \qquad 10.3.1$$

Example 10.3.1 Showing That Two Graphs Are Isomorphic

Show that the following two graphs are isomorphic.





It is not hard to show that graph isomorphism is an equivalence relation on a set of graphs; in other words, it is reflexive, symmetric, and transitive.

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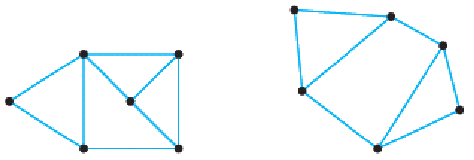
Example 10.3.3 Showing That Two Graphs Are Not Isomorphic

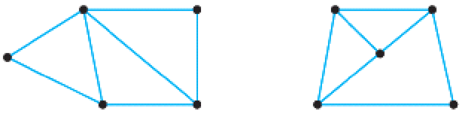
Show that the following pairs of graphs are not isomorphic by finding an isomorphic invariant that they do not share.

a.



G $\qquad$ G'

b.



H $\qquad$ H'

Solution

a. G has nine edges; G' has only eight.

b. H has a vertex of degree 4; H' does not. ■

Definition

If G and G' are simple graphs, then **G is isomorphic to G'** if, and only if, there exists a one-to-one correspondence g from the vertex set $V(G)$ of G to the vertex set $V(G')$ of G' that preserves the edge-endpoint functions of G and G' in the sense that for all vertices u and v of G,

$$\{u, v\} \text{ is an edge in } G \iff \{g(u), g(v)\} \text{ is an edge in } G'. \qquad 10.3.2$$

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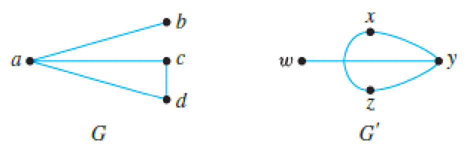
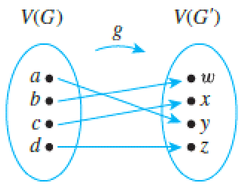
Example 10.3.5 **Isomorphism of Simple Graphs**

Are the two graphs shown below isomorphic? If so, define an isomorphism.



Solution Yes. Define $g: V(G) \to V(G')$ by the arrow diagram shown below.



Then g is one-to-one and onto by inspection. The fact that g preserves the edge-endpoint functions of G and G' is shown by the following table:

Edges of G	Edges of G'
$\{a, b\}$	$\{y, w\} = \{g(a), g(b)\}$
$\{a, c\}$	$\{y, x\} = \{g(a), g(c)\}$
$\{a, d\}$	$\{y, z\} = \{g(a), g(d)\}$
$\{c, d\}$	$\{x, z\} = \{g(c), g(d)\}$