Sequence

Wednesday, 4 September 2024 9:29 am

5.1 Sequences

A mathematician, like a painter or poet, is a maker of patterns.
—G. H. Hardy, *A Mathematician's Apology*, 1940

Imagine that a person decides to count his ancestors. He has two parents, four grandparents, eight great-grandparents, and so forth. These numbers can be written in a row as

Note The symbol "..." is called an *ellipsis*. It is shorthand for "and so forth."

$$2, 4, 8, 16, 32, 64, 128, \ldots$$

To express the pattern of the numbers, suppose that each is labeled by an integer giving its position in the row.

Position in the row	1	2	3	4	5	6	7...
Number of ancestors	2	4	8	16	32	64	128...

Note Strictly speaking, the true value of A_k is less than 2^k when k is large, because ancestors from one branch of the family tree may also appear on other branches of the tree.

The number corresponding to position 1 is 2, which equals 2^1. The number corresponding to position 2 is 4, which equals 2^2. For positions 3, 4, 5, 6, and 7, the corresponding numbers are 8, 16, 32, 64, and 128, which equal 2^3, 2^4, 2^5, 2^6, and 2^7, respectively. For a general value of k, let A_k be the number of ancestors in the kth generation back. The pattern of computed values strongly suggests the following for each k:

$$A_k = 2^k.$$

Definition

A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

We typically represent a sequence as a set of elements written in a row. In the sequence denoted

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n,$$

each individual element a_k (read "a sub k") is called a **term**. The k in a_k is called a **subscript** or **index**, m (which may be any integer) is the subscript of the **initial term**, and n (which must be an integer that is greater than or equal to m) is the subscript of the **final term**. The notation

$$a_m, a_{m+1}, a_{m+2}, \ldots$$

denotes an **infinite sequence**. An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

The following example shows that it is possible for two different formulas to give sequences with the same terms.

Example 5.1.1 **Finding Terms of Sequences Given by Explicit Formulas**

Define sequences $a_1, a_2, a_3, \ldots$ and $b_2, b_3, b_4, \ldots$ by the following explicit formulas:

$$a_k = \frac{k}{k+1} \quad \text{for every integer } k \geq 1,$$

$$b_i = \frac{i-1}{i} \quad \text{for every integer } i \geq 2.$$

Compute the first five terms of both sequences.

Solution

$$a_1 = \frac{1}{1+1} = \frac{1}{2} \qquad b_2 = \frac{2-1}{2} = \frac{1}{2}$$

$$a_2 = \frac{2}{2+1} = \frac{2}{3} \qquad b_3 = \frac{3-1}{3} = \frac{2}{3}$$

$$a_3 = \frac{3}{3+1} = \frac{3}{4} \qquad b_4 = \frac{4-1}{4} = \frac{3}{4}$$

$$a_4 = \frac{4}{4+1} = \frac{4}{5} \qquad b_5 = \frac{5-1}{5} = \frac{4}{5}$$

$$a_5 = \frac{5}{5+1} = \frac{5}{6} \qquad b_6 = \frac{6-1}{6} = \frac{5}{6}$$

As you can see, the first terms of both sequences are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$; in fact, it can be shown that all terms of both sequences are identical. ■

Example 5.1.2 **An Alternating Sequence**

Compute the first six terms of the sequence $c_0, c_1, c_2, \ldots$ defined as follows:

$$c_j = (-1)^j \quad \text{for every integer } j \geq 0.$$

Example 5.1.3 **Finding an Explicit Formula to Fit Given Initial Terms**

Find an explicit formula for a sequence with the following initial terms:

$$1, \quad -\frac{1}{4}, \quad \frac{1}{9}, \quad -\frac{1}{16}, \quad \frac{1}{25}, \quad -\frac{1}{36}, \ldots.$$

Solution Denote the general term of the sequence by a_k and suppose the first term is a_1. Next observe that the denominator of each term is a perfect square. Thus the terms can be rewritten as

$$\begin{array}{cccccc} \frac{1}{1^2}, & \frac{(-1)}{2^2}, & \frac{1}{3^2}, & \frac{(-1)}{4^2}, & \frac{1}{5^2}, & \frac{(-1)}{6^2}. \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array}$$

Now note that the denominator of each term equals the square of the subscript of that term, and that the numerator equals ± 1. Hence

$$a_k = \frac{\pm 1}{k^2}.$$

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Note that making the first term a_0 would have led to the alternative formula

$$a_k = \frac{(-1)^k}{(k+1)^2} \quad \text{for every integer } k \geq 0.$$

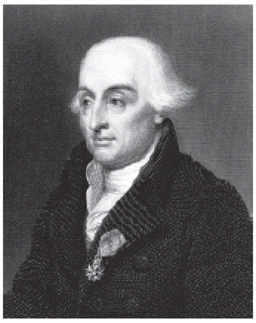
You should check that this formula also gives the correct first six terms. ■

later on. See exercise 5 at the end of this section.

Summation Notation

Consider again the example in which $A_k = 2^k$ represents the number of ancestors a person has in the kth generation back. What is the total number of ancestors for the past six generations? The answer is

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 126.$$



Historical/Getty Images

Joseph Louis Lagrange (1736–1813)

It is convenient to use a shorthand notation to write such sums. In 1772 the French mathematician Joseph Louis Lagrange introduced the capital Greek letter sigma, Σ, to denote the word *sum* (or *summation*), and defined the summation notation as follows:

Definition

If m and n are integers and $m \leq n$, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from k equals m to n of a-sub-k**, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

Example 5.1.4 Computing Summations

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$. Compute the following:

a. $\sum_{k=1}^{5} a_k$ b. $\sum_{k=2}^{2} a_k$ c. $\sum_{k=1}^{2} a_{2k}$

Solution

a. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = (-2) + (-1) + 0 + 1 + 2 = 0$

b. $\sum_{k=2}^{2} a_k = a_2 = -1$

c. $\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2} = a_2 + a_4 = -1 + 1 = 0$ ■

Oftentimes, the terms of a summation are expressed using an explicit formula. For instance, it is common to see summations such as

$$\sum_{k=1}^{5} k^2 \quad \text{or} \quad \sum_{i=0}^{8} \frac{(-1)^i}{i+1}.$$

Example 5.1.5 When the Terms of a Summation Are Given by a Formula

Compute $\sum_{k=1}^{5} k^2$.

Solution

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

When the upper limit of a summation is a variable, an ellipsis is used to write the summation in expanded form.

Example 5.1.6 Changing from Summation Notation to Expanded Form

Write $\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$ in expanded form:

Solution

$$\begin{aligned}\sum_{i=0}^{n} \frac{(-1)^i}{i+1} &= \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1} \\ &= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1} \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}\end{aligned}$$

Example 5.1.7 Changing from Expanded Form to Summation Notation

Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}.$$

Solution The general term of this summation can be expressed as $\frac{i+1}{n+i}$ for each integer i from 0 to n. Hence

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{i=0}^{n} \frac{i+1}{n+i}.$$

Example 5.1.9 Using a Single Summation Sign and Separating Off a Final Term

a. Write $\sum_{k=0}^{n} 2^k + 2^{n+1}$ as a single summation.

b. Rewrite $\sum_{i=1}^{n+1} \frac{1}{i^2}$ by separating off the final term.

Solution

a. $\sum_{k=0}^{n} 2^k + 2^{k+1} = (2^0 + 2^1 + 2^2 + \cdots + 2^n) + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$

b. $\sum_{i=1}^{n+1} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$

Example 5.1.10 **A Telescoping Sum**

Some sums can be transformed so that successive cancellation of terms collapses the final result like a telescope. For instance, observe that for every integer $k \geq 1$,

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}.$$

Use this identity to find a simple expression for $\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Solution

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}\end{aligned}$$

■

Product Notation

The notation for the product of a sequence of numbers is analogous to the notation for their sum. The Greek capital letter pi, Π, denotes a product. For example,

$$\prod_{k=1}^{5} a_k = a_1 a_2 a_3 a_4 a_5.$$

Definition

If m and n are integers and $m \leq n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

A recursive definition for the product notation is the following: If m is any integer, then

$$\prod_{k=m}^{m} a_k = a_m \quad \text{and} \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for every integer } n > m.$$

Example 5.1.11 **Computing Products**

Compute the following products:

a. $\prod_{k=1}^{5} k$

b. $\prod_{k=1}^{1} \frac{k}{k+1}$

Solution

a. $\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

b. $\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$

■

Properties of Summations and Products

The following theorem states general properties of summations and products. The proof of the theorem is discussed in Section 5.6.

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$
2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law
3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$.

Example 5.1.12 **Using Properties of Summation and Product**

Let $a_k = k + 1$ and $b_k = k - 1$ for every integer k. Write each of the following expressions as a single summation or product:

a. $\displaystyle\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$

b. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right)$

Solution

a.
$$\begin{aligned}
\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k &= \sum_{k=m}^{n} (k+1) + 2 \cdot \sum_{k=m}^{n} (k-1) && \text{by substitution} \\
&= \sum_{k=m}^{n} (k+1) + \sum_{k=m}^{n} 2 \cdot (k-1) && \text{by Theorem 5.1.1 (2)} \\
&= \sum_{k=m}^{n} ((k+1) + 2 \cdot (k-1)) && \text{by Theorem 5.1.1 (1)} \\
&= \sum_{k=m}^{n} (3k-1) && \text{by algebraic simplification}
\end{aligned}$$

b.
$$\begin{aligned}
\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) &= \left(\prod_{k=m}^{n} (k+1)\right) \cdot \left(\prod_{k=m}^{n} (k-1)\right) && \text{by substitution} \\
&= \prod_{k=m}^{n} ((k+1) \cdot (k-1)) && \text{by Theorem 5.1.1 (3)} \\
&= \prod_{k=m}^{n} (k^2 - 1) && \text{by algebraic simplification}
\end{aligned}$$
■

Change of Variable

Observe that

$$\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$$

and also that $$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2.$$

Hence $$\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2.$$

The symbol used to represent an index of a summation is an example of a local variable, often called a **dummy variable**, because, as illustrated above, it can be replaced by any other symbol as long as the replacement is made in each location where it occurs. Outside of that context (both before and after), the symbol may have another meaning entirely. In the same way, a symbol used to represent a variable in a universally or existentially quantified state can be replaced by any other symbol as long as the replacements are made consistently.

The appearance of a summation can be altered by more complicated changes of variable as well. For example, observe that

$$\begin{aligned}\sum_{j=2}^{4} (j-1)^2 &= (2-1)^2 + (3-1)^2 + (4-1)^2 \\ &= 1^2 + 2^2 + 3^2 \\ &= \sum_{k=1}^{3} k^2.\end{aligned}$$

Factorial and "n Choose r" Notation

The product of all consecutive integers up to a given integer occurs so often in mathematics that it is given a special notation—*factorial* notation.

Definition

For each positive integer n, the quantity ***n* factorial** denoted $n!$, is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted $0!$, is defined to be 1:

$$0! = 1.$$

The definition of zero factorial as 1 may seem odd, but, as you will see when you read Chapter 9, it is convenient for many mathematical formulas.

Example 5.1.15 **The First Ten Factorials**

$$\begin{aligned} 0! &= 1 & 1! &= 1 \\ 2! &= 2 \cdot 1 = 2 & 3! &= 3 \cdot 2 \cdot 1 = 6 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 & 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\ 6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 & 7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040 \\ 8! &= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 & 9! &= 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 40{,}320 & &= 362{,}880 \end{aligned}$$

■

Example 5.1.16 **Computing with Factorials**

Simplify the following expressions:

a. $\dfrac{8!}{7!}$ b. $\dfrac{5!}{2!\cdot 3!}$ c. $\dfrac{1}{2!\cdot 4!}+\dfrac{1}{3!\cdot 3!}$ d. $\dfrac{(n+1)!}{n!}$ e. $\dfrac{n!}{(n-3)!}$

Solution

a. $\dfrac{8!}{7!}=\dfrac{8\cdot \cancel{7!}}{\cancel{7!}}=8$

b. $\dfrac{5!}{2!\cdot 3!}=\dfrac{5\cdot 4\cdot \cancel{3!}}{2!\cdot \cancel{3!}}=\dfrac{5\cdot 4}{2\cdot 1}=10$

Formula for Computing $\binom{n}{r}$

For all integers n and r with $0 \le r \le n$,

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}.$$

In this chapter, we show how to compute its values. Because n choose r is always an integer, you can be sure that all the factors in the denominator of the formula will be canceled out by factors in the numerator. Many electronic calculators have keys for computing values of $\binom{n}{r}$. These are denoted in various ways such as nCr, $C(n, r)$, ${}^nC_r$, and $C_{n,r}$. The letter C is used because the quantities $\binom{n}{r}$ are also called *combinations*. Sometimes they are referred to as *binomial coefficients* because of the connection with the binomial theorem discussed in Section 9.7.

Example 5.1.17 **Computing $\binom{n}{r}$**

Use the formula for computing $\binom{n}{r}$ to evaluate the following expressions:

a. $\binom{8}{5}$ b. $\binom{4}{4}$ c. $\binom{n+1}{n}$

Solution

a. $$\binom{8}{5}=\frac{8!}{5!(8-5)!}$$

$$=\frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3\cdot 2\cdot 1}}{(\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1)\cdot(\cancel{3\cdot 2\cdot 1})}$$

always cancel common factors before multiplying

$$= 56.$$

Sequences in Computer Programming

An important data type in computer programming consists of finite sequences. In computer programming contexts, these are usually referred to as *one-dimensional arrays*. For example, consider a program that analyzes the wages paid to a sample of 50 workers. Such a program might compute the average wage and the difference between each individual wage and the average. This would require that each wage be stored in memory for retrieval later in the calculation. To avoid the use of entirely separate variable names for all of the 50 wages, each is written as a term of a one-dimensional array:

$$W[1], W[2], W[3], \ldots, W[50].$$

Note that the subscript labels are written inside square brackets. The reason is that until relatively recently, it was impossible to type actual dropped subscripts on most computer keyboards.

Example 5.1.18 **Dummy Variable in a Loop**

The index variable for a **for-next** loop is a local, or dummy, variable. For example, the following three algorithm segments all produce the same output:

1. **for** $i := 1$ **to** n
   print $a[i]$
   next i

2. **for** $j := 0$ **to** $n-1$
   print $a[j+1]$
   next j

3. **for** $k := 2$ **to** $n+1$
   print $a[k-1]$
   next k

■

the sum to equal $a[1]$; the one on the right initializes the sum to equal 0. In both cases the output is $\sum_{k=1}^{n} a[k]$.

$s := a[1]$	$s := 0$
for $k := 2$ **to** n	**for** $k := 1$ **to** n
$s := s + a[k]$	$s := s + a[k]$
next k	**next** k

TEST YOURSELF

Answers to Test Yourself questions are located at the end of each section.

1. The notation $\sum_{k=m}^{n} a_k$ is read "_____."
2. The expanded form of $\sum_{k=m}^{n} a_k$ is _____.
3. The value of $a_1 + a_2 + a_3 + \cdots + a_n$ when $n = 2$ is "_____."
4. The notation $\prod_{k=m}^{n} a_k$ is read "_____."
5. If n is a positive integer, then $n! =$ _____.
6. $\sum_{k=m}^{n} a_k + c\sum_{k=m}^{n} b_k =$ _____.
7. $\left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right) =$ _____.

EXERCISE SET 5.1*

Write the first four terms of the sequences defined by the formulas in 1–6.

1. $a_k = \dfrac{k}{10+k}$, for every integer $k \geq 1$.
2. $b_j = \dfrac{5-j}{5+j}$, for every integer $j \geq 1$.
3. $c_i = \dfrac{(-1)^i}{3^i}$, for every integer $i \geq 0$.

Find explicit formulas for sequences of the form a_1, a_2, $a_3, \ldots$ with the initial terms given in 10–16.

10. $-1, 1, -1, 1, -1, 1$

11. $0, 1, -2, 3, -4, 5$

12. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

13. $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \frac{1}{6} - \frac{1}{7}$

18. Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$, $a_4 = 0$, $a_5 = -1$, and $a_6 = -2$. Compute each of the summations and products below.

a. $\sum_{i=0}^{6} a_i$ **b.** $\sum_{i=0}^{0} a_i$ **c.** $\sum_{j=1}^{3} a_{2j}$ **d.** $\prod_{k=0}^{6} a_k$ **e.** $\prod_{k=2}^{2} a_k$

Write each of 43–52 using summation or product notation.

43. $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2$

44. $(1^3 - 1) - (2^3 - 1) + (3^3 - 1) - (4^3 - 1) + (5^3 - 1)$

45. $(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$