Function & Relation

Saturday, 10 August 2024 12:24 pm

Definition

A **function f from a set X to a set Y**, denoted $f: X \to Y$, is a relation from X, the **domain** of f, to Y, the **co-domain** of f, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

Arrow Diagrams

Recall from Section 1.3 that if X and Y are finite sets, you can define a function f from X to Y by drawing an arrow diagram. You make a list of elements in X and a list of elements in Y, and draw an arrow from each element in X to the corresponding element in Y, as shown in Figure 7.1.1.

This arrow diagram does define a function because:

1. Every element of X has an arrow that points to an element in Y.
2. No element of X has two arrows that point to two different elements of Y.

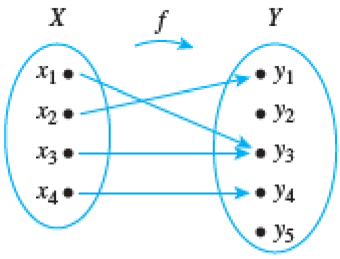

FIGURE 7.1.1

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Example 7.1.1 **Functions and Nonfunctions**

Which of the arrow diagrams in Figure 7.1.2 define functions from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?

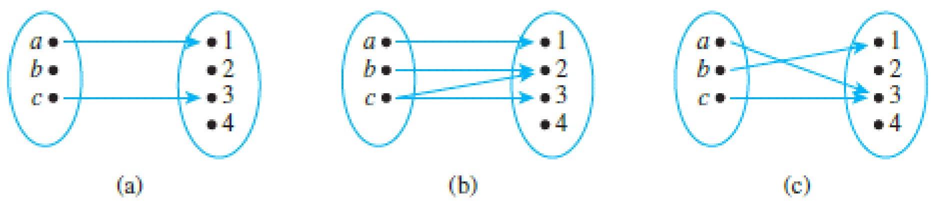

FIGURE 7.1.2

A Function Defined by an Arrow Diagram

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.

a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

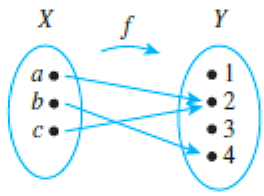

FIGURE 7.1.3

Solution

a. domain of $f = \{a, b, c\}$, co-domain of $f = \{1, 2, 3, 4\}$

b. $f(a) = 2$, $f(b) = 4$, $f(c) = 2$

c. range of $f = \{2, 4\}$

d. yes, no

e. inverse image of $2 = \{a, c\}$
inverse image of $4 = \{b\}$
inverse image of $1 = \varnothing$ (*since no arrows point to* 1)

f. $\{(a, 2), (b, 4), (c, 2)\}$ ■

Definition

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

and

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

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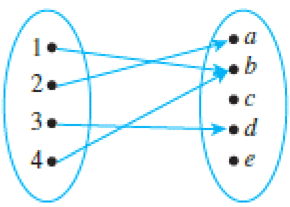
The Action of a Function on Subsets of a Set

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$, and define $F: X \to Y$ by the following arrow diagram:



Let $A = \{1, 4\}$, $C = \{a, b\}$, and $D = \{c, e\}$. Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.

Solution

$$F(A) = \{b\} \quad F(X) = \{a, b, d\} \quad F^{-1}(C) = \{1, 2, 4\} \quad F^{-1}(D) = \varnothing$$

■

One-to-One Functions

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Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically:

$$F\colon X \to Y \text{ is one-to-one} \Leftrightarrow \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

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A function $F\colon X \to Y$ is *not* one-to-one $\Leftrightarrow$ $\exists$ elements x_1 and x_2 in X with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

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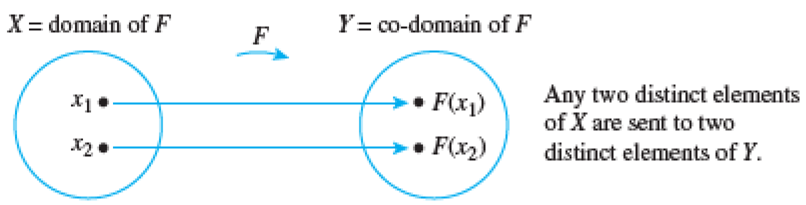
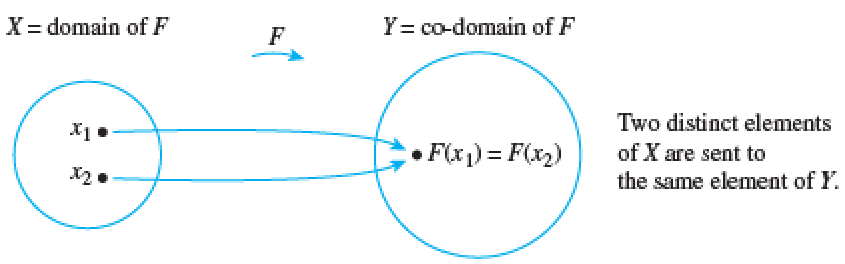


FIGURE 7.2.1(a) A One-to-One Function Separates Points



IGURE 7.2.1(b) A Function That Is Not One-to-One Collapses Points Together

Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F\colon X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X,\ F(x) \neq y.$$

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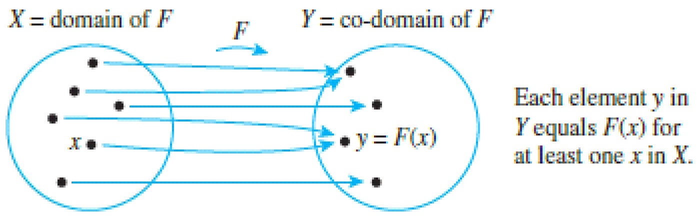

FIGURE 7.2.3(a) A Function That Is Onto

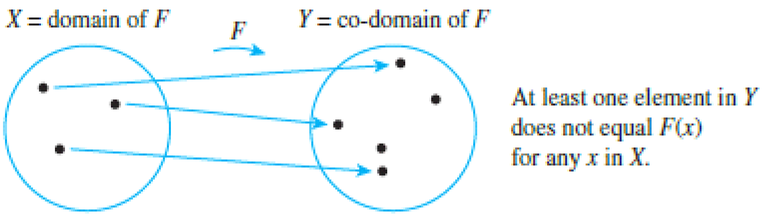

FIGURE 7.2.3(b) A Function That Is Not Onto

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One-to-One Correspondences

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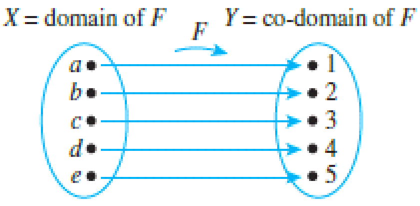

FIGURE 7.2.5 An Arrow Diagram for a One-to-One Correspondence

Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.

Theorem 7.2.2

Suppose $F: X \to Y$ is a one-to-one correspondence; in other words, suppose F is one-to-one and onto. Then there is a function $F^{-1}: Y \to X$ that is defined as follows: Given any element y in Y,

$$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

Or, equivalently,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

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