Induction for Inequality

Sunday, 14 December 2025 9:50 pm

PROVING AN INEQUALITY:

Use mathematical induction to prove that for all integers $n \geq 3$.

$$2n + 1 < 2^n$$

SOLUTION:

1. Basis Step:

For $n = 3$

L.H.S $= 2(3) + 1 = 6 + 1 = 7$

R.H.S $= 2^3 = 8$

Since $7 < 8$, so the statement is true for $n = 3$.

2. Inductive Step:

Suppose the statement is true for $n = k$, i.e.,

$2k + 1 < 2^k$………………….(1) $k \geq 3$

We need to show that the statement is true for $n = k+1$,

i.e.;

$2(k+1) + 1 < 2^{k+1}$…………..(2)

Consider L.H.S of (2)

$= 2(k+1) + 1$

$= 2k + 2 + 1$

$= (2k + 1) + 2$

$< 2^k + 2$ using (1)

$< 2^k + 2^k$ (since $2 < 2^k$ for $k \geq 3$)

$< 2 \cdot 2^k = 2^{k+1}$

Thus $2(k+1)+1 < 2k+1$ (proved)

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EXERCISE:

Show by mathematical induction

$$1 + nx \leq (1+x)^n$$

for all real numbers $x > -1$ and integers $n \geq 2$

SOLUTION:

1. Basis Step:

For $n = 2$

L.H.S $= 1 + (2)x = 1 + 2x$

RHS $= (1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$ $(x^2 > 0)$

$\Rightarrow$ statement is true for $n = 2$.

2. Inductive Step:

Suppose the statement is true for $n = k$.

That is, for $k \geq 2$, $1 + k\,x \leq (1 + x)^k$....................(1)

We want to show that the statement is also true for $n = k + 1$ i.e.,

$$1 + (k + 1)x \leq (1 + x)^{k+1}$$

Since $x > -1$, therefore $1 + x > 0$.

Multiplying both sides of (1) by $(1+x)$ we get

$$\begin{aligned}(1+x)(1+x)^k &\geq (1 + x)(1 + kx)\\ &= 1 + kx + x + kx^2\\ &= 1 + (k + 1)x + kx^2\end{aligned}$$

but

so $\begin{bmatrix} x > -1, & \text{so } x^2 \geq 0 \\ \& k \geq 2, & \text{so } kx^2 \geq 0 \end{bmatrix}$

$$(1+x)(1+x)^k \geq 1 + (k + 1)x$$

Thus $1 + (k+1)x \leq (1+x)^{k+1}$. Hence by mathematical induction, the inequality is true.

<u>EXERCISE:</u>

Prove by mathematical induction that

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$$

Whenever n is a positive integer greater than 1.

<u>SOLUTION:</u>

1. Basis Step: for $n = 2$

L.H.S $=1+\frac{1}{4}=\frac{5}{4}=1.25$

R.H.S $=2-\frac{1}{2}=\frac{3}{2}=1.5$

Clearly LHS < RHS
Hence the statement is true for n = 2.

2.Inductive Step:

Suppose that the statement is true for some integers k > 1, i.e.;

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}<2-\frac{1}{k} \qquad (1)$$

We need to show that the statement is true for n = k + 1. That is

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}<2-\frac{1}{k+1} \qquad (2)$$

Consider the LHS of (2)

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}=1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

$$<\left(2-\frac{1}{k}\right)+\frac{1}{(k+1)^2}$$

$$=2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)$$

We need to prove that

$$2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)\leq 2-\frac{1}{k+1}$$

or $$-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)\leq -\frac{1}{k+1}$$

or $$\frac{1}{k}-\frac{1}{(k+1)^2}\geq \frac{1}{k+1}$$

or $$\frac{1}{k}-\frac{1}{k+1}\geq \frac{1}{(k+1)^2}$$

Now $$\frac{1}{k}-\frac{1}{k+1}=\frac{k+1-k}{k(k+1)}$$

$$=\frac{1}{k(k+1)}>\frac{1}{(k+1)^2}$$