Relation

Saturday, 24 August 2024 5:54 pm

CHAPTER 8 PROPERTIES OF RELATIONS

Example 8.1.1 **The Less-than Relation for Real Numbers**

Define a relation L from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all real numbers x and y,

$$x \, L \, y \Leftrightarrow x < y.$$

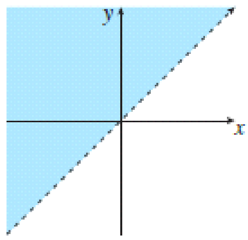
a. Is $57 \, L \, 53$? b. Is $(-17) \, L \, (-14)$? c. Is $143 \, L \, 143$? d. Is $(-35) \, L \, 1$?

e. Draw the graph of L as a subset of the Cartesian plane $\mathbf{R} \times \mathbf{R}$.

Solution

a. No, $57 > 53$. b. Yes, $-17 < -14$. c. No, $143 = 143$. d. Yes, $-35 < 1$.

e. For each value of x, all the points (x, y) with $y > x$ are on the graph. So the graph consists of all the points above the line $x = y$.



Example 8.1.2 **The Congruence Modulo 2 Relation**

Define a relation E from $\mathbf{Z}$ to $\mathbf{Z}$ as follows: For every $(m, n) \in \mathbf{Z} \times \mathbf{Z}$,

$$m\,E\,n \quad \Leftrightarrow \quad m - n \text{ is even.}$$

a. Is $4\,E\,0$? Is $2\,E\,6$? Is $3\,E\,(-3)$? Is $5\,E\,2$?

b. List five integers that are related by E to 1.

c. Prove that if n is any odd integer, then $n\,E\,1$.

Solution

a. Yes, $4\,E\,0$ because $4 - 0 = 4$ and 4 is even.
Yes, $2\,E\,6$ because $2 - 6 = -4$ and -4 is even.
Yes, $3\,E\,(-3)$ because $3 - (-3) = 6$ and 6 is even.
No, $5 \not{E}\, 2$ because $5 - 2 = 3$ and 3 is not even.

b. There are many such lists. One is

$$\begin{array}{rl} 1 & \text{because } 1 - 1 = 0 \text{ is even.} \\ 3 & \text{because } 3 - 1 = 2 \text{ is even.} \\ 5 & \text{because } 5 - 1 = 4 \text{ is even.} \\ -1 & \text{because } -1 - 1 = -2 \text{ is even.} \\ -3 & \text{because } -3 - 1 = -4 \text{ is even.} \end{array}$$

c. **Proof:** Suppose n is any odd integer. Then $n = 2k + 1$ for some integer k. Now by definition of E, $n\,E\,1$ if, and only if, $n - 1$ is even. But by substitution,

$$n - 1 = (2k + 1) - 1 = 2k,$$

and since k is an integer, $2k$ is even. Hence $n\,E\,1$ *[as was to be shown]*.

It can be shown (see exercise 2 at the end of this section) that integers m and n are related by E if, and only if, $m \bmod 2 = n \bmod 2$ (that is, both are even or both are odd). When this occurs m and n are said to be **congruent modulo 2**. ■

Example 8.1.3 **A Relation on a Power Set**

Let $X = \{a, b, c\}$. Then $\mathscr{P}(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a relation $\mathbf{S}$ from $\mathscr{P}(X)$ to $\mathscr{P}(X)$ as follows: For all sets A and B in $\mathscr{P}(X)$ (that is, for all subsets A and B of X),

$$A\,\mathbf{S}\,B \Leftrightarrow A \text{ has at least as many elements as } B.$$

a. Is $\{a, b\}\,\mathbf{S}\,\{b, c\}$? b. Is $\{a\}\,\mathbf{S}\,\varnothing$? c. Is $\{b, c\}\,\mathbf{S}\,\{a, b, c\}$? d. Is $\{c\}\,\mathbf{S}\,\{a\}$?

Solution

a. Yes, both sets have two elements.

b. Yes, $\{a\}$ has one element and $\varnothing$ has zero elements, and $1 \geq 0$.

c. No, $\{b, c\}$ has two elements and $\{a, b, c\}$ has three elements and $2 < 3$.

d. Yes, both sets have one element. ■

The Inverse of a Relation

If R is a relation from A to B, then a relation R^{-1} from B to A can be defined by interchanging the elements of all the ordered pairs of R.

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Definition

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

This definition can be written operationally as follows:

$$\boxed{\text{For all } x \in A \text{ and } y \in B, \quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R.}$$

Example 8.1.4 **The Inverse of a Finite Relation**

Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$, and let R be the "divides" relation from A to B: For every ordered pair $(x, y) \in A \times B$,

$$x\, R\, y \quad \Leftrightarrow \quad x \mid y \qquad x \text{ divides } y.$$

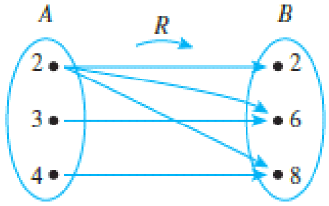
a. State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}.

b. Describe R^{-1} in words.

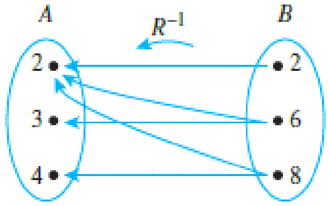
Solution

a. $R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$

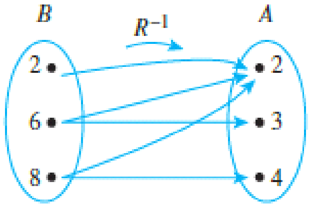
$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$



To draw the arrow diagram for R^{-1}, you can copy the arrow diagram for R but reverse the directions of the arrows.



Or you can redraw the diagram so that B is on the left.



b. R^{-1} can be described in words as follows: For every ordered pair $(y, x) \in B \times A$,

$$y\, R^{-1} x \quad \Leftrightarrow \quad y \text{ is a multiple of } x.$$

■

Example 8.1.5 **The Inverse of an Infinite Relation**

Define a relation R from $\mathbf{R}$ to $\mathbf{R}$ as follows: For every ordered pair $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$x\, R\, y \quad \Leftrightarrow \quad y = 2|x|.$$

Draw the graphs of R and R^{-1} in the Cartesian plane. Is R^{-1} a function?

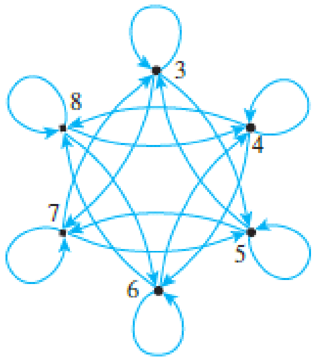
Solution A point (v, u) is on the graph of R^{-1} if, and only if, (u, v) is on the graph of R. Note that if $x \geq 0$, then the graph of $y = 2|x| = 2x$ is a straight line with slope 2. And if $x < 0$, then the graph of $y = 2|x| = 2(-x) = -2x$ is a straight line with slope -2. Some sample values are tabulated and the graphs are shown below.

Directed Graph of a Relation

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For every $x, y \in A$,

$$x\, R\, y \quad \Leftrightarrow \quad 2 \mid (x - y).$$

Draw the directed graph of R.



TEST YOURSELF

Answers to Test Yourself questions are located at the end of each section.

1. If R is a relation from A to B, $x \in A$, and $y \in B$, the notation $x\,R\,y$ means that ______.
2. If R is a relation from A to B, $x \in A$, and $y \in B$, the notation $x \not R\, y$ means that ______.
3. If R is a relation from A to B, $x \in A$, and $y \in B$, then $(y, x) \in R^{-1}$ if, and only if, ______.
4. A relation on a set A is a relation from ______ to ______.
5. If R is a relation on a set A, the directed graph of R has an arrow from x to y if, and only if, ______.

ANSWERS FOR TEST YOURSELF

1. x is related to y by R 2. x is not related to y by R 3. $(x, y) \in R$ 4. A; A 5. x is related to y by R

Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let R be the "less than" relation. That is, for every ordered pair $(x, y) \in A \times B$,

$$x\,R\,y \quad \Leftrightarrow \quad x < y.$$

State explicitly which ordered pairs are in R and R^{-1}.

Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let S be the "divides" relation. That is, for every ordered pair $(x, y) \in A \times B$,

$$x\,S\,y \quad \Leftrightarrow \quad x \mid y.$$

State explicitly which ordered pairs are in S and S^{-1}.

Draw the directed graphs of the relations defined in 13–18.

13. Define a relation R on $A = \{0, 1, 2, 3\}$ by $R = \{(0, 0), (1, 2), (2, 2)\}$.

14. Define a relation S on $B = \{a, b, c, d\}$ by $S = \{(a, b), (a, c), (b, c), (d, d)\}$.

15. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For every $x, y \in A$,

$$x\,R\,y \quad \Leftrightarrow \quad x \mid y.$$

Definition

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for every $x \in A$, $x\, R\, x$.
2. R is **symmetric** if, and only if, for every $x, y \in A$, if $x\, R\, y$ then $y\, R\, x$.
3. R is **transitive** if, and only if, for every $x, y, z \in A$, if $x\, R\, y$ and $y\, R\, z$ then $x\, R\, z$.

1. R is reflexive	$\Leftrightarrow$	for every x in A, $(x, x) \in R$.
2. R is symmetric	$\Leftrightarrow$	for every x and y in A, *if* $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive	$\Leftrightarrow$	for every x, y, and z in A, *if* $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Reflexivity, Symmetry, and Transitivity

Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation R on A as follows: For every $x, y \in A$,

$$x\, R\, y \quad \Leftrightarrow \quad 3 \mid (x - y).$$