Lecture No.3 Laws of Logic

APPLYING LAWS OF LOGIC

Using law of logic, simplify the statement form

$$p \vee [\sim(\sim p \wedge q)]$$

Solution:

$p \vee [\sim(\sim p \wedge q)]$	$\equiv p \vee [\sim(\sim p) \vee (\sim q)]$	DeMorgan's Law
	$\equiv p \vee [p \vee (\sim q)]$	Double Negative Law: $\sim(\sim p) \equiv p$
	$\equiv [p \vee p] \vee (\sim q)$	Associative Law for $\vee$
	$\equiv p \vee (\sim q)$	Idempotent Law: $p \vee p \equiv p$

That is the simplified statement form.

Example: Using Laws of Logic, verify the logical equivalence

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

Solution:

$\sim(\sim p \wedge q) \wedge (p \vee q)$	$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	DeMorgan's Law
	$\equiv (p \vee \sim q) \wedge (p \vee q)$	Double Negative Law
	$\equiv p \vee (\sim q \wedge q)$	Distributive Law
	$\equiv p \vee c$	Negation Law
	$\equiv p$	Identity Law

SIMPLIFYING A STATEMENT:

"You will get an A if you are hardworking and the sun shines, or you are hardworking and it rains." Rephrase the condition more simply.

Solution:

Let p = "You are hardworking'
q = "The sun shines"
r = "It rains" .

The condition is $(p \wedge q) \vee (p \wedge r)$

Using distributive law in reverse,

$$(p \wedge q) \vee (p \wedge r) \equiv p \wedge (q \vee r)$$

Putting $p \wedge (q \vee r)$ back into English, we can rephrase the given sentence as

"You will get an A if you are hardworking and the sun shines or it rains.

EXERCISE:

Use Logical Equivalence to rewrite each of the following sentences more simply.

1. **It is not true that I am tired and you are smart.**
   {I am **not** tired **or** you are **not** smart.}
2. **It is not true that I am tired or you are smart.**
   {I am **not** tired **and** you are **not** smart.}
3. **I forgot my pen or my bag and I forgot my pen or my glasses.**
   {I forgot my pen **or** I forgot my bag **and** glasses.

4. **It is raining and I have forgotten my umbrella, or it is raining and I have forgotten my hat.**
{It is raining **and** I have forgotten my umbrella **or** my hat.}

CONDITIONAL STATEMENTS:

Introduction

Consider the statement:

"**If you earn an A in Math, then I'll buy you a computer**."

This statement is made up of two simpler statements:

p: "You earn an A in Math"
q: "I will buy you a computer."

The original statement is then saying :

if p is true, then q is true, or, more simply, **if** p, **then** q.

We can also phrase this as p **implies** q. It is denoted by $\mathbf{p \rightarrow q}$.

CONDITIONAL STATEMENTS OR IMPLICATIONS:

If p and q are statement variables, the conditional of q by p is "If p then q" or "p implies q" and is denoted $p \rightarrow q$.

$\mathbf{p \rightarrow q}$ is false when p is true and q is false; otherwise it is true.
The arrow "$\rightarrow$ " is the **conditional** operator.
In $p \rightarrow q$, the statement **p** is called **the hypothesis (or antecedent)** and q is called the **conclusion** (or **consequent**).

TRUTH TABLE:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

PRACTICE WITH CONDITIONAL STATEMENTS:

Determine the truth value of each of the following conditional statements:

1. "If 1 = 1, then 3 = 3." **TRUE**
2. "If 1 = 1, then 2 = 3." **FALSE**
3. "If 1 = 0, then 3 = 3." **TRUE**
4. "If 1 = 2, then 2 = 3." **TRUE**
5. "If 1 = 1,then 1 = 2 and 2 = 3." **FALSE**
6. "If 1 = 3 or 1 = 2 then 3 = 3." **TRUE**

ALTERNATIVE WAYS OF EXPRESSING IMPLICATIONS:

The implication **p → q** could be expressed in many alternative ways as:

•"if p then q"	•"not p unless q"
•"p implies q"	•"q follows from p"
•"if p, q"	•"q if p"
•"p only if q"	•"q whenever p"
•"p is sufficient for q"	•"q is necessary for p"

EXERCISE:

Write the following statements in the form "if p, then q" in English.

a) ***Your guarantee is good only if you bought your CD less than 90 days ago.***

If your guarantee is good, then you must have bought your CD player less than 90 days ago.

b) ***To get tenure as a professor, it is sufficient to be world-famous.***

If you are world-famous, then you will get tenure as a professor.

c) ***That you get the job implies that you have the best credentials.***

If you get the job, then you have the best credentials.

d) ***It is necessary to walk 8 miles to get to the top of the Peak.***

If you get to the top of the peak, then you must have walked 8 miles.

TRANSLATING ENGLISH SENTENCES TO SYMBOLS:

Let p and q be propositions:

p = "you get an A on the final exam"
q = "you do every exercise in this book"
r = "you get an A in this class"

Write the following propositions using p, q, and r and logical connectives.

1. To get an A in this class it is necessary for you to get an A on the final.

SOLUTION $\mathbf{p \to r}$

2. You do every exercise in this book; You get an A on the final, implies, you get an A in the class.

SOLUTION $\mathbf{p \wedge q \to r}$

3. Getting an A on the final and doing every exercise in this book is sufficient For getting an A in this class.

SOLUTION $\mathbf{p \wedge q \to r}$

TRANSLATING SYMBOLIC PROPOSITIONS TO ENGLISH:

Let **p**, **q**, and **r** be the propositions:

p = "you have the flu"
q = "you miss the final exam"
r = "you pass the course"

Express the following propositions as an English sentence.

1. $\mathbf{p \to q}$

If you have flu, then you will miss the final exam.

2. $\mathbf{\sim q \to r}$

If you don't miss the final exam, you will pass the course.

3. **$\sim p \wedge \sim q \rightarrow r$**

If you neither have flu nor miss the final exam, then you will pass the course.

HIERARCHY OF OPERATIONS
FOR LOGICAL CONNECTIVES

- $\sim$ (negation)
- $\wedge$ (conjunction), $\vee$ (disjunction)
- $\rightarrow$ (conditional)

Example: Construct a truth table for the statement form **$p \vee \sim q \rightarrow \sim p$**

p	q	~ q	~p	$p \vee \sim q$	**$p \vee \sim q \rightarrow \sim p$**
T	T	F	F	T	F
T	F	T	F	T	F
F	T	F	T	F	T
F	F	T	T	T	T

Example: Construct a truth table for the statement form **$(p \rightarrow q) \wedge (\sim p \rightarrow r)$**

p	q	r	$p \rightarrow q$	~p	$\sim p \rightarrow r$	$(p \rightarrow q) \wedge (\sim p \rightarrow r)$
T	T	T	T	F	T	T
T	T	F	T	F	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

LOGICAL EQUIVALENCE INVOLVING IMPLICATION

Use truth table to show $\mathbf{p \to q \equiv \sim q \to \sim p}$

p	q	~q	~p	p→q	~q → ~p
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

same truth values

Hence the given two expressions are equivalent.

IMPLICATION LAW

$$\mathbf{p \to q \equiv \sim p \lor q}$$

p	q	p→q	~p	~p∨q
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same truth values

NEGATION OF A CONDITIONAL STATEMENT:

Since $p \to q \equiv \sim p \lor q$

So $\sim(p \to q) \equiv \sim(\sim p \lor q)$

$\equiv \sim(\sim p) \land (\sim q)$ by De Morgan's law

$\equiv p \land \sim q$ by the Double Negative law

Thus the negation of "**if p then q**" is logically equivalent to "**p and not q**".
Accordingly, the negation of an **if-then** statement does not start with the word **if**.

EXAMPLES

Write negations of each of the following statements:

1. If Ali lives in Pakistan then he lives in Lahore.
2. If my car is in the repair shop, then I cannot get to class.
3. If x is prime then x is odd **or** x is 2.
4. If n is divisible by 6, then n is divisible by 2 **and** n is divisible by 3.

SOLUTIONS:

1. Ali lives in Pakistan and he does not live in Lahore.
2. My car is in the repair shop and I can get to class.
3. x is prime but x is not odd **and** x is not 2.
4. n is divisible by 6 but n is not divisible by 2 **or** by 3.

INVERSE OF A CONDITIONAL STATEMENT:

The inverse of the conditional statement **p → q** is **~p → ~q**

A conditional and its inverse are not equivalent as could be seen from the truth table.

p	q	p→q	~p	~q	~p →~q
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

different truth values in rows 2 and 3

WRITING INVERSE:

1. ***If today is Friday, then 2 + 3 = 5.***
   If today is not Friday, then $2 + 3 \neq 5$.

2. ***If it snows today, I will ski tomorrow.***
   If it does not snow today I will not ski tomorrow.

3. ***If P is a square, then P is a rectangle.***
   If P is not a square then P is not a rectangle.

4. ***If my car is in the repair shop, then I cannot get to class.***
   If my car is not in the repair shop, then I shall get to the class.

CONVERSE OF A CONDITIONAL STATEMENT:

The converse of the conditional statement **p → q** is **q →p.**

A conditional and its converse are not equivalent. i.e., → is not a commutative operator.

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

not the same

WRITING CONVERSE:

1. ***If today is Friday, then 2 + 3 = 5.***
   If 2 + 3 = 5, then today is Friday.

2. ***If it snows today, I will ski tomorrow.***
   I will ski tomorrow only if it snows today.

3. ***If P is a square, then P is a rectangle.***
   If P is a rectangle then P is a square.

4. ***If my car is in the repair shop, then I cannot get to class.***
   If I cannot get to the class, then my car is in the repair shop.

CONTRAPOSITIVE OF A CONDITIONAL STATEMENT:

The contra-positive of the conditional statement $\mathbf{p \to q}$ is $\mathbf{\sim q \to \sim p}$
A conditional and its contra-positive are equivalent.

Symbolically $\mathbf{p \to q \equiv \sim q \to \sim p}$

1. ***If today is Friday, then 2 + 3 = 5.***
   If $2 + 3 \neq 5$, then today is not Friday.
2. ***If it snows today, I will ski tomorrow.***
   I will not ski tomorrow only if it does not snow today.
3. ***If P is a square, then P is a rectangle.***
   If P is not a rectangle then P is not a square.
4. ***If my car is in the repair shop, then I cannot get to class.***
   If I can get to the class, then my car is not in the repair shop.

EXERCISE:

1. Show that $p \to q \equiv \sim q \to \sim p$ (Use the truth table.)
2. Show that $q \to p \equiv \sim p \to \sim q$ (Use the truth table.)

Lecture No.4 Biconditional

BICONDITIONAL

If p and q are statement variables, the biconditional of p and q is "**p if and only if q**".
It is denoted **p↔q**. "*if and only if"* is abbreviated as **iff.**
The double headed arrow " ↔" is the **biconditional operator.**

TRUTH TABLE FOR **p↔q**.

p	q	p ↔q
T	T	T
T	F	F
F	T	F
F	F	T

Remark:

- p ↔ q is true only when p and q both are true or both are false.
- p ↔ q is false when either p or q is false.

EXAMPLES:

Identify which of the following are True or false?

1."**1+1 = 3** if and only if **earth is flat**"
TRUE
2. "**Sky is blue** iff **1 = 0**"
FALSE
3. "**Milk is white** iff **birds lay eggs**"
TRUE
4. "**33 is divisible by 4** if and only if **horse has four legs**"
FALSE
5. "**$x > 5$ iff $x^2 > 25$**"
FALSE

REPHRASING BICONDITIONAL:

p↔q is also expressed as:

- "p is necessary and sufficient for q"
- "If p then q, and conversely"
- "p is equivalent to q"

Example: Show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

same truth values

<u>EXERCISE:</u>

Rephrase the following propositions in the form "p if and only if q" in English.

1. If it is hot outside, you buy an ice cream cone, and if you buy an ice cream cone, it is hot outside.

<u>Sol</u> You buy an ice cream cone if and only if it is hot outside.

2. For you to win the contest it is necessary and sufficient that you have the only winning ticket.

<u>Sol</u> You win the contest if and only if you hold the only winning ticket.

3. If you read the news paper every day, you will be informed and conversely.

<u>Sol</u> You will be informed if and only if you read the news paper every day.

4. It rains if it is a weekend day, and it is a weekend day if it rains.

<u>Sol</u> It rains if and only if it is a weekend day.

5. The train runs late on exactly those days when I take it.

<u>Sol</u> The train runs late if and only if it is a day I take the train.

6. This number is divisible by 6 precisely when it is divisible by both 2 and 3.

<u>Sol</u> This number is divisible by 6 if and only if it is divisible by both 2 and 3.

TRUTH TABLE FOR $(p \to q) \leftrightarrow (\sim q \to \sim p)$

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

TRUTH TABLE FOR $(p\leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q)\leftrightarrow(r\leftrightarrow q)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	T
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	T	T	T

TRUTH TABLE FOR $\mathbf{p \wedge {\sim}r \leftrightarrow q \vee r}$

Here $p \wedge \sim r \leftrightarrow q \vee r$ means $(p \wedge (\sim r)) \leftrightarrow (q \vee r)$

p	q	r	~r	$p\wedge\sim r$	$q\vee r$	$p \wedge \sim r \leftrightarrow q\vee r$
T	T	T	F	F	T	F
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	F	T

LOGICAL EQUIVALENCE INVOLVING BICONDITIONAL

Example: Show that **$\sim p \leftrightarrow q$** and **$p \leftrightarrow \sim q$** are logically equivalent.

p	q	~p	~q	~p↔q	p ↔ ~q
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

same truth values

Hence **$\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$**

EXERCISE:

Show that **$\sim(p \oplus q)$** and **$p \leftrightarrow q$** are logically equivalent.

p	q	p⊕q	~(p⊕q)	p↔q
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

same truth values

Hence **$\sim(p \oplus q) \equiv p \leftrightarrow q$**

LAWS OF LOGIC:

1. Commutative Law: $p \leftrightarrow q \equiv q \leftrightarrow p$
2. Implication Laws: $p \rightarrow q \equiv \sim p \vee q$
   $\equiv \sim (p \wedge \sim q)$
3. Exportation Law: $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$
4. Equivalence: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
5. Reductio ad absurdum $p \rightarrow q \equiv (p \wedge \sim q) \rightarrow c$

APPLICATION:

Example: Rewrite the statement forms without using the symbols $\rightarrow$ or $\leftrightarrow$

1. $p \wedge \sim q \rightarrow r$
2. $(p \rightarrow r) \leftrightarrow (q \rightarrow r)$

Solution:

1. $p\wedge\sim q\to r \equiv (p \wedge \sim q)\to r$ Order of operations

$\equiv \sim (p \wedge \sim q) \vee r$ Implication law

2. $(p\to r)\leftrightarrow(q \to r) \equiv (\sim p \vee r) \leftrightarrow (\sim q \vee r)$ Implication law

$\equiv [(\sim p \vee r) \to(\sim q \vee r)] \wedge [(\sim q \vee r) \to(\sim p \vee r)]$

Equivalence of biconditional

$\equiv [\sim(\sim p \vee r) \vee (\sim q \vee r)] \wedge [\sim(\sim q \vee r) \vee (\sim p \vee r)]$

Implication law

Example: Rewrite the statement form $\mathbf{\sim p \vee q \to r \vee \sim q}$ to a logically equivalent form that uses only **~** and $\wedge$**.**

Solution:

STATEMENT	**REASON**
$\sim p \vee q \to r \vee \sim q$	Given statement form
$\equiv (\sim p \vee q) \to (r \vee \sim q)$	Order of operations
$\equiv \sim[(\sim p \vee q) \wedge \sim (r \vee \sim q)]$	Implication law $p\to q \equiv \sim(p\wedge\sim q)$
$\equiv \sim[\sim(p \wedge \sim q) \wedge (\sim r \wedge q)]$	De Morgan's law

Example: Show that $\mathbf{\sim(p\to q) \to p}$ is a tautology without using truth tables.

Solution:

STATEMENT	**REASON**
$\sim(p\to q) \to p$	Given statement form
$\equiv \sim[\sim(p \wedge \sim q)] \to p$	Implication law $p\to q \equiv \sim(p \wedge \sim q)$
$\equiv (p \wedge \sim q) \to p$	Double negation law
$\equiv \sim(p \wedge \sim q) \vee p$	Implication law $p\to q \equiv \sim p \vee q$
$\equiv (\sim p \vee q) \vee p$	De Morgan's law
$\equiv (q \vee \sim p) \vee p$	Commutative law of $\vee$
$\equiv q \vee (\sim p \vee p)$	Associative law of $\vee$
$\equiv q \vee t$	Negation law
$\equiv t$	Universal bound law

EXERCISE:

Suppose that p and q are statements so that $p\to q$ is false. Find the truth values of each of the following:

1.$\sim p \to q$

2.$p \vee q$

3.$q \leftrightarrow p$

SOLUTION

Hint: ($p\to q$ is false when p is true and q is false.)

1.TRUE

2.TRUE

3.FAL