Functions

Thursday, 22 August 2024 9:47 am

TEST YOURSELF

1. If F is a function from a set X to a set Y, then F is one-to-one if, and only if, ______.
2. If F is a function from a set X to a set Y, then F is not one-to-one if, and only if, ______.
3. If F is a function from a set X to a set Y, then F is onto if, and only if, ______.
4. If F is a function from a set X to a set Y, then F is not onto if, and only if, ______.
5. The following two statements are ______:

   $\forall u, v \in U$, if $H(u) = H(v)$ then $u = v$.

   $\forall u, v \in U$, if $u \neq v$ then $H(u) \neq H(v)$.
6. Given a function $F: X \to Y$ where X is an infinite set, to prove that F is one-to-one, you suppose that ______ and then you show that ______.
7. Given a function $F: X \to Y$ where X is an infinite set, to prove that F is onto, you suppose that ______ and then you show that ______.
8. Given a function $F: X \to Y$, to prove that F is not one-to-one, you ______.
9. Given a function $F: X \to Y$, to prove that F is not onto, you ______.
10. A one-to-one correspondence from a set X to a set Y is a ______ that is ______.
11. If F is a one-to-one correspondence from a set X to a set Y and y is in Y, then $F^{-1}(y)$ is ______.

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a. Define $f: \mathbf{Z} \to \mathbf{Z}$ by the rule $f(n) = 2n$, for every integer n.
(i) Is f one-to-one? Prove or give a counterexample.
(ii) Is f onto? Prove or give a counterexample.

b. Let $2\mathbf{Z}$ denote the set of all even integers. That is, $2\mathbf{Z} = \{n \in \mathbf{Z} \mid n = 2k$, for some integer $k\}$. Define $h: \mathbf{Z} \to 2\mathbf{Z}$ by the rule $h(n) = 2n$, for each integer n. Is h onto? Prove or give a counterexample.

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a. Define $g: \mathbf{Z} \to \mathbf{Z}$ by the rule $g(n) = 4n - 5$, for each integer n.
(i) Is g one-to-one? Prove or give a counterexample.
(ii) Is g onto? Prove or give a counterexample.

b. Define $G: \mathbf{R} \to \mathbf{R}$ by the rule $G(x) = 4x - 5$ for every real number x. Is G onto? Prove or give a counterexample.

15. $f(x) = \dfrac{x+1}{x}$, for each number $x \neq 0$

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17. $f(x) = \dfrac{3x-1}{x}$, for each real number $x \neq ($

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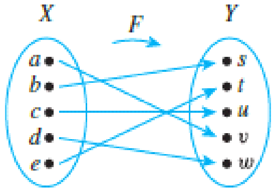

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Composition of Functions

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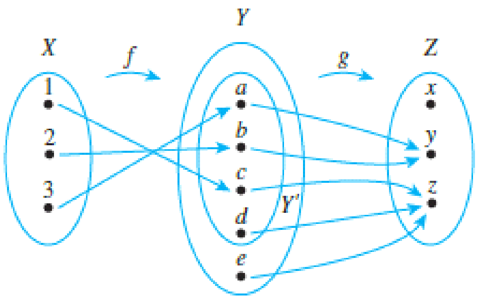
Composition of Functions Defined on Finite Sets

Let $X = \{1, 2, 3\}$, $Y' = \{a, b, c, d\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Define functions $f: X \to Y'$ and $g: Y \to Z$ by the arrow diagrams below.



Draw the arrow diagram for $g \circ f$. What is the range of $g \circ f$?

Recall that the identity function on a set X, I_X, is the function from X to X defined by the formula

$$I_X(x) = x \quad \text{for every } x \in X.$$

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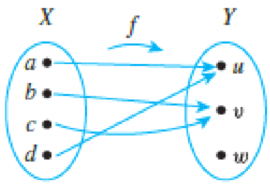
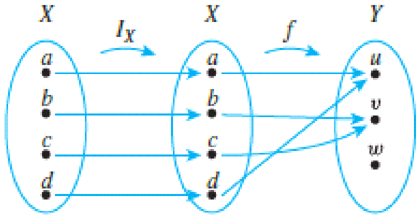
Composition with the Identity Function

Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, and suppose $f: X \to Y$ is given by the arrow diagram shown below.



Find $f \circ I_X$ and $I_Y \circ f$.

Solution The values of $f \circ I_X$ are obtained by tracing through the arrow diagram shown below.



$$(f \circ I_X)(a) = f(I_X(a)) = f(a) = u$$
$$(f \circ I_X)(b) = f(I_X(b)) = f(b) = v$$
$$(f \circ I_X)(c) = f(I_X(c)) = f(c) = v$$
$$(f \circ I_X)(d) = f(I_X(d)) = f(d) = u$$

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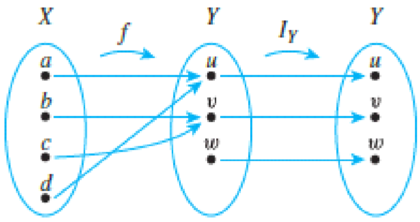
Thus, for every element x in X,

$$(f \circ I_X)(x) = f(x).$$

By definition of equality of functions, this means that $f \circ I_X = f$.

Similarly, the equality $I_Y \circ f = f$ can be verified by tracing through the arrow diagram below for each x in X and noting that in each case, $(I_Y \circ f)(x) = f(x)$.



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Theorem 7.3.1 Composition with an Identity Function

If f is a function from a set X to a set Y, and I_X is the identity function on X, and I_Y is the identity function on Y, then

$$\text{(a) } f \circ I_X = f \quad \text{and} \quad \text{(b) } I_Y \circ f = f.$$

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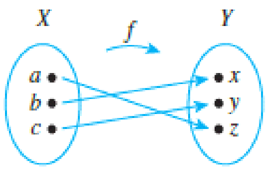
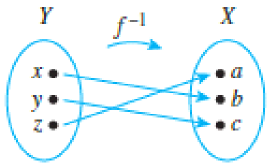
Composing a Function with Its Inverse

Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Define $f: X \to Y$ by the following arrow diagram.



You can see from the diagram that f is one-to-one and onto. Thus f^{-1} exists and is found by tracing the arrows backwards, as shown below.



Now $f^{-1} \circ f$ is found by following the arrows from X to Y by f and back to X by f^{-1}. If you do this, you will see that

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(z) = a$$
$$(f^{-1} \circ f)(b) = f^{-1}(f(b)) = f^{-1}(x) = b$$

and

$$(f^{-1} \circ f)(c) = f^{-1}(f(c)) = f^{-1}(y) = c.$$

Thus the composition of f and f^{-1} sends each element to itself. So by definition of the identity function,

$$f^{-1} \circ f = I_X.$$

In a similar way, you can see that

$$f \circ f^{-1} = I_Y. \quad ■$$

Theorem 7.3.2 Composition of a Function with Its Inverse

If $f: X \to Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \to X$, then

$$\text{(a) } f^{-1} \circ f = I_X \quad \text{and} \quad \text{(b) } f \circ f^{-1} = I_Y.$$

Composition of One-to-One Functions

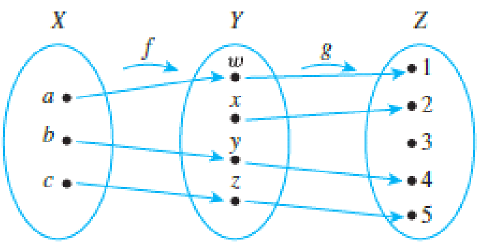


FIGURE 7.3.1

Then $g \circ f$ is the function with the arrow diagram shown in Figure 7.3.2.

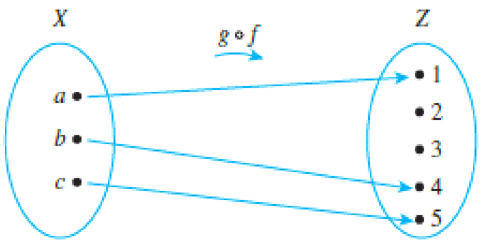


FIGURE 7.3.2

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Theorem 7.3.3

If $f: X \to Y$ and $g: Y \to Z$ are both one-to-one functions, then $g \circ f$ is one-to-one.

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Composition of Onto Functions

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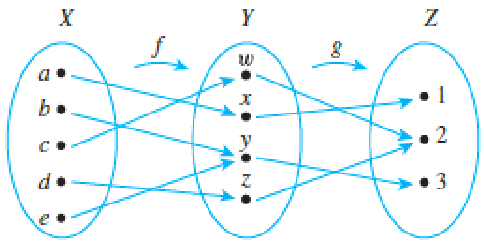

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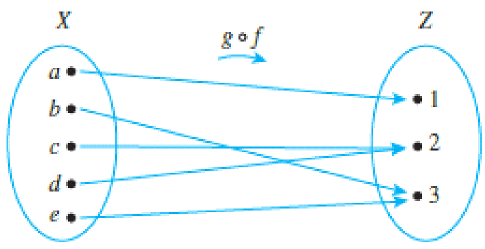

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Theorem 7.3.4

If $f: X \to Y$ and $g: Y \to Z$ are both onto functions, then $g \circ f$ is onto.

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TEST YOURSELF

1. If f is a function from X to Y', g is a function from Y to Z, and $Y' \subseteq Y$, then $g \circ f$ is a function from ______ to ______, and $(g \circ f)(x) =$ ______ for every x in X.

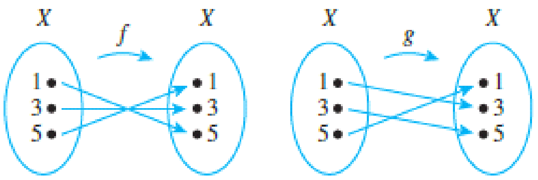
2. If f is a function from X to Y and I_x and I_y are the identity functions from X to X and Y to Y, respectively, then $f \circ I_x =$ ______ and $I_y \circ f =$ ______.

3. If f is a one-to-one correspondence from X to Y, then $f^{-1} \circ f =$ ______ and $f \circ f^{-1} =$ ______.

4. If f is a one-to-one function from X to Y and g is a one-to-one function from Y to Z, you prove that $g \circ f$ is one-to-one by supposing that ______ and then showing that ______.

5. If f is an onto function from X to Y and g is an onto function from Y to Z, you prove that $g \circ f$ is onto by supposing that ______ and then showing that ______.

In each of 1 and 2, functions f and g are defined by arrow diagrams. Find $g \circ f$ and $f \circ g$ and determine whether $g \circ f$ equals $f \circ g$.

1.


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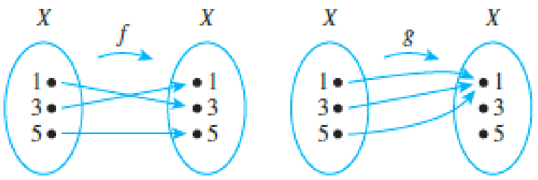

In 3 and 4, functions F and G are defined by formulas. Find $G \circ F$ and $F \circ G$ and determine whether $G \circ F$ equals $F \circ G$.

3. $F(x) = x^3$ and $G(x) = x - 1$, for each real number x.

4. $F(x) = x^5$ and $G(x) = x^{1/5}$ for each real number x.

5. Define $f: \mathbf{R} \to \mathbf{R}$ by the rule $f(x) = -x$ for every real number x. Find $(f \circ f)(x)$.

6. Define $F: \mathbf{Z} \to \mathbf{Z}$ and $G: \mathbf{Z} \to \mathbf{Z}$ by the rules $F(a) = 7a$ and $G(a) = a \bmod 5$ for each integer a. Find $(G \circ F)(0)$, $(G \circ F)(1)$, $(G \circ F)(2)$, $(G \circ F)(3)$, and $(G \circ F)(4)$.

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12. $F: \mathbf{R} \to \mathbf{R}$ and $F^{-1}\mathbf{R} \to \mathbf{R}$ are defined by

$$F(x) = 3x + 2 \quad \text{and} \quad F^{-1}(y) = \frac{y-2}{3},$$

for every $y \in \mathbf{R}$.

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14. H and H^{-1} are both defined from $\mathbf{R} - \{1\}$ to $\mathbf{R} - \{1\}$ by the formula

$$H(x) = H^{-1}(x) = \frac{x+1}{x-1}, \quad \text{for each } x \in \mathbf{R} - \{1\}.$$

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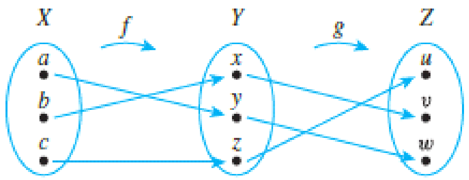
In 26 and 27 find $(g \circ f)^{-1}$, g^{-1}, f^{-1}, and $f^{-1} \circ g^{-1}$, and state how $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are related.

26. Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$, and $Z = \{u, v, w\}$. Define $f: X \to Y$ and $g: Y \to Z$ by the arrow diagrams below.



27. Define $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ by the formulas

$$f(x) = x + 3 \quad \text{and} \quad g(x) = -x \quad \text{for each } x \in \mathbf{R}.$$

17. Prove Theorem 7.3.2(b): If $f: X \to Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \to X$, then $f \circ f^{-1} = I_Y$, where I_Y is the identity function on Y.

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