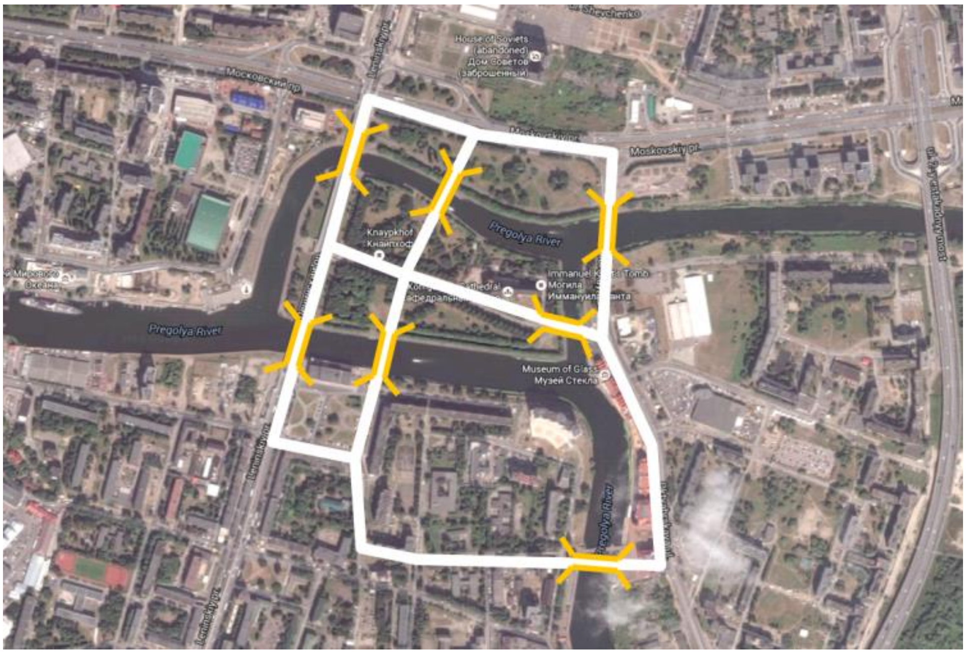
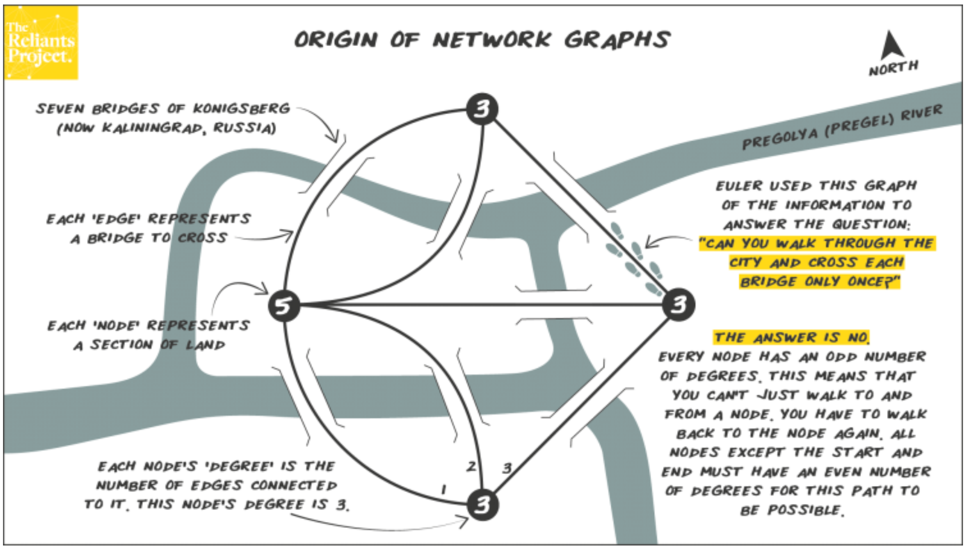
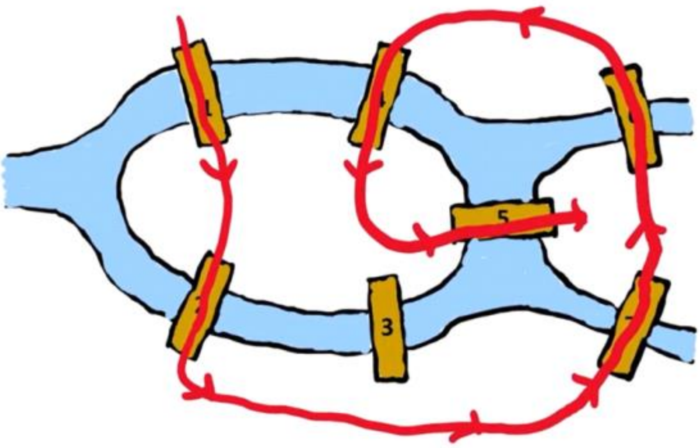
Graph Theory

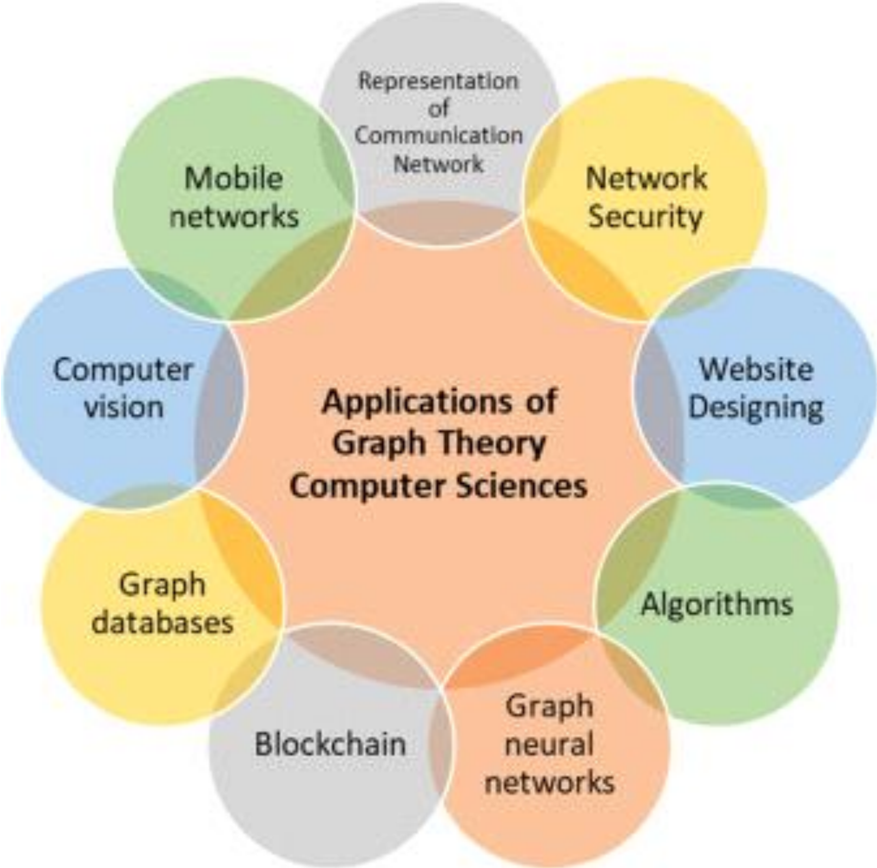
Sunday, 21 December 2025 11:58 am



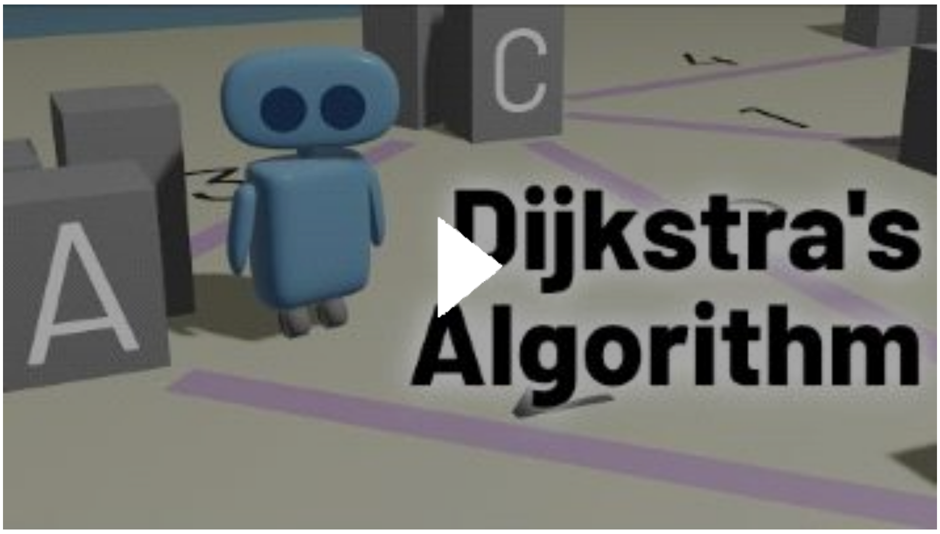




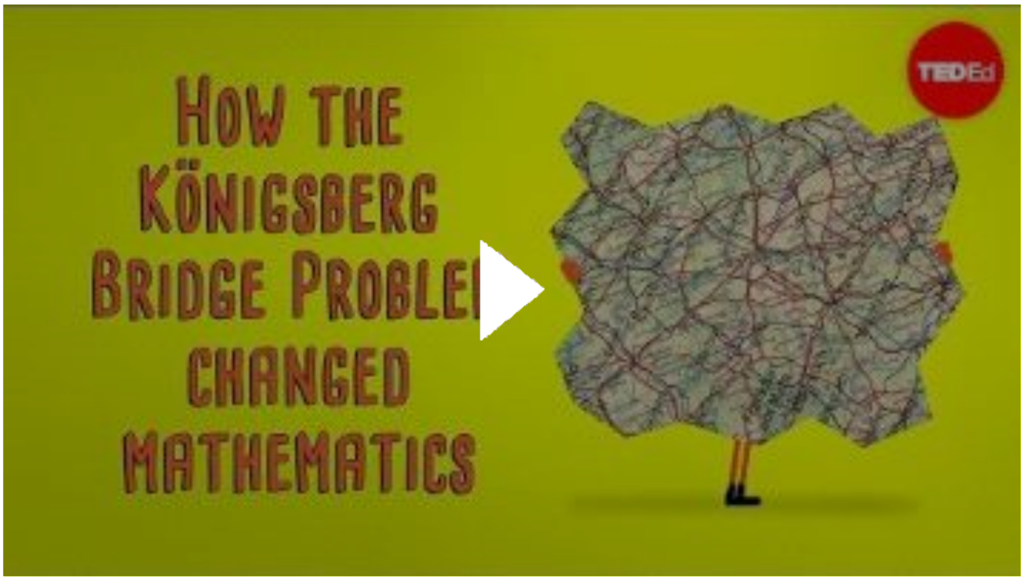
Bridge 3 not used.



[How Dijkstra's Algorithm Works]



[How the Königsberg bridge problem changed mathematics - Dan Van der Vieren]


HOW THE
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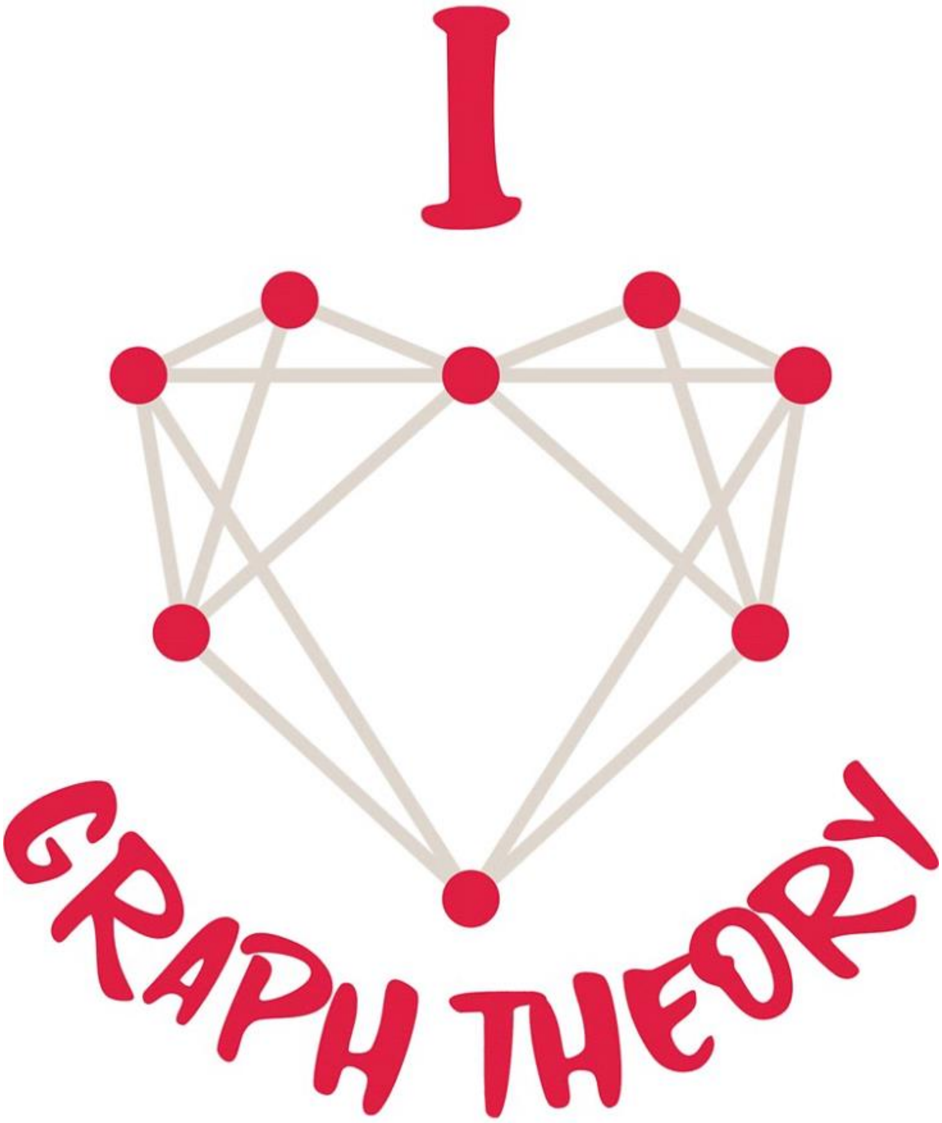

I
GRAPH THEORY

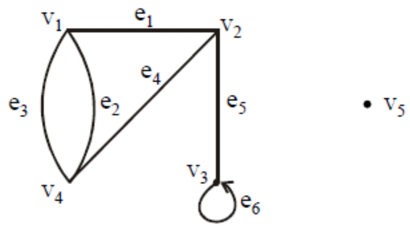
INTRODUCTION:

Graph theory plays an important role in several areas of computer science such as:

- switching theory and logical design
- artificial intelligence
- formal languages
- computer graphics
- operating systems
- compiler writing
- information organization and retrieval.

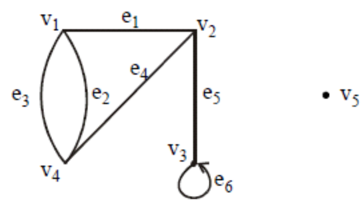
GRAPH:

A graph is a non-empty set of points called vertices and a set of line segments joining pairs of vertices called edges.
Formally, a graph G consists of two finite sets:
(i) A set V=V(G) of vertices (or points or nodes)
(ii)A set E=E(G) of edges; where each edge corresponds to a pair of vertices.



The graph G with
$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and
$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

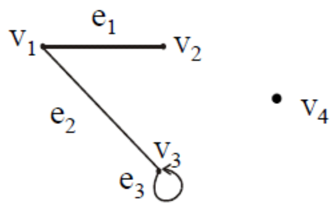
SOME TERMINOLOGY:



1. An edge connects either one or two vertices called its **endpoints** (edge e_1 connects vertices v_1 and v_2 described as $\{v_1, v_2\}$ i.e v_1 and v_2 are the endpoints of an edge e_1).
2. An edge with just one endpoint is called a **loop**. Thus a loop is an edge that connects a vertex to itself (e.g., edge e_6 makes a loop as it has only one endpoint v_3).
3. Two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be adjacent to itself.
4. An edge is said to be **incident** on each of its endpoints(i.e. e_1 is incident on v_1 and v_2).
5. A vertex on which no edges are incident is called **isolated** (e.g., v_5)
6. Two distinct edges with the same set of end points are said to be **parallel** (i.e. e_2 & e_3).

EXAMPLE:

Define the following graph formally by specifying its vertex set, its edge set, and a table giving the edge endpoint function.



SOLUTION:

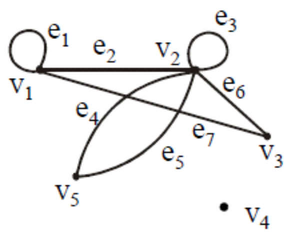
Vertex Set = $\{v_1, v_2, v_3, v_4\}$
Edge Set = $\{e_1, e_2, e_3\}$

Edge - endpoint function is:

Edge	Endpoint
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3\}$

EXAMPLE:

For the graph shown below
(i) find all edges that are incident on v_1;
(ii)find all vertices that are adjacent to v_3;
(iii)find all loops;
(iv)find all parallel edges;
(v)find all isolated vertices;



SOLUTION:

(i) v_1 is incident with edges e_1, e_2 and e_7
(ii) vertices adjacent to v_3 are v_1 and v_2
(iii) loops are e_1 and e_3
(iv) only edges e_4 and e_5 are parallel
(v) The only isolated vertex is v_4 in this Graph.

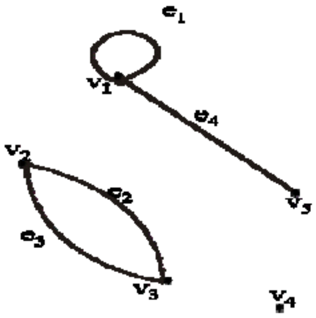
DRAWING PICTURE FOR A GRAPH:

Draw picture of Graph H having vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$ with edge endpoint function

Edge	Endpoint
e_1	$\{v_1\}$
e_2	$\{v_2,v_3\}$
e_3	$\{v_2,v_3\}$
e_4	$\{v_1,v_5\}$

SOLUTION:

Given $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$
and $E(H) = \{e_1, e_2, e_3, e_4\}$
with edge endpoint function



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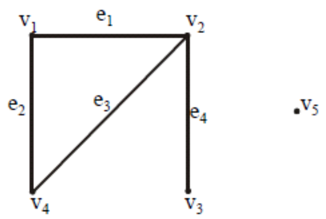
SIMPLE GRAPH

A simple graph is a graph that does not have any loop or parallel edges.

EXAMPLE:



It is a simple graph H
$V(H) = \{v_1, v_2, v_3, v_4, v_5\}$ & $E(H) = \{e_1, e_2, e_3, e_4\}$

EXERCISE:

Draw all simple graphs with the four vertices $\{u, v, w, x\}$ and two edges, one of which is $\{u, v\}$.

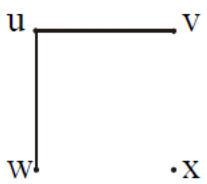
SOLUTION:

There are $C(4,2) = 6$ ways of choosing two vertices from 4 vertices. These edges may be listed as:

$$\{u,v\}, \{u,w\}, \{u,x\}, \{v,w\}, \{v,x\}, \{w,x\}$$

One edge of the graph is specified to be $\{u,v\}$, so any of the remaining five from this list may be chosen to be the second edge. This required graphs are:

1.



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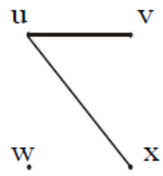
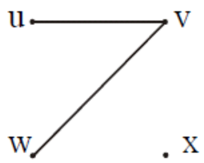


3.



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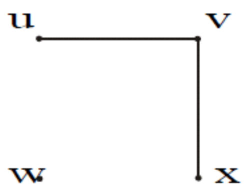
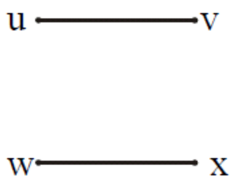


5.



DEGREE OF A VERTEX:

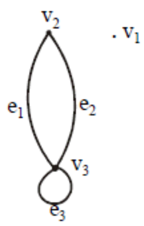
Let G be a graph and v a vertex of G. The degree of v, denoted **deg(*v*),** equals the number of edges that are incident on v, with an edge that is a loop counted twice.

Note:(i)The total degree of G is the sum of the degrees of all the vertices of G.

(ii) The **degree** of a loop is counted twice.

EXAMPLE:

For the graph shown



deg (v_1) = 0, since v_1 is isolated vertex.

deg (v_2) = 2, since v_2 is incident on e_1 and e_2 .

deg (v_3) = 4, since v_3 is incident on e_1,e_2 and the loop e_3.

Total degree of G = deg(v_1) + deg(v_2) + deg(v_3)

= 0 + 2 + 4

= 6

REMARK:

The total degree of G, which is 6, equals twice the number of edges of G, which is 3.

THE HANDSHAKING THEOREM:

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G.

Specifically, if the vertices of G are v_1, v_2, …, v_n, where n is a positive integer, then

the total degree of G $= \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n)$
$= 2$. (the number of edges of G)

EXERCISE:

Draw a graph with the specified properties or explain why no such graph exists.

(i) Graph with four vertices of degrees 1, 2, 3 and 3
(ii) Graph with four vertices of degrees 1, 2, 3 and 4
(iii)Simple graph with four vertices of degrees 1, 2, 3 and 4

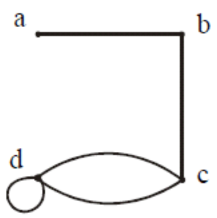
SOLUTION:

(i) Total degree of graph $= 1 + 2 + 3 + 3$
$= 9$ an odd integer

Since, the total degree of a graph is always even, hence no such graph is possible.
Note: As we know that "for any graph,the sum of the degrees of all the vertices of G equals twice the number of edges of G **or** the total degree of G is an even number".

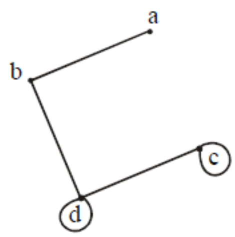
(ii) Two graphs with four vertices of degrees 1, 2, 3 & 4 are

1.



or

2.



The vertices a, b, c, d have degrees 1,2,3, and 4 respectively(i.e graph exists).

EXERCISE:

Suppose a graph has vertices of degrees 1, 1, 4, 4 and 6. How many edges does the graph have?

SOLUTION:

The total degree of graph $= 1 + 1 + 4 + 4 + 6$
$= 16$

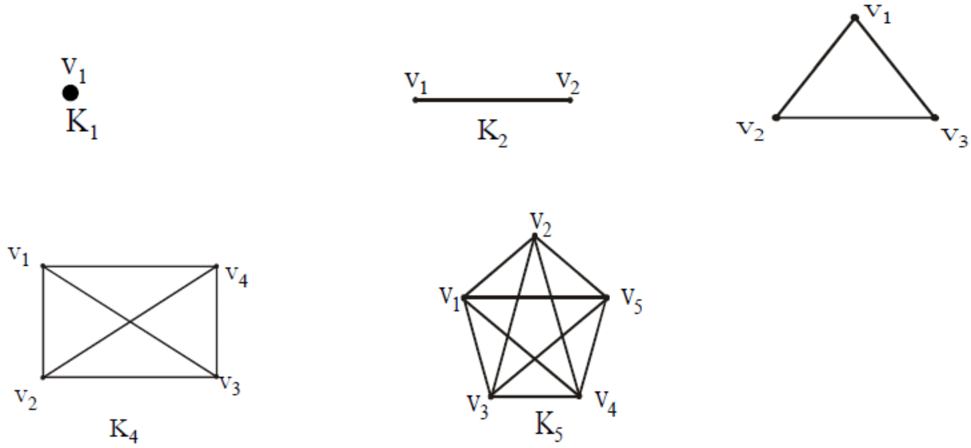
Since, the total degree of graph = 2.(number of edges of graph) [by using Handshaking theorem]

$\Rightarrow \quad 16 = 2.$(number of edges of graph)

$$\Rightarrow \text{Number of edges of graph} = \frac{16}{2} = 8$$

COMPLETE GRAPH:

A complete graph on n vertices is a simple graph in which each vertex is connected to every other vertex and is denoted by K_n (K_n means that there are n vertices).
The following are complete graphs K_1, K_2,K_3, K_4 and K_5.



EXERCISE:

For the complete graph K_n, find

(i) the degree of each vertex

(ii)the total degrees

(iii)the number of edges

SOLUTION:

(i) Each vertex v is connected to the other (n-1) vertices in K_n; hence deg (v) = n - 1 for every v in K_n.

(ii) Each of the n vertices in K_n has degree n - 1; hence, the total degree in

$K_n = (n - 1) + (n - 1) + \ldots + (n - 1)$ n times

$= n (n - 1)$

(iii) Each pair of vertices in K_n determines an edge, and there are C(n, 2) ways of selecting two vertices out of n vertices. Hence,

Number of edges in K_n = C(n, 2)

$$= \frac{n(n-1)}{2}$$

Alternatively,

The total degrees in graph K_n = 2 (number of edges in K_n)

$\Rightarrow$ $n(n-1)$ = 2(number of edges in K_n)

$\Rightarrow$

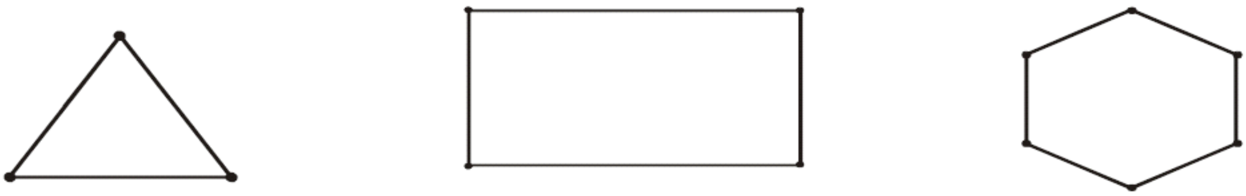
Number of edges in K_n $= \dfrac{n(n-1)}{2}$

REGULAR GRAPH:

A graph G is regular of degree k or k-regular if every vertex of G has degree k. In other words, a graph is regular if every vertex has the same degree. Following are some regular graphs.

(i) 0-regular

(ii) 1-regular



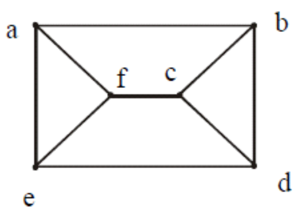
(iii) 2-regular

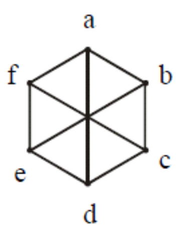
REMARK: The complete graph K_n is (n-1) regular.

EXERCISE:

Draw two 3-regular graphs with six vertices.

SOLUTION:

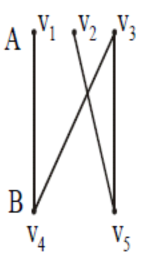


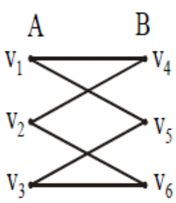


BIPARTITE GRAPH:

A bipartite graph G is a simple graph whose vertex set can be partitioned into two mutually disjoint non empty subsets A and B such that the vertices in A may be connected to vertices in B, but no vertices in A are connected to vertices in A and no vertices in B are connected to vertices in B.

The following are bipartite graphs





1. WALK

A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G.
Thus a walk has the form

$$v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n$$

where the v's represent vertices, the e's represent edges $v_0=v$, $v_n=w$, and for all $i = 1, 2 \ldots n$, v_{i-1} and v_i are endpoints of e_i.

The trivial walk from v to v consists of the single vertex v.

2. CLOSED WALK

A closed walk is a walk that starts and ends at the same vertex.

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3. CIRCUIT

A circuit is a closed walk that does not contain a repeated edge. Thus a circuit is a walk of the form

$$v_0e_1v_1e_2 \ldots v_{n-1}\ e_n\ v_n$$

where $v_0 = v_n$ and all the e_i,s are distinct.

4. SIMPLE CIRCUIT

A simple circuit is a circuit that does not have any other repeated vertex except the first and last.
Thus a simple circuit is a walk of the form

$$v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n$$

where all the e_i,s are distinct and all the v_j,s are distinct except that $v_0 = v_n$

5. PATH

A path from v to w is a walk from v to w that does not contain a repeated edge.
Thus a path from v to w is a walk of the form

$$v = v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n = w$$

where all the e_i,s are distinct (that is $e_i \neq e_k$ for any $i \neq k$).

6. SIMPLE PATH

A simple path from v to w is a path that does not contain a repeated vertex.
Thus a simple path is a walk of the form

$$v = v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n = w$$

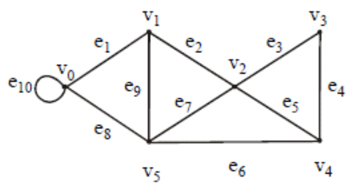
where all the e_i,s are distinct and all the v_j,s are also distinct (that is, $v_j \neq v_m$ for any $j \neq m$).

SUMMARY

	Repeated Edge	Repeated Vertex	Starts and Ends at Same Point
walk	allowed	Allowed	allowed
closed walk	allowed	Allowed	yes(means, where it starts also ends at that point)
circuit	no	Allowed	yes
simple circuit	no	first and last only	yes
path	no	Allowed	allowed
simple path	no	no	No

EXERCISE:

In the graph below, determine whether the following walks are paths, simple paths, closed walks, circuits, simple circuits, or are just walks.



(a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$

(b) $v_1v_2v_3v_4v_5v_2$

(c) $v_4v_2v_3v_4v_5v_2v_4$

(d) $v_2v_1v_5v_2v_3v_4v_2$

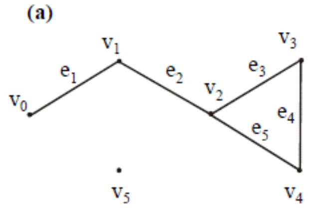
(e) $v_0v_5v_2v_3v_4v_2v_1$

(f) $v_5v_4v_2v_1$

SOLUTION:

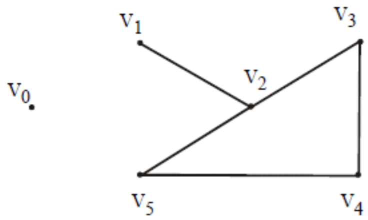
(a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$

(a)



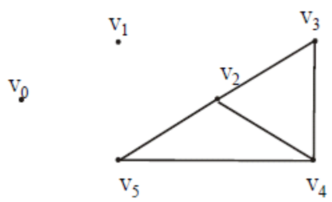
This graph starts at vertex v_1,then goes to v_2 along edge e_2,and moves continuously, at the end it goes from v_1 to v_0 along e_1.Note it that the vertex v_2 and the edge e_2 is repeated twice, and starting and ending, not at the same points.Hence The graph is just a walk.

(b) $v_1v_2v_3v_4v_5v_2$



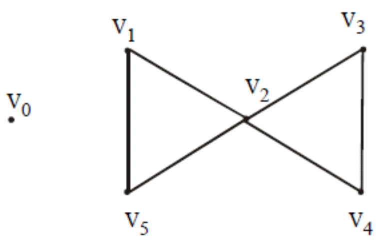
In this graph vertex v_2 is repeated twice.As no edge is repeated so the graph is a path.

(c) $v_4v_2v_3v_4v_5v_2v_4$



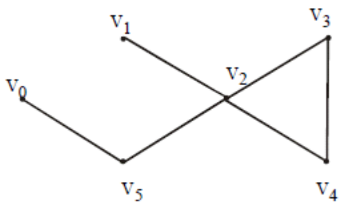
As vertices v_2 & v_4 are repeated and graph starts and ends at the same point v_4,also the edge(i.e. e_5)connecting v_2 & v_4 is repeated, so the graph is a closed walk.

(d) $v_2v_1v_5v_2v_3v_4v_2$



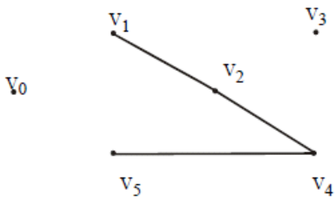
In this graph, vertex v_2 is repeated and the graph starts and end at the same vertex (i.e. at v_2) and no edge is repeated, hence the above graph is a circuit.

(e) $v_0v_5v_2v_3v_4v_2v_1$



Here vertex v_2 is repeated and no edge is repeated so the graph is a path.

(f) $v_5v_4v_2v_1$



Neither any vertex nor any edge is repeated so the graph is a simple path.

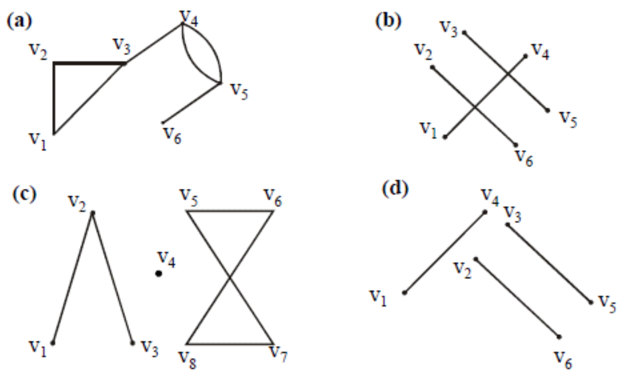
CONNECTEDNESS:

Let G be a graph. Two vertices *v* and *w* of G are connected if, and only if, there is a walk from *v* to *w*. The graph G is connected if, and only if, given any two vertices *v* and *w* in G, there is a walk from *v* to *w*. Symbolically:

G is connected $\Leftrightarrow \forall$ vertices v, w $\in$ V (G), $\exists$ a walk from v to w:

EXAMPLE:

Which of the following graphs have a connectedness?



EULER CIRCUITS

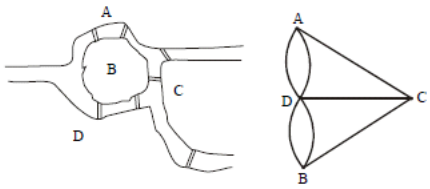
DEFINITION:

Let G be a graph. An Euler circuit for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is sequence of adjacent vertices and edges in G that starts and ends at the same vertex uses every vertex of G at least once, and used every edge of G exactly once.

THEOREM:

A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has an even degree.

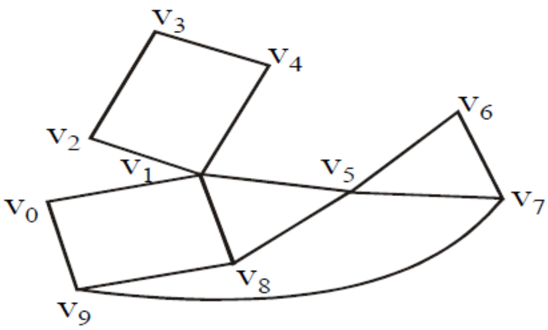
KONIGSBERG BRIDGES PROBLEM



We try to solve Konigsberg bridges problem by Euler method.
Here deg(a)=3,deg(b)=3,deg(c)=3 and deg(d)=5 as the vertices have odd degree so there is no possibility of an Euler circuit.

EXERCISE:

Determine whether the following graph has an Euler circuit.

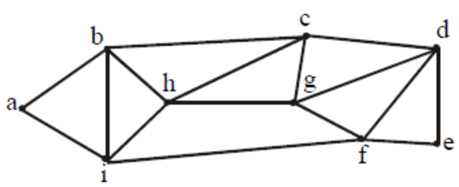


SOLUTION:

As deg (v_1) =5, an odd degree so the following graph has not an Euler circuit.

EXERCISE:

Determine whether the following graph has Euler circuit.



SOLUTION:

From above clearly deg(a)=2, deg(b)=4, deg(c)=4, deg(d)=4, deg(e)=2, deg(f)=4, deg(g)=4, deg(h)=4, deg(i)=4
Since the degree of each vertex is even, and the graph has Euler Circuit. One such circuit is:

a b c d e f g d f i h c g h b i a

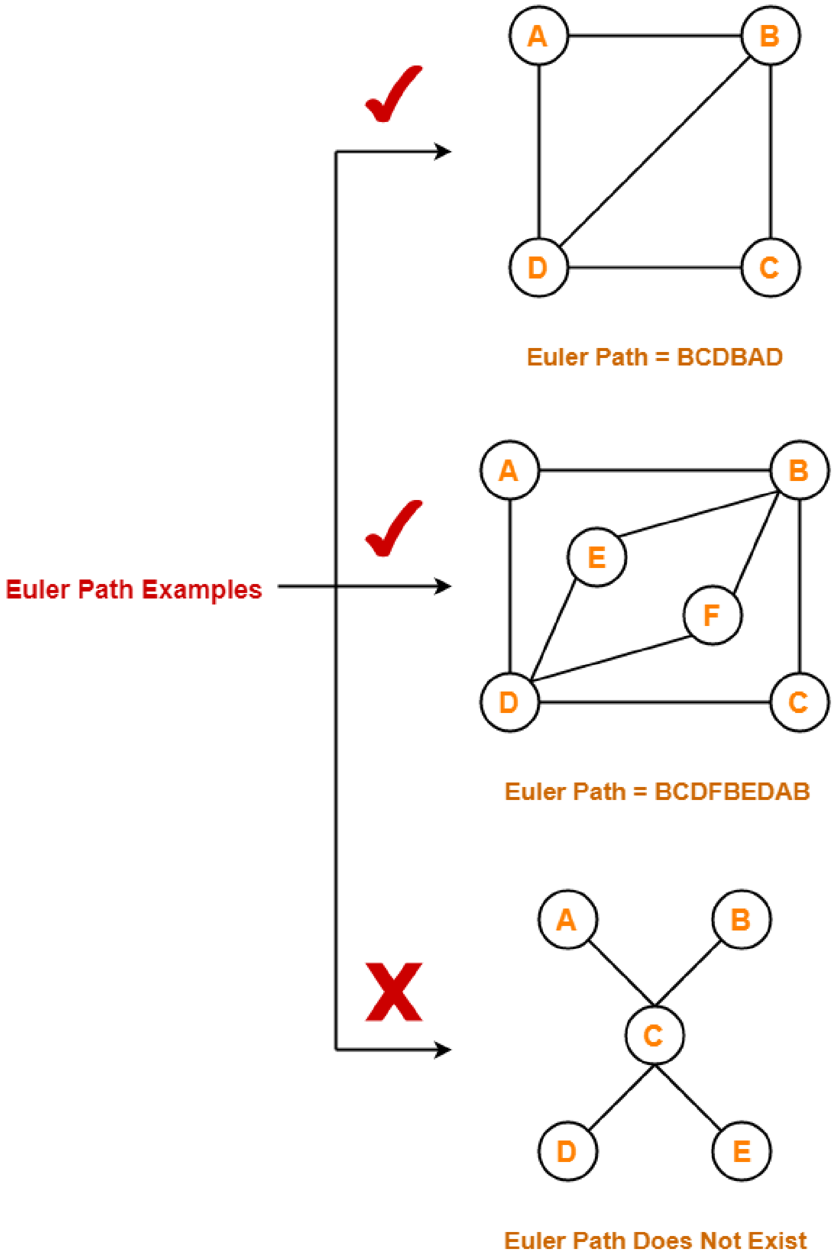
EULER PATH

DEFINITION:

Let G be a graph and let v and w be two vertices of G. An Euler path from v to w is a sequence of adjacent edges and vertices that starts at v, end and w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

COROLLARY

Let G be a graph and let v and w be two vertices of G. There is an Euler path from v to w if, and only if, G is connected, v and w have odd degree and all other vertices of G have even degree.



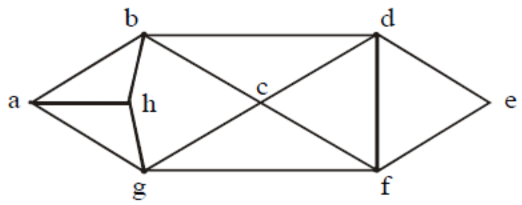
HAMILTONIAN CIRCUITS

DEFINITION:

Given a graph G, a Hamiltonian circuit for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once.

EXERCISE:

Find Hamiltonian Circuit for the following graph.



SOLUTION:

The Hamiltonian Circuit for the following graph is:

a b d e f c g h a

Another Hamiltonian Circuit for the following graph could be:

a b c d e f g h a

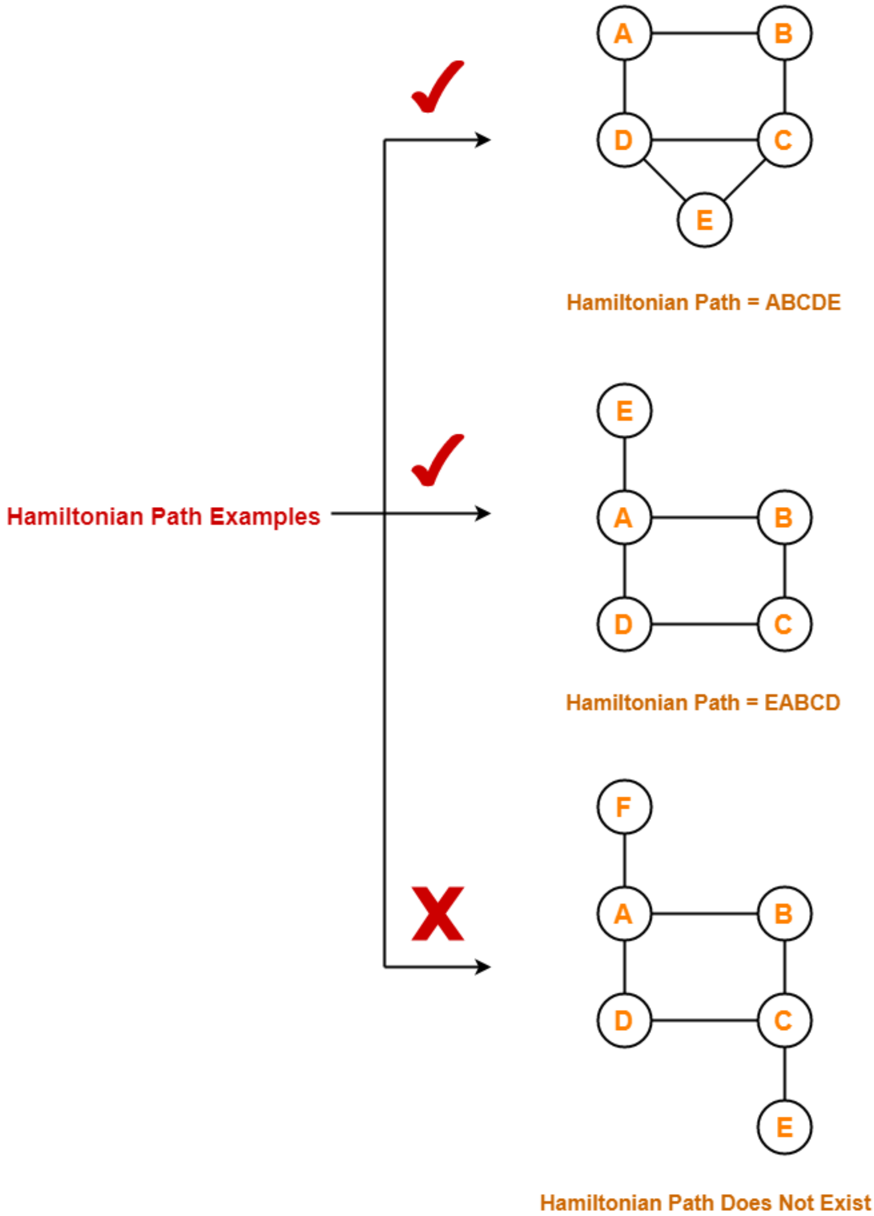
PROPOSITION:

If a graph G has a Hamiltonian circuit then G has a sub-graph H with the following properties:

1. H contains every vertex G
2. H is connected
3. H has the same number of edges as vertices
4. Every vertex of H has degree 2

What is Hamiltonian Path?

Hamiltonian Path in a graph **G** is a path that visits every vertex of G exactly once and it doesn't have to return to the starting vertex. It's an open path.


Hamiltonian Path Examples
A
B
D
C
E
Hamiltonian Path = ABCDE
E
A
B
D
C
Hamiltonian Path = EABCD
F
A
B
D
C
E
Hamiltonian Path Does Not Exist