Residue Harmonic Balance Method for Buckling and Self-Contact of a Flexible Loop under Uniform Pressure

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Outline

Outline

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Problem

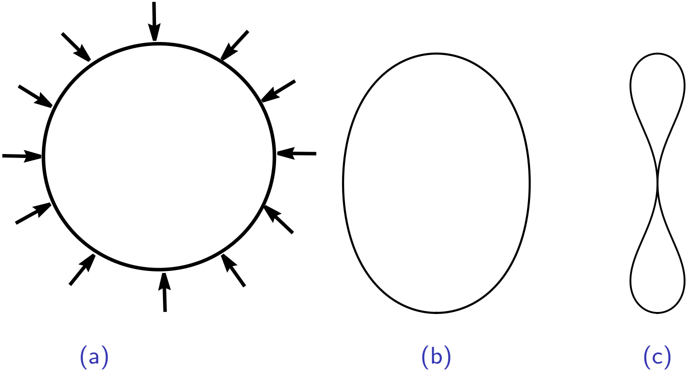

Figure: Inextensible elastic ring and rod deforms under symmetric uniform external force p.

Applications

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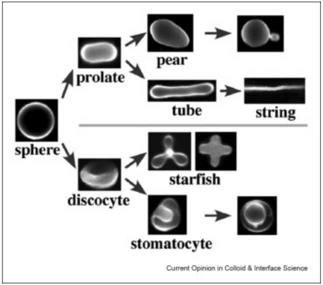
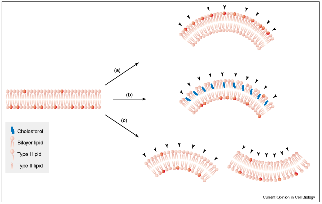
- Soft Structure & Robotics : Shapes of vesicles, cell membrane, microtubles.
  Packing & folding problem, rubber/ribbon deformation.



(a) Vesicle deformation



(b) Membrane deformation

Abstract

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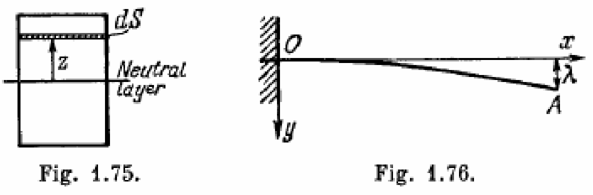
- Computing the equilibrium shapes and corresponding force value for specific configurations, particularly the contact shape where the elastica gets self-contact.
- Comparison with the established methods and draw conclusion.
- The residue harmonic balance is straightforward to implement and has significant computational advantages over several methods previously used for this problem.

Introduction to Euler's Elastica

Euler's Elastica Model: Historic Background

Jacob Bernoulli (1691): The *Elastica problem* (what shape of elastica, an ideal thin elastic rod on a plane, is allowed?)

(a) Jacob Bernoulli



(b) Beam Bending

Euler's Elastica Model: Historic Background

James Bernoulli (1744): Requested Euler to solve the following problem in energy functional E, as a minimization of the bending strain energy.

$$E := \int \kappa^2 ds. \tag{1}$$

where κ denotes curvature and s denotes arc length.

Figure: James Bernoulli

Euler's Elastica Model: Historic Background

Leonard Euler (1746): Solved the Elastica problem by developing the variational method and Elliptic integral solution[1].

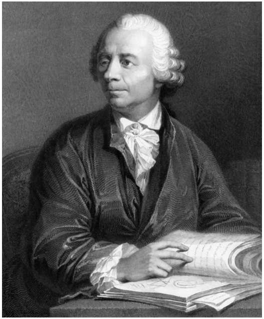


Figure: Leonard Euler

1 .

[1] "The elastica: a mathematical history"

Euler’s Elastica Model: Equation

Equation of Euler’s model:

- **Energy minimization**(Calculus of Variation).

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- **Force Balance Method**. (Sum of all forces are equal to zero)

We have following equation

$$\kappa''(s) + \frac{1}{2}\kappa^3(s) - c\,\kappa(s) - p = 0. \tag{2}$$

where c is an integration constant.
The final form of Euler's elastica equation.

$$v''(s) + \mu v(s) - \beta + \frac{1}{2}v(s)^3 + \frac{3}{2}v(s)^2 = 0. \tag{3}$$

where $\mu = \frac{3}{2} - c, \quad \beta = c + p - \frac{1}{2}$ and $v(s) = 1 - \kappa(s)$.
The equation of applied force is

$$p = \mu + \beta - 1. \tag{4}$$

Configuration of a ring

The closure conditions are

$$v'(0) = v'(\pi/n) = 0, \qquad \int_0^{\pi/n} v(s)\, ds = 0. \tag{5}$$

the following periodicity conditions.

$$v(-s) = v(s), \qquad v(s - \frac{2\pi}{n}) = v(s). \tag{6}$$

the parametric coordinates $x_1(s)$ and $x_2(s)$ of the curve

$$x_1(s) = \int_0^s \cos(\theta(\varepsilon))\, d\varepsilon, \quad x_2(s) = \int_0^s \sin(\theta(\varepsilon))\, d\varepsilon. \tag{7}$$

$$\theta(s) = s + \int_0^s (v(\varepsilon)) d\varepsilon, \quad 0 \leq s \leq 2\pi. \tag{8}$$

Solution to the Euler's Elastica model

Djondjorov, Vassilev & Mladenov's work

Djondjorov at el. in their paper [2], have solved the transcendental equations through Jacobi Elliptic function.

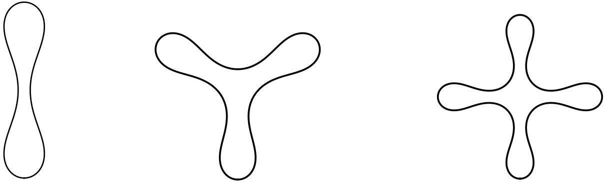

Figure: Closed ring shapes corresponding to (a) $p = 4.75$ (two fold symmetry); (b) $p = 16.25$ (three fold symmetry); (c) $p = 35.25$ (four fold symmetry)

[2] Analytic description and explicit parametrisation of the equilibrium shapes of elastic rings and tubes under uniform hydrostatic pressure." International Journal of Mechanical Sciences 53.5 (2011): 355-364.

Djondjorov, Vassilev & Mladenov's work

Considering contact point(external loading/force p_{cp} value) and shape as a criterion.

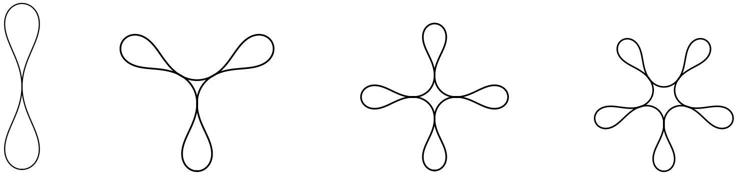

Figure: Equilibrium ring shapes corresponding to (a) $p_{n=2} = 5.24$; (b) $p_{n=3} = 21.65$; (c) $p_{n=4} = 51.84$; (d) $p_{n=5} = 97.83$

Elliptic Function							
2	3	4	5	6	7	8	9
5.24	21.65	51.84	97.83	161.07	242.68	343.51	464.27

Solution methods to the Euler's elastica model

- **Newton-Harmonic balance method.**

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- **Homotopy Perturbation Method**

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- **HBM** is an advantageous technique for solving the strongly nonlinear equation, especially when an approximate solution is periodic.
- The theory suggests that the existence of a solution should be in the form of a Fourier series, a finite sum of trigonometric functions.

$$u(t) = \sum_{k=0}^{M} (A_k \cos(k\,w\,t + k\,\phi) + B_k \sin(k\,w\,t + k\,\phi)). \tag{9}$$

After substituting in the equation, now it is required to determine the coefficients by balancing the harmonic $(\cos, \sin, etc.)$. Final step is to form the algebraic equations and solve.

- This method has been applied to nonlinear electrical circuits and radio-frequency integrated circuits (RFICs).

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2. Let's recall Euler's elastica equation

$$v''(s) + \mu v(s) - \beta + \frac{1}{2}v(s)^3 + \frac{3}{2}v(s)^2 = 0. \tag{10}$$

3. Substitute series in model

$$v(s) = \sum_{n=0}^{\infty} q^n v_n(s), \qquad \mu = \sum_{n=0}^{\infty} q^n \mu_n \quad \& \quad \beta = \sum_{n=0}^{\infty} q^n \beta_n$$

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4. Where coefficients $q^0, q^1, q^2, \ldots$ for zeroth, first, second approximation, and so on.

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3. We get this

$$R_0 = \frac{1}{8}((3A^3+8A\mu_0-32)\cos(2s)+A^3\cos(6s)+6A^2\cos(4s)+6A^2-8\beta_0)$$

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4. Solving by using harmonic balance, we get the following

$$\beta_0 = \frac{3A^2}{4}, \mu_0 = \frac{1}{8}\left(32 - 3A^2\right)$$

.

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1. Taking the first order terms as $\phi(v_1, \mu_1, \beta_1)$ with R_0, we have next residual R_1. Mathematically, $R_1 = \phi(v_1, \mu_1, \beta_1) + R_0$.

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3. Let $v_1(s) = c_1(\cos(2s) - \cos(4s))$
4. We get $v_1 = \frac{2A^2(\cos(2s)-\cos(4s))}{A^2-4A-32}, \mu_1 = -\frac{3(A-2)A^2}{2(A^2-4A-32)}, \beta_1 = -\frac{3(A-4)A^3}{4(A^2-4A-32)}$.
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5. The procedure repeats upto required order of approximation.
6. Finally, combine terms for the solution.

Residue Harmonic Balance Method

Here is the third order approximation terms

$$
\begin{aligned}
\mu^3 &= \mu_0 + \mu_1 + \mu_2 + \mu_3, \\
\mu^3 &= \Big(-99A^{15} - 4746A^{14} + 92304A^{13} + 1763160A^{12} - 31006080A^{11} \\
&\quad - 120823680A^{10} + 2995821568A^{9} + 2929586176A^{8} \\
&\quad - 128872284160A^{7} - 28084011008A^{6} + 3193677283328A^{5} \\
&\quad + 1284732092416A^{4} - 42434276884480A^{3} - 43843026157568A^{2} \\
&\quad + 219902325555200A + 351843720888320 \Big) \frac{1}{8192N}.
\end{aligned} \tag{11}
$$

$$
N = (A-8)^4 (A+4)^4 (3A^3 - 396A^2 + 1280A + 10240).
$$

Residue Harmonic Balance Method

$$\beta^3 = \beta_0 + \beta_1 + \beta_2 + \beta_3,$$

$$\begin{aligned}\beta^3 = \frac{3}{32768\,N}\Big(&3A^{17} + 58A^{16} - 1200A^{15} + 12840A^{14} - 57344A^{13} - 3548800A^{12} \\ &+ 36221952A^{11} + 208818176A^{10} - 3644260352A^{9} - 1651245056A^{8} \\ &+ 174386577408A^{7} + 29997662208A^{6} - 4256312590336A^{5} \\ &- 3917010173952A^{4} + 43980465111040A^{3} + 87960930222080A^{2}\Big).\end{aligned}$$

$$N = (A-8)^4(A+4)^4(3A^3 - 396A^2 + 1280A + 10240).$$

Residue Harmonic Balance Method

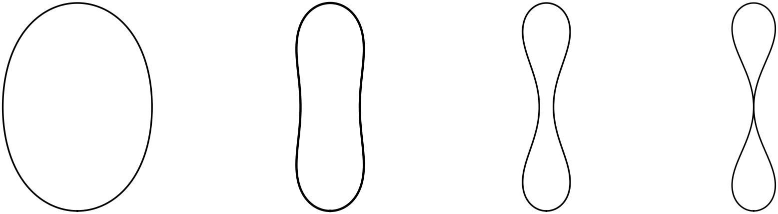
$$
\begin{aligned}
v^3 &= v_0 + v_1 + v_2 + v_3, \\
v^3 &= \frac{A}{512N}\Big(15A^{13} + 774A^{12} - 14264A^{11} - 296952A^{10} + 5392096A^9 \\
&\quad + 12434304A^8 - 458665472A^7 - 44531712A^6 + 17231773696A^5 \\
&\quad - 140509184A^4 - 346080411648A^3 - 250181844992A^2 \\
&\quad + 3092376453120A + 5497558138880\Big)\cos(2s) \\
&\quad + \frac{A^2}{4096N}\Big(3A^{13} + 58A^{12} - 1680A^{11} - 11544A^{10} \\
&\quad + 370688A^9 + 2836608A^8 - 82752512A^7 + 68829184A^6 \\
&\quad + 4964745216A^5 - 675282944A^4 - 130996502528A^3 \\
&\quad - 117037858816A^2 + 1374389534720A + 2748779069440\Big)\cos(4s)\ldots
\end{aligned}
\tag{12}
$$

Residue Harmonic Balance Method

Equation continues

$$
\begin{aligned}
v^3 = \ldots - \frac{A^3}{512N}\Big(&15A^{11} + 774A^{10} - 13912A^9 - 85048A^8 + 1694176A^7 \\
&+ 3472512A^6 - 75837952A^5 - 103399424A^4 + 1746665472A^3 \\
&+ 2795503616A^2 - 16441671680A - 37580963840\Big)\cos(6x) \\
- \frac{A^4}{4096N}\Big(&3A^{11} + 58A^{10} - 1680A^9 - 26648A^8 + 494080A^7 \\
&+ 1618048A^6 - 36748288A^5 - 81895424A^4 + 985530368A^3 \\
&+ 2141192192A^2 - 10066329600A - 26843545600\Big)\cos(8x).
\end{aligned}
\tag{13}
$$

Equilibrium Shapes by Harmonic Balance Method



(a) $p = 3.06$ (b) $p = 3.57$ (c) $p = 4.63$ (d) $p = 5.247$

Figure: Equilibrium loops for rotational symmetry $n = 2$ using Residue HB method

Figure

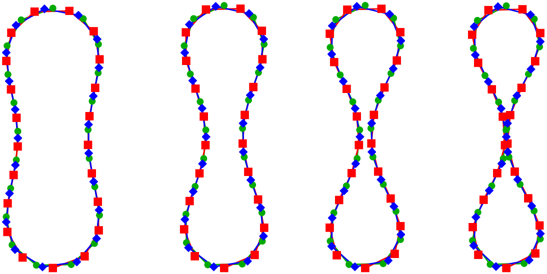


Figure: Equilibrium loops for rotational symmetry $n = 2$ for $p = 4.0, 4.5, 5.0$ and $p_c = 5.247$ using different methods.

p	Ar_{Res}	Ar_{New}	Ar_{Res}/Ar_{New}	Ar_{Elp}	Ar_{Res}/Ar_{Elp}
3	3.14154	3.14154	1.000	3.14154	1.000
4	1.78106	1.77857	1.00140	1.77937	1.00094
4.5	1.34717	1.34113	1.00450	1.34094	1.00464
5	1.00707	0.99620	1.01091	0.99590	1.01121
5.247	0.86526	0.85151	1.01614	0.85120	1.01651

Table: Enclosed area Ar comparision using 3rd order residual harmonic(Res), 3rd order Newton-harmonic balance (New) and elliptic function (Elp)

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- Various setting of elastic loops and open rod can be considered.

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- Stability range fits well comparing with numerical solution

Discussion & Conclusion

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Thanking to listener

Thank you so much for your attention!

Bibliography

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2 The Elastica: A Mathematical History

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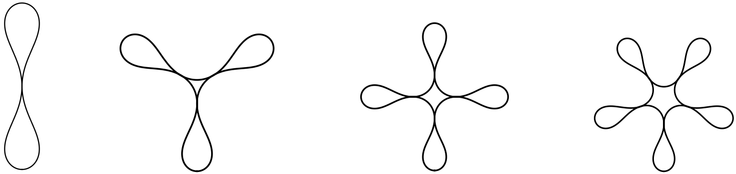
Appendix 1: Contact Shape using Newton Harmonic Balance Method(3rd Order)



Results Comparison					
Mode	Third-Approx	Elliptic	Mode	Third-Approx	Elliptic
2	5.247	5.247	6	160.157	161.077
3	21.65	21.65	7	240.74	242.682
4	51.843	51.844	8	341.38	343.517
5	97.455	97.834	9	459.90	464.276

Appendix 2: Self–Intersection Equilibrium Shapes by HBM

When loading parameter crosses over contact point, it will form self intersection shape due to two dimensional planar condition.

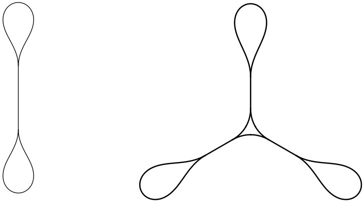


Figure: Plot of $v_{n=2}(s)$ at $p = 10.34$, $v_{n=3}(s)$ at $p = 70.34$ and $v_{n=6}(s)$ at $p = 600$

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Any question, comment and feedback.