

Residue Harmonic Balance Method for Buckling and Self-Contact of a Flexible Loop under Uniform Pressure

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- 4 Euler's Elastica Solution

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Problem

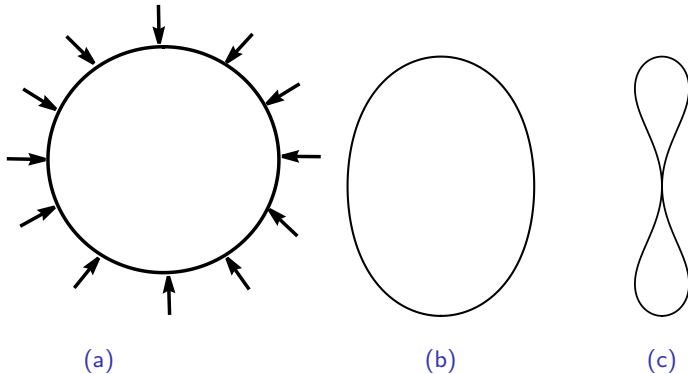


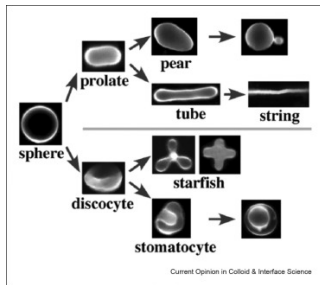
Figure: Inextensible elastic ring and rod deforms under symmetric uniform external force p .

Applications

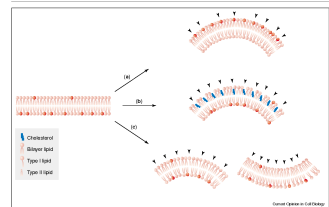
- **Engineering:** Beam or 2D-rod problem and Pole-vaulting, borehole/pipes and cylindrical shell like structures.

Applications

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- **Soft Structure & Robotics :** Shapes of vesicles, cell membrane, microtubules.
Packing & folding problem, rubber/ribbon deformation.



(a) Vesicle deformation



(b) Membrane deformation

Abstract

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- Finding the solution to Euler's elastica equation by using Residue Harmonic methods, which describes the shape of an elastic curve under the action of external forces.
- Computing the equilibrium shapes and corresponding force value for specific configurations, particularly the contact shape where the elastica gets self-contact.
- Comparison with the established methods and draw conclusion.

Introduction to Euler's Elastica

Euler's Elastica Model: Historic Background

Jacob Bernoulli (1691): The *Elastica problem* (what shape of elastica, an ideal thin elastic rod on a plane, is allowed?)



(a) Jacob Bernoulli

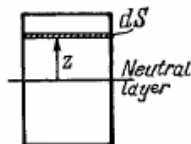


Fig. 1.75.



Fig. 1.76.

(b) Beam Bending

Euler's Elastica Model: Historic Background

James Bernoulli (1744): Requested Euler to solve the following problem in energy functional E , as a minimization of the bending strain energy.

$$E := \int \kappa^2 ds. \quad (1)$$

where κ denotes curvature and s denotes arc length.



Figure: James Bernoulli

Euler's Elastica Model: Historic Background

Leonard Euler (1746): Solved the Elastica problem by developing the variational method and Elliptic integral solution^[1].

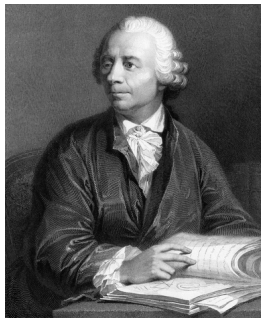


Figure: Leonard Euler

¹.

¹"The elastica: a mathematical history"

Euler's Elastica Model: Equation

Equation of Euler's model:

- **Energy minimization**(Calculus of Variation).

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- **Force Balance Method.** (Sum of all forces are equal to zero)

We have following equation

$$\kappa''(s) + \frac{1}{2}\kappa^3(s) - c\kappa(s) - p = 0. \quad (2)$$

where c is an integration constant.

The final form of Euler's elastica equation.

$$v''(s) + \mu v(s) - \beta + \frac{1}{2}v(s)^3 + \frac{3}{2}v(s)^2 = 0. \quad (3)$$

where $\mu = \frac{3}{2} - c$, $\beta = c + p - \frac{1}{2}$ and $v(s) = 1 - \kappa(s)$.

The equation of applied force is

$$p = \mu + \beta - 1. \quad (4)$$

Configuration of a ring

The closure conditions are

$$v'(0) = v'(\pi/n) = 0, \quad \int_0^{\pi/n} v(s) ds = 0. \quad (5)$$

the following periodicity conditions.

$$v(-s) = v(s), \quad v(s - \frac{2\pi}{n}) = v(s). \quad (6)$$

the parametric coordinates $x_1(s)$ and $x_2(s)$ of the curve

$$x_1(s) = \int_0^s \cos(\theta(\varepsilon)) d\varepsilon, \quad x_2(s) = \int_0^s \sin(\theta(\varepsilon)) d\varepsilon. \quad (7)$$

$$\theta(s) = s + \int_0^s (v(\varepsilon)) d\varepsilon, \quad 0 \leq s \leq 2\pi. \quad (8)$$

Solution to the Euler's Elastica model

Djondjorov, Vassilev & Mladenov's work

Djondjorov et al. in their paper ², have solved the transcendental equations through Jacobi Elliptic function.

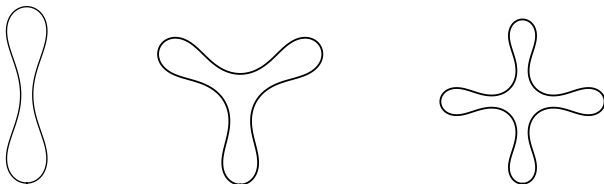


Figure: Closed ring shapes corresponding to (a) $p = 4.75$ (two fold symmetry); (b) $p = 16.25$ (three fold symmetry); (c) $p = 35.25$ (four fold symmetry)

²Analytic description and explicit parametrisation of the equilibrium shapes of elastic rings and tubes under uniform hydrostatic pressure." International Journal of Mechanical Sciences 53.5 (2011): 355-364.

Djondjorov, Vassilev & Mladenov's work

Considering **contact point**(external loading/force p_{cp} value) and **shape** as a criterion.

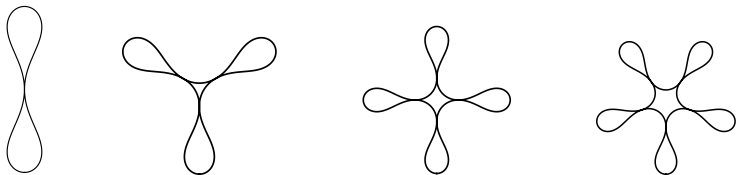


Figure: Equilibrium ring shapes corresponding to (a) $p_{n=2} = 5.24$; (b) $p_{n=3} = 21.65$; (c) $p_{n=4} = 51.84$; (d) $p_{n=5} = 97.83$

Elliptic Function							
2	3	4	5	6	7	8	9
5.24	21.65	51.84	97.83	161.07	242.68	343.51	464.27

Solution methods to the Euler's elastica model

- **Newton-Harmonic balance method.**

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- **HBM** is an advantageous technique for solving the strongly nonlinear equation, especially when an approximate solution is periodic.
- The theory suggests that the existence of a solution should be in the form of a Fourier series, a finite sum of trigonometric functions.

$$u(t) = \sum_{k=0}^M (A_k \cos(k \omega t + k \phi) + B_k \sin(k \omega t + k \phi)). \quad (9)$$

After substituting in the equation, now it is required to determine the coefficients by balancing the harmonic (cos, sin, etc.). Final step is to form the algebraic equations and solve.

- This method has been applied to nonlinear electrical circuits and radio-frequency integrated circuits (RFICs).

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- 5 Update approximation by adding Δ terms.

Harmonic Balance Equation

Zeroth Approximation:

$$v_0 = A \cos(2ns) \quad (10)$$

$$\mu_0 = \frac{1}{8} (8n^2 - 3A^2), \quad \beta_0 = \frac{3A^2}{4} \quad (11)$$

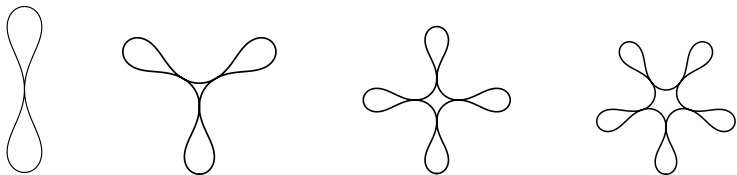
First Approximation:

$$v(s) = \frac{2A^2(\cos(ns) - \cos(2ns))}{A^2 - 4A - 8n^2} + A \cos(ns) \quad (12)$$

$$\mu = \frac{1}{8} (8n^2 - 3A^2) - \frac{3(A - 2)A^2}{2(A^2 - 4A - 8n^2)} \quad (13)$$

$$\beta = \frac{3A^2}{4} - \frac{3(A - 4)A^3}{4(A^2 - 4A - 8n^2)} \quad (14)$$

Contact Shape using Newton Harmonic Balance Method(3rd Order)



Results Comparison					
Mode	Third-Approx	Elliptic	Mode	Third-Approx	Elliptic
2	5.247	5.247	6	160.157	161.077
3	21.65	21.65	7	240.74	242.682
4	51.843	51.844	8	341.38	343.517
5	97.455	97.834	9	459.90	464.276

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- ③ Substitute series in model

$$v(s) = \sum_{n=0}^{\infty} v_n(s), \quad \mu = \sum_{n=0}^{\infty} \mu_n \quad \& \quad \beta = \sum_{n=0}^{\infty} \beta_n$$

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- ③ We get this

$$R_0 = \frac{1}{8}((3A^3 + 8A\mu_0 - 32) \cos(2s) + A^3 \cos(6s) + 6A^2 \cos(4s) + 6A^2 - 8\beta_0)$$

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- ④ Solving by using harmonic balance, we get the following

$$\beta_0 = \frac{3A^2}{4}, \mu_0 = \frac{1}{8}(32 - 3A^2)$$

Residual Harmonic Balance Algorithm

- 1 Taking the first order terms as $\phi(v_1, \mu_1, \beta_1)$ with R_0 , we have next residual R_1 . Mathematically, $R_1 = \phi(v_1, \mu_1, \beta_1) + R_0$.

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$$v_1 = \frac{2A^2(\cos(2s) - \cos(4s))}{A^2 - 4A - 32}, \mu_1 = -\frac{3(A-2)A^2}{2(A^2 - 4A - 32)}, \beta_1 = -\frac{3(A-4)A^3}{4(A^2 - 4A - 32)}.$$
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- ④ The procedure repeats upto required order of approximation.
- ⑤ Finally, combine terms for the solution.

Residue Harmonic Balance Method

Here is the third order approximation terms

$$\mu^3 = \mu_0 + \mu_1 + \mu_2 + \mu_3,$$

$$\begin{aligned} \mu^3 = & \left(-99A^{15} - 4746A^{14} + 92304A^{13} + 1763160A^{12} - 31006080A^{11} \right. \\ & - 120823680A^{10} + 2995821568A^9 + 2929586176A^8 \\ & - 128872284160A^7 - 28084011008A^6 + 3193677283328A^5 \\ & + 1284732092416A^4 - 42434276884480A^3 - 43843026157568A^2 \\ & \left. + 219902325555200A + 351843720888320 \right) \frac{1}{8192N}. \end{aligned} \quad (16)$$

$$N = (A - 8)^4 (A + 4)^4 (3A^3 - 396A^2 + 1280A + 10240).$$

Residue Harmonic Balance Method

$$\beta^3 = \beta_0 + \beta_1 + \beta_2 + \beta_3,$$

$$\beta^3 = \frac{3}{32768 N} \left(3A^{17} + 58A^{16} - 1200A^{15} + 12840A^{14} - 57344A^{13} - 3548800A^{12} \right. \\ \left. + 36221952A^{11} + 208818176A^{10} - 3644260352A^9 - 1651245056A^8 \right. \\ \left. + 174386577408A^7 + 29997662208A^6 - 4256312590336A^5 \right. \\ \left. - 3917010173952A^4 + 43980465111040A^3 + 87960930222080A^2 \right).$$

$$N = (A - 8)^4 (A + 4)^4 (3A^3 - 396A^2 + 1280A + 10240).$$

Residue Harmonic Balance Method

$$v^3 = v_0 + v_1 + v_2 + v_3,$$

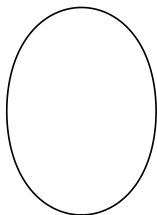
$$v^3 = \frac{A}{512N} \left(15A^{13} + 774A^{12} - 14264A^{11} - 296952A^{10} + 5392096A^9 \right. \\ + 12434304A^8 - 458665472A^7 - 44531712A^6 + 17231773696A^5 \\ - 140509184A^4 - 346080411648A^3 - 250181844992A^2 \\ + 3092376453120A + 5497558138880 \Big) \cos(2s) \\ + \frac{A^2}{4096N} \left(3A^{13} + 58A^{12} - 1680A^{11} - 11544A^{10} \right. \\ + 370688A^9 + 2836608A^8 - 82752512A^7 + 68829184A^6 \\ + 4964745216A^5 - 675282944A^4 - 130996502528A^3 \\ - 117037858816A^2 + 1374389534720A + 2748779069440 \Big) \cos(4s) \dots \quad (17)$$

Residue Harmonic Balance Method

Equation continues

$$\begin{aligned}
 v^3 = \dots & - \frac{A^3}{512N} \left(15A^{11} + 774A^{10} - 13912A^9 - 85048A^8 + 1694176A^7 \right. \\
 & + 3472512A^6 - 75837952A^5 - 103399424A^4 + 1746665472A^3 \\
 & + 2795503616A^2 - 16441671680A - 37580963840 \left. \right) \cos(6x) \\
 & - \frac{A^4}{4096N} \left(3A^{11} + 58A^{10} - 1680A^9 - 26648A^8 + 494080A^7 \right. \\
 & + 1618048A^6 - 36748288A^5 - 81895424A^4 + 985530368A^3 \\
 & + 2141192192A^2 - 10066329600A - 26843545600 \left. \right) \cos(8x).
 \end{aligned}
 \tag{18}$$

Equilibrium Shapes by Harmonic Balance Method



(a) $p = 3.06$



(b) $p = 3.57$



(c) $p = 4.63$



(d) $p = 5.247$

Figure: Equilibrium loops for rotational symmetry $n = 2$ using Residue HB method

Figure

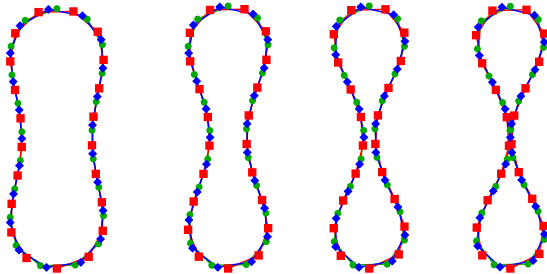


Figure: Equilibrium loops for rotational symmetry $n = 2$ for $p = 4.0, 4.5, 5.0$ and $p_c = 5.247$ using different methods.

p	Ar_{Res}	Ar_{New}	Ar_{Res}/Ar_{New}	Ar_{Elp}	Ar_{Res}/Ar_{Elp}
3	3.14154	3.14154	1.000	3.14154	1.000
4	1.78106	1.77857	1.00140	1.77937	1.00094
4.5	1.34717	1.34113	1.00450	1.34094	1.00464
5	1.00707	0.99620	1.01091	0.99590	1.01121
5.247	0.86526	0.85151	1.01614	0.85120	1.01651

Table: Enclosed area Ar comparison using 3rd order residual harmonic (Res), 3rd order Newton-harmonic balance (New) and elliptic function (Elp)

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- Various setting of elastic loops and open rod can be considered.

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- Stability range fits well comparing with numerical solution

Discussion & Conclusion

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- Computational cost is low comparing with other methods.

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- Eventually bifurcate at very large loading.

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Cons

- Compute shape at large error for more two mode loading
- Eventually bifurcate at very large loading.
- Different rod configurations are not aligned according to boundary conditions.

Bibliography

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Thanking to listener

Thank you so much for your attention!