Introduction to Mathematics

Quadratic Functions

Week 9

Learning Outcomes

- Students will be able to plot Quadratic Function.
- Students will be able to solve Quadratic Function Applications.

DEFINITION: QUADRATIC FUNCTION
A *quadratic function* involving the independent variable x and the dependent variable y has the general form

$$y = f(x) = ax^2 + bx + c \tag{6.1}$$

where a, b, and c are constants, $a \neq 0$.

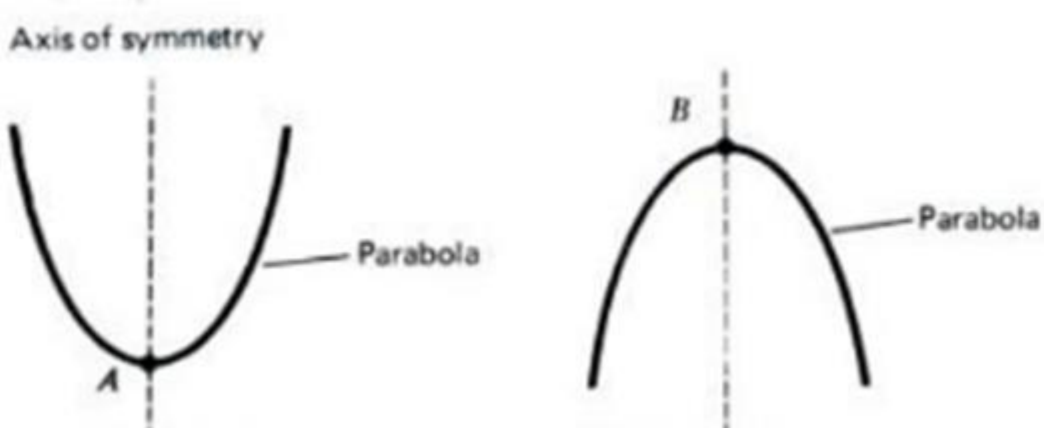
For example:

$$y = 4x^2 + 16x - 4$$

Here a=4 ; b=16 ; c=-14

Graphical Representation

All quadratic functions having the form of Eq. (6.1) graph as ***parabolas.*** Figure 6.1 illustrates two parabolas which have different orientations. A parabola which "opens" upward, such as that in Fig. 6.1*a*, is said to be ***concave up.*** A parabola which "opens" downward, such as that in Fig. 6.1*b*, is said to be ***concave down.**** The point at which a parabola either "bottoms out" when it is concave up or "peaks out" when it is concave down is called the ***vertex*** of the parabola. Points *A* and *B* are the respective ***vertices*** for the two parabolas in Fig. 6.1.



(*a*) Concave up (*b*) Concave down

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Co-ordinates of Vertex (x, y) of Parabola:

$$V(x,y) = \left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$$

where *a*, *b*, and *c* are the parameters or constants in Eq. (6.1).

A parabola is a curve having a particular symmetry. In Fig. 6.1, the dashed vertical line which passes through the vertex is called the **axis of symmetry.** This line separates the parabola into two symmetrical halves. If you were able to fold one side of the parabola using the axis of symmetry as a hinge, you would find that the two halves coincide (i.e., they are mirror images of each other).

Sketching the graph of a quadratic function can be accomplished by using the "brute force" methods discussed in Chap. 4. However, there are certain key attributes of quadratic functions which enable us to sketch the corresponding parabola with relative ease. If the ***concavity*** of the parabola, ***y intercept, x intercept(s),*** and ***vertex*** can be determined, a reasonable sketch of the parabola can be drawn.

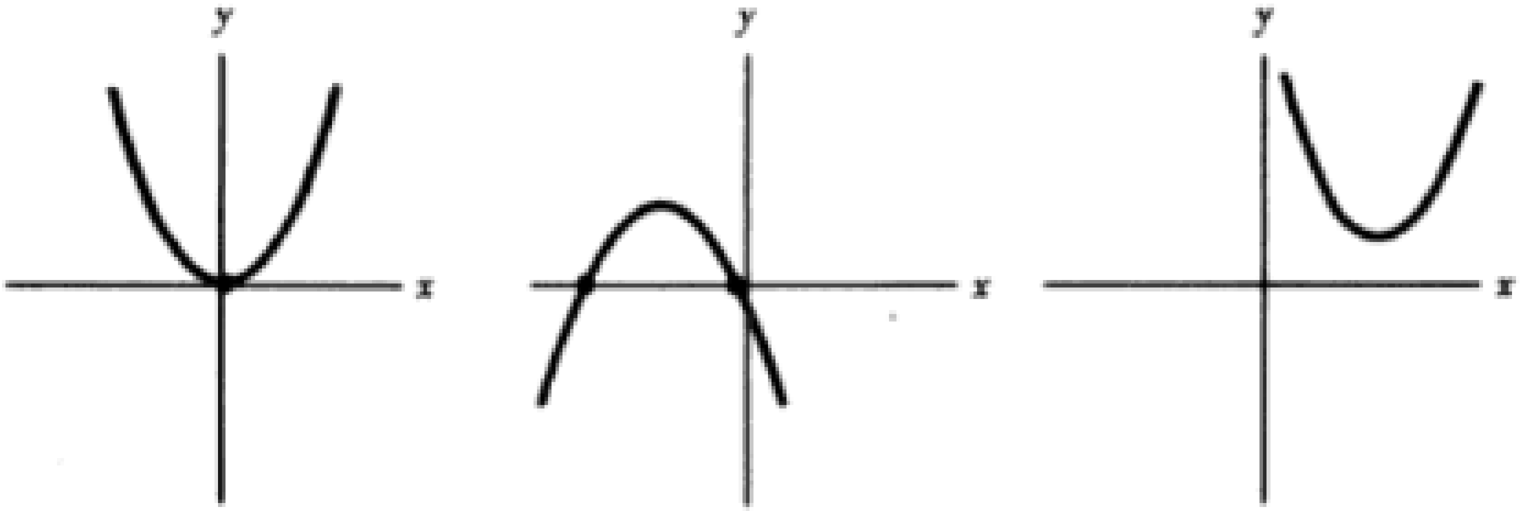
Concavity Once a function has been recognized as having the general quadratic form of Eq. (6.1), the concavity of the parabola can be determined by the sign of the coefficient on the x^2 term. If $a > 0$, the function will graph as a parabola which is ***concave up.*** If $a < 0$, the parabola is ***concave down.***

y intercept The ***y intercept*** for a function was defined in Chap. 2. Graphically, the y intercept was identified as the point at which the line intersects the y axis. Algebraically, the y intercept was identified as the value of y when x equals 0, or $f(0)$. Thus, for quadratic functions having the form

$$y = f(x) = ax^2 + bx + c$$

$f(0) = c$, or the y intercept for the corresponding parabola occurs at $(0, c)$.

x intercept(s) The x ***intercept*** for a function was also defined in Chap. 2 as the point(s) at which the graph of a line crosses the x axis. Equivalently, the x intercept represents the value(s) of x when y equals 0. For quadratic functions, there may be one x intercept, two x intercepts, or *no* x intercept. These possibilities are shown in Fig. 6.2.



(a) One x intercept (b) Two x intercepts (c) No x intercept

Figure 6.2 x intercept possibilities for quadratic functions.

Q: Sketch the following quadratic function:

$$f(x) = 3x^2 + 6x - 45$$

$a = 3$; $b = 6$; $c = -45$

Step 1:

Check Concavity:

i.e. $a > 0$ or $a < 0$

Here $a = 3$ which is greater than zero.

Therefore, Parabola will be Concave-Up.

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Step 2: Finding y-intercept

To do so, put $\boxed{x=0}$ in the given expression i.e. $f(x) = y = 3x^2 + 6x - 45$.

$$\Rightarrow y = \cancel{3(0)^2}^{\,0} + \cancel{6(0)}^{\,0} - 45$$

$$\Rightarrow y = -45$$

$$\boxed{\therefore \text{ y-intercept} = (0, -45)}$$

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Step 3: Finding x-intercept

To do so, put $\boxed{y=0}$ in the given expression

i.e. $f(x) = y = 3x^2 + 6x - 45$

$\Rightarrow 3x^2 + 6x - 45 = 0$

$\Rightarrow 3(x^2 + 2x - 15) = 0$

$\Rightarrow x^2 + 2x - 15 = 0$

$\Rightarrow x^2 + 5x - 3x - 15 = 0$

$\Rightarrow x(x+5) - 3(x+5) = 0$

$\Rightarrow (x+5)(x-3) = 0$

$\Rightarrow x + 5 = 0$ or $x - 3 = 0$

$\Rightarrow \boxed{x = -5}$ or $\boxed{x = +3}$

Therefore, x-intercepts are:
$(-5, 0)$ & $(3, 0)$

Step 4: Finding Vertex Coordinates $V(x,y)$

$\because$ We know that $V(x,y) = \left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$

Here, $a = 3$, $b = 6$ & $c = -45$

$$\Rightarrow V(x,y) = \left\{\frac{-\cancel{6}}{\cancel{2}(\cancel{3})}, \frac{4(3)(-45)-6^2}{4(3)}\right\}$$

$$\Rightarrow V(x,y) = \left(-1, \frac{-540-36}{12}\right)$$

$$\Rightarrow \boxed{V(x,y) = (-1, -48)}$$

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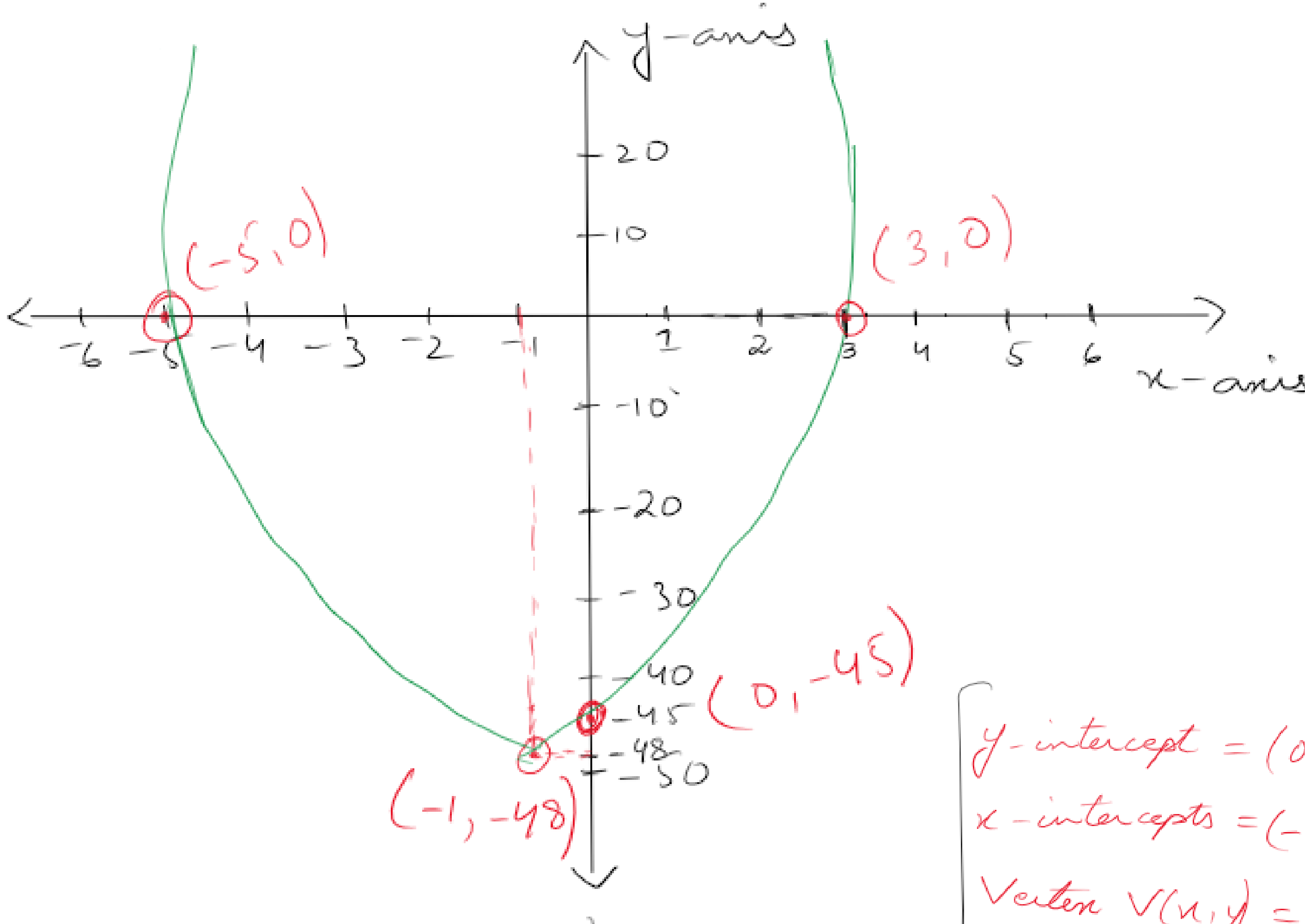
Graphing Parabola:



y-intercept = (0, -45)

x-intercepts = (-5, 0) & (3, 0)

Vertex V(x, y) = (-1, -48)

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PRACTICE EXERCISE

Given $f(x) = 2x^2 - x - 15$, determine: (*a*) concavity, (*b*) y intercept, (*c*) x intercept(s), and (*d*) location of vertex. *Answer:* (*a*) up, (*b*) (0, −15), (*c*) (−2.5, 0) and (3, 0), (*d*) (0.25, −15.125).

For the above question, Repeat all the steps and plot a parabola of the given quadratic function.

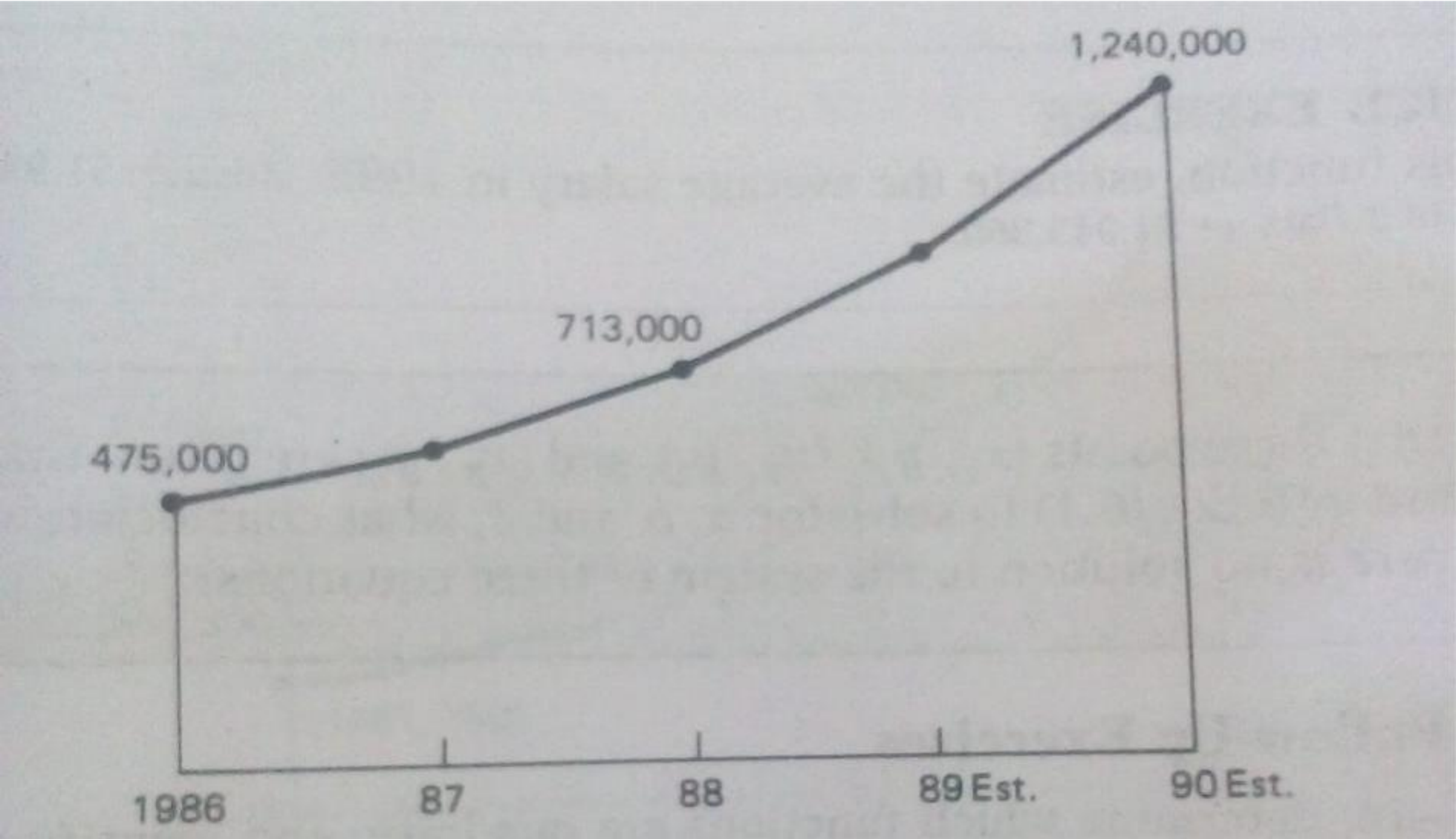
Answers are given to verify your working.

Quadratic applications

Problem 36:

The number of persons working out of their homes has been increasing rapidly in recent years. Fig illustrates data gathered in a study regarding the number of persons working at home 35 or more hours per week. the data appears to be almost quadratic in appearance. Using the data for 1986 and 1988 and the projected value for 1990, determine the quadratic estimating function $n = f(t)$, where n equals the number of persons working 35 or more hours per week at home (stated in thousands) and t equals time measured in years since 1986. According to this function, what is the number of persons working at home expected to equal in 1995 ? in 2000 ?

Graph



Solution

Let the quadratic function be p = f(t) = $at^2 + bt + c$ ………………….equation (I)

Where p= number of persons working at home, stated in thousands and t= time measured in years since 1986 (i.e. 1986 →t=1, 1987 →t=2, etc.). The data for three years of interest is translated as (1, 475), (3, 713) and (5,1240).

Substitute each of the data point in equation I

Solution

Using point (1, 475)

$475 = a(1)^2 + b(1) + c$

$475 = a + b + c$ ……………………………equation A

Using point (3, 713)

$713 = a(3)^2 + b(3) + c$ or $\rightarrow 713 = 9a + 3b + c$

$713 = 9a + 3b + c$ ………………………… .equation B

Solution

Using point (5, 1240)

$1240 = a(5)^2 + b(5) + c$

1240 = 25a +5b +c ....equation C

Using equation A and B (eliminate c)

We get 238=8a +2b (divide by 2 whole equation)

We get 119 = 4a + b ....equation D

Solution

Now using equation B and C (eliminate c again so we get equation of two varaibles)

$527 = 16a + 2b$.................................equation E

From equation D, we can convert $b = 119 - 4a$

Put the value of b in equatioin E

So eqyation E becomes $527 = 16a + 2(119 - 4a)$

$$527 = 16a + 238 - 8a$$

Solution

$a = 36.125$, put the value of a in equation D

$119 = 4(36.125) + b$

$b = -25.5$, put the value of a and b in equation A to get c

$475 = 36.125 - 25.5 + C$

$c = 464.375$

So the equation I becomes, $P = 36.125t^2 - 25.5t + 464.375$

Solution

For 1995 put t=10

$$p = 36.125(10)^2 - 25.5(t) + 464.375$$

$$p = 36334.375$$

P=36334.375*1000

P=36334375 approx.

Solution

For 2000 put t = 15

$$p = 36.125(15)^2 - 25.5(15) + 464.375$$

$$p = 8210$$

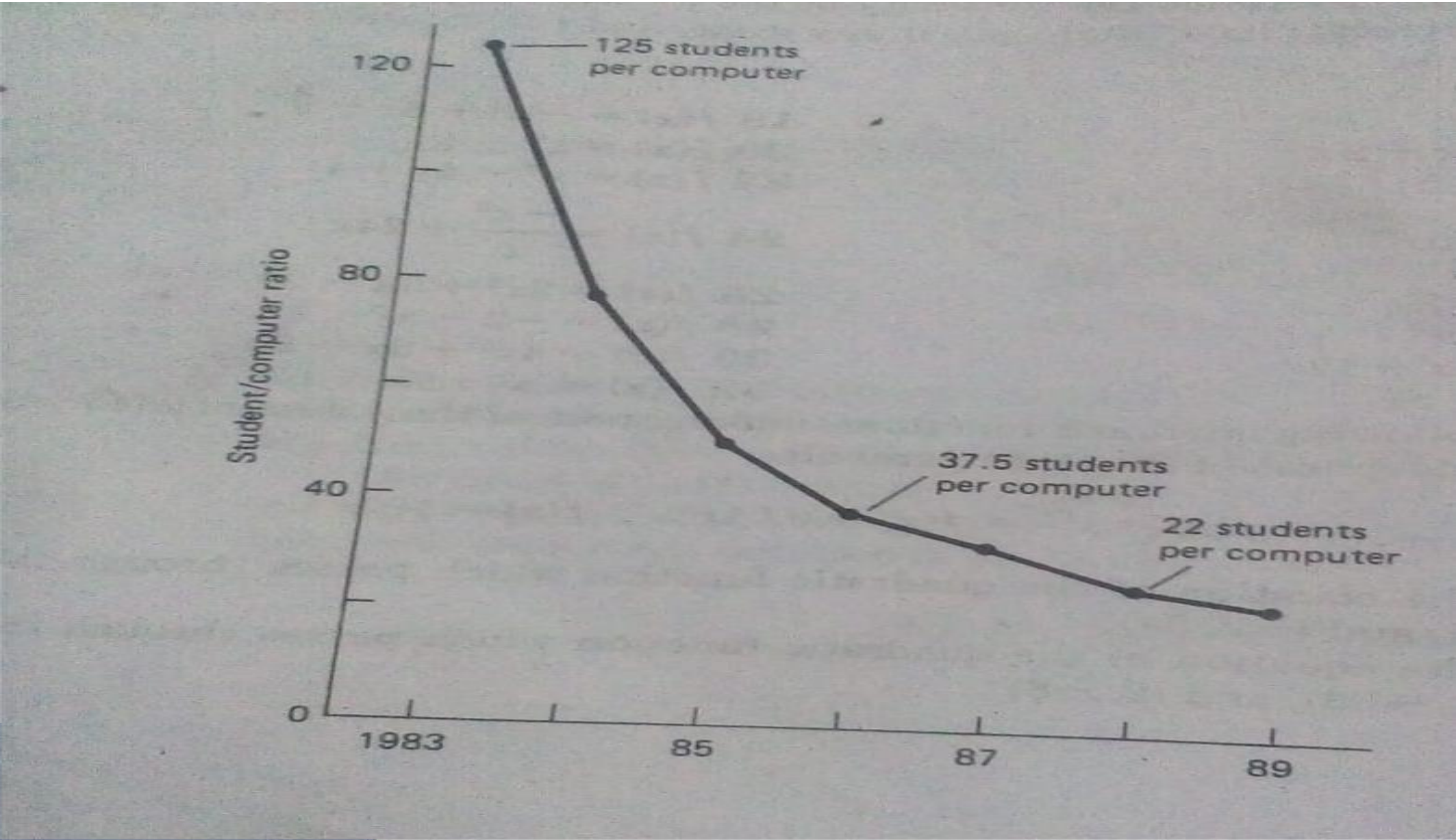
P=8210*1000

P=8210000 approx.

Problem 37

A survey taken during 1990 indicated the increased availability of computers in U.S. public class rooms. Fig displays results of this survey. During the 1983-1984 academic years, the number of student per computer was 125. For the 1986-1987 academic years, the number had dropped to 37.5. For the 1989-1990 academic years, the number was 22 students per computer. Using these three data points, determine the quadratic estimating function n=f(t), where n equals the number of students per computer and t equals time measured in years since the 1983-1984 academic year (i.e., t=0 corresponds to1983-1984). Using this function, estimate the number of students per computer during the 1990-1991 academic years. Based on this result, what conclusion can you reach?

Graph


125 students per computer
37.5 students per computer
22 students per computer
Student/computer ratio
120
80
40
0
1983
85
87
89

Solution

The data from the graph is collected as:

(0, 125), (3, 37.5), (5, 22)

Let the quadratic function be $n = f(t) = at^2 + bt + c$ ....equation (I)

Where n= number of students per computer.

Using point (0,125)

Put in equation I

Solution

$125 = a(0)^2 + b(0) + c$

$c = 125$

Using point (3, 37.5) we get $37.5 = 9a + 3b + 125$

$9a + 3b = -87.5$ ……………………………equation 1

Using point (5, 22) we get $22 = a(5)^2 + b(5) + 125$

$22 = 25a + 5b + 125$

Solution

$25a + 5b = -103$ ……………………….equation 2

Equation 2 can be rewritten as $b = -20.6 - 5a$ ……equation 3

Put the above value of b in equation 1 we get

$a = 4.283$, put this value in 3 we get

$b = -42.016$

The equation **I** become $n = 4.283t^2 - 42.016t + 125$

Problem 38

The cellular telephone industry has grown rapidly since the late 1980's.Fig. presents data regarding the number of subscribers between 1985 and 1990. The pattern of growth in the number of subscribers looks as if it could be approximated reasonably well by using a quadratic function. Using the data points for 1986,1987 and 1989 (200,000, 950,000, and 2600,000 subscribers respectively), determine the quadratic estimating function n=f(t), where n equals the number of subscribers (stated in millions) and t equals time measured in years since 1985. According to this estimating function, how many subscribers are expected in the year 1995?

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graph

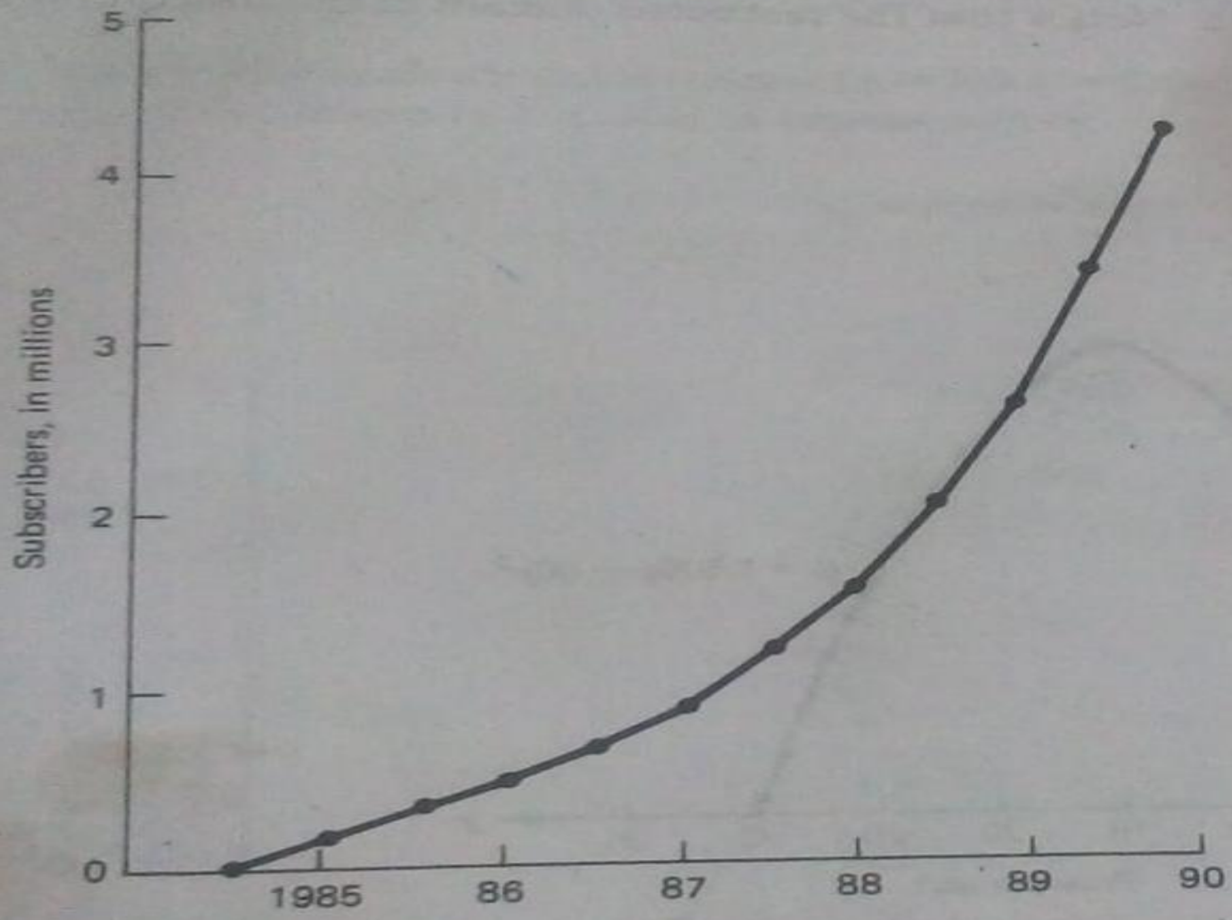

Figure 6.8 Cellular phone use in the United States.
(Source: Cellular Telecommunications Industry Association)

Solution

Let the quadratic function be $n = at^2 + bt + c$

Where n = number of subscribers in millions

Consider 1985 as t→1, 1987 as →t=3 and 1989 as t→5

Convert 200,000, 950000 and 2600000 into millions we get

0.2, 0.95 and 2.6 millions respectively.

Solution

So the data points are (1, 0.2), (3, 0.95) and (5, 2.6)

Using point (1, 0.2)

$0.2 = a + b + c$ ....equation 1

Using point (3, 0.95) we get

Solution

$$0.95 = a(3)^2 + b(3) + c$$

$$0.95 = 9a + 3b + c \ldots\ldots\ldots\ldots\ldots \text{equation 2}$$

$$using\ point\ (5, 2.6)$$

We get $2.6 = a(5)^2 + b(5) + c$

$$2.6 = 25a + 5b + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{equation 3}$$

Subtract equation 1 from equation 2 we get

Solution

0.75= 8a + 2b dividing by 2 we get

0.375 = 4a +b…………..equation 4

Now subtract equation 1 from equation 3 we get

$2.4 = 24a + 4b$ ………………..equation 5

$from\ equation\ 4\ , b = 0.375 - 4a$ ……………..equation 6

Use the above value and put in equation 5

Solution

So equation 5 becomes $2.4 = 24a + 4(0.375 - 4a)$

$2.4 = 24a + 1.5 - 16a \rightarrow 8a = 2.4 - 1.5$

$a = 0.1125$, use the value in equation 6

$b = 0.375 - 4(0.1125)$

$b = 0.2625$ now put a and b in equation 1

$0.2 = 0.1125 + 0.2625 + c, so\ c = -0.175$

Solution

So the quadratic function can be written as

$$n = 0.1125t^2 + 0.2625t - 0.175$$

For 1995 $put\ t = 11$

So, $n = 0.1125(11)^2 + 0.2625(11) - 0.175$

$n = 16.325$ millions

Review Question

Sketch the following function and compare with the one which we have plotted earlier in this lecture

$$f(x) = -3x^2 + 6x - 45$$

Learning Material

- https://study.com/academy/topic/quadratic-equations-and-functions-business-math-lesson-plans.html
- http://www.montereyinstitute.org/courses/Algebra1/COURSE_TEXT_RESOURCE/U10_L2_T1_text_container.html

Thank you