Introduction to Mathematics

Matrix Algebra

Week 11

Welcome!!

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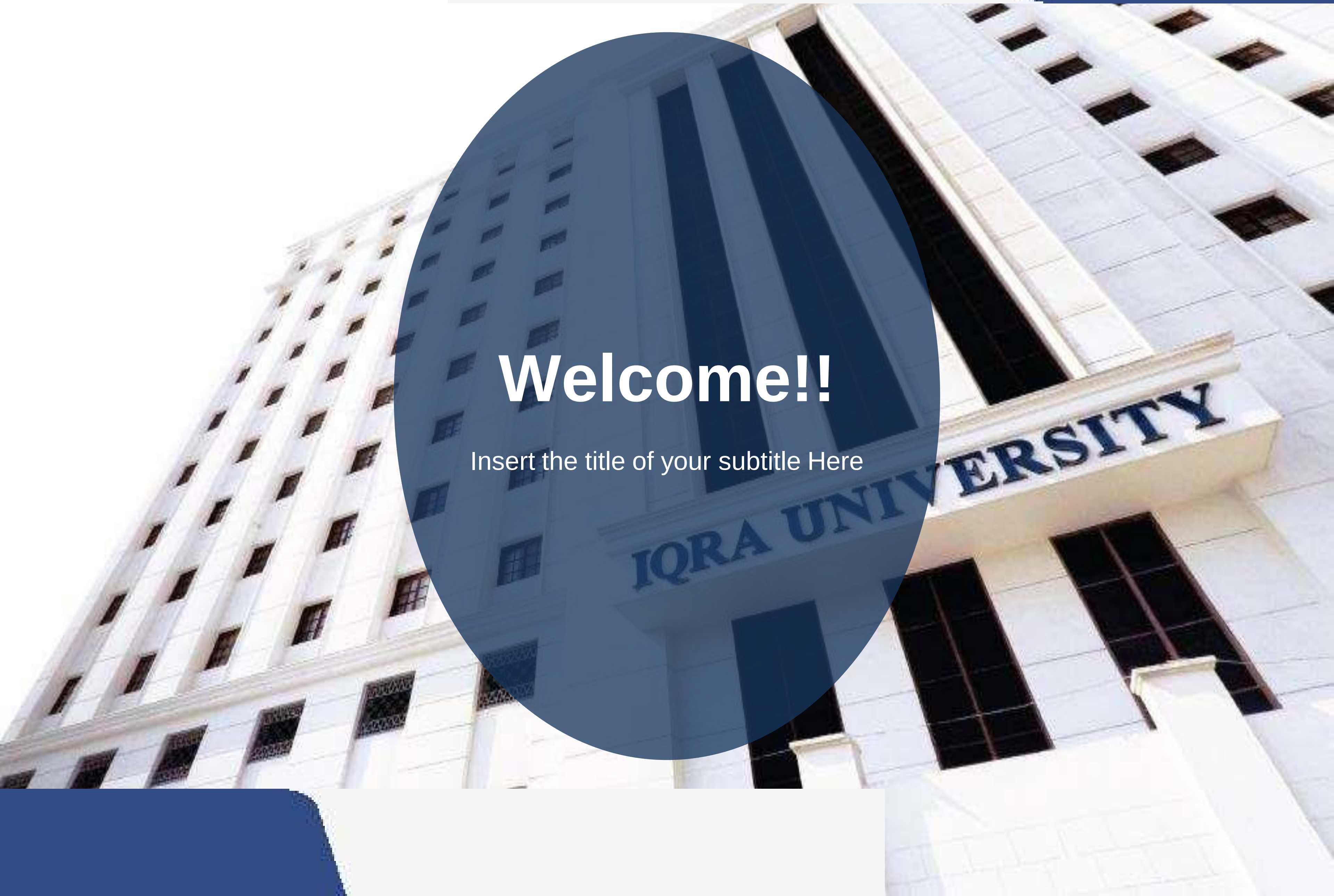

Matrix Algebra

- Determinant of Matrix
- Inverse of Matrix
- Crammer’s Rule

Learning Outcomes

- Know how to solve linear equations using Cramer’s Rule.
- To Compute matrix addition and multiplication.
- To Compute Inverse Matrix.

Matrices - Operations

INVERSE OF A MATRIX

Consider a scalar k. The inverse is the reciprocal or division of 1 by the scalar.

Example:

k=7 the inverse of k or $k^{-1} = 1/k = 1/7$

Division of matrices is not defined since there may be **AB** = **AC** while **B** = **C**

Instead matrix inversion is used.

The inverse of a square matrix, **A**, if it exists, is the unique matrix $\mathbf{A}^{-1}$ where:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Matrices - Operations

Example:

$$A = {}_2A^2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Because:

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrices - Operations

Properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{-1})^{-1} = A$$

$$(A^{T})^{-1} = (A^{-1})^{T}$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

A square matrix that has an inverse is called a nonsingular matrix

A matrix that does not have an inverse is called a singular matrix

Square matrices have inverses except when the determinant is zero

When the determinant of a matrix is zero the matrix is singular

Matrices - Operations

DETERMINANT OF A MATRIX

To compute the inverse of a matrix, the determinant is required

Each square matrix **A** has a unit scalar value called the determinant of **A**, denoted by det **A** or **|A|**

If $$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

then $$|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$$

Matrices - Operations

If **A** = [**A**] is a single element (1x1), then the determinant is defined as the value of the element

Then |**A**| =det **A** = a_{11}

If **A** is (n x n), its determinant may be defined in terms of order (n-1) or less.

Matrices - Operations

MINORS

If **A** is an n x n matrix and one row and one column are deleted, the resulting matrix is an (n-1) x (n-1) submatrix of **A**.

The determinant of such a submatrix is called a minor of **A** and is designated by m_{ij} , where *i* and *j* correspond to the deleted

row and column, respectively.

m_{ij} is the minor of the element a_{ij} in **A**.

Matrices - Operations

eg.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor

Delete first row and column from **A** .

The determinant of the remaining 2 x 2 sub-matrix is the minor of a_{11}

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Matrices - Operations

Therefore the minor of a_{12} is:

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Matrices - Operations

COFACTORS

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number i and column j is even, $c_{ij} = m_{ij}$ and when $i+j$ is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

Matrices - Operations

DETERMINANTS CONTINUED

The determinant of an n x n matrix **A** can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$$

The determinant of **A** is therefore the sum of the products of the elements of the first row of **A** and their corresponding cofactors.

(It is possible to define |**A**| in terms of any other row or column but for simplicity, the first row only is used)

Matrices - Operations

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Matrices - Operations

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A| = (3)(2) - (1)(1) = 5$$

Matrices - Operations

For a 3 x 3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Matrices - Operations

The determinant of a matrix A is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Matrices - Operations

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

Matrices - Operations

ADJOINT MATRICES

A cofactor matrix **C** of a matrix **A** is the square matrix of the same order as **A** in which each element a_{ij} is replaced by its cofactor c_{ij} .

Example:

If

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The cofactor C of A is

$$C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

Matrices - Operations

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^T$$

It can be shown that:

A(adj **A**) = (adj**A**) **A** = |**A**| **I**

Example:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$adjA = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Matrices - Operations

$$A(adjA) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

$$(adjA)A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

Matrices - Operations

USING THE ADJOINT MATRIX IN MATRIX INVERSION

Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

and

$$\mathbf{A}(\text{adj }\mathbf{A}) = (\text{adj}\mathbf{A})\,\mathbf{A} = |\mathbf{A}|\,\mathbf{I}$$

then

$$A^{-1} = \frac{adjA}{|A|}$$

Matrices - Operations

Example

A = $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

To check

AA^{-1} = A^{-1} A = I

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices - Operations

Example 2

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of **A** is

|**A**| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2

The elements of the cofactor matrix are

$$c_{11} = +(-1), \qquad c_{12} = -(-2), \qquad c_{13} = +(3),$$
$$c_{21} = -(-1), \qquad c_{22} = +(-4), \qquad c_{23} = -(7),$$
$$c_{31} = +(-1), \qquad c_{32} = -(-2), \qquad c_{33} = +(5),$$

Matrices - Operations

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$adjA = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

Matrices - Operations

The result can be checked using

$$\mathbf{AA^{-1}} = \mathbf{A^{-1}A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants

Matrix Inversion

Simple 2 x 2 case

Simple 2 x 2 case

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Since it is known that

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Simple 2 x 2 case

Multiplying gives

$$aw + by = 1$$

$$ax + bz = 0$$

$$cw + dy = 0$$

$$cx + dz = 1$$

It can simply be shown that

$$|A| = ad - bc$$

Simple 2 x 2 case

thus

$$y = \frac{1 - aw}{b}$$

$$y = \frac{-cw}{d}$$

$$\frac{1 - aw}{b} = \frac{-cw}{d}$$

$$w = \frac{d}{da - bc} = \frac{d}{|A|}$$

Simple 2 x 2 case

$$z = \frac{-ax}{b}$$

$$z = \frac{1 - cx}{d}$$

$$\frac{-ax}{b} = \frac{1 - cx}{d}$$

$$x = \frac{b}{-da + bc} = -\frac{b}{|A|}$$

Simple 2 x 2 case

$$w = \frac{1 - by}{a}$$

$$w = \frac{-dy}{c}$$

$$\frac{1 - by}{a} = \frac{-dy}{c}$$

$$y = \frac{c}{-ad + cb} = -\frac{c}{|A|}$$

Simple 2 x 2 case

$$x = \frac{-bz}{a}$$

$$x = \frac{1-dz}{c}$$

$$\frac{-bz}{a} = \frac{1-dz}{c}$$

$$z = \frac{a}{ad-bc} = \frac{a}{|A|}$$

Simple 2 x 2 case

So that for a 2 x 2 matrix the inverse can be constructed in a simple fashion as

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} \frac{d}{|A|} & \frac{b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Exchange elements of main diagonal
- Change sign in elements off main diagonal
- Divide resulting matrix by the determinant

Simple 2 x 2 case

Example

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{bmatrix}$$

Check inverse

A^{-1} A=I

$$-\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices and Linear Equations

Linear Equations

Linear Equations

Linear equations are common and important for survey problems

Matrices can be used to express these linear equations and aid in the computation of unknown values

Example

n equations in *n* unknowns, the a_{ij} are numerical coefficients, the b_i are constants and the x_j are unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Linear Equations

The equations may be expressed in the form

AX = **B**

where

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n1} \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

n x n n x 1 n x 1

Number of unknowns = number of equations = n

Linear Equations

If the determinant is nonzero, the equation can be solved to produce n numerical values for x that satisfy all the simultaneous equations

To solve, premultiply both sides of the equation by $\mathbf{A}^{-1}$ which exists because $|\mathbf{A}| = \mathbf{0}$

/

$$\mathbf{A}^{-1}\,\mathbf{AX} = \mathbf{A}^{-1}\,\mathbf{B}$$

Now since

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

We get

$$\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$$

So if the inverse of the coefficient matrix is found, the unknowns, **X** would be determined

Linear Equations

Example

$$3x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 - x_3 = 3$$

The equations can be expressed as

$$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Linear Equations

When **A**$^{-1}$ is computed the equation becomes

$$X = A^{-1}B = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$$

Therefore

$$x_1 = 2,$$
$$x_2 = -3,$$
$$x_3 = -7$$

Linear Equations

The values for the unknowns should be checked by substitution back into the initial equations

$$x_1 = 2, \quad x_2 = -3, \quad x_3 = -7$$

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 - x_3 &= 3 \end{aligned}$$

$$\begin{aligned} 3\times(2) - (-3) + (-7) &= 2 \\ 2\times(2) + (-3) &= 1 \\ (2) + 2\times(-3) - (-7) &= 3 \end{aligned}$$

Cramer's Rule

Solution of System of 2 Linear Equations (Cramer's Rule)

Let the system of linear equations be

$$a_1 x + b_1 y = c_1 \qquad \ldots(i)$$

$$a_2 x + b_2 y = c_2 \qquad \ldots(ii)$$

Then $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ provided $D \neq 0$,

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Cramer's Rule

Note :

(1) If $D \neq 0$,

then the system is consistent and has unique solution.

(2) If $D = 0$ and $D_1 = D_2 = 0$,

then the system is consistent and has infinitely many solutions.

(3) If $D = 0$ and one of D_1, $D_2 \neq 0$,

then the system is inconsistent and has no solution.

Cramer's Rule

Example

Using Cramer's rule , solve the following system of equations 2x-3y=7, 3x+y=5

Solution :

$$D=\begin{vmatrix}2 & -3\\ 3 & 1\end{vmatrix}=2+9=11\neq 0$$

$$D_1=\begin{vmatrix}7 & -3\\ 5 & 1\end{vmatrix}=7+15=22$$

$$D_2=\begin{vmatrix}2 & 7\\ 3 & 5\end{vmatrix}=10-21=-11$$

$$\because D\neq 0$$

$$\therefore \text{By Cramer's Rule } x=\frac{D_1}{D}=\frac{22}{11}=2 \text{ and } y=\frac{D_2}{D}=\frac{-11}{11}=-1$$

Cramer's Rule

Solution of System of 3 Linear Equations (Cramer's Rule)

Let the system of linear equations be

$$a_1x + b_1y + c_1z = d_1 \qquad \ldots(i)$$

$$a_2x + b_2y + c_2z = d_2 \qquad \ldots(ii)$$

$$a_3x + b_3y + c_3z = d_3 \qquad \ldots(iii)$$

Then $x = \dfrac{D_1}{D},\ y = \dfrac{D_2}{D},\ z = \dfrac{D_3}{D}$ provided $D \neq 0$,

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Cramer's Rule

Note:

(1) If $D \neq 0$, then the system is consistent and has a unique solution.

(2) If $D=0$ and $D_1 = D_2 = D_3 = 0$, then the system has infinite solutions or no solution.

(3) If $D = 0$ and one of $D_1, D_2, D_3 \neq 0$, then the system is inconsistent and has no solution.

(4) If $d_1 = d_2 = d_3 = 0$, then the system is called the system of homogeneous linear equations.

(i) If $D \neq 0$, then the system has only trivial solution $x = y = z = 0$.

(ii) If $D = 0$, then the system has infinite solutions.

Cramer's Rule

Example

Using Cramer's rule , solve the following system of equations

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

Solution :

$$D = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

$= 5(18+10) + 1(12-25)+4(-4 -15)$

$= 140 – 13 – 76 = 140 - 89$

$= 51 \neq 0$

$$D_1 = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ -1 & -2 & 6 \end{vmatrix}$$

$= 5(18+10)+1(12+5)+4(-4 +3)$

$= 140 +17 – 4$

$= 153$

Cramer’s Rule

$$D_2 = \begin{vmatrix} 5 & 5 & 4 \\ 2 & 2 & 5 \\ 5 & -1 & 6 \end{vmatrix}$$

= 5(12 +5)+5(12 - 25)+ 4(-2 - 10)
= 85 + 65 – 48 = 150 - 48
= 102

$$D_3 = \begin{vmatrix} 5 & -1 & 5 \\ 2 & 3 & 2 \\ 5 & -2 & -1 \end{vmatrix}$$

= 5(-3 +4)+1(-2 - 10)+5(-4-15)
= 5 – 12 – 95 = 5 - 107
= - 102

$\because D \neq 0$

$\therefore$ By Cramer's Rule $x = \frac{D_1}{D} = \frac{153}{51} = 3,\ y = \frac{D_2}{D} = \frac{102}{51} = 2$

and $z = \frac{D_3}{D} = \frac{-102}{51} = -2$

Cramer's Rule

Example

Solve the following system of homogeneous linear equations:

$x + y - z = 0$, $x - 2y + z = 0$, $3x + 6y + -5z = 0$

Solution:

$$\text{We have } D = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 3 & 6 & -5 \end{bmatrix} = 1(10-6) - 1(-5-3) - 1(6+6)$$

$$= 4 + 8 - 12 = 0$$

$\therefore$ The system has infinitely many solutions.

Putting $z = k$, in first two equations, we get

$x + y = k$, $x - 2y = -k$

Cramer's Rule

$$\therefore \text{By Cramer's rule} \quad x = \frac{D_1}{D} = \frac{\begin{vmatrix} k & 1 \\ -k & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-2k + k}{-2 - 1} = \frac{k}{3}$$

$$y = \frac{D_2}{D} = \frac{\begin{vmatrix} 1 & k \\ 1 & -k \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-k - k}{-2 - 1} = \frac{2k}{3}$$

These values of x, y and z = k satisfy (iii) equation.

$$\therefore x = \frac{k}{3}, \; y = \frac{2k}{3}, \; z = k, \quad \text{where } k \in R$$

Learning Material

- https://ebookbou.edu.bd/Books/Text/SOB/MBA/mba_2306/Unit-11.pdf
- https://nptel.ac.in/courses/122104018/
- http://www.ilectureonline.com/lectures/subject/MATH/36/272

Activity

1. Find the inverse of the matrix, $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & 2 \end{bmatrix}$

2. Find the inverse of the matrix, $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$

3. Solve the following system of equations by using Gaussian method.

$$2x - 5y + 7z = 6$$
$$x - 3y + 4z = 3$$
$$3x - 8y + 11z = 11$$

4. Use matrix inversion to solve the following system of equations:

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

5. Use matrix inversion to solve the following system of equations:

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + 2y + 9z = 14$$

Thank you

Insert the title of your subtitle Here