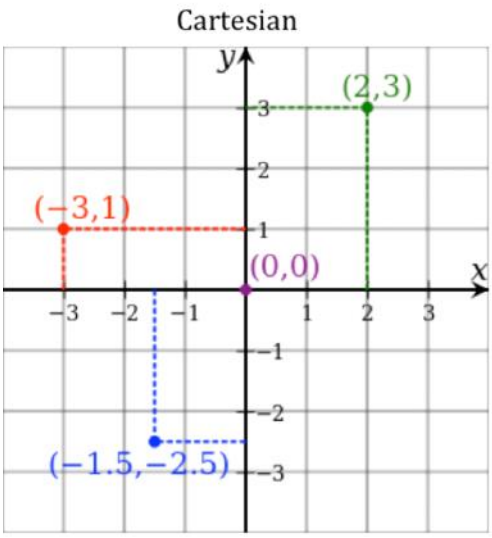
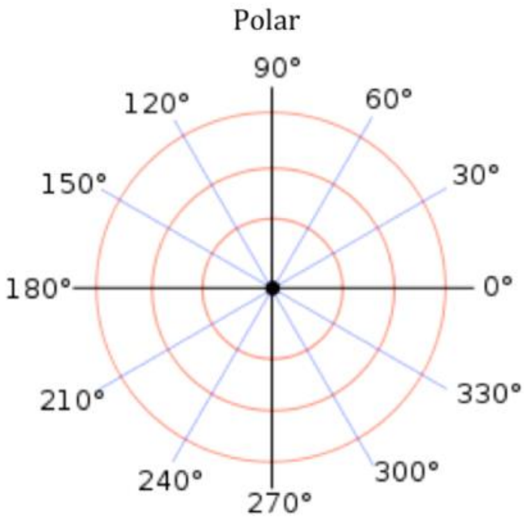
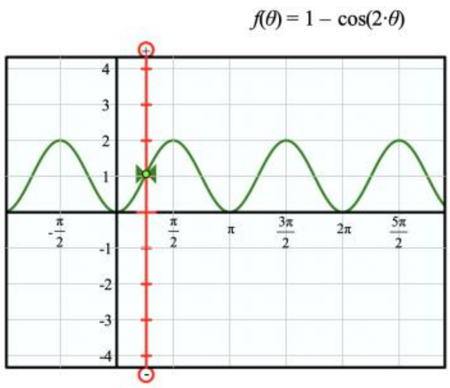
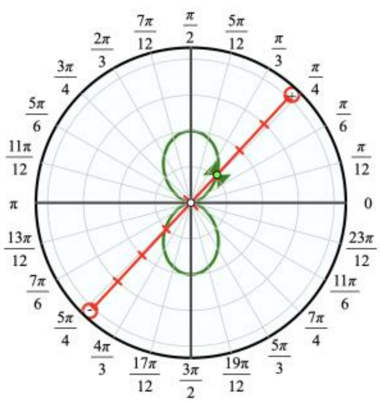
Geometry

Monday, 5 January 2026 11:05 am









Convert between Rectangular and Polar Coordinates

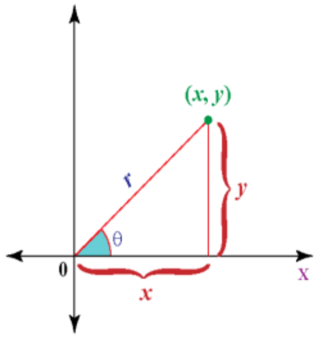
Rectangular Coordinates: (x, y) ⟷ Polar Coordinates: (r,θ)

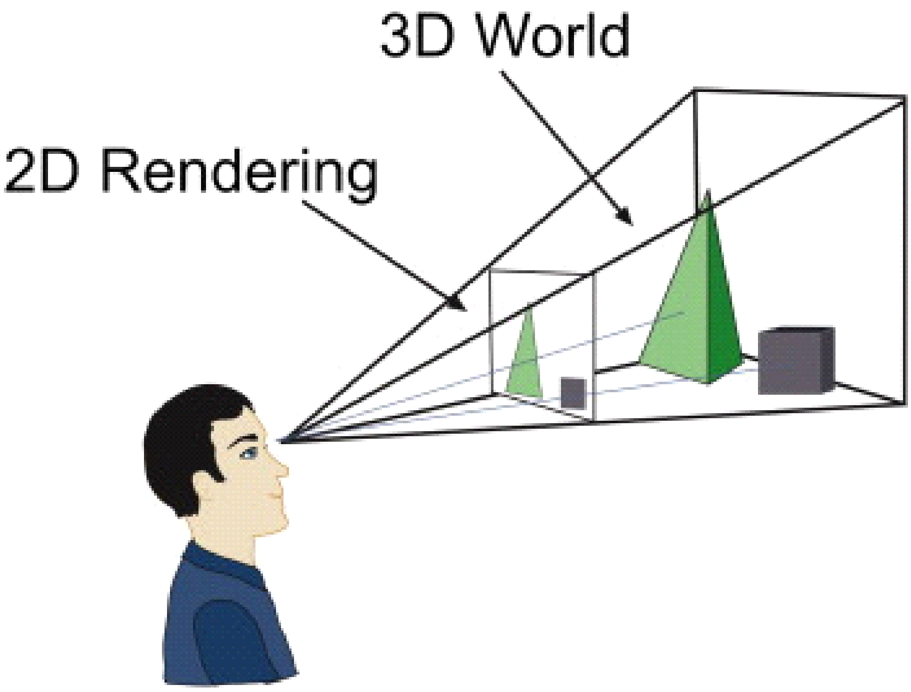
Convert from rectangular to polar coordinates

$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$

Convert from rectangular to polar coordinates

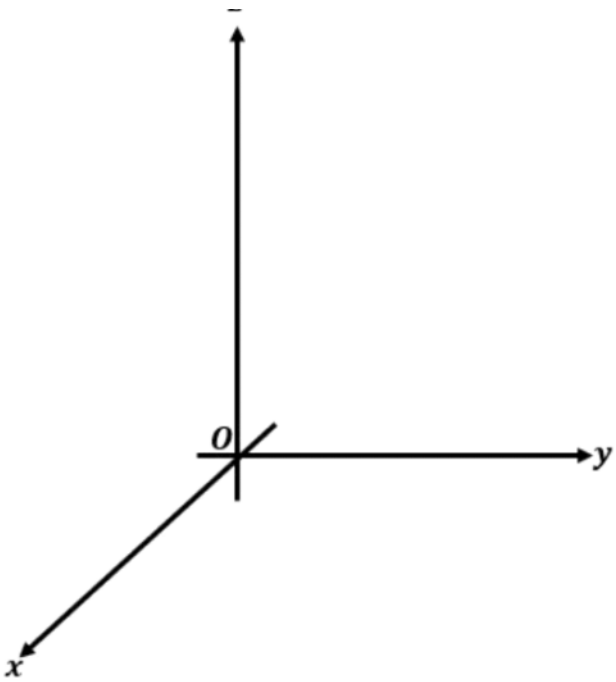
$x = r\cos\theta \qquad y = r\sin\theta$



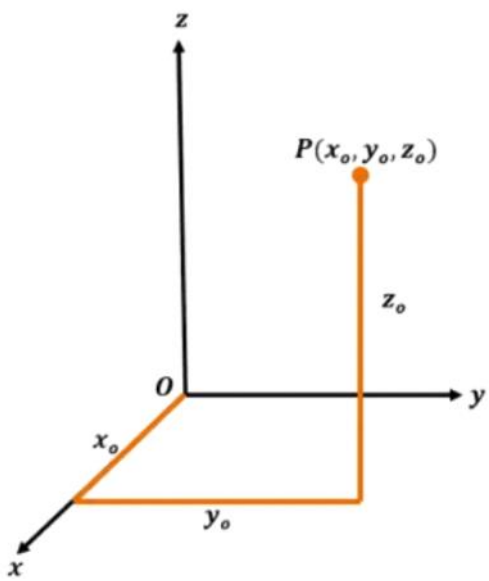


What is the three-dimensional coordinate system?

The three-dimensional coordinate system contains an origin (normally denoted by O) and formed by three mutually perpendicular coordinate axes: the x-axis, y-axis, and the z-axis.

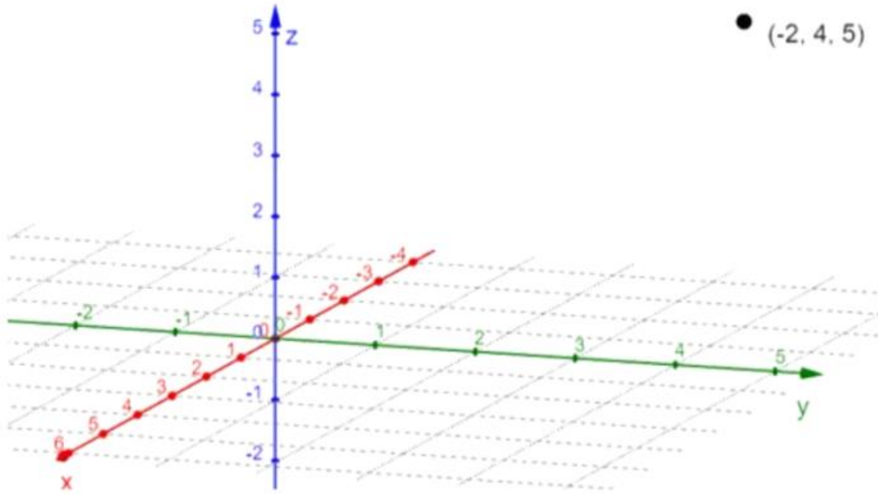


Now, if we want to locate the point in space, we'll need three numbers :x, y, and z. We call (x, y, z), the **ordered triple**. This is how we established the origin of the 3D coordinate system. Instead of two axes, we now have three **coordinate axes** that are mutually perpendicular to each other. In 3D coordinate systems, we normally view the x **and** y**-axis as the two horizontal axes that are perpendicular to each other**. The z**-axis** becomes the **sole vertical axis** in three dimensions.

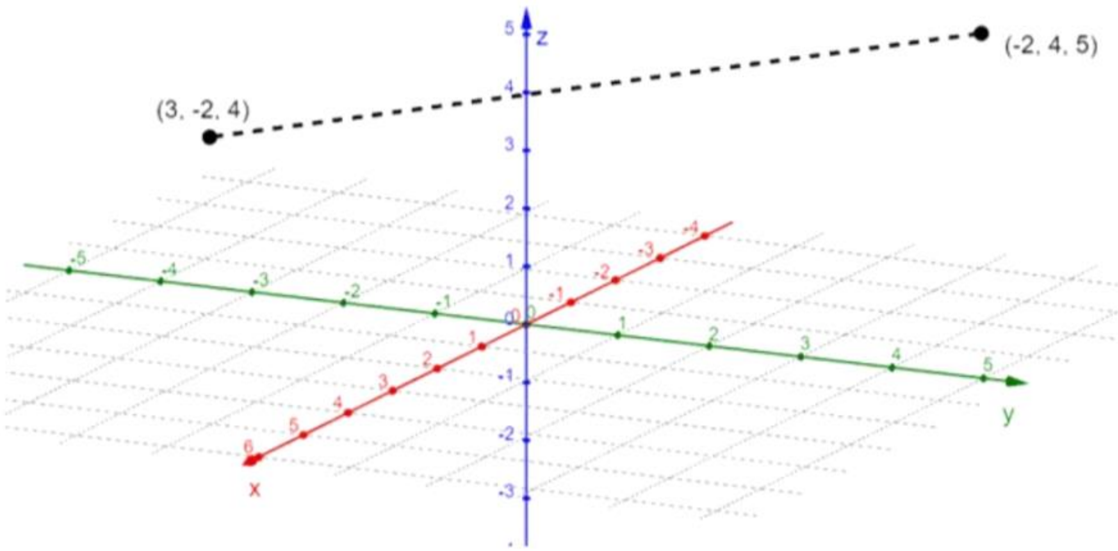


Let's try plotting the coordinate point, $(-2, 4, 5)$, on our 3D rectangular coordinate system.

- From the origin, move -2 units along the x-axis. This means that the point will be behind the first octant, where x is negative.
- Now, the point must be 4 units parallel to the y-axis and 5 units parallel to the z-axis.
- You can construct a rectangular box to guide in plotting $(-2, 4, 5)$.



This graph shows the point, $(-2, 4, 5)$ plotted in the three-dimensional rectangular coordinate system. Now, in the same space, graph $(3, -2, 4)$ as shown below.



DISTANCE FORMULA FOR THE 3D COORDINATE SYSTEM

Suppose that we have two points in space: $P(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. The distance between the two points can be calculated using the distance formula shown below:

$$|P_1P_2| = \sqrt{(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2}$$

Let's try to apply this formula to calculate the distance between the two points: $P = (3, -2, 4)$ and $Q = (-2, 4, 5)$.

$$\begin{aligned}|PQ| &= \sqrt{(3 – -2)^2 + (-2 – 4)^2 + (4 – 5)^2}\\ &= \sqrt{25 + 36 + 1}\\ &= \sqrt{62}\end{aligned}$$

Example 1 Graph $x = 3$ in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

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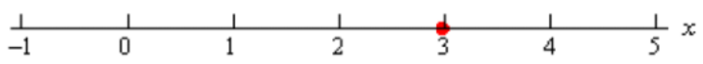
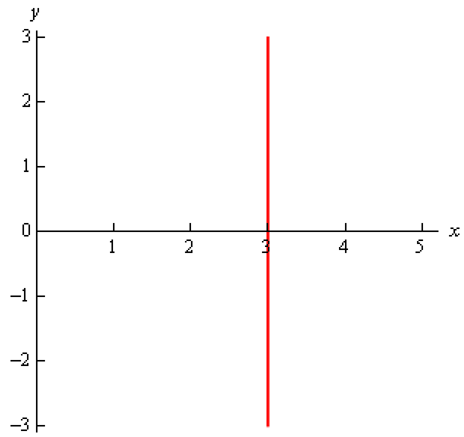
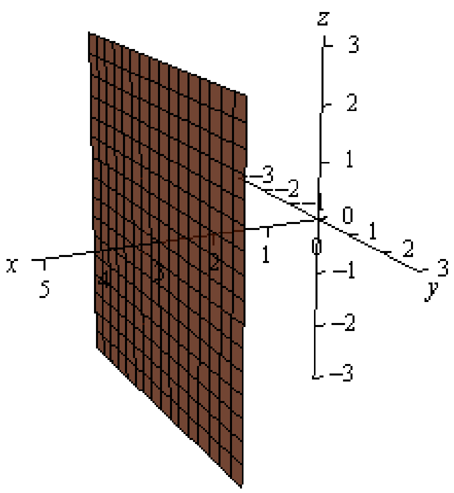
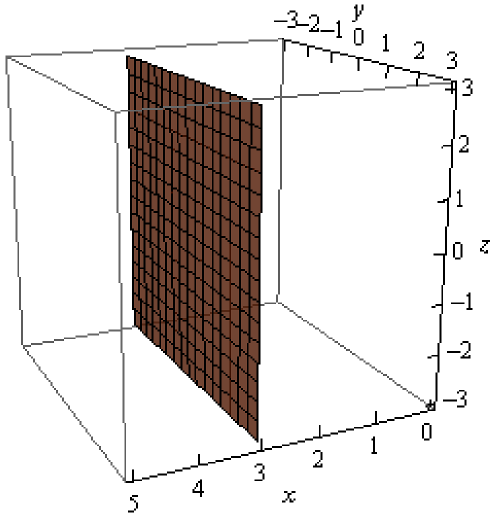
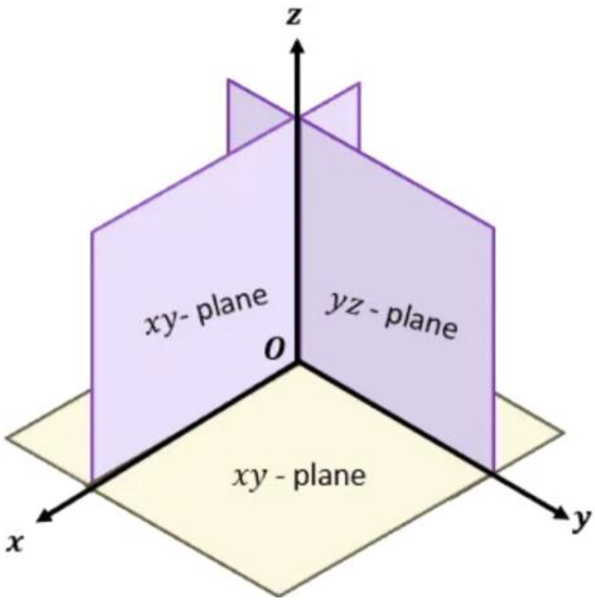


Here is the graph of $x = 3$ in $\mathbb{R}^2$.









COORDINATES IN THREE-DIMENSIONAL SYSTEM

Since our point is now positioned in a space, the point $P(x_o, y_o, z_o)$ will be located:

- x_o units from the yz-plane
- y_o units from the xz-plane
- z_o units from the xy-plane

The point P represented by the ordered triple, (x_o, y_o, z_o), we call x_o, y_o, and z_o the coordinates of P. Just like in the rectangular coordinate system, we call x_o the x-coordinate, y_o the y-coordinate, and z_o the z-coordinate.

Example 6 What do the graphs of the equations $z = 0$, $z = 3$, and $z = -1$ look like?

Solution To graph $z = 0$, we visualize the set of points whose z-coordinate is zero. If the z-coordinate is 0, then we must be at the same vertical level as the origin, that is, we are in the horizontal plane containing the origin. So the graph of $z = 0$ is the middle plane in Figure 12.6. The graph of $z = 3$ is a plane parallel to the graph of $z = 0$, but three units above it. The graph of $z = -1$ is a plane parallel to the graph of $z = 0$, but one unit below it.

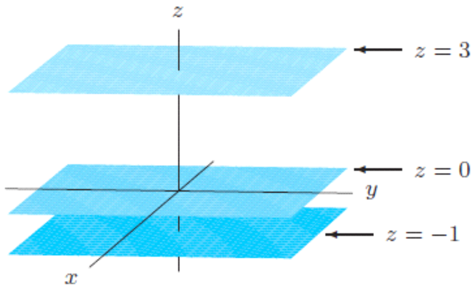


Figure 12.6: The planes $z = -1$, $z = 0$, and $z = 3$

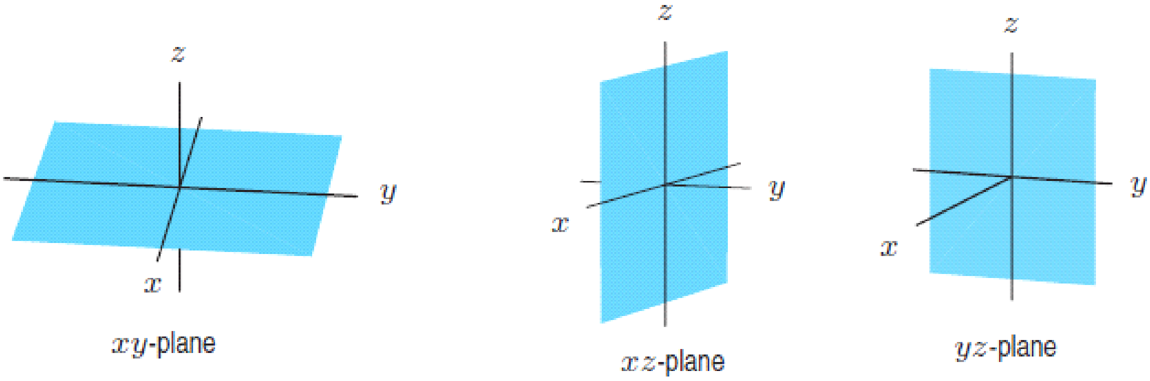
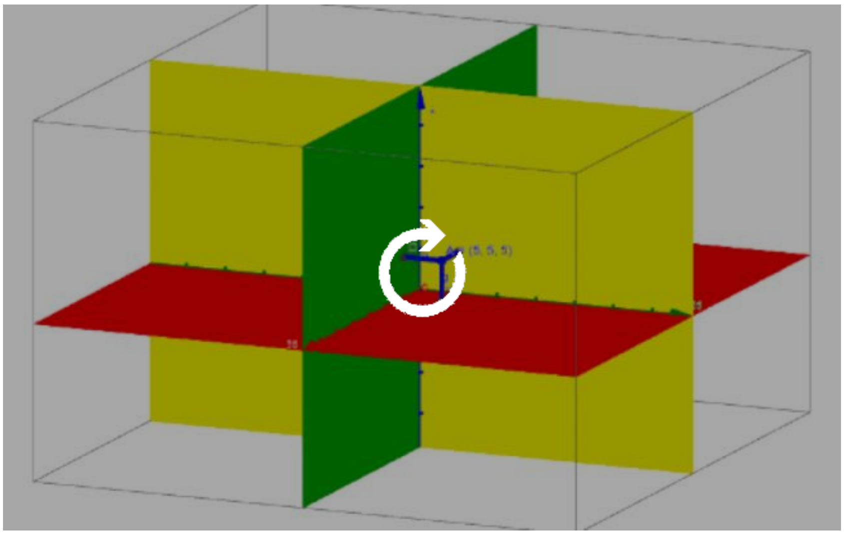


Figure 12.7: The three coordinate planes

[Three Dimensional Coordinate System with Axes Planes]



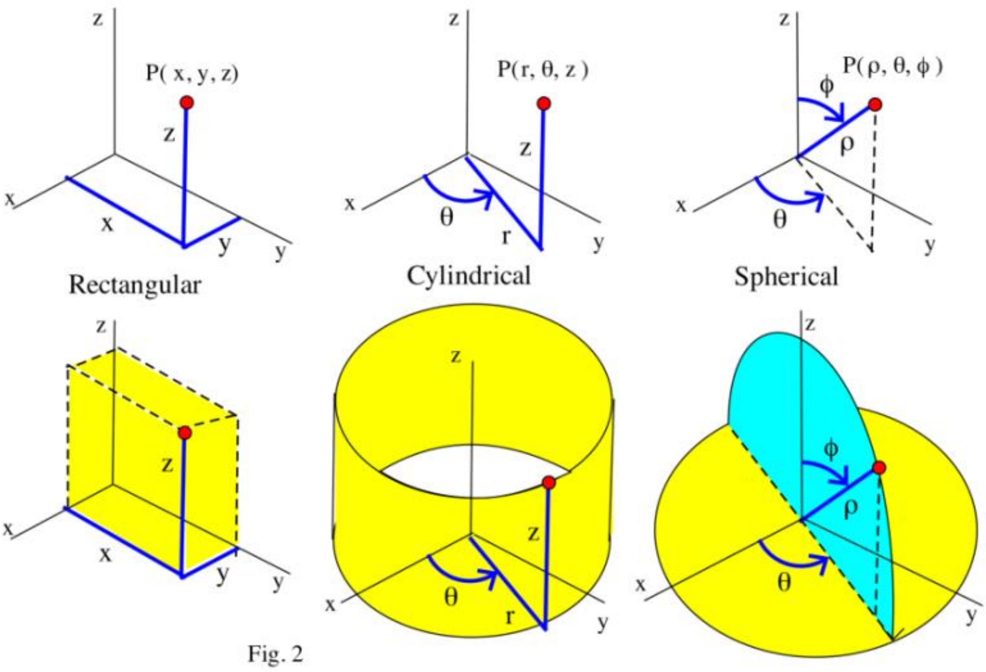


Fig. 2

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$\begin{aligned} r &= \rho \sin \phi, \qquad & x &= r \cos \theta = \rho \sin \phi \cos \theta, \\ z &= \rho \cos \phi, \qquad & y &= r \sin \theta = \rho \sin \phi \sin \theta, \end{aligned} \tag{1}$$

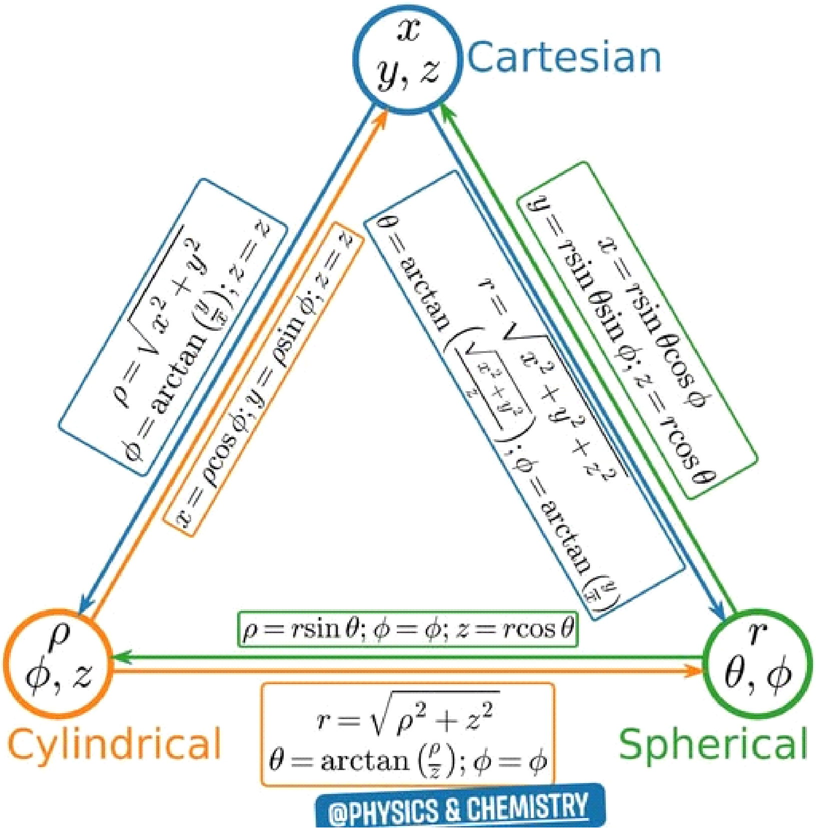
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r \cos \theta$	$x = \rho \sin \phi \cos \theta$	$r = \rho \sin \phi$
$y = r \sin \theta$	$y = \rho \sin \phi \sin \theta$	$z = \rho \cos \phi$
$z = z$	$z = \rho \cos \phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\, dy\, dz \\ &= dz\, r\, dr\, d\theta \\ &= \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta \end{aligned}$$



x y, z
Cartesian
ρ ϕ, z
Cylindrical
r θ, ϕ
Spherical
$\rho = \sqrt{x^2+y^2}$
$\phi = \arctan\left(\frac{y}{x}\right); z = z$
$x = \rho\cos\phi; y = \rho\sin\phi; z = z$
$r = \sqrt{x^2+y^2+z^2}$
$\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right); \phi = \arctan\left(\frac{y}{x}\right)$
$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi; z = r\cos\theta$
$\rho = r\sin\theta; \phi = \phi; z = r\cos\theta$
$r = \sqrt{\rho^2+z^2}$
$\theta = \arctan\left(\frac{\rho}{z}\right); \phi = \phi$
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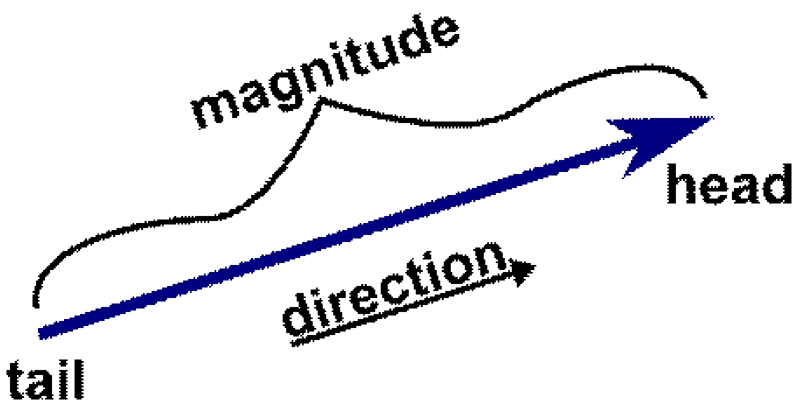
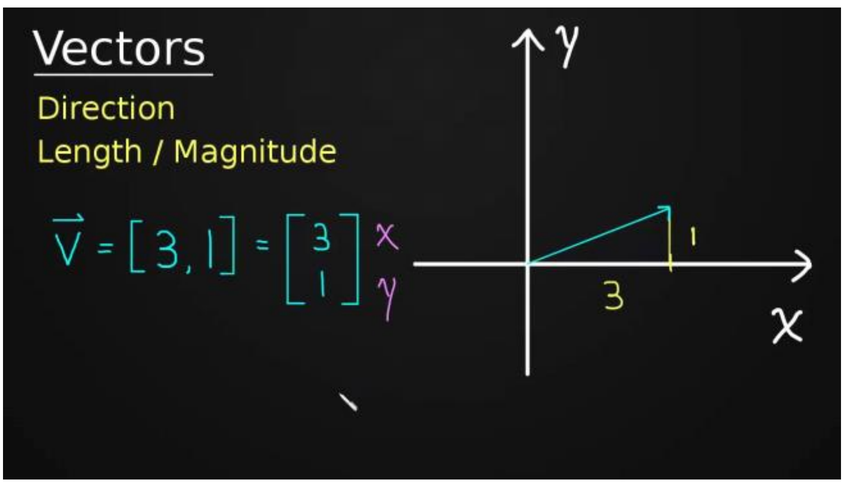
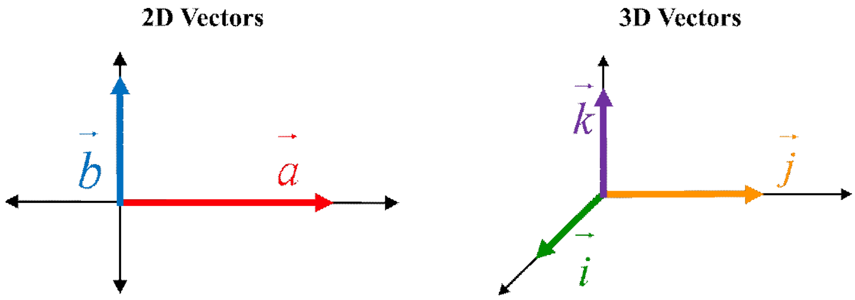
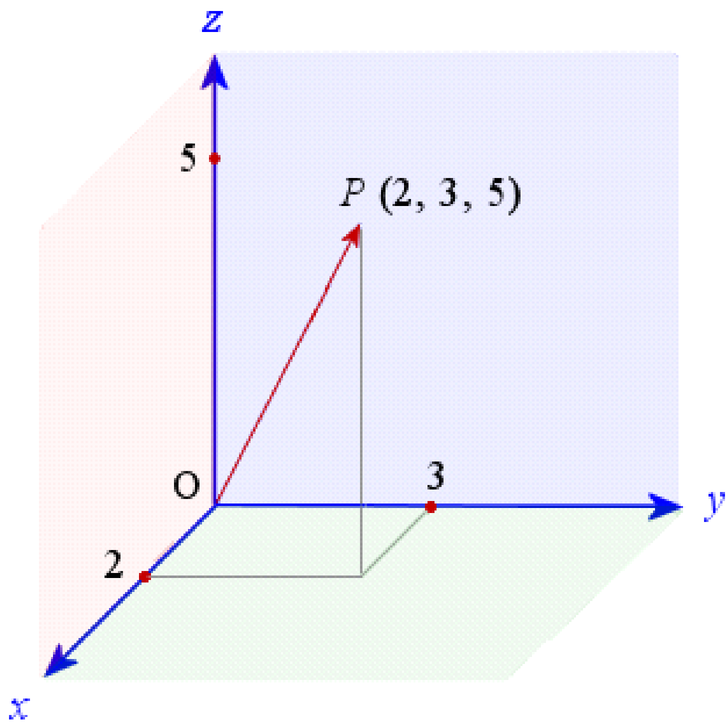


magnitude
head
direction
tail



Vectors
Direction
Length / Magnitude
$\vec{V} = [3, 1] = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{matrix} x \\ y \end{matrix}$
y
x
3
1





2.1 Direction Cosines

Consider a vector $\mathbf{v} = \langle a, b, c \rangle$. Let α, β, and γ be the angles that $\mathbf{v}$ makes with the x-, y-, and z-axes respectively.

The **direction cosines** are defined as:

$$\cos\alpha = \frac{a}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{b}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{c}{\|\mathbf{v}\|}$$

where

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}.$$

They satisfy the identity:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Example 1:

Determine the direction cosine of a line joining the point $(-4, 2, 3)$ with the origin.

Solution:

Given that, the line joins the origin $(0, 0, 0)$ and the point $(-4, 2, 3)$. Hence, the direction ratios are $-4, 2, 3$.

Also, the magnitude of a line = $\sqrt{[(-4)^2+(2)^2+(3)^2]}$

$= \sqrt{(16+4+9)} = \sqrt{29}$.

Therefore, the direction cosines are $((-4/\sqrt{29}), (2/\sqrt{29}), (3/\sqrt{29}))$.

Question 1. If a line makes angles of 90°, 60°, and 30° with the positive direction of x, y, and z-axis respectively. Find its direction cosines.

Solution:

Let us considered the direction cosines of line be l, m, n.

As we know that direction cosines of a line will be cosines

of the angle made with x, y and z axis.

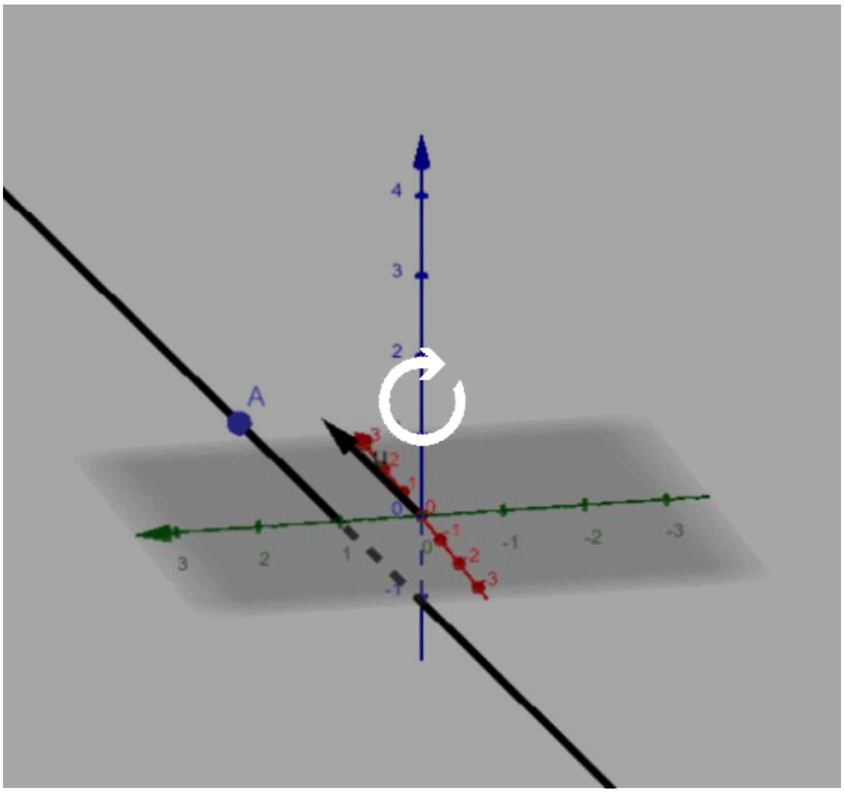
$l = \cos 90° = 0$

$m = \cos 60° = 1/2$

$n = \cos 30° = \sqrt{3}/2$

Hence, the direction cosines of line are $0, 1/2, \sqrt{3}/2$.

Analytic Geometry in 3D

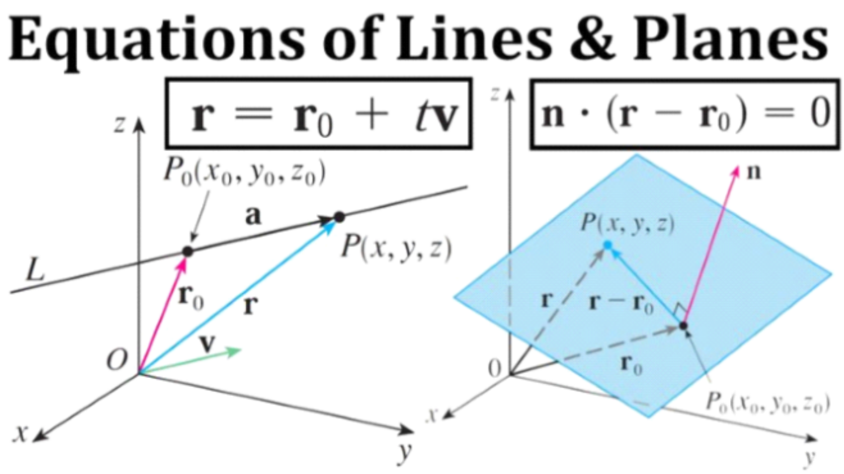


A
u
4
3
2
0
1
-1
-2
-3

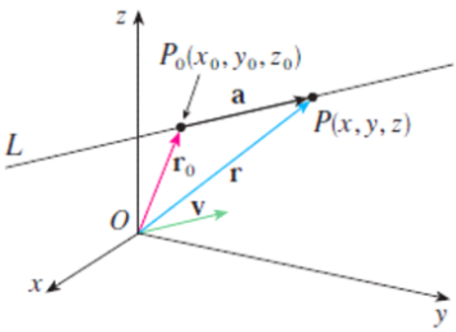
Equation of Line & Plane

Monday, 9 June 2025 5:39 pm

Equations of Lines and Planes



$$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$



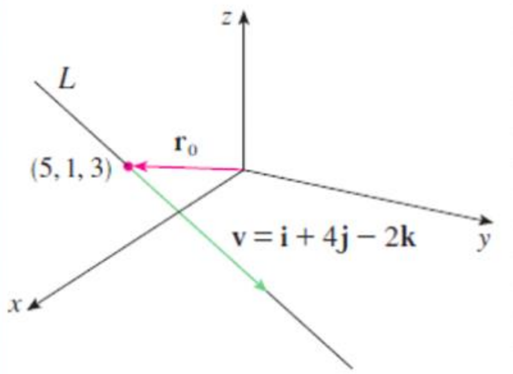
$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$\boxed{x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct}$$

EXAMPLE 1

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

(b) Find two other points on the line.



SOLUTION

(a) Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation $\boxed{1}$ becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or
$$\mathbf{r} = (5 + t)\,\mathbf{i} + (1 + 4t)\,\mathbf{j} + (3 - 2t)\,\mathbf{k}$$

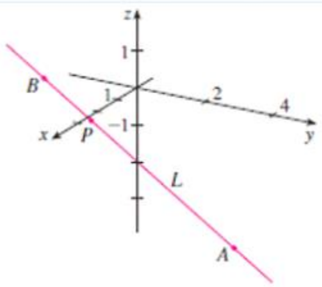
Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

(b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so $(6, 5, 1)$ is a point on the line. Similarly, $t = -1$ gives the point $(4, -3, 5)$.

EXAMPLE 2

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
(b) At what point does this line intersect the xy-plane?



SOLUTION

(a) We are not explicitly given a vector parallel to the line, but observe that the vector $\mathbf{v}$ with representation $\overrightarrow{AB}$ is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus direction numbers are $a = 1$, $b = -5$, and $c = 4$. Taking the point $(2, 4, -3)$ as P_0, we see that parametric equations $\boxed{2}$ are

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t$$

and symmetric equations $\boxed{3}$ are

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

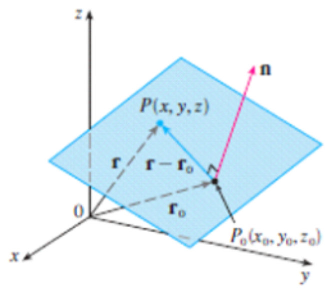
$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

(b) The line intersects the xy-plane when $z = 0$, so we put $z = 0$ in the symmetric equations and obtain

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{3}{4}$$

This gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$, so the line intersects the xy-plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

Planes



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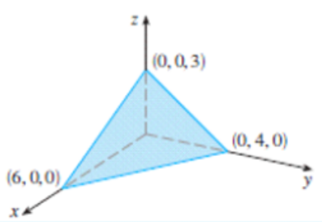
$$\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0}$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

V EXAMPLE 4 Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.



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SOLUTION Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 7, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

To find the x-intercept we set $y = z = 0$ in this equation and obtain $x = 6$. Similarly, the y-intercept is 4 and the z-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7).

$$\boxed{ax + by + cz + d = 0}$$

EXAMPLE 5 Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

SOLUTION The vectors **a** and **b** corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \qquad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both **a** and **b** lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\,\mathbf{i} + 20\,\mathbf{j} + 14\,\mathbf{k}$$

With the point $P(1, 3, 2)$ and the normal vector **n**, an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

EXAMPLE 6 Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

SOLUTION We substitute the expressions for x, y, and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

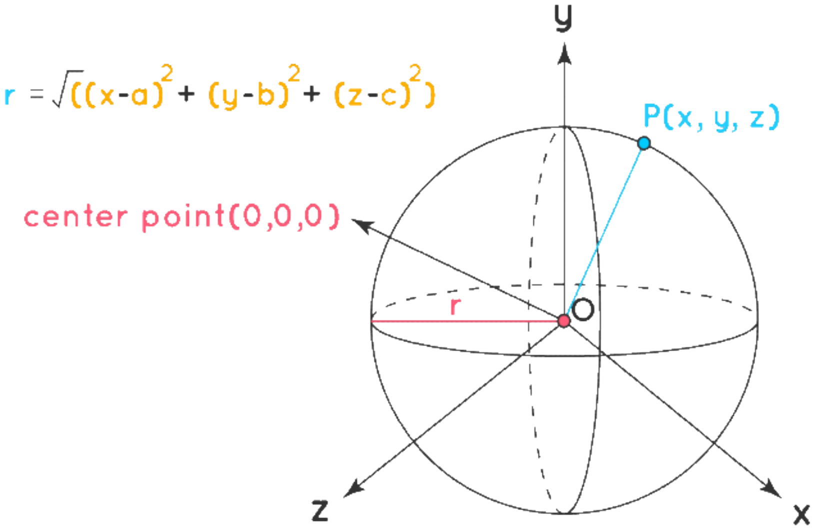
This simplifies to $-10t = 20$, so $t = -2$. Therefore the point of intersection occurs when the parameter value is $t = -2$. Then $x = 2 + 3(-2) = -4$, $y = -4(-2) = 8$, $z = 5 - 2 = 3$ and so the point of intersection is $(-4, 8, 3)$.

Two planes are **parallel** if their normal vectors are parallel. For instance, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors are $\mathbf{n}_1 = \langle 1, 2, -3 \rangle$ and $\mathbf{n}_2 = \langle 2, 4, -6 \rangle$ and $\mathbf{n}_2 = 2\mathbf{n}_1$. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle θ in Figure 9).

V EXAMPLE 7

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

(b) Find symmetric equations for the line of intersection L of these two planes.



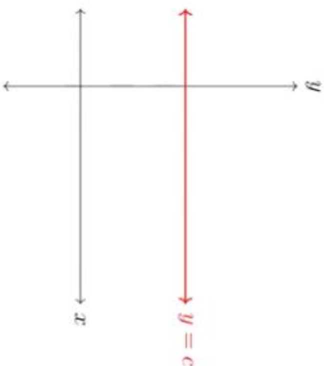
Definition 9.1.12: A *trace* of a function f of two independent variables x and y in the x direction is a curve of the form $z = f(x, c)$, where c is a constant.

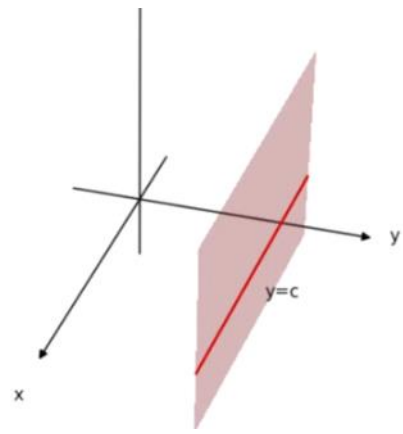
Similarly, a *trace* of a function f of two independent variables x and y in the y direction is a curve of the form $z = f(c, y)$ where c is a constant.

$z = f(x, c)$

Free (the x)

Fixed (the c)

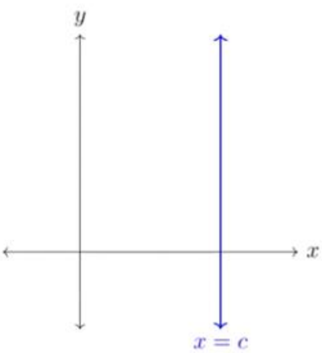


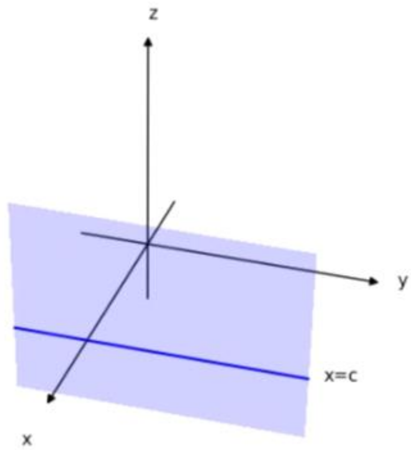


$z = f(c, y)$

Fixed (the c)

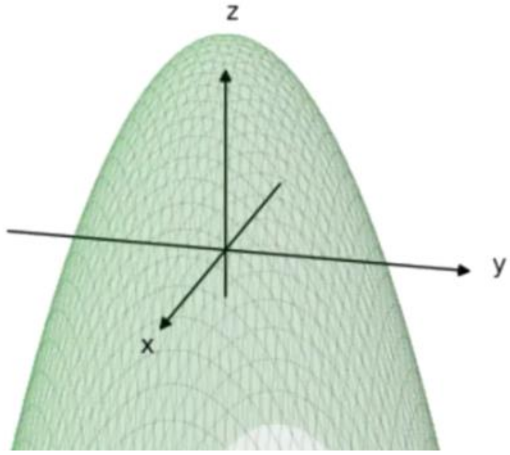
Free (the y)





$$f(x,y) = 4 - x^2 - y^2$$

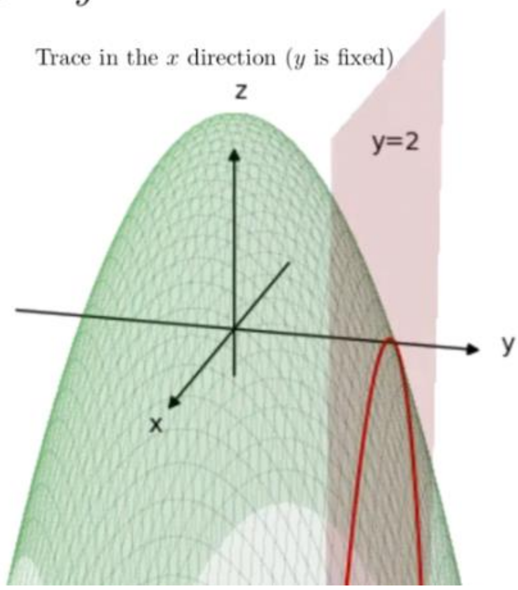
$x \backslash y$	-3	-2	-1	0	1	2	3
-3	-14	-9	-6	-5	-6	-9	-14
-2	-9	-4	-1	0	-1	-4	-9
-1	-6	-1	2	3	2	-1	-6
0	-5	0	3	4	3	0	-5
1	-6	-1	2	3	2	-1	-6
2	-9	-4	-1	0	-1	-4	-9
3	-14	-9	-6	-5	-6	-9	-14



$$f(x,y) = 4 - x^2 - y^2$$

$x \backslash y$	-3	-2	-1	0	1	2	3
-3	-14	-9	-6	-5	-6	-9	-14
-2	-9	-4	-1	0	-1	-4	-9
-1	-6	-1	2	3	2	-1	-6
0	-5	0	3	4	3	0	-5
1	-6	-1	2	3	2	-1	-6
2	-9	-4	-1	0	-1	-4	-9
3	-14	-9	-6	-5	-6	-9	-14

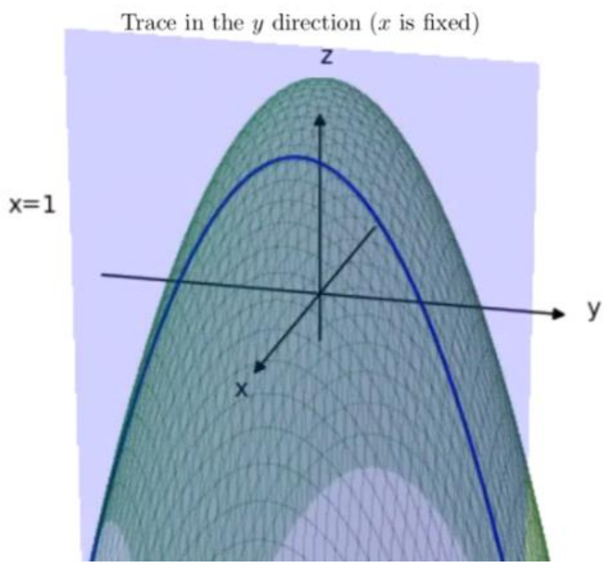
$$f(x,\color{red}{2}) = 4 - x^2 - \color{red}{2}^2 = -x^2$$

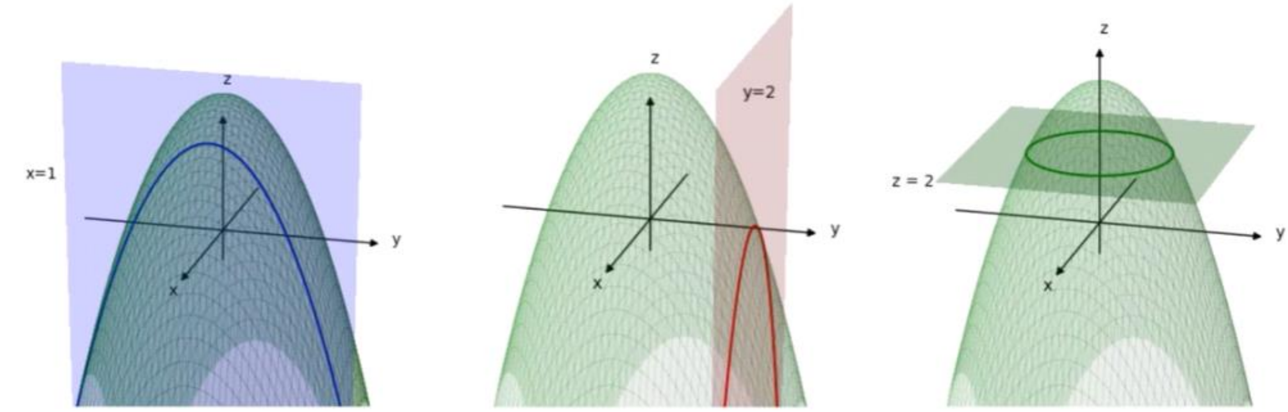


$$f(x,y) = 4 - x^2 - y^2$$

$x \backslash y$	-3	-2	-1	0	1	2	3
-3	-14	-9	-6	-5	-6	-9	-14
-2	-9	-4	-1	0	-1	-4	-9
-1	-6	-1	2	3	2	-1	-6
0	-5	0	3	4	3	0	-5
1	-6	-1	2	3	2	-1	-6
2	-9	-4	-1	0	-1	-4	-9
3	-14	-9	-6	-5	-6	-9	-14

$$f(\color{blue}{1},y) = 4 - \color{blue}{1}^2 - y^2 = 3 - y^2$$



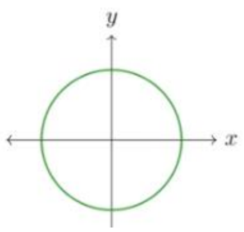


Definition 9.1.15: A *level curve (or contour)* of a function f of two independent variables x and y is a curve of the form $k = f(x, y)$, where k is a constant.

$$f(x, y) = 4 - x^2 - y^2$$

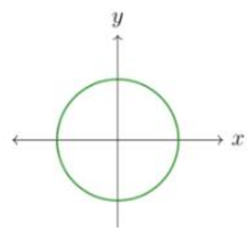
$\underline{k = 0}$

$0 = 4 - x^2 - y^2 \implies x^2 + y^2 = 4$



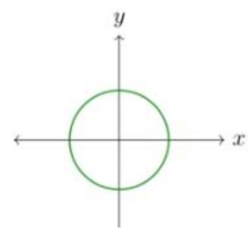
$\underline{k = 1}$

$1 = 4 - x^2 - y^2 \implies x^2 + y^2 = 3$



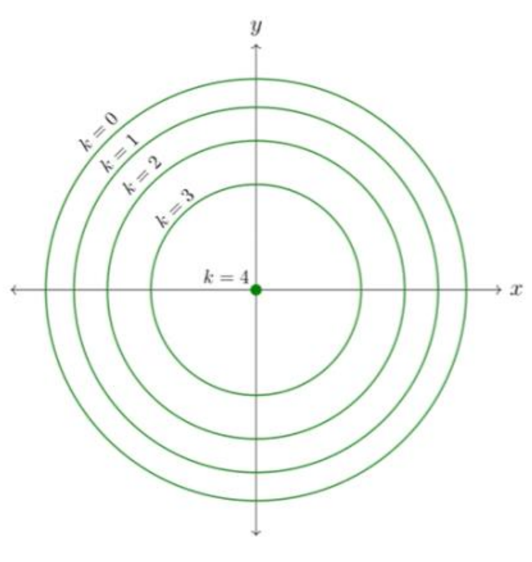
$\underline{k = 2}$

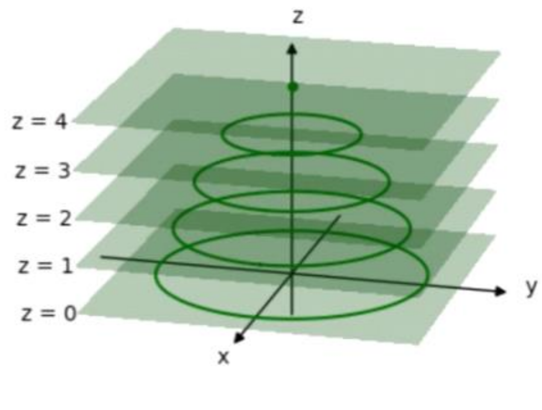
$2 = 4 - x^2 - y^2 \implies x^2 + y^2 = 2$

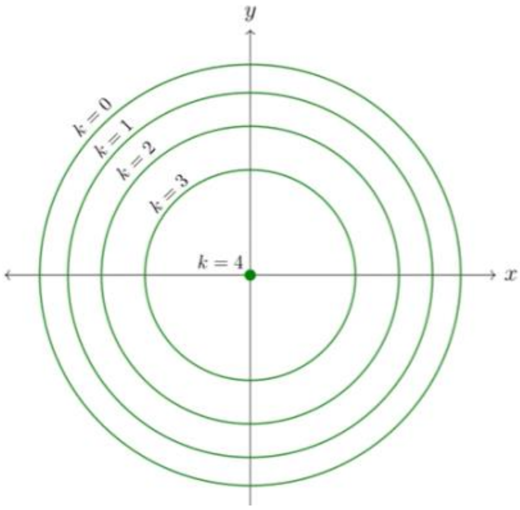


9.1.5: Contour Maps and Level Curves

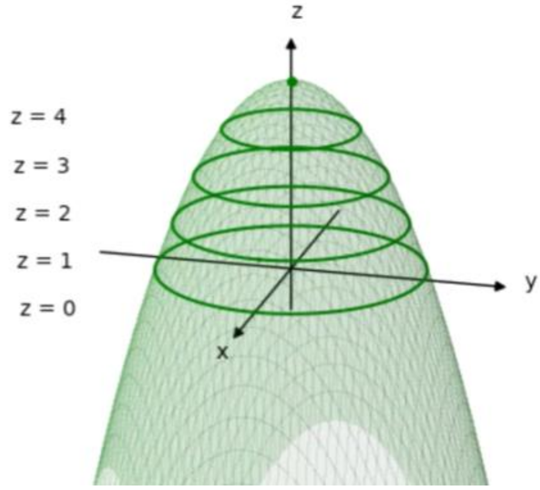
$$f(x, y) = 4 - x^2 - y^2$$



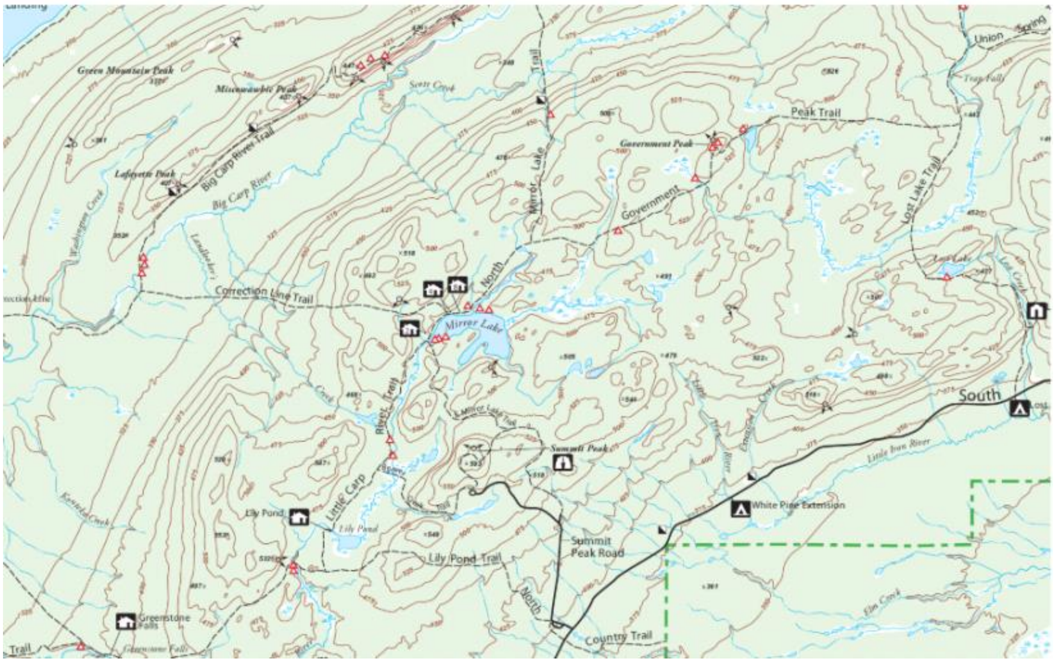




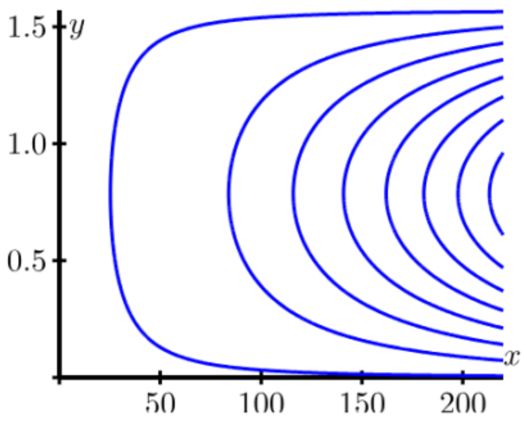
y
x
k = 0
k = 1
k = 2
k = 3
k = 4



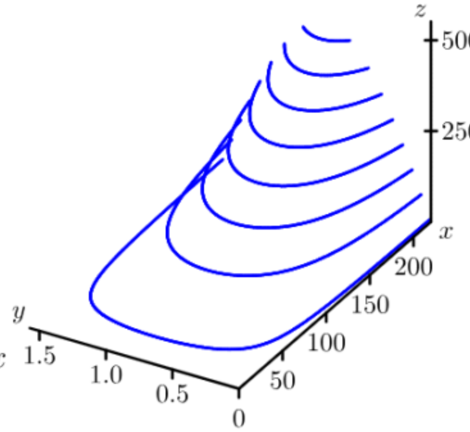
z
z = 4
z = 3
z = 2
z = 1
z = 0
y
x



Peak Trail
Government Peak
Government
Correction Line Trail
Mirror Lake
Lost Lake Trail
South
White Pine Extension
Summit Peak
Summit Peak Road
Lily Pond
Lily Pond Trail
Little Carp
North
Country Trail
Greenstone Falls
Big Carp River



y
1.5
1.0
0.5
x
50
100
150
200



z
500
250
x
200
150
100
50
0
y
1.5
1.0
0.5

`