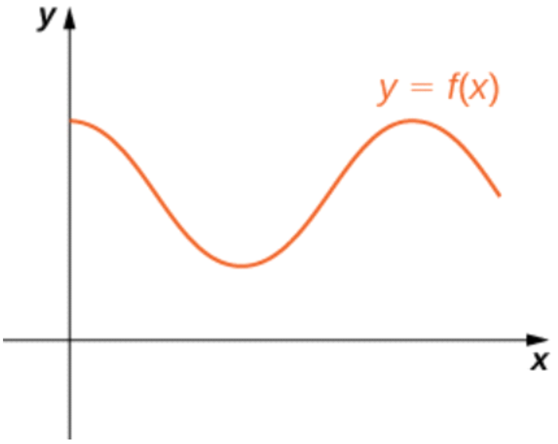
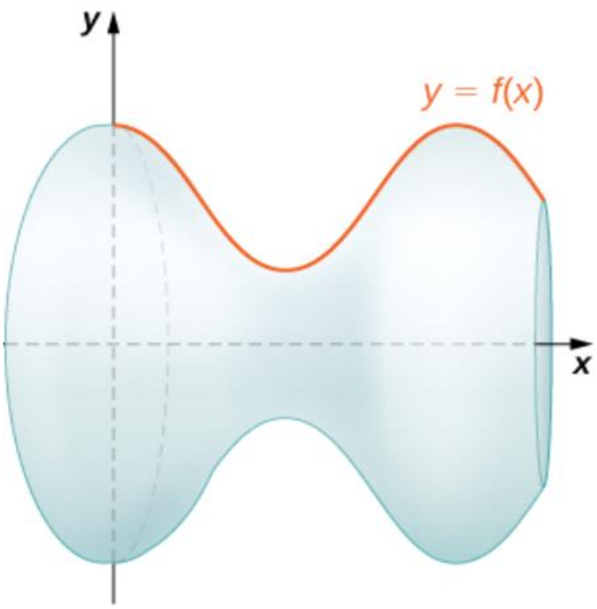
Surface of Revolution

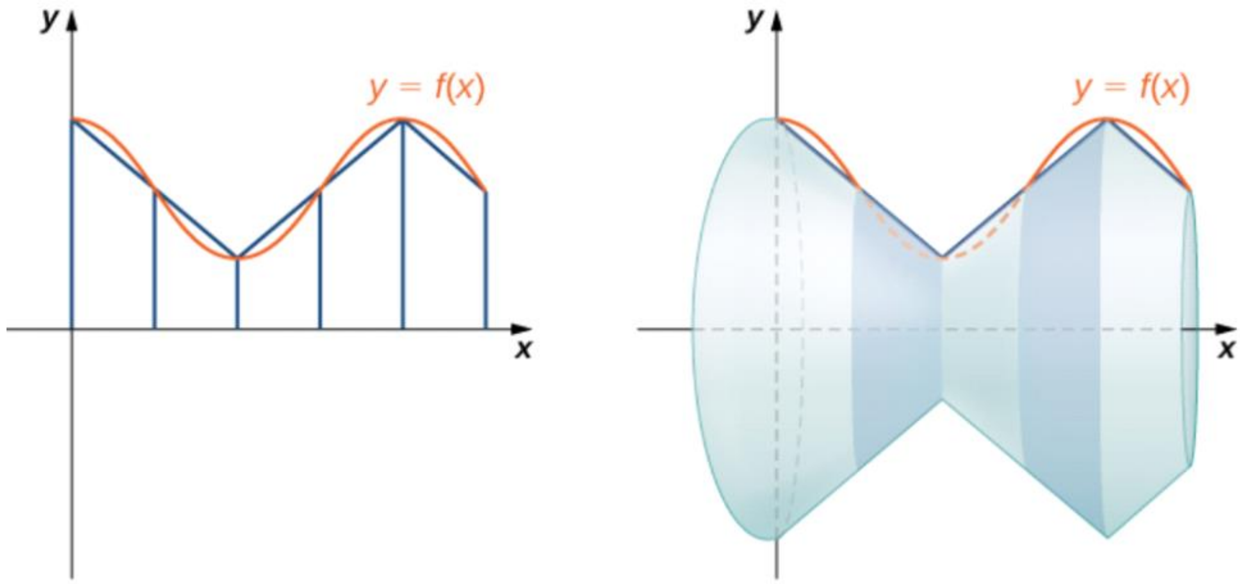
Monday, 5 January 2026 1:37 pm

https://mathinsight.org/surfaces_revolution



The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex.





Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\right) dx$$

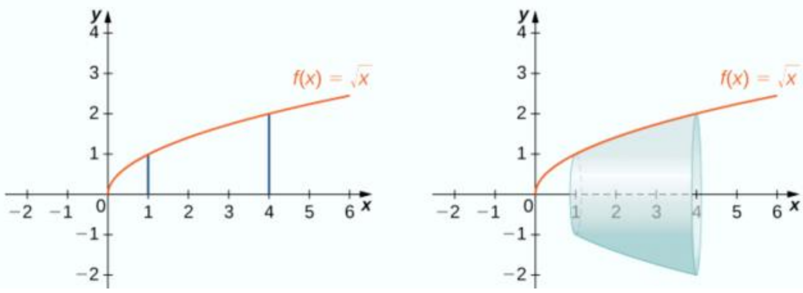
Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y-axis is given by

$$\text{Surface Area} = \int_c^d \left(2\pi g(y)\sqrt{1 + \left(g'(y)\right)^2}\right) dy$$

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Solution

The graph of $f(x)$ and the surface of rotation are shown in Figure $6.4.10$.



We have $f(x) = \sqrt{x}$. Then, $f'(x) = \frac{1}{2\sqrt{x}}$ and $\left(f'(x)\right)^2 = \frac{1}{4x}$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b \left(2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\right) dx \\ &= \int_1^4 \left(2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\right) dx \\ &= \int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right) dx.\end{aligned}$$

Let $u = x + 1/4$. Then, $du = dx$. When $x = 1$, $u = 5/4$, and when $x = 4$, $u = 17/4$. This gives us

$$\begin{aligned}\int_0^1 \left(2\pi\sqrt{x+\frac{1}{4}}\right) dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}\, du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right] \Big|_{5/4}^{17/4} \\ &= \frac{\pi}{6}[17\sqrt{17} - 5\sqrt{5}] \approx 30.846 \text{ units}^2\end{aligned}$$

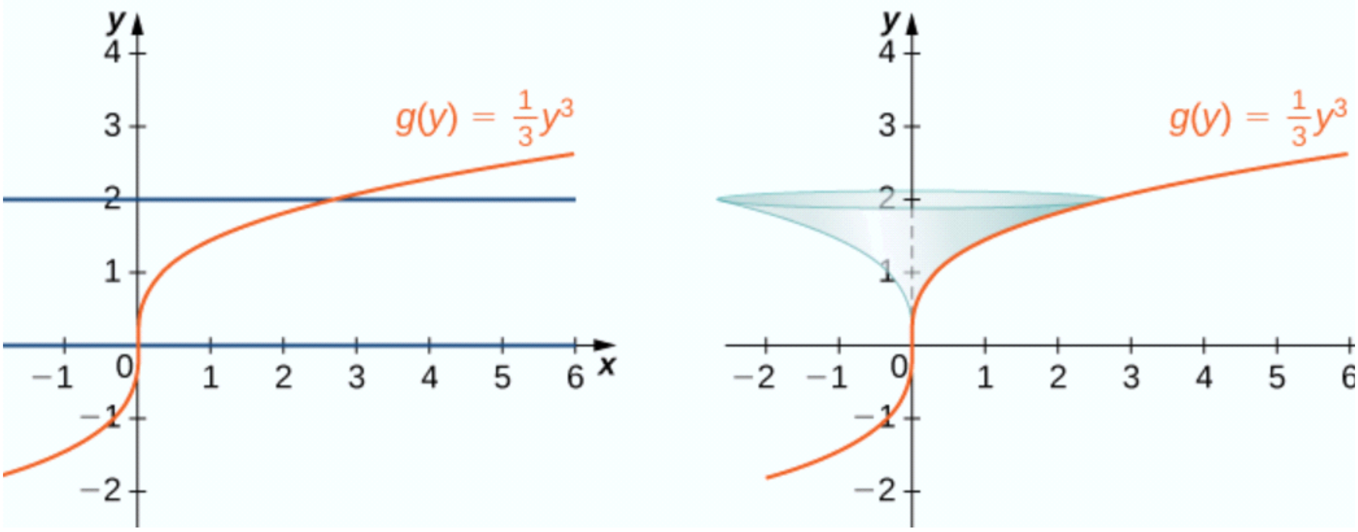
Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Hint

Answer

$$\frac{\pi}{6}\left(5\sqrt{5} - 3\sqrt{3}\right) \approx 3.133 \text{ units}^2$$

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \le y \le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.



We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $\left(g'(y)\right)^2 = y^4$. Then

$$\begin{aligned}\text{Surface Area} &= \int_c^d \left(2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\right) dy \\ &= \int_0^2 \left(2\pi\left(\frac{1}{3}y^3\right)\sqrt{1+y^4}\right) dy \\ &= \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right) dy.\end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3\,dy$. When $y = 0$, $u = 1$, and when $y = 2$, $u = 17$. Then

$$\begin{aligned}\frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right) dy &= \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}\,du \\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]\Big|_1^{17} = \frac{\pi}{9}\left[(17)^{3/2}-1\right] \approx 24.118\,\text{units}^2.\end{aligned}$$

We will rotate the parametric curve given by,

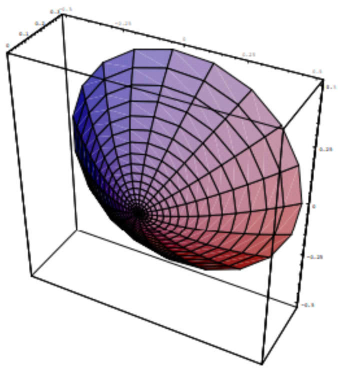
$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

$$\begin{aligned}S &= \int 2\pi y\, ds \qquad \text{rotation about } x-\text{axis} \\ S &= \int 2\pi x\, ds \qquad \text{rotation about } y-\text{axis}\end{aligned}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \quad \text{if } x = f(t), y = g(t),\ \alpha \le t \le \beta$$

Example. Find the area of the surface generated by revolving the following curve about the y-axis:

$$x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad 0 \le t \le 1.$$



The curve is revolved about the y-axis, so $R = x = \frac{1}{2}t^2$. This is positive, so no absolute values are needed.

$$\frac{dx}{dt} = t, \quad \frac{dy}{dt} = t^2.$$

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^2 + t^4\\ &= t^2(1+t^2)\end{aligned}$$

Hence,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t\sqrt{t^2+1}.$$

The area is

$$S = \int_0^1 2\pi \cdot \frac{1}{2}t^2 \cdot t\sqrt{t^2+1}\,dt = \frac{\pi}{2}\left[\frac{2}{5}(t^2+1)^{5/2} - \frac{2}{3}(t^2+1)^{3/2}\right]_0^1 = \left(\frac{2}{15} + \frac{2\sqrt{2}}{15}\right)\pi = 0.32189\ldots.$$

Here's the work for the antiderivative:

$$\int t^2 \cdot t\sqrt{t^2+1}\,dt = \frac{1}{2}\int (u-1)\sqrt{u}\,du = \frac{1}{2}\int (u^{3/2} - u^{1/2})\,du = \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + c =$$

$$\left[u = t^2+1, \quad t^2 = u-1, \quad du = 2t\,dt, \quad dt = \frac{du}{2t}\right]$$

$$\frac{1}{2}\left(\frac{2}{5}(t^2+1)^{5/2} - \frac{2}{3}(t^2+1)^{3/2}\right) + c. \quad \square$$

Example. Find the area of the surface generated by revolving the following curve about the x-axis:

$$x = \frac{1}{3}t^3 - 4t + 1, \quad y = 2t^2, \quad 0 \le t \le 1.$$