System of Linear Equation

Monday, 5 January 2026 4:06 pm

$$\begin{cases} 3x_1 + 4x_2 + 10x_3 + 19x_4 - 2x_5 - 3x_6 = 141 \\ 7x_1 + 2x_2 - 13x_3 - 7x_4 + 21x_5 + 8x_6 = 2567 \\ -x_1 + 9x_2 + \frac{3}{2}x_3 + x_4 + 14x_5 + 27x_6 = 26 \\ \frac{1}{2}x_1 + 4x_2 + 10x_3 + 11x_4 + 2x_5 + x_6 = -15. \end{cases}$$

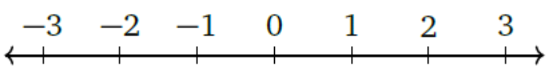
1.1 Systems of Linear Equations

Objectives

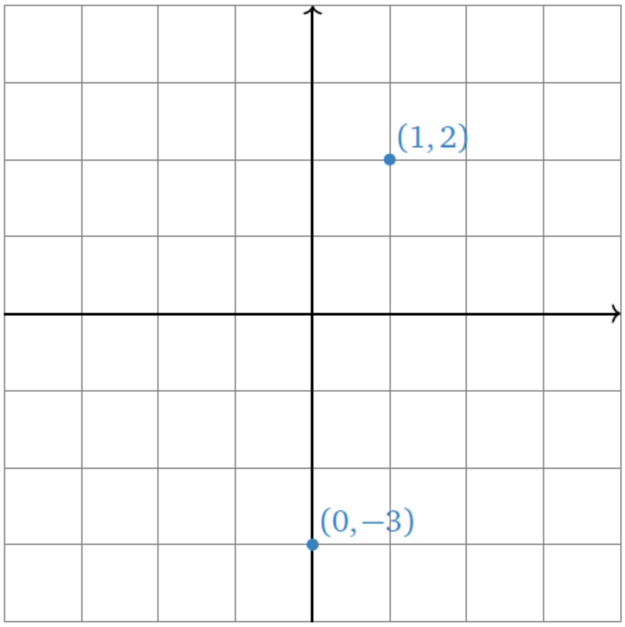
1. Understand the definition of $\mathbf{R}^n$, and what it means to use $\mathbf{R}^n$ to label points on a geometric object.
2. *Pictures:* solutions of systems of linear equations, parameterized solution sets.
3. *Vocabulary words:* **consistent**, **inconsistent**, **solution set**.

Line, Plane, Space, Etc.

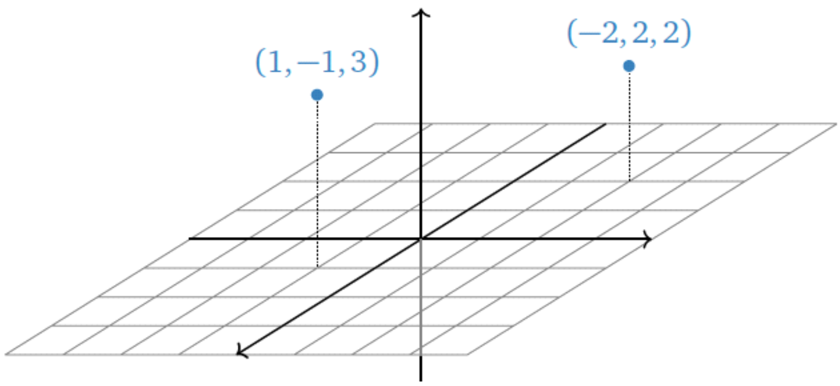
Example (The number line)**.** When $n = 1$, we just get $\mathbf{R}$ back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the number line.



Example (The Euclidean plane)**.** When $n = 2$, we can think of $\mathbf{R}^2$ as the xy-plane. We can do so because every point on the plane can be represented by an ordered pair of real numbers, namely, its x- and y-coordinates.



Example (3-Space)**.** When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. We can do so because every point in space can be represented by an ordered triple of real numebrs, namely, its x-, y-, and z-coordinates.



So what is $\mathbf{R}^4$? or $\mathbf{R}^5$? or $\mathbf{R}^n$? These are harder to visualize, so you have to go back to the definition: $\mathbf{R}^n$ is the set of all ordered n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$.

They are still "geometric" spaces, in the sense that our intuition for $\mathbf{R}^2$ and $\mathbf{R}^3$ often extends to $\mathbf{R}^n$.

We will make definitions and state theorems that apply to any $\mathbf{R}^n$, but we will only draw pictures for $\mathbf{R}^2$ and $\mathbf{R}^3$.

The power of using these spaces is the ability to *label* various objects of interest, such as geometric objects and solutions of systems of equations, by the points of $\mathbf{R}^n$.

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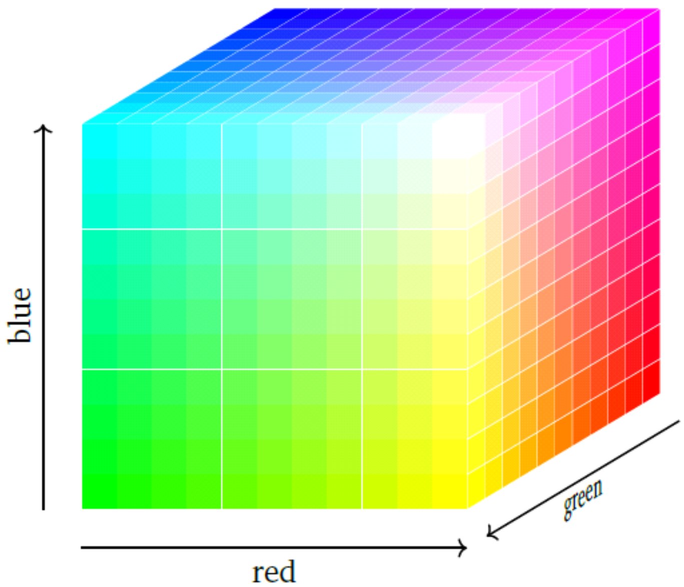
Example (Color Space). All colors you can see can be described by three quantities: the amount of red, green, and blue light in that color. (Humans are trichromatic.) Therefore, we can use the points of $\mathbf{R}^3$ to *label* all colors: for instance, the point $(.2, .4, .9)$ labels the color with 20% red, 40% green, and 90% blue intensity.



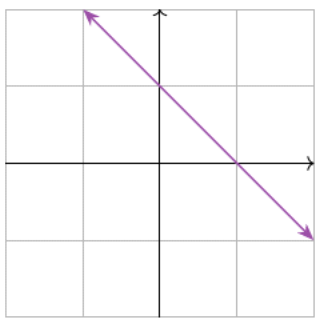
Example (QR Codes). A QR code is a method of storing data in a grid of black and white squares in a way that computers can easily read. A typical QR code is a 29×29 grid. Reading each line left-to-right and reading the lines top-to-bottom (like you read a book) we can think of such a QR code as a sequence of $29 \times 29 = 841$ digits, each digit being 1 (for white) or 0 (for black). In such a way, the entire QR code can be regarded as a point in $\mathbf{R}^{841}$. As in the previous example, it is very useful from a psychological perspective to view a QR code as a *single* piece of data in this way.

The QR code for this textbook is a 29×29 *array of black/white squares.*

1.1.2 Pictures of Solution Sets

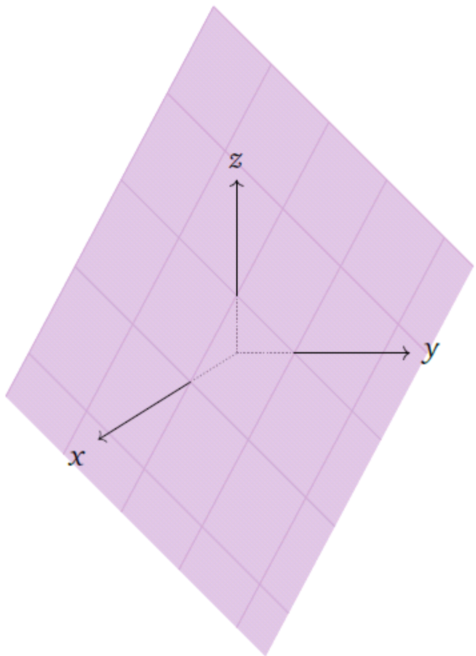
Before discussing how to solve a system of linear equations below, it is helpful to see some pictures of what these solution sets look like geometrically.

One Equation in Two Variables. Consider the linear equation $x + y = 1$. We can rewrite this as $y = 1 - x$, which defines a line in the plane: the slope is -1, and the x-intercept is 1.



Definition (Lines)**.** For our purposes, a **line** is a ray that is *straight* and *infinite* in both directions.

One Equation in Three Variables. Consider the linear equation $x + y + z = 1$. This is the **implicit equation** for a plane in space.



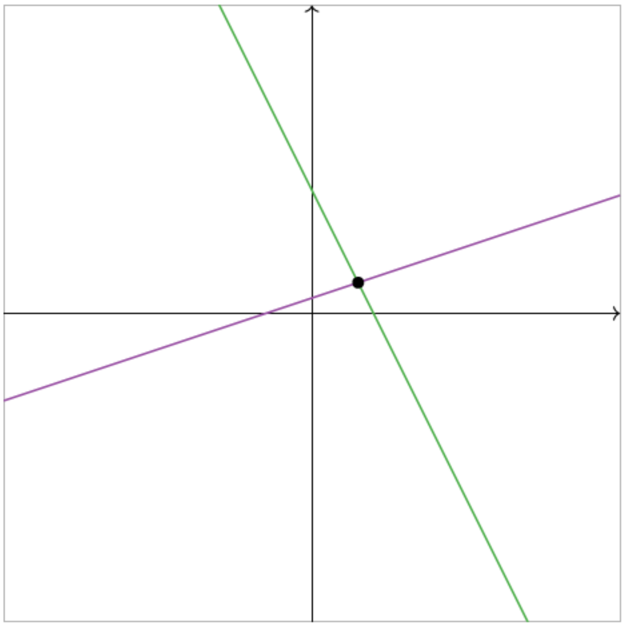
Definition (Planes). A **plane** is a flat sheet that is infinite in all directions.

Remark. The equation $x + y + z + w = 1$ defines a "3-plane" in 4-space, and more generally, a single linear equation in n variables defines an "$(n-1)$-plane" in n-space. We will make these statements precise in Section 2.7.

Two Equations in Two Variables. Now consider the system of two linear equations

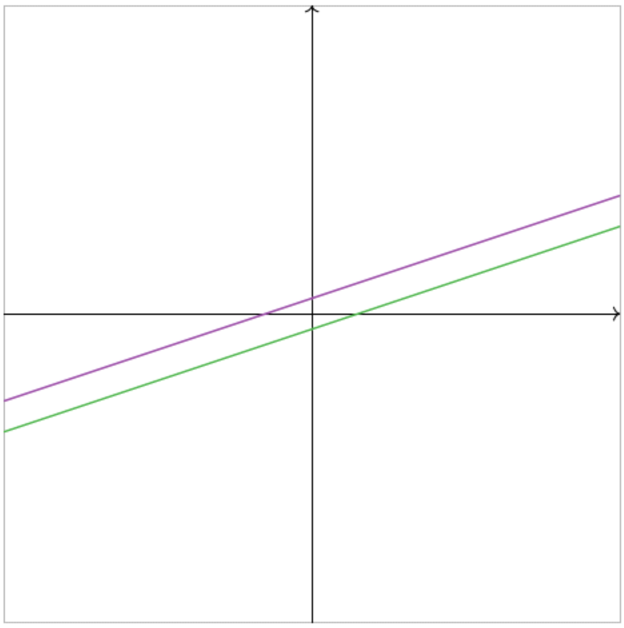
$$\begin{cases} x - 3y = -3 \\ 2x + y = 8. \end{cases}$$

Each equation individually defines a line in the plane, pictured below.



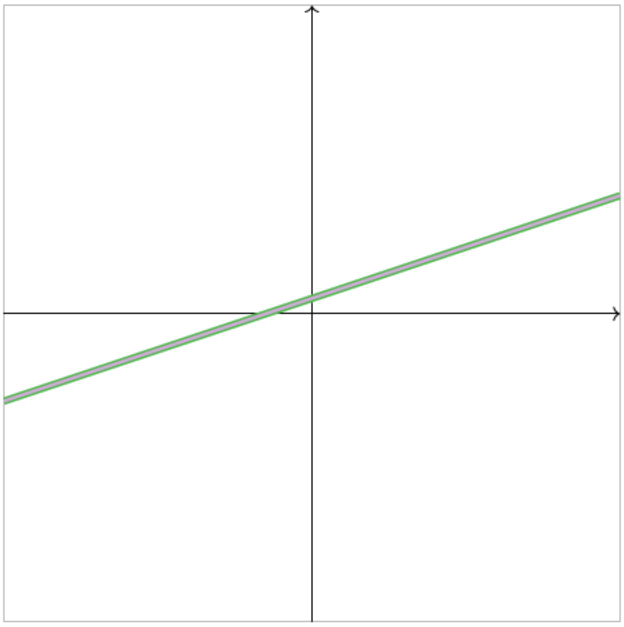
$$\begin{cases} x - 3y = -3 \\ x - 3y = 3. \end{cases}$$

These define *parallel* lines in the plane.



$$\begin{cases} x - 3y = -3 \\ 2x - 6y = -6. \end{cases}$$

The second equation is a multiple of the first, so these equations define the *same* line in the plane.



1.2 Row Reduction

Objectives

1. Learn to replace a system of linear equations by an augmented matrix.
2. Learn how the elimination method corresponds to performing row operations on an augmented matrix.
3. Understand when a matrix is in (reduced) row echelon form.
4. Learn which row reduced matrices come from inconsistent linear systems.
5. *Recipe:* the row reduction algorithm.
6. *Vocabulary words:* **row operation**, **row equivalence**, **matrix**, **augmented matrix**, **pivot**, **(reduced) row echelon form**.

$$\left\{\begin{array}{rcr} x + 2y + 3z &=& 6 \\ 2x - 3y + 2z &=& 14 \\ 3x + y - z &=& -2 \end{array}\right. \xrightarrow{\text{becomes}} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right).$$

This is called an **augmented matrix**. The word "augmented" refers to the vertical line, which we draw to remind ourselves where the equals sign belongs; a **matrix** is a grid of numbers without the vertical line. In this notation, our three valid ways of manipulating our equations become **row operations**:

- ***Scaling:*** multiply all entries in a row by a nonzero number.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_1 = R_1 \times -3} \left(\begin{array}{rrr|r} -3 & -6 & -9 & -18 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

  Here the notation R_1 simply means "the first row", and likewise for R_2, R_3, etc.

- ***Replacement:*** add a multiple of one row to another, replacing the second row with the result.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2 \times R_1} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

- ***Swap:*** interchange two rows.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_3} \left(\begin{array}{rrr|r} 3 & 1 & -1 & -2 \\ 2 & -3 & 2 & 14 \\ 1 & 2 & 3 & 6 \end{array}\right)$$

Example. Solve (1.1.1) using row operations.

Solution. We start by forming an augmented matrix:

$$\left\{\begin{array}{rcr} x + 2y + 3z &=& 6 \\ 2x - 3y + 2z &=& 14 \\ 3x + y - z &=& -2 \end{array}\right. \xrightarrow{\text{becomes}} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right).$$

Eliminating a variable from an equation means producing a zero to the left of the line in an augmented matrix. First we produce zeros in the first column (i.e. we eliminate x) by subtracting multiples of the first row.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right) \xrightarrow{R_3=R_3\div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_2\longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+7R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

$$\xrightarrow{R_3=R_3\div 10} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{\text{becomes}} \begin{cases} x + 2y + 3z = 6 \\ \quad\;\; y + 2z = 4 \\ \qquad\quad\; z = 3 \end{cases}$$

Hence $z = 3$; back-substituting as in this example gives $(x, y, z) = (1, -2, 3)$.

Example (An Inconsistent System). Solve the following system of equations using row operations:

$$\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases}$$

Solution. First we put our system of equations into an augmented matrix.

$$\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases} \xrightarrow{\text{augmented matrix}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right)$$

We clear the entries below the top-left using row replacement.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right) \xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 5 & 9 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 4R_1} \left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

Now we clear the second entry from the last row.

$$\left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 - R_2} \left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right)$$

This translates back into the system of equations

$$\begin{cases} x + y = 2 \\ y = -1 \\ 0 = 2. \end{cases}$$

Our original system has the same solution set as this system. But this system has no solutions: there are no values of x, y making the third equation true! We conclude that our original equation was inconsistent.

Definition. A matrix is in **row echelon form** if:

1. All zero rows are at the bottom.
2. The first nonzero entry of a row is to the *right* of the first nonzero entry of the row above.
3. Below the first nonzero entry of a row, all entries are zero.

Here is a picture of a matrix in row echelon form:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{r l} \star & = \text{any number} \\ \boxed{\star} & = \text{any nonzero number} \end{array}$$

Definition. A **pivot** is the first nonzero entry of a row of a matrix in row echelon form.

A matrix in row-echelon form is generally easy to solve using back-substitution. For example,

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{\text{becomes}} \begin{cases} x + 2y + 3z = 6 \\ \quad\;\; y + 2z = 4 \\ \qquad\;\; 10z = 30. \end{cases}$$

We immediately see that $z = 3$, which implies $y = 4 - 2 \cdot 3 = -2$ and $x = 6 - 2(-2) - 3 \cdot 3 = 1$. See this example.

Definition. A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition:

5. Each pivot is equal to 1.

6. Each pivot is the only nonzero entry in its column.

Here is a picture of a matrix in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \star = \text{any number} \\ 1 = \text{pivot} \end{array}$$

A matrix in reduced row echelon form is in some sense completely solved. For example,

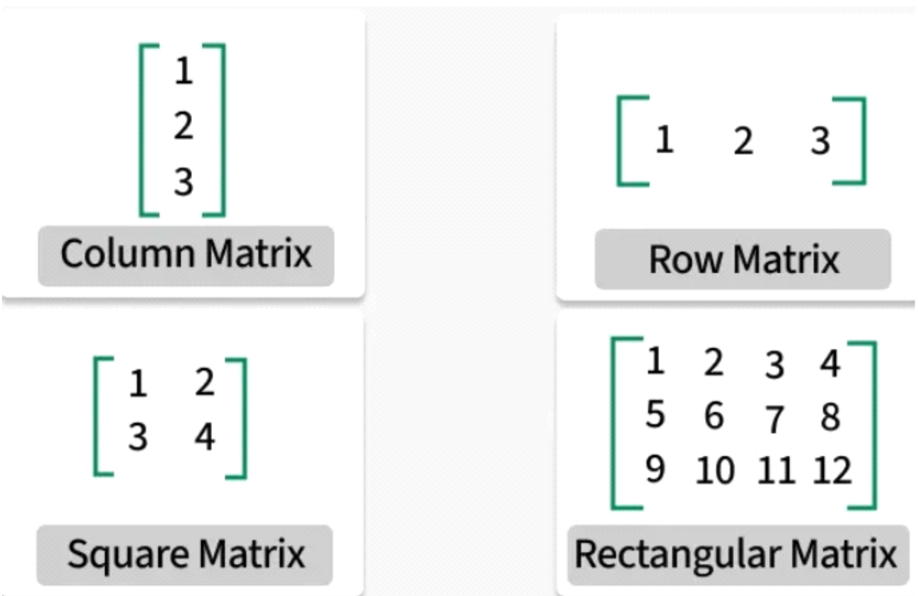
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{\text{becomes}} \begin{cases} x = & 1 \\ y = & -2 \\ z = & 3. \end{cases}$$

Example. The following matrices are in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right) \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The following matrices are in row echelon form but not reduced row echelon form:

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \qquad \left(\begin{array}{ccc|c} 2 & 7 & 1 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$



$$\begin{bmatrix} 1 \end{bmatrix}$$

Singleton Matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Null Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Diagonal Matrix

Upper Triangular Matrix

A square matrix in which all the elements below the diagonal are zero and the elements from the diagonal and above are non-zero elements is called an Upper Triangular Matrix. In an Upper Triangular Matrix, the non-zero elements form a triangular-like shape.

$$\begin{bmatrix} 8 & 5 & 6 \\ 0 & 4 & 7 \\ 0 & 0 & 9 \end{bmatrix}_{3\times 3}$$

Lower Triangular Matrix

A square matrix in which all the elements above the diagonal are zero and the elements from the diagonal and below are non-zero elements is called a Lower Triangular Matrix. In a Lower Triangular Matrix, the non-zero elements form a triangular-like shape from the diagonal and below.

$$\begin{bmatrix} 8 & 0 & 0 \\ 6 & 4 & 0 \\ 5 & 7 & 9 \end{bmatrix}_{3\times 3}$$

Singular Matrix

A singular matrix is referred to as a square matrix whose determinant is zero and is not invertible.
If det A = 0, a square matrix "A" is said to be singular; otherwise, it is said to be non-singular.

$$A = \begin{bmatrix} 3 & 6 & 9 \\ 6 & 12 & 18 \\ 2 & 4 & 6 \end{bmatrix}$$

$\Rightarrow |A| = 3(12 \times 6 - 18 \times 4) - 6(6 \times 6 - 18 \times 2) + 9(6 \times 4 - 12 \times 2)$

$\Rightarrow |A| = 3(72 - 72) - 6(36 - 36) + 9(24 - 24)$

$\Rightarrow |A| = 3 \times 0 - 6 \times 0 + 9 \times 0 = 0$

Non Singular Matrix

A Non-Singular matrix is defined as a square matrix whose determinant is not equal to zero and is invertible.

$$|A| = \begin{bmatrix} 1 & 5 \\ 9 & 8 \end{bmatrix}$$

$\Rightarrow |A| = 8 \times 1 - 9 \times 5 = 8 - 45 = -37$

Symmetric Matrix

A square matrix "A" of any order is defined as a symmetric matrix if the transpose of the matrix is equal to the original matrix itself, i.e., $A^T = A$.

$$|A| = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Skew Symmetric Matrix

A square matrix "A" of any order is defined as a skew-symmetric matrix if the transpose of the matrix is equal to the negative of the original matrix itself, i.e., $A^T = -A$.

$$\begin{bmatrix} 0 & 3 & 5 \\ -3 & 0 & -2 \\ -5 & 2 & 0 \end{bmatrix}$$

Orthogonal Matrix

A square matrix whose transpose is equal to its inverse is called Orthogonal Matrix. In an Orthogonal Matrix if $A^T = A^{-1}$ then $AA^T = I$ where I is the Identity Matrix.

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \text{ and } A^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$A \times A^T = \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) \\ \sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta) & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix}$$

$$\Rightarrow\ A \times A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{(2\times 2)}$$

Idempotent Matrix

An idempotent matrix is a special type of square matrix that remains unchanged when multiplied by itself, i.e., $A^2 = A$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Hence,\ A \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$$

Nilpotent Matrix

A Nilpotent is a square matrix that when raised to some positive power results in a zero matrix. The least power let's say 'p' for which the matrix yields zero matrices, then it is called the Nilpotent Matrix of power 'p'.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $A^3 = A \,.\, A^2$

$$\Rightarrow A^3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, A is a Nilpotent Matrix of index 3.

Involutory Matrix

An involutory matrix is a special type of square matrix whose inverse is the original matrix itself, i.e., $P = P^{-1}$, or, in other words, its square is equal to an identity matrix i.e. $P^2 = I$.

$$A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$$

Periodic Matrix

A periodic matrix is a square matrix that exhibits periodicity, meaning there exists a positive integer **p** such that when the matrix is raised to the power **p+1**, it equals the original matrix (**$A^{p+1} = A$**). If p = 1 then $A^2 = A$ it means A is an Idempotent Matrix. Thus we can say that the Idempotent Matrix is a case of the Periodic Matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The above square matrix is a Periodic Matrix of Period 2, where p = 1.