Lect 1: System of Linear Equation

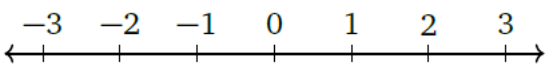
Tuesday, 6 January 2026 4:32 pm

Lecture 1

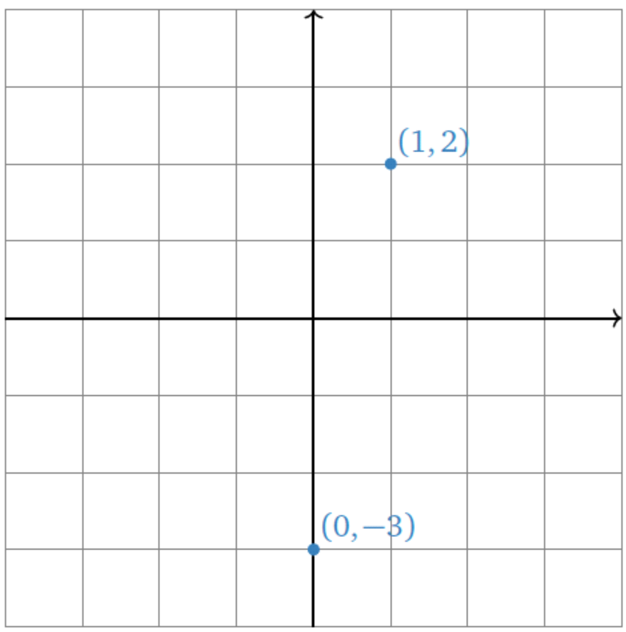
Systems of Linear Equations

Line, Plane, Space, Etc.

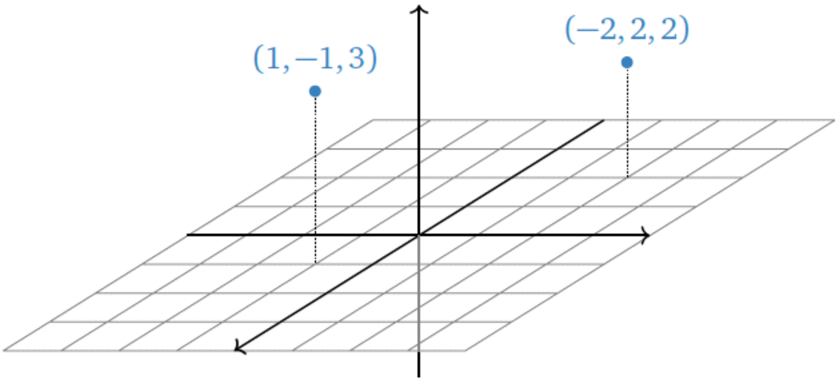
Example (The number line). When $n = 1$, we just get $\mathbf{R}$ back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the number line.



Example (The Euclidean plane). When $n = 2$, we can think of $\mathbf{R}^2$ as the xy-plane. We can do so because every point on the plane can be represented by an ordered pair of real numbers, namely, its x- and y-coordinates.



Example (3-Space)**.** When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. We can do so because every point in space can be represented by an ordered triple of real numebrs, namely, its x-, y-, and z-coordinates.



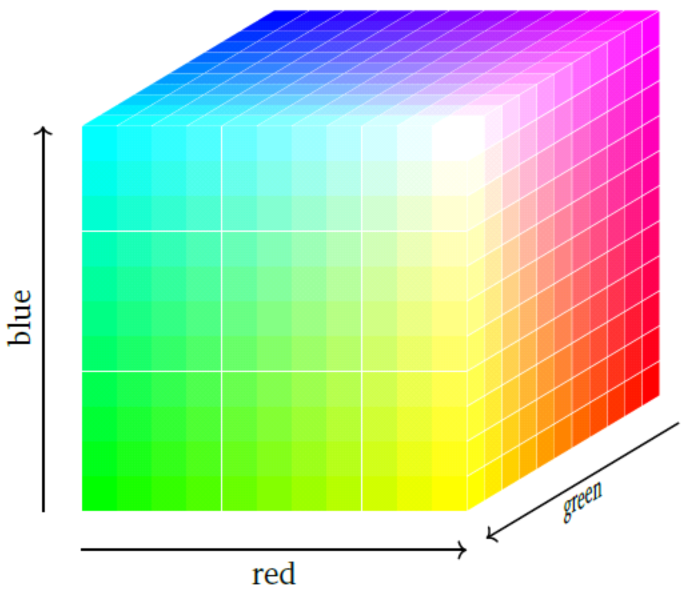
So what is $\mathbf{R}^4$? or $\mathbf{R}^5$? or $\mathbf{R}^n$? These are harder to visualize, so you have to go back to the definition: $\mathbf{R}^n$ is the set of all ordered n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$.

They are still "geometric" spaces, in the sense that our intuition for $\mathbf{R}^2$ and $\mathbf{R}^3$ often extends to $\mathbf{R}^n$.

We will make definitions and state theorems that apply to any $\mathbf{R}^n$, but we will only draw pictures for $\mathbf{R}^2$ and $\mathbf{R}^3$.

The power of using these spaces is the ability to *label* various objects of interest, such as geometric objects and solutions of systems of equations, by the points of $\mathbf{R}^n$.

Example (Color Space)**.** All colors you can see can be described by three quantities: the amount of red, green, and blue light in that color. (Humans are trichromatic.) Therefore, we can use the points of $\mathbf{R}^3$ to *label* all colors: for instance, the point $(.2, .4, .9)$ labels the color with 20% red, 40% green, and 90% blue intensity.



Example (QR Codes). A QR code is a method of storing data in a grid of black and white squares in a way that computers can easily read. A typical QR code is a 29×29 grid. Reading each line left-to-right and reading the lines top-to-bottom (like you read a book) we can think of such a QR code as a sequence of $29 \times 29 = 841$ digits, each digit being 1 (for white) or 0 (for black). In such a way, the entire QR code can be regarded as a point in $\mathbf{R}^{841}$. As in the previous example, it is very useful from a psychological perspective to view a QR code as a *single* piece of data in this way.

The QR code for this textbook is a 29×29 array of black/white squares.

1.1 What is a system of linear equations?

Definition 1.1: A **system of m linear equations** in n unknown variables $x_1, x_2, \ldots, x_n$ is a collection of m equations of the form

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{1.1}$$

The numbers a_{ij} are called the **coefficients** of the linear system; because there are m equations and n unknown variables there are thefore $m \times n$ coefficients. The main problem with a linear system is of course to solve it:

Problem: Find a list of n numbers $(s_1, s_2, \ldots, s_n)$ that satisfy the system of linear equations (1.1).

system consisting of $m = 2$ equations and $n = 3$ unknowns:

$$\begin{aligned} x_1 - 5x_2 - 7x_3 &= 0 \\ 5x_2 + 11x_3 &= 1 \end{aligned}$$

Here is a linear system consisting of $m = 3$ equations and $n = 2$ unknowns:

$$\begin{aligned} -5x_1 + x_2 &= -1 \\ \pi x_1 - 5x_2 &= 0 \\ 63x_1 - \sqrt{2}x_2 &= -7 \end{aligned}$$

And finally, below is a linear system consisting of $m = 4$ equations and $n = 6$ unknowns:

$$\begin{aligned} -5x_1 + x_3 - 44x_4 - 55x_6 &= -1 \\ \pi x_1 - 5x_2 - x_3 + 4x_4 - 5x_5 + \sqrt{5}x_6 &= 0 \\ 63x_1 - \sqrt{2}x_2 - \tfrac{1}{5}x_3 + \ln(3)x_4 + 4x_5 - \tfrac{1}{33}x_6 &= 0 \\ 63x_1 - \sqrt{2}x_2 - \frac{1}{5}x_3 - \tfrac{1}{8}x_4 - 5x_6 &= 5 \end{aligned}$$

Example 1.2. Verify that $(1, 2, -4)$ is a solution to the system of equations

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 2 \\ x_1 + 3x_2 - x_3 &= 11. \end{aligned}$$

Is $(1, -1, 2)$ a solution to the system?

INCONSISTENT $\Leftrightarrow$ NO SOLUTION

A linear system is called **consistent** if it has at least one solution:

CONSISTENT $\Leftrightarrow$ AT LEAST ONE SOLUTION

Example 1.3. Show that the linear system does not have a solution.

$$\begin{aligned} -x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1. \end{aligned}$$

Solution. If we add the two equations we get

$$0 = 4$$

which is a contradiction. Therefore, there does not exist a list (s_1, s_2) that satisfies the system because this would lead to the contradiction $0 = 4$. $\square$

Example 1.4. Let t be an arbitrary real number and let

$$\begin{aligned} s_1 &= -\tfrac{3}{2} - 2t \\ s_2 &= \tfrac{3}{2} + t \\ s_3 &= t. \end{aligned}$$

Show that for any choice of the parameter t, the list (s_1, s_2, s_3) is a solution to the linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 3x_2 - x_3 &= 3. \end{aligned}$$

Solution. Substitute the list (s_1, s_2, s_3) into the left-hand-side of the first equation

$$\left(-\tfrac{3}{2} - 2t\right) + \left(\tfrac{3}{2} + t\right) + t = 0$$

and in the second equation

$$\left(-\tfrac{3}{2} - 2t\right) + 3\left(\tfrac{3}{2} + t\right) - t = -\tfrac{3}{2} + \tfrac{9}{2} = 3$$

Both equations are satisfied for any value of t. Because we can vary t arbitrarily, we get an infinite number of solutions parameterized by t. For example, compute the list (s_1, s_2, s_3) for $t = 3$ and confirm that the resulting list is a solution to the linear system. $\square$

$$\begin{aligned} 5x_1 - 3x_2 + 8x_3 &= -1 \\ x_1 + 4x_2 - 6x_3 &= 0 \\ 2x_2 + 4x_3 &= 3 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 5 & -3 & 8 \\ 1 & 4 & -6 \\ 0 & 2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \quad [\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 5 & -3 & 8 & -1 \\ 1 & 4 & -6 & 0 \\ 0 & 2 & 4 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 4 & -2 & 8 & 12 \\ 0 & 1 & -7 & 2 & -4 \\ 0 & 0 & 5 & -1 & 7 \end{bmatrix}$$

$$\begin{aligned} x_1 + 4x_2 - 2x_3 + 8x_4 &= 12 \\ x_2 - 7x_3 + 2x_4 &= -4 \\ 5x_3 - x_4 &= 7. \end{aligned}$$

Solving linear systems

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of one equation to another.

Example 1.6. Solve the linear system using elementary row operations.

$$\begin{aligned} -3x_1 + 2x_2 + 4x_3 &= 12 \\ x_1 - 2x_3 &= -4 \\ 2x_1 - 3x_2 + 4x_3 &= -3 \end{aligned}$$

Solution. Our goal is to perform elementary row operations to obtain a triangular structure and then use back substitution to solve. The augmented matrix is

$$\begin{bmatrix} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{bmatrix}.$$

Interchange Row 1 (R_1) and Row 2 (R_2):

$$\begin{bmatrix} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{bmatrix}$$

As you will see, this first operation will simplify the next step. Add $3R_1$ to R_2:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{bmatrix} \xrightarrow{3R_1 + R_2} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{bmatrix}$$

Add $-2R_1$ to R_3:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix}$$

Multiply R_2 by $\frac{1}{2}$:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix}$$

Add $3R_2$ to R_3:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 8 & 5 \end{bmatrix} \xrightarrow{3R_2 + R_3} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

Multiply R_3 by $\frac{1}{5}$:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix} \xrightarrow{\frac{1}{5}R_3} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Example 1.7. Using elementary row operations, show that the linear system is inconsistent.

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 + x_3 &= 0 \\ 2x_1 + 4x_3 &= 1 \end{aligned}$$

Solution. The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}$$

Perform the operation $-2R_1 + R_3$:

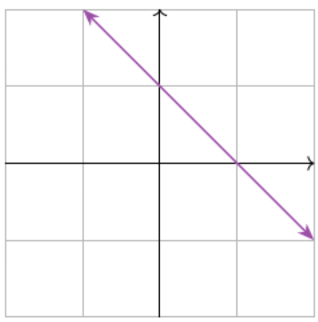
$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Geometric interpretation of the solution set

1.1.2 Pictures of Solution Sets

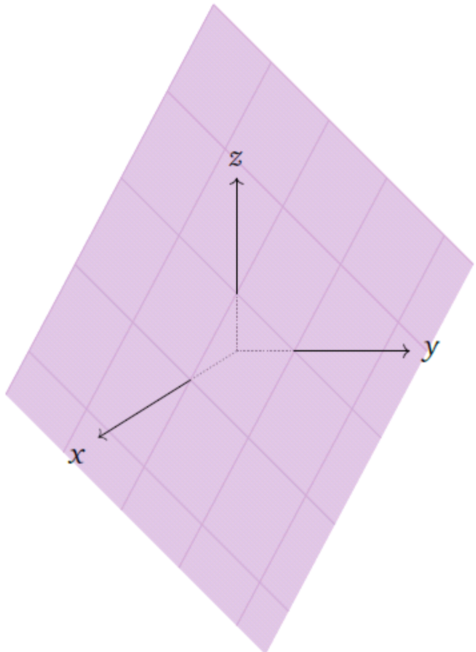
Before discussing how to solve a system of linear equations below, it is helpful to see some pictures of what these solution sets look like geometrically.

One Equation in Two Variables. Consider the linear equation $x + y = 1$. We can rewrite this as $y = 1 - x$, which defines a line in the plane: the slope is -1, and the x-intercept is 1.



Definition (Lines). For our purposes, a **line** is a ray that is *straight* and *infinite* in both directions.

One Equation in Three Variables. Consider the linear equation $x + y + z = 1$. This is the **implicit equation** for a plane in space.



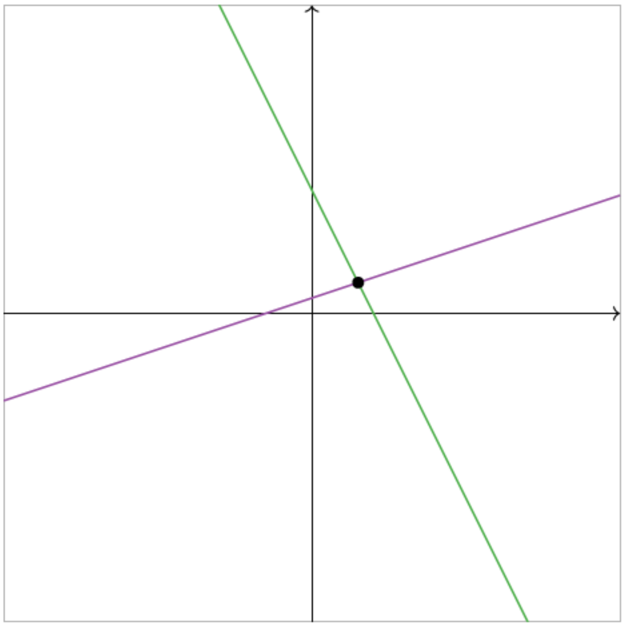
Definition (Planes). A **plane** is a flat sheet that is infinite in all directions.

Remark. The equation $x + y + z + w = 1$ defines a "3-plane" in 4-space, and more generally, a single linear equation in n variables defines an "$(n-1)$-plane" in n-space. We will make these statements precise in Section 2.7.

Two Equations in Two Variables. Now consider the system of two linear equations

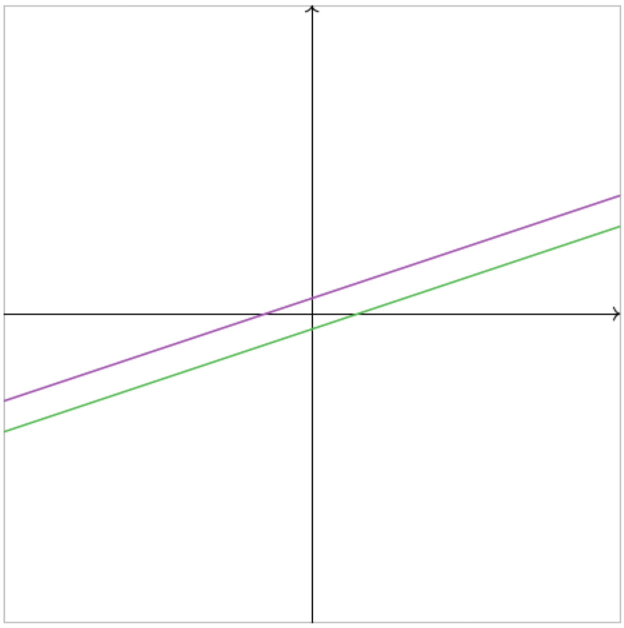
$$\begin{cases} x - 3y = -3 \\ 2x + y = 8. \end{cases}$$

Each equation individually defines a line in the plane, pictured below.



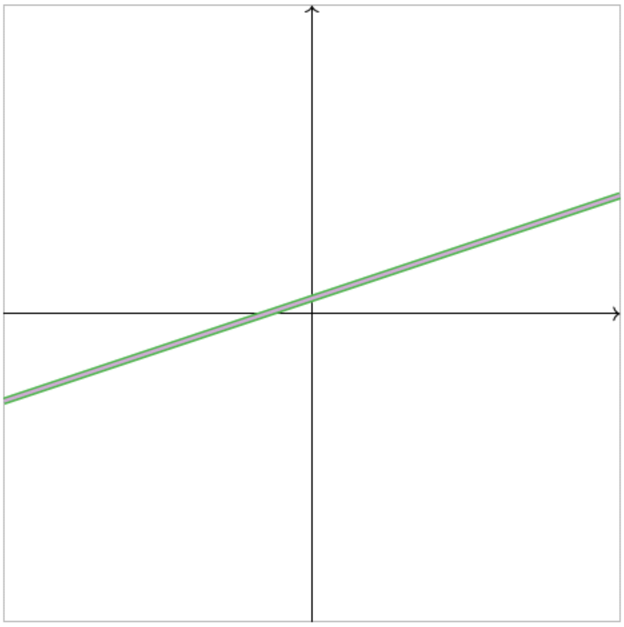
$$\begin{cases} x - 3y = -3 \\ x - 3y = 3. \end{cases}$$

These define *parallel* lines in the plane.



$$\begin{cases} x - 3y = -3 \\ 2x - 6y = -6. \end{cases}$$

The second equation is a multiple of the first, so these equations define the *same* line in the plane.



The set of points (x_1, x_2) that satisfy the linear system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \tag{1.2}$$

is the intersection of the two lines determined by the equations of the system. The solution for this system is $(3, 2)$. The two lines intersect at the point $(x_1, x_2) = (3, 2)$, see Figure 1.1.

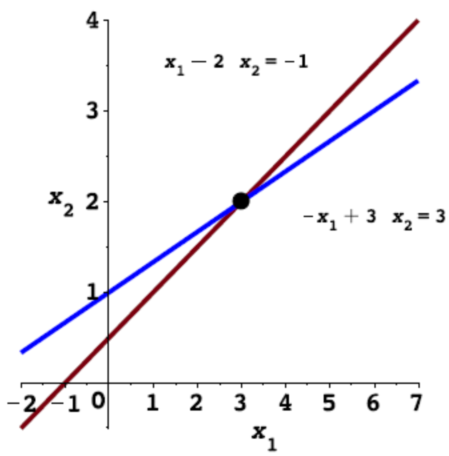
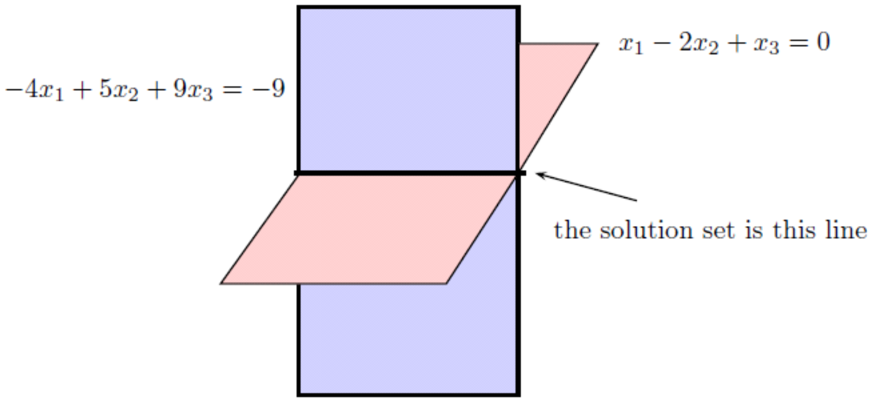


Figure 1.1: The intersection point of the two lines is the solution of the linear system (1.2)

Similarly, the solution of the linear system

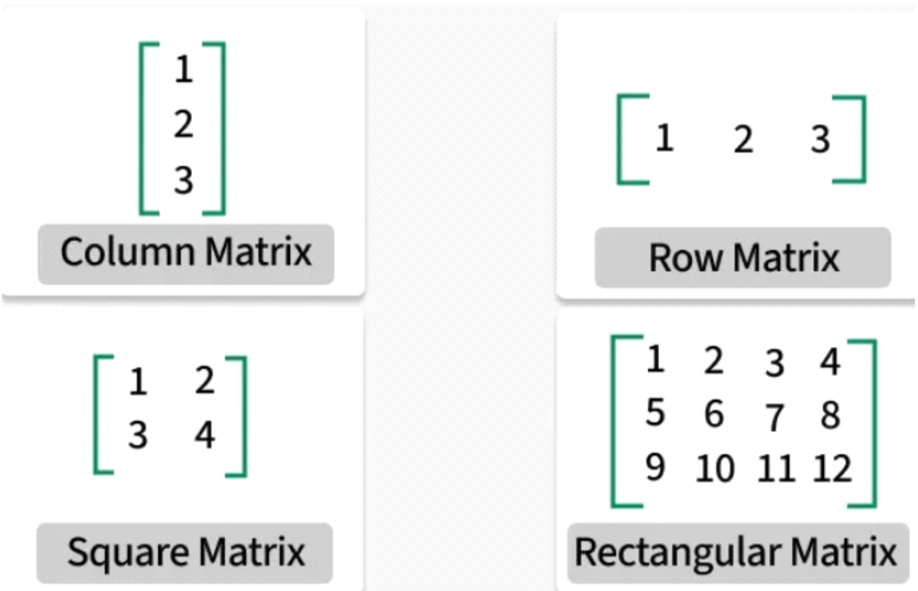
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

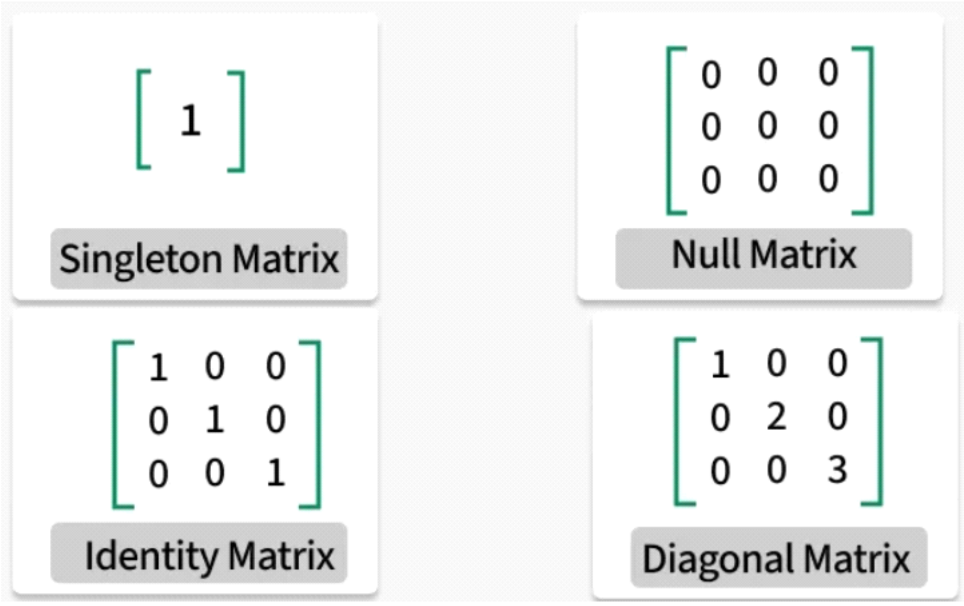


After this lecture you should know the following:

- what a linear system is
- what it means for a linear system to be consistent and inconsistent
- what matrices are
- what are the matrices associated to a linear system
- what the elementary row operations are and how to apply them to simplify a linear system
- what it means for two matrices to be row equivalent
- how to use the method of back substitution to solve a linear system
- what an inconsistent row is
- how to identify using elementary row operations when a linear system is inconsistent
- the geometric interpretation of the solution set of a linear system

Types of Matrix





Upper Triangular Matrix

A square matrix in which all the elements below the diagonal are zero and the elements from the diagonal and above are non-zero elements is called an Upper Triangular Matrix. In an Upper Triangular Matrix, the non-zero elements form a triangular-like shape.

$$\begin{bmatrix} 8 & 5 & 6 \\ 0 & 4 & 7 \\ 0 & 0 & 9 \end{bmatrix}_{3\times 3}$$

Lower Triangular Matrix

A square matrix in which all the elements above the diagonal are zero and the elements from the diagonal and below are non-zero elements is called a Lower Triangular Matrix. In a Lower Triangular Matrix, the non-zero elements form a triangular-like shape from the diagonal and below.

$$\begin{bmatrix} 8 & 0 & 0 \\ 6 & 4 & 0 \\ 5 & 7 & 9 \end{bmatrix}_{3\times 3}$$

Singular Matrix

A singular matrix is referred to as a square matrix whose determinant is zero and is not invertible.
If det A = 0, a square matrix "A" is said to be singular; otherwise, it is said to be non-singular.

$$A = \begin{bmatrix} 3 & 6 & 9 \\ 6 & 12 & 18 \\ 2 & 4 & 6 \end{bmatrix}$$

$\Rightarrow |A| = 3(12 \times 6 - 18 \times 4) - 6(6 \times 6 - 18 \times 2) + 9(6 \times 4 - 12 \times 2)$

$\Rightarrow |A| = 3(72 - 72) - 6(36 - 36) + 9(24 - 24)$

$\Rightarrow |A| = 3 \times 0 - 6 \times 0 + 9 \times 0 = 0$

Non Singular Matrix

A Non-Singular matrix is defined as a square matrix whose determinant is not equal to zero and is invertible.

$$|A| = \begin{bmatrix} 1 & 5 \\ 9 & 8 \end{bmatrix}$$

$\Rightarrow |A| = 8 \times 1 - 9 \times 5 = 8 - 45 = -37$

Symmetric Matrix

A square matrix "A" of any order is defined as a symmetric matrix if the transpose of the matrix is equal to the original matrix itself, i.e., $A^T = A$.

$$|A| = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Skew Symmetric Matrix

A square matrix "A" of any order is defined as a skew-symmetric matrix if the transpose of the matrix is equal to the negative of the original matrix itself, i.e., $A^T = -A$.

$$\begin{bmatrix} 0 & 3 & 5 \\ -3 & 0 & -2 \\ -5 & 2 & 0 \end{bmatrix}$$

Orthogonal Matrix

A square matrix whose transpose is equal to its inverse is called Orthogonal Matrix. In an Orthogonal Matrix if $A^T = A^{-1}$ then $AA^T = I$ where I is the Identity Matrix.

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \text{ and } A^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$A \times A^T = \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) \\ \sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta) & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix}$$

$$\Rightarrow A \times A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{(2\times 2)}$$

Idempotent Matrix

An idempotent matrix is a special type of square matrix that remains unchanged when multiplied by itself, i.e., $A^2 = A$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Hence,\ A \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$$

Nilpotent Matrix

A Nilpotent is a square matrix that when raised to some positive power results in a zero matrix. The least power let's say 'p' for which the matrix yields zero matrices, then it is called the Nilpotent Matrix of power 'p'.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $A^3 = A \cdot A^2$

$$\Rightarrow A^3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, A is a Nilpotent Matrix of index 3.

Involutory Matrix

An involutory matrix is a special type of square matrix whose inverse is the original matrix itself, i.e., $P = P^{-1}$, or, in other words, its square is equal to an identity matrix i.e. $P^2 = I$.

$$A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$$

Periodic Matrix

A periodic matrix is a square matrix that exhibits periodicity, meaning there exists a positive integer **p** such that when the matrix is raised to the power **p+1**, it equals the original matrix (**$A^{p+1} = A$**). If p = 1 then $A^2 = A$ it means A is an Idempotent Matrix. Thus we can say that the Idempotent Matrix is a case of the Periodic Matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The above square matrix is a Periodic Matrix of Period 2, where p = 1.