Lect 4: Matrix Eq 1

Monday, 12 January 2026 12:44 pm

The Matrix Equation $\mathbf{Ax} = \mathbf{b}$

Definition 4.1: Given a matrix $\mathbf{A} \in M_{m\times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

we define the product of $\mathbf{A}$ and $\mathbf{x}$ as the vector $\mathbf{Ax}$ in $\mathbb{R}^m$ given by

$$\mathbf{Ax} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

$$(m \times n) \cdot (n \times 1) \to m \times 1$$

Example 4.2. Compute $\mathbf{Ax}$.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ -4 \\ -3 \\ 8 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 3 & 3 & -2 \\ 4 & -4 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & 1 & -2 \\ 3 & -3 & 3 \\ 0 & -2 & -3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$
\begin{aligned}
\mathbf{A}\mathbf{x} &= \begin{bmatrix} 1 & -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} (1)(2) + (-1)(-4) + (3)(-3) + (0)(8) \end{bmatrix} = \begin{bmatrix} -3 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}\mathbf{x} &= \begin{bmatrix} 3 & 3 & -2 \\ 4 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} (3)(1) + (3)(0) + (-2)(-1) \\ (4)(1) + (-4)(0) + (-1)(-1) \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 5 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}\mathbf{x} &= \begin{bmatrix} -1 & 1 & 0 \\ 4 & 1 & -2 \\ 3 & -3 & 3 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} (-1)(-1) + (1)(2) + (0)(-2) \\ (4)(-1) + (1)(2) + (-2)(-2) \\ (3)(-1) + (-3)(2) + (3)(-2) \\ (0)(-1) + (-2)(2) + (-3)(-2) \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 2 \\ -15 \\ 2 \end{bmatrix}
\end{aligned}
$$

Theorem 4.3: Let $\mathbf{A}$ be an $m \times n$ a matrix.
(a) For any vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ it holds that

$$\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v}.$$

(b) For any vector $\mathbf{u}$ and scalar c it holds that

$$\mathbf{A}(c\mathbf{u}) = c(\mathbf{A}\mathbf{u}).$$

Example 4.4. For the given data, verify that the properties of Theorem 4.3 hold:

$$\mathbf{A} = \begin{bmatrix} 3 & -3 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad c = -2.$$

To find a vector $\mathbf{x} \in \mathbb{R}^n$ that solves the matrix equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

we solve the linear system whose augmented matrix is

$$\begin{bmatrix}\mathbf{A} & \mathbf{b}\end{bmatrix} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathbf{b}\end{bmatrix}.$$

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

ı the compact form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Theorem 4.6: Let $\mathbf{A} \in M_{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The following statements are equivalent:
(a) The equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution.
(b) The vector $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$.
(c) The linear system represented by the augmented matrix $\begin{bmatrix}\mathbf{A} & \mathbf{b}\end{bmatrix}$ is consistent.

Solve, if possible, the matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & -6 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}.$$

$$[\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & -6 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & -6 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -12 & 0 \end{bmatrix}.$$

Here $r = \operatorname{rank}(\mathbf{A}) = 3$ and therefore $d = 0$, i.e., no free parameters. Peforming back substitution we obtain that $x_1 = -11$, $x_2 = 3$, and $x_3 = 0$. Thus, the solution to the matrix equation is unique (no free parameters) and is given by

$$\mathbf{x} = \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}$$

Let's verify that $\mathbf{Ax} = \mathbf{b}$:

$$\mathbf{Ax} = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & -6 \end{bmatrix} \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -11+9+0 \\ -11+15+0 \\ 33-21 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix} = \mathbf{b}$$

In other words, $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$:

$$-11 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$$

Is the vector $\mathbf{b}$ in the span of the vectors $\mathbf{v}_1, \mathbf{v}_2$?

$$\mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}, \; \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \; \mathbf{v}_2 = \begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix}$$

$2.5\mathbf{v}_1 + 1.5\mathbf{v}_2 = \mathbf{b}$

Theorem 4.11: Let $\mathbf{A} \in M_{m\times n}$ be a matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, that is, $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$. The following are equivalent:

(a) $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^m$

(b) Every $\mathbf{b} \in \mathbb{R}^m$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

(c) The matrix equation $\mathbf{Ax} = \mathbf{b}$ has a solution for *any* $\mathbf{b} \in \mathbb{R}^m$.

(d) The rank of $\mathbf{A}$ is m.

Example 4.12. Do the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Solution. From Theorem 4.11, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$ if the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ has rank $r = 3$ (leading entries in its REF/RREF). The RREF of $\mathbf{A}$ is

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which does indeed have $r = 3$ leading entries. Therefore, regardless of the choice of $\mathbf{b} \in \mathbb{R}^3$, the augmented matrix $[\mathbf{A}\ \mathbf{b}]$ will be consistent. Therefore, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3.$$

In other words, every vector $\mathbf{b} \in \mathbb{R}^3$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. $\square$

After this lecture you should know the following:

- how to multiply a matrix $\mathbf{A}$ with a vector $\mathbf{x}$
- that the product $\mathbf{Ax}$ is a linear combination of the columns of $\mathbf{A}$
- how to solve the matrix equation $\mathbf{Ax} = \mathbf{b}$ if $\mathbf{A}$ and $\mathbf{b}$ are known
- how to determine if a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ spans all of $\mathbb{R}^m$
- the relationship between the equation $\mathbf{Ax} = \mathbf{b}$, when $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$, and when the augmented matrix $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$ is consistent (Theorem 4.6)
- when the columns of a matrix $\mathbf{A} \in M_{m \times n}$ span all of $\mathbb{R}^m$ (Theorem 4.11)
- the basic properties of matrix-vector multiplication Theorem 4.3