Lect. 5. Vectors

Friday, 9 January 2026 11:05 am

3.1 Vectors in $\mathbb{R}^n$

Recall that a **column vector** in $\mathbb{R}^n$ is a $n \times 1$ matrix. From now on, we will drop the "column" descriptor and simply use the word **vectors**. It is important to emphasize that a vector in $\mathbb{R}^n$ is simply a list of n numbers; you are safe (and highly encouraged!) to forget the idea that a vector is an object with an arrow. Here is a vector in $\mathbb{R}^2$:

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Here is a vector in $\mathbb{R}^3$:

$$\mathbf{v} = \begin{bmatrix} -3 \\ 0 \\ 11 \end{bmatrix}.$$

Here is a vector in $\mathbb{R}^6$:

$$\mathbf{v} = \begin{bmatrix} 9 \\ 0 \\ -3 \\ 6 \\ 0 \\ 3 \end{bmatrix}.$$

We can add/subtract vectors, and multiply vectors by numbers or **scalars**. For example, here is the addition of two vectors:

$$\begin{bmatrix} 0 \\ -5 \\ 9 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ 9 \\ 3 \end{bmatrix}.$$

And the multiplication of a scalar with a vector:

$$3 \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 15 \end{bmatrix}.$$

And here are both operations combined:

$$-2 \begin{bmatrix} 4 \\ -8 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 9 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ -6 \end{bmatrix} + \begin{bmatrix} -6 \\ 27 \\ 12 \end{bmatrix} = \begin{bmatrix} -14 \\ 43 \\ 6 \end{bmatrix}.$$

These operations constitute "the algebra" of vectors. As the following example illustrates, vectors can be used in a natural way to represent the solution of a linear system.

Example 3.1. Write the general solution in vector form of the linear system represented by the augmented matrix

$$[\mathbf{A} \ \ \mathbf{b}] = \begin{bmatrix} 1 & -7 & 2 & -5 & 8 & 10 \\ 0 & 1 & -3 & 3 & 1 & -5 \\ 0 & 0 & 0 & 1 & -1 & 4 \end{bmatrix}$$

Solution. The number of unknowns is $n = 5$ and the associated coefficient matrix $\mathbf{A}$ has rank $r = 3$. Thus, the solution set is parametrized by $d = n - r = 2$ parameters. This system was considered in Example 2.4 and the general solution was found to be

$$\begin{aligned} x_1 &= -89 - 31t_1 + 19t_2 \\ x_2 &= -17 - 4t_1 + 3t_2 \\ x_3 &= t_2 \\ x_4 &= 4 + t_1 \\ x_5 &= t_1 \end{aligned}$$

where t_1 and t_2 are arbitrary real numbers. The solution in vector form therefore takes the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -89 - 31t_1 + 19t_2 \\ -17 - 4t_1 + 3t_2 \\ t_2 \\ 4 + t_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} -89 \\ -17 \\ 0 \\ 4 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} -31 \\ -4 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 19 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\square$

A fundamental problem in **linear algebra** is solving vector equations for an unknown vector. As an example, suppose that you are given the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -8 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -2 \\ 9 \\ 4 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -14 \\ 43 \\ 6 \end{bmatrix},$$

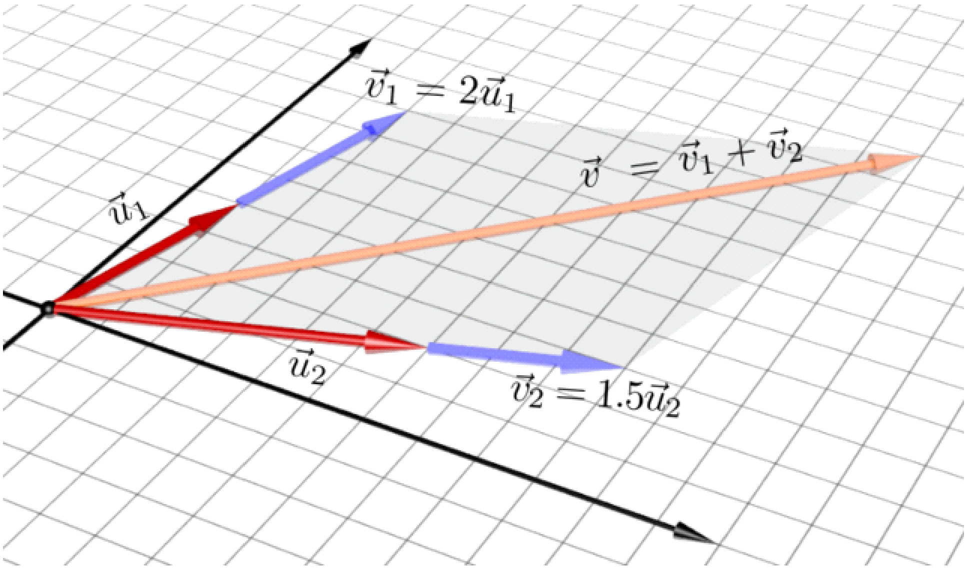
and asked to find numbers x_1 and x_2 such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$, that is,

$$x_1 \begin{bmatrix} 4 \\ -8 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 9 \\ 4 \end{bmatrix} = \begin{bmatrix} -14 \\ 43 \\ 6 \end{bmatrix}.$$

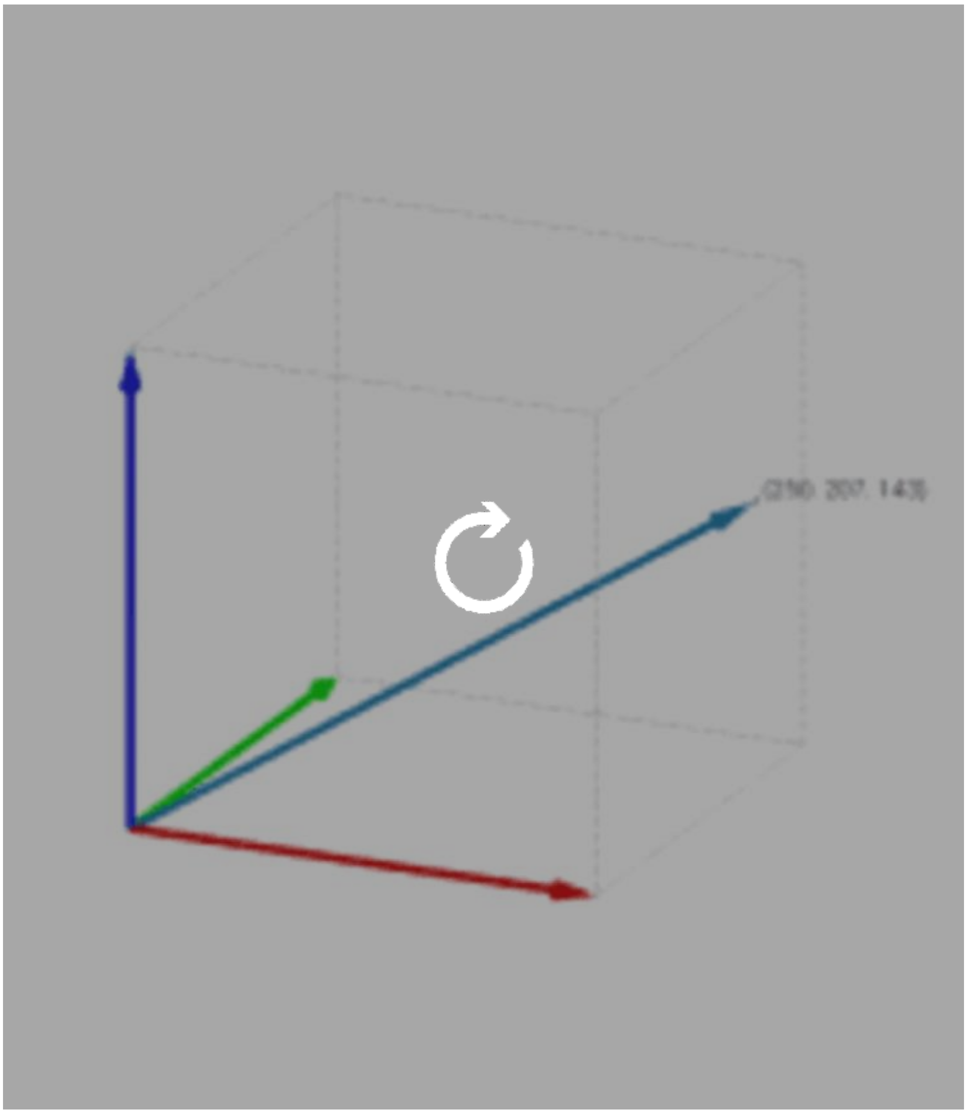
Here the unknowns are the scalars x_1 and x_2. After some guess and check, we find that $x_1 = -2$ and $x_2 = 3$ is a solution to the problem since

$$-2 \begin{bmatrix} 4 \\ -8 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 9 \\ 4 \end{bmatrix} = \begin{bmatrix} -14 \\ 43 \\ 6 \end{bmatrix}.$$

In some sense, the vector $\mathbf{b}$ is a combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. This motivates the following definition.



[Linear combination of vectors and RGB colours]



Definition 3.2: Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. A vector $\mathbf{b}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ if there exists scalars $x_1, x_2, \ldots, x_p$ such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$.

The scalars in a linear combination are called the **coefficients** of the linear combination. As an example, given the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -3 \\ 0 \\ -27 \end{bmatrix}$$

you can verify (and you should!) that

$$3\mathbf{v}_1 + 4\mathbf{v}_2 - 2\mathbf{v}_3 = \mathbf{b}.$$

Therefore, we can say that $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ with coefficients $x_1 = 3$, $x_2 = 4$, and $x_3 = -2$.

3.2 The linear combination problem

The linear combination problem is the following:

Problem: Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ and $\mathbf{b}$, is $\mathbf{b}$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$?

For example, say you are given the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

and also

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

Does there exist scalars x_1, x_2, x_3 such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}? \tag{3.1}$$

For obvious reasons, equation (3.1) is called a **vector equation** and the unknowns are x_1, x_2, and x_3. To gain some intuition with the linear combination problem, let's do an example by inspection.

Example 3.3. Let $\mathbf{v}_1 = (1, 0, 0)$, let $\mathbf{v}_2 = (0, 0, 1)$, let $\mathbf{b}_1 = (0, 2, 0)$, and let $\mathbf{b}_2 = (-3, 0, 7)$. Are $\mathbf{b}_1$ and $\mathbf{b}_2$ linear combinations of $\mathbf{v}_1, \mathbf{v}_2$?

Solution. For any scalars x_1 and x_2

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

and thus no, $\mathbf{b}_1$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. On the other hand, by inspection we have that

$$-3\mathbf{v}_1 + 7\mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} = \mathbf{b}_2$$

and thus yes, $\mathbf{b}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. These examples, of low dimension, were more-or-less obvious. Going forward, we are going to need a systematic way to solve the linear combination problem that does not rely on pure inspection. □

We now describe how the linear combination problem is connected to the problem of solving a system of linear equations. Consider again the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

Does there exist scalars x_1, x_2, x_3 such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}? \tag{3.2}$$

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \begin{bmatrix} x_1 \\ 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ x_3 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + 2x_3 \end{bmatrix}.$$

$$\begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_3 &= -2. \end{aligned}$$

$$[\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & -2 \end{bmatrix}$$

$$[\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b} \end{bmatrix}$$

Applying the row reduction algorithm, the solution is

$$x_1 = 0,\ x_2 = 2,\ x_3 = -1$$

and thus these coefficients solve the linear combination problem. In other words,

$$0\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{b}$$

Example 3.4. Is the vector $\mathbf{b} = (7, 4, -3)$ a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} ?$$

Solution. Form the augmented matrix:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

The RREF of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and therefore the solution is $x_1 = 3$ and $x_2 = 2$. Therefore, yes, $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$:

$$3\mathbf{v}_1 + 2\mathbf{v}_2 = 3 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} = \mathbf{b}$$

Notice that the solution set does not contain any free parameters because $n = 2$ (unknowns) and $r = 2$ (rank) and so $d = 0$. Therefore, the above linear combination is the only way to write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. $\square$

Example 3.5. Is the vector $\mathbf{b} = (1, 0, 1)$ a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} ?$$

Solution. The augmented matrix of the corresponding linear system is

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}.$$

After row reducing we obtain that

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The last row is inconsistent, and therefore the linear system does not have a solution. Therefore, no, $\mathbf{b}$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. $\square$

Example 3.6. Is the vector $\mathbf{b} = (8, 8, 12)$ a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix}?$$

Solution. The augmented matrix is

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} x_1 &= -8 - 2t \\ x_2 &= t \\ x_3 &= 4 \end{aligned}$$

$$-10\mathbf{v}_1 + \mathbf{v}_2 + 4\mathbf{v}_3 = -10 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + 4 \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 12 \end{bmatrix} = \mathbf{b}$$

Or, choosing $t = -2$ we obtain $x_1 = -4$, $x_2 = -2$, and $x_3 = 4$, and you can verify that

$$-4\mathbf{v}_1 - 2\mathbf{v}_2 + 4\mathbf{v}_3 = -4 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + 4 \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 12 \end{bmatrix} = \mathbf{b}$$

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p.$$

$$\mathbf{v}_2 = 0\mathbf{v}_1 + (1)\mathbf{v}_2 + 0\mathbf{v}_3 + \cdots + 0\mathbf{v}_p.$$

$$x\mathbf{v}_2 = 0\mathbf{v}_1 + x\mathbf{v}_2 + 0\mathbf{v}_3 + \cdots + 0\mathbf{v}_p.$$

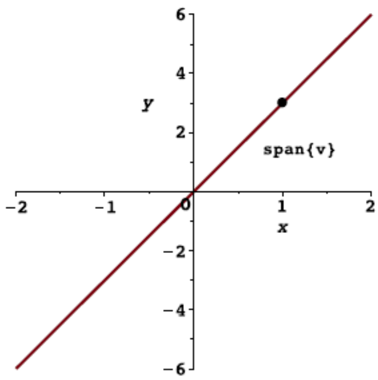
The span of a set of vectors

$$-10\mathbf{v}_1 + \mathbf{v}_2 + 4\mathbf{v}_3 = -10\begin{bmatrix}2\\1\\3\end{bmatrix} + \begin{bmatrix}4\\2\\6\end{bmatrix} + 4\begin{bmatrix}6\\4\\9\end{bmatrix} = \begin{bmatrix}8\\8\\12\end{bmatrix}.$$

Definition 3.7: Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors. The set of all vectors that are a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is called the **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, and we denote it by

$$S = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}.$$

$\mathbf{b} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.



1: The span of a single non-zero vector in $\mathbb{R}^2$.

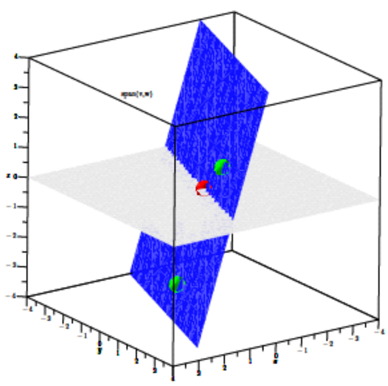


Figure 3.2: The span of two vectors, not multiples of each other, in $\mathbb{R}^3$.

Example 3.8. Is the vector $\mathbf{b} = (7, 4, -3)$ in the span of the vectors $\mathbf{v}_1 = (1, -2, -5)$, $\mathbf{v}_2 = (2, 5, 6)$? In other words, is $\mathbf{b} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

Example 3.9. Is the vector $\mathbf{b} = (1,0,1)$ in the span of the vectors $\mathbf{v}_1 = (1,0,2), \mathbf{v}_2 = (0,1,0), \mathbf{v}_3 = (2,1,4)$?

Solution. From Example 3.5, we have that

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b}\end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The last row is inconsistent and therefore $\mathbf{b}$ is not in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. □

Example 3.10. Is the vector $\mathbf{b} = (8,8,12)$ in the span of the vectors $\mathbf{v}_1 = (2,1,3), \mathbf{v}_2 = (4,2,6), \mathbf{v}_3 = (6,4,9)$?

Solution. From Example 3.6, we have that

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b}\end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system is consistent and therefore $\mathbf{b} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In this case, the solution set contains $d = 1$ free parameters and therefore, it is possible to write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in infinitely many ways. □

Example 3.11. Answer the following with True or False, and explain your answer.

(a) The vector $\mathbf{b} = (1,2,3)$ is in the span of the set of vectors

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix} \right\}.$$

(b) The solution set of the linear system whose augmented matrix is $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b}\end{bmatrix}$ is the same as the solution set of the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$.

(c) Suppose that the augmented matrix $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b}\end{bmatrix}$ has an inconsistent row. Then either $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ or $\mathbf{b} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(d) The span of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ (at least one of which is nonzero) contains only the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and the zero vector $\mathbf{0}$.

After this lecture you should know the following:

- what a vector is
- what a linear combination of vectors is
- what the linear combination problem is
- the relationship between the linear combination problem and the problem of solving linear systems of equations
- how to solve the linear combination problem
- what the span of a set of vectors is
- the relationship between what it means for a vector $\mathbf{b}$ to be in the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ and the problem of writing $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$
- the geometric interpretation of the span of a set of vectors