Lect. 7: Linear Transformation

Monday, 19 January 2026 4:38 pm

Introduction to Linear Mappings

Vector mappings

By a **vector mapping** we mean simply a function

$$\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^m.$$

The **domain** of T is $\mathbb{R}^n$ and the **co-domain** of T is $\mathbb{R}^m$. The case $n = m$ is allowed of course. In engineering or physics, the domain is sometimes called the **input space** and the co-domain is called the **output space**. Using this terminology, the points $\mathbf{x}$ in the domain are called the **inputs** and the points $\mathsf{T}(\mathbf{x})$ produced by the mapping are called the **outputs**.

Definition 7.1: The vector $\mathbf{b} \in \mathbb{R}^m$ is in the **range** of T, or in the **image** of T, if there exists some $\mathbf{x} \in \mathbb{R}^n$ such that $\mathsf{T}(\mathbf{x}) = \mathbf{b}$.

In other words, $\mathbf{b}$ is in the range of T if there is an input $\mathbf{x}$ in the domain of T that outputs $\mathbf{b} = \mathsf{T}(\mathbf{x})$. In general, not every point in the co-domain of T is in the range of T. For example, consider the vector mapping $\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$\mathsf{T}(\mathbf{x}) = \begin{bmatrix} x_1^2 \sin(x_2) - \cos(x_1^2 - 1) \\ x_1^2 + x_2^2 + 1 \end{bmatrix}.$$

The vector $\mathbf{b} = (3, -1)$ is not in the range of T because the second component of $\mathsf{T}(\mathbf{x})$ is positive. On the other hand, $\mathbf{b} = (-1, 2)$ is in the range of T because

$$\mathsf{T}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1^2 \sin(0) - \cos(1^2 - 1) \\ 1^2 + 0^2 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \mathbf{b}.$$

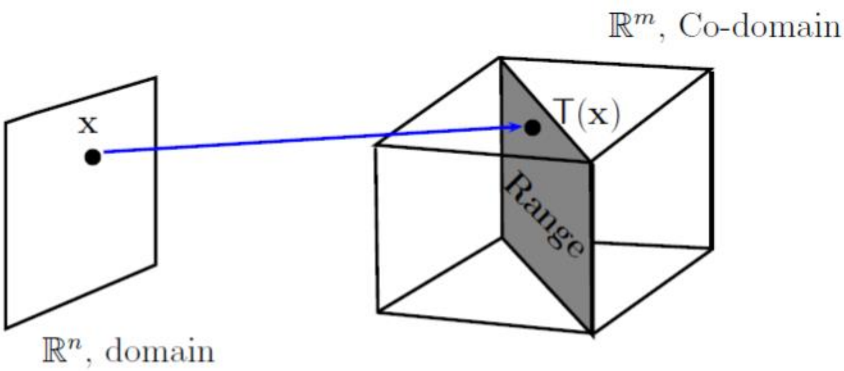


Figure 7.1: The domain, co-domain, and range of a mapping.

Linear mappings

For our purposes, vector mappings $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^m$ can be organized into two categories: (1) linear mappings and (2) nonlinear mappings.

Definition 7.2: The vector mapping $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **linear** if the following conditions hold:

- For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, it holds that $\mathsf{T}(\mathbf{u} + \mathbf{v}) = \mathsf{T}(\mathbf{u}) + \mathsf{T}(\mathbf{v})$.
- For any $\mathbf{u} \in \mathbb{R}^n$ and any scalar c, it holds that $\mathsf{T}(c\mathbf{u}) = c\mathsf{T}(\mathbf{u})$.

If T is not linear then it is said to be **nonlinear**.

As an example, the mapping

$$\mathsf{T}(\mathbf{x}) = \begin{bmatrix} x_1^2 \sin(x_2) - \cos(x_1^2 - 1) \\ x_1^2 + x_2^2 + 1 \end{bmatrix}$$

is **nonlinear**. To see this, previously we computed that

$$\mathsf{T}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Example 7.3. Is the vector mapping $\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}^3$ linear?

$$\mathsf{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \\ -x_1 - 3x_2 \end{bmatrix}$$

Solution. We must verify that the two conditions in Definition 7.2 hold. For the first condition, take arbitrary vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. We compute:

$$\begin{aligned}
\mathsf{T}(\mathbf{u} + \mathbf{v}) &= \mathsf{T}\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2(u_1 + v_1) - (u_2 + v_2) \\ (u_1 + v_1) + (u_2 + v_2) \\ -(u_1 + v_1) - 3(u_2 + v_2) \end{bmatrix} \\
&= \begin{bmatrix} 2u_1 + 2v_1 - u_2 - v_2 \\ u_1 + v_1 + u_2 + v_2 \\ -u_1 - v_1 - 3u_2 - 3v_2 \end{bmatrix} \\
&= \begin{bmatrix} 2u_1 - u_2 + 2v_1 - v_2 \\ u_1 + u_2 + v_1 + v_2 \\ -u_1 - 3u_2 - v_1 - 3v_2 \end{bmatrix} \\
&= \begin{bmatrix} 2u_1 - u_2 \\ u_1 + u_2 \\ -u_1 - 3u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 - v_2 \\ v_1 + v_2 \\ -v_1 - 3v_2 \end{bmatrix} \\
&= \mathsf{T}(\mathbf{u}) + \mathsf{T}(\mathbf{v})
\end{aligned}$$

Therefore, for arbitrary $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, it holds that

$$\mathsf{T}(\mathbf{u}+\mathbf{v}) = \mathsf{T}(\mathbf{u}) + \mathsf{T}(\mathbf{v}).$$

To prove the second condition, let $c \in \mathbb{R}$ be an arbitrary scalar. Then:

$$\begin{aligned}
\mathsf{T}(c\mathbf{u}) = \mathsf{T}\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) &= \begin{bmatrix} 2(cu_1) - (cu_2) \\ (cu_1) + (cu_2) \\ -(cu_1) - 3(cu_2) \end{bmatrix} \\
&= \begin{bmatrix} c(2u_1 - u_2) \\ c(u_1 + u_2) \\ c(-u_1 - 3u_2) \end{bmatrix} \\
&= c \begin{bmatrix} 2u_1 - u_2 \\ u_1 + u_2 \\ -u_1 - 3u_2 \end{bmatrix} \\
&= c\mathsf{T}(\mathbf{u})
\end{aligned}$$

Therefore, both conditions of Definition 7.2 hold, and thus T is a linear map.

Example 7.4. Let $\alpha \geq 0$ and define the mapping $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^n$ by the formula $\mathsf{T}(\mathbf{x}) = \alpha\mathbf{x}$. If $0 \leq \alpha \leq 1$ then T is called a **contraction** and if $\alpha > 1$ then T is called a **dilation**. In either case, show that T is a linear mapping.

Solution. Let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary. Then

$$\mathsf{T}(\mathbf{u}+\mathbf{v}) = \alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v} = \mathsf{T}(\mathbf{u}) + \mathsf{T}(\mathbf{v}).$$

This shows that condition (1) in Definition 7.2 holds. To show that the second condition holds, let c is any number. Then

$$\mathsf{T}(c\mathbf{x}) = \alpha(c\mathbf{x}) = \alpha c\mathbf{x} = c(\alpha\mathbf{x}) = c\mathsf{T}(\mathbf{x}).$$

Therefore, both conditions of Definition 7.2 hold, and thus T is a linear mapping. To see a particular example, consider the case $\alpha = \frac{1}{2}$ and $n = 3$. Then,

$$\mathsf{T}(\mathbf{x}) = \tfrac{1}{2}\mathbf{x} = \begin{bmatrix} \frac{1}{2}x_1 \\ \frac{1}{2}x_2 \\ \frac{1}{2}x_3 \end{bmatrix}.$$

Matrix mappings

$\mathsf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

1. $\mathbf{A}(\mathbf{u}+\mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v}$

2. $\mathbf{A}(c\mathbf{u}) = c\mathbf{A}\mathbf{u}$.

Theorem 7.5: To a given matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ associate the mapping $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^m$ defined by the formula $\mathsf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Then T is a linear mapping.

$$\mathsf{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Therefore, T is a matrix mapping corresponding to the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & -3 \end{bmatrix}$$

that is, $\mathsf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$. By Theorem 7.5, T is a linear mapping.

Example 7.9. Let $\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u}$ to $\mathsf{T}(\mathbf{u}) = (3, 4)$ and maps $\mathbf{v}$ to $\mathsf{T}(\mathbf{v}) = (-2, 5)$. Find $\mathsf{T}(2\mathbf{u} + 3\mathbf{v})$.

Solution. Because T is a linear mapping we have that

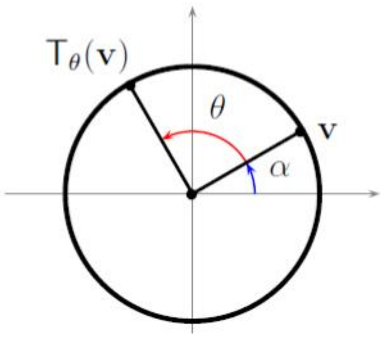
$$\mathsf{T}(2\mathbf{u} + 3\mathbf{v}) = \mathsf{T}(2\mathbf{u}) + \mathsf{T}(3\mathbf{v}) = 2\mathsf{T}(\mathbf{u}) + 3\mathsf{T}(\mathbf{v}).$$

We know that $\mathsf{T}(\mathbf{u}) = (3, 4)$ and $\mathsf{T}(\mathbf{v}) = (-2, 5)$. Therefore,

$$\mathsf{T}(2\mathbf{u} + 3\mathbf{v}) = 2\mathsf{T}(\mathbf{u}) + 3\mathsf{T}(\mathbf{v}) = 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 23 \end{bmatrix}.$$

$\square$

Example 7.10. (Rotations) Let $\mathsf{T}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping on the 2D plane that rotates every $\mathbf{v} \in \mathbb{R}^2$ by an angle θ. Write down a formula for T_θ and show that T_θ is a linear mapping.



Solution. If $\mathbf{v} = (\cos(\alpha), \sin(\alpha))$ then

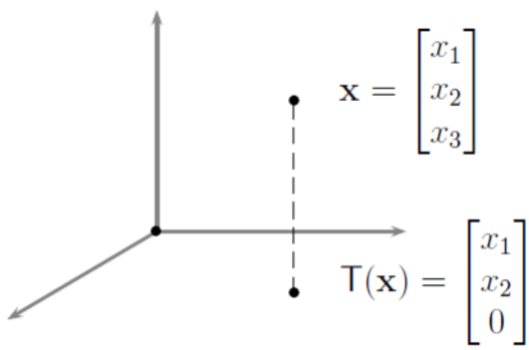
$$\mathsf{T}_\theta(\mathbf{v}) = \begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Example 7.11. (Projections) Let $\mathsf{T} : \mathbb{R}^3 \to \mathbb{R}^2$ be the vector mapping

$$\mathsf{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

$$\mathsf{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



e 7.2: Projection onto the (x_1, x_2) plane