Lect 5: Matrix Eq 2

Monday, 12 January 2026 3:28 pm

Homogeneous and Nonhomogeneous Systems

Definition 5.1: A linear system of the form $\mathbf{Ax} = \mathbf{0}$ is called a **homogeneous** linear system.

A homogeneous system $\mathbf{Ax} = \mathbf{0}$ always has at least one solution, namely, the zero solution because $\mathbf{A0} = \mathbf{0}$. A homogeneous system is therefore always consistent. The zero solution $\mathbf{x} = \mathbf{0}$ is called the **trivial solution** and any non-zero solution is called a **nontrivial solution**. From the existence and uniqueness theorem (Theorem 2.5), we know that a consistent linear system will have either one solution or infinitely many solutions. Therefore, a homogeneous linear system has nontrivial solutions if and only if its solution set has at least one parameter.

Recall that the number of parameters in the solution set is $d = n - r$, where r is the rank of the coefficient matrix $\mathbf{A}$ and n is the number of unknowns.

Example 5.2. Does the linear homogeneous system have any nontrivial solutions?

$$\begin{aligned} 3x_1 + x_2 - 9x_3 &= 0 \\ x_1 + x_2 - 5x_3 &= 0 \\ 2x_1 + x_2 - 7x_3 &= 0 \end{aligned}$$

The RREF is:

$$\begin{bmatrix} 3 & 1 & -9 & 0 \\ 1 & 1 & -5 & 0 \\ 2 & 1 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent. The rank of the coefficient matrix is $r = 2$ and thus there will be $d = 3 - 2 = 1$ free parameter in the solution set. If we let x_3 be the free parameter, say $x_3 = t$, then from the row equivalent augmented matrix

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we obtain that $x_2 = 3x_3 = 3t$ and $x_1 = 2x_3 = 2t$. Therefore, the general solution of the linear system is

$$x_1 = 2t$$

linear system is

$$x_1 = 2t$$
$$x_2 = 3t$$
$$x_3 = t$$

Or more compactly if we let $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ then $\mathbf{x} = \mathbf{v}t$. Hence, any solution $\mathbf{x}$ to the linear system can be written as a linear combination of the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. In other words, the solution set of the linear system is the span of the vector $\mathbf{v}$:

$$\mathrm{span}\{\mathbf{v}\}.$$

□

Example 5.3. Find the general solution of the homogenous system $\mathbf{Ax} = \mathbf{0}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 1 & 4 \\ 3 & 7 & 7 & 3 & 13 \\ 2 & 5 & 5 & 2 & 9 \end{bmatrix}.$$

Solution. After row reducing we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 1 & 4 \\ 3 & 7 & 7 & 3 & 13 \\ 2 & 5 & 5 & 2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here $n = 5$, and $r = 2$, and therefore the number of parameters in the solution set is $d = n - r = 3$. The second row of $\mathtt{rref}(\mathbf{A})$ gives the equation

$$x_2 + x_3 + x_5 = 0.$$

Setting $x_5 = t_1$ and $x_3 = t_2$ as free parameters we obtain that

$$x_2 = -x_3 - x_5 = -t_2 - t_1.$$

From the first row we obtain the equation

$$x_1 + x_4 + 2x_5 = 0$$

The unknown x_5 has already been assigned, so we must now choose either x_1 or x_4 to be a parameter. Choosing $x_4 = t_3$ we obtain that

$$x_1 = -x_4 - 2x_5 = -t_3 - 2t_1$$

In summary, the general solution can be written as

$$\mathbf{x} = \begin{bmatrix} -t_3 - 2t_1 \\ -t_2 - t_1 \\ t_2 \\ t_3 \\ t_1 \end{bmatrix} = t_1 \underbrace{\begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_1} + t_2 \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_2} + t_3 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_3} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3$$

where t_1, t_2, t_3 are arbitrary parameters. In other words, any solution $\mathbf{x}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$\mathbf{x} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

□

Nonhomogeneous systems

Example 5.6. Write the general solution, in parametric vector form, of the linear system

$$\begin{aligned} 3x_1 + x_2 - 9x_3 &= 2 \\ x_1 + x_2 - 5x_3 &= 0 \\ 2x_1 + x_2 - 7x_3 &= 1. \end{aligned}$$

Solution. The RREF of the augmented matrix is:

$$\begin{bmatrix} 3 & 1 & -9 & 2 \\ 1 & 1 & -5 & 0 \\ 2 & 1 & -7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent and the rank of the coefficient matrix is $r = 2$. Therefore, there are $d = 3 - 2 = 1$ parameters in the solution set. Letting $x_3 = t$ be the parameter, from the second row of the RREF we have

$$x_2 = 3t - 1$$

And from the first row of the RREF we have

$$x_1 = 2t + 1$$

Therefore, the general solution of the system in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} 2t + 1 \\ 3t - 1 \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{p}} + t \underbrace{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

$\mathbf{Ap} = \mathbf{b}$

Example 5.7. Write the general solution, in parametric vector form, of the linear system represented by the augmented matrix

$$\begin{bmatrix} 3 & -3 & 6 & 3 \\ -1 & 1 & -2 & -1 \\ 2 & -2 & 4 & 2 \end{bmatrix}.$$

Solution. Write the general solution, in parametric vector form, of the linear system represented by the augmented matrix

$$\begin{bmatrix} 3 & -3 & 6 & 3 \\ -1 & 1 & -2 & -1 \\ 2 & -2 & 4 & 2 \end{bmatrix}$$

The RREF of the augmented matrix is

$$\begin{bmatrix} 3 & -3 & 6 & 3 \\ -1 & 1 & -2 & -1 \\ 2 & -2 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here $n = 3$, $r = 1$ and therefore the solution set will have $d = 2$ parameters. Let $x_3 = t_1$ and $x_2 = t_2$. Then from the first row we obtain

$$x_1 = 1 + x_2 - 2x_3 = 1 + t_2 - 2t_1$$

The general solution in parametric vector form is therefore

$$\mathbf{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{p}} + t_1 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_1} + t_2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_2}$$

You should verify that $\mathbf{p}$ is a solution to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$:

$$\mathbf{A}\mathbf{p} = \mathbf{b}$$

And that $\mathbf{v}_1$ and $\mathbf{v}_2$ are solutions to the homogeneous linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$:

$$\mathbf{A}\mathbf{v}_1 = \mathbf{A}\mathbf{v}_2 = \mathbf{0}$$

Summary

The material in this lecture is so important that we will summarize the main results. The solution set of a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be written in the form

$$\mathbf{x} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_d\mathbf{v}_d$$

where $\mathbf{A}\mathbf{p} = \mathbf{b}$ and where each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$ satisfies $\mathbf{A}\mathbf{v}_i = \mathbf{0}$. Loosely speaking,

$$\{\text{Solution set of } \mathbf{A}\mathbf{x} = \mathbf{b}\} = \mathbf{p} + \{\text{Solution set of } \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

or

$$\{\text{Solution set of } \mathbf{A}\mathbf{x} = \mathbf{b}\} = \mathbf{p} + \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$$

where $\mathbf{p}$ satisfies $\mathbf{A}\mathbf{p} = \mathbf{b}$ and $\mathbf{A}\mathbf{v}_i = 0$.

After this lecture you should know the following:

- what a homogeneous/nonhomogenous linear system is
- when a homogeneous linear system has nontrivial solutions
- how to write the general solution set of a homogeneous system in parametric vector form Theorem 5.4)
- how to write the solution set of a nonhomogeneous system in parametric vector form Theorem 5.5)
- the relationship between the solution sets of the nonhomogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ and the homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{0}$