Lect 6: Vectors II

Monday, 12 January 2026 12:26 pm

Linear Independence

{EAST, NORTH, NORTH-EAST}

Definition 16.1: Let V be a vector space and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set of vectors in V. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is **linearly independent** if the only scalars $c_1, c_2, \ldots, c_p$ that satisfy the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

are the trivial scalars $c_1 = c_2 = \cdots = c_p = 0$. If the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is not linearly independent then we say that it is **linearly dependent**.

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We now describe the redundancy in a set of linear dependent vectors. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ are linearly dependent, it follows that there are scalars $c_1, c_2, \ldots, c_p$, **at least one of which is nonzero**, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}. \tag{$\star$}$$

For example, suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ are linearly dependent. Then there are scalars c_1, c_2, c_3, c_4, not all of them zero, such that equation $(\star)$ holds. Suppose, for the sake of argument, that $c_3 \neq 0$. Then,

$$\mathbf{v}_3 = -\frac{c_1}{c_3}\mathbf{v}_1 - \frac{c_2}{c_3}\mathbf{v}_2 - \frac{c_4}{c_3}\mathbf{v}_4.$$

Theorem 16.2: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Example 16.3. Show that the following set of 2×2 matrices is linearly dependent:

$$\left\{ \mathbf{A}_1 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} 5 & 0 \\ -2 & -3 \end{bmatrix} \right\}.$$

Solution. It is clear that $\mathbf{A}_1$ and $\mathbf{A}_2$ are linearly independent, i.e., $\mathbf{A}_1$ cannot be written as a scalar multiple of $\mathbf{A}_2$, and vice-versa. Since the $(2,1)$ entry of $\mathbf{A}_1$ is zero, the only way to get the -2 in the $(2,1)$ entry of $\mathbf{A}_3$ is to multiply $\mathbf{A}_2$ by -2. Similary, since the $(2,2)$ entry of $\mathbf{A}_2$ is zero, the only way to get the -3 in the $(2,2)$ entry of $\mathbf{A}_3$ is to multiply $\mathbf{A}_1$ by 3. Hence, we suspect that $3\mathbf{A}_1 - 2\mathbf{A}_2 = \mathbf{A}_3$. Verify:

$$3\mathbf{A}_1 - 2\mathbf{A}_2 = \begin{bmatrix} 3 & 6 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -2 & -3 \end{bmatrix} = \mathbf{A}_3$$

Therefore, $3\mathbf{A}_1 - 2\mathbf{A}_2 - \mathbf{A}_3 = \mathbf{0}$ and thus we have found scalars c_1, c_2, c_3 not all zero such that $c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + c_3\mathbf{A}_3 = \mathbf{0}$. □

Bases

Definition 16.4: Let W be a subspace of a vector space V. A set of vectors $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in W is said to be a **basis** for W if

(a) the set $\mathcal{B}$ spans all of W, that is, $\mathsf{W} = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, and

(b) the set $\mathcal{B}$ is linearly independent.

Example 16.5. Show that the standard unit vectors form a basis for $\mathsf{V} = \mathbb{R}^3$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solution. Any vector $\mathbf{x} \in \mathbb{R}^3$ can be written as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

Therefore, $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \mathbb{R}^3$. The set $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent. Indeed, if there are scalars c_1, c_2, c_3 such that

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = \mathbf{0}$$

then clearly they must all be zero, $c_1 = c_2 = c_3 = 0$. Therefore, by definition, $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$. This basis is called the **standard basis** for $\mathbb{R}^3$. Analogous arguments hold for $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$. □

Example 16.6. Is $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for $\mathbb{R}^3$?

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ -2 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -6 \\ -6 \end{bmatrix}$$

Solution. Form the matrix $\mathbf{A} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ and row reduce:

$$\mathbf{A} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 16.7. In $\mathsf{V} = \mathbb{R}^4$, consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 4 \\ 2 \\ -3 \end{bmatrix}.$$

Let $\mathsf{W} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Is $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for W?

Solution. By definition, $\mathcal{B}$ is a spanning set for W, so we need only determine if $\mathcal{B}$ is linearly independent. Form the matrix, $\mathbf{A} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ and row reduce to obtain

$$\mathbf{A} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\operatorname{rank}(\mathbf{A}) = 2$ and thus $\mathcal{B}$ is linearly dependent. Notice $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3$. Therefore, $\mathcal{B}$ is not a basis of W. $\square$

Example 16.8. Find a basis for the vector space of 2×2 matrices.

Dimension of a Vector Space

Theorem 16.10: Let V be a vector space. Then all bases of V have the same number of vectors.

Definition 16.11: Let V be a vector space. The **dimension** of V, denoted $\dim V$, is the number of vectors in any basis of V. The dimension of the trivial vector space $V = \{0\}$ is defined to be zero.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Theorem 16.12: Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be vectors in $\mathbb{R}^n$. If $\mathcal{B}$ is linearly independent then $\mathcal{B}$ is a basis for $\mathbb{R}^n$. Or if $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^n$ then $\mathcal{B}$ is a basis for $\mathbb{R}^n$.

Example 16.13. Do the columns of the matrix $\mathbf{A}$ form a basis for $\mathbb{R}^4$?

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 3 & -2 \\ 4 & 7 & 8 & -6 \\ 0 & 0 & 1 & 0 \\ -4 & -6 & -6 & 3 \end{bmatrix}$$

Solution. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ denote the columns of $\mathbf{A}$. Since we have $n = 4$ vectors in $\mathbb{R}^n$, we need only check that they are linearly independent. Compute

$$\det \mathbf{A} = -2 \neq 0$$

Hence, $\operatorname{rank}(\mathbf{A}) = 4$ and thus the columns of $\mathbf{A}$ are linearly independent. Therefore, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ form a basis for $\mathbb{R}^4$. $\square$