Optimization: Second Derivative Test

Friday, 16 August 2024 6:29 pm

15.1 CRITICAL POINTS: LOCAL EXTREMA AND SADDLE POINTS

Functions of several variables, like functions of one variable, can have *local* and *global* extrema. (That is, local and global maxima and minima.) A function has a local extremum at a point where it takes on the largest or smallest value in a small region around the point. Global extrema are the largest or smallest values anywhere on the domain under consideration. (See Figures 15.1 and 15.2.)

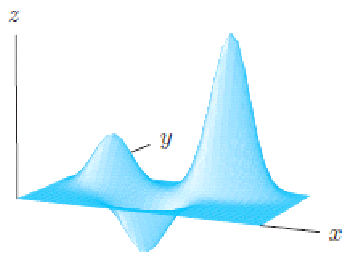


Figure 15.1: Local and global extrema for a function of two variables on $0 \le x \le a$, $0 \le y \le b$

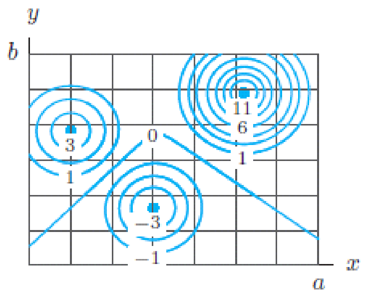


Figure 15.2: Contour map of the function in Figure 15.1

More precisely, considering only points at which f is defined, we say:

- f has a **local maximum** at the point P_0 if $f(P_0) \ge f(P)$ for all points P near P_0.
- f has a **local minimum** at the point P_0 if $f(P_0) \le f(P)$ for all points P near P_0.

Points where the gradient is either $\vec{0}$ or undefined are called **critical points** of the function.

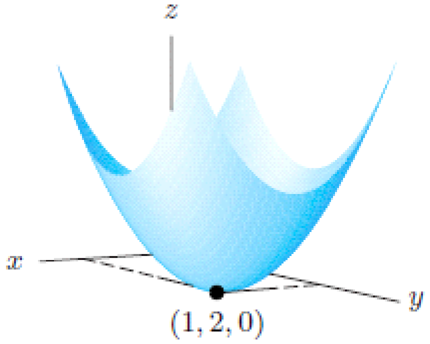
Example 1 Find and analyze the critical points of $f(x, y) = x^2 - 2x + y^2 - 4y + 5$.

Solution To find the critical points, we set both partial derivatives equal to zero:

$$f_x(x, y) = 2x - 2 = 0$$
$$f_y(x, y) = 2y - 4 = 0.$$

Solving these equations gives $x = 1$, $y = 2$. Hence, f has only one critical point, namely $(1, 2)$.

$$f(x, y) = x^2 - 2x + y^2 - 4y + 5 = (x - 1)^2 + (y - 2)^2.$$



EXAMPLE : Find local extrema of ...
for $f(x,y) = x^2 + y^2 - 2x - 6y + 14$


Figure 15.13: Local minimum: $D > 0$ and $a > 0$

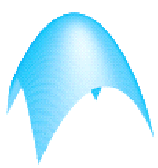


Figure 15.14: Local maximum: $D > 0$ and $a < 0$


Figure 15.15: Saddle point: $D < 0$

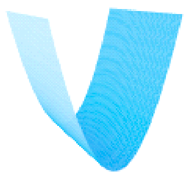
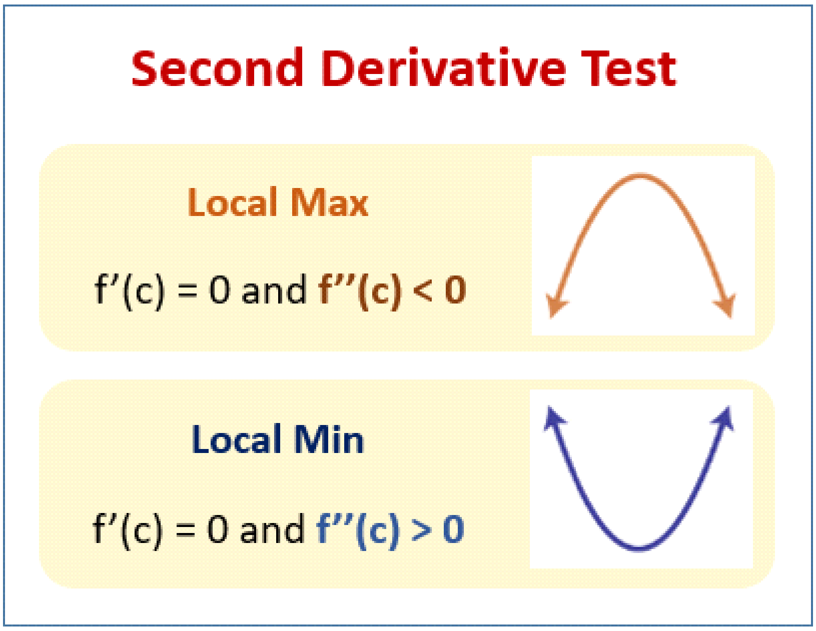


Figure 15.16: Parabolic cylinder: $D = 0$



Second-Derivative Test for Functions of Two Variables

Suppose (x_0, y_0) is a point where grad $f(x_0, y_0) = \vec{0}$. Let

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
- If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
- If $D < 0$, then f has a saddle point at (x_0, y_0).
- If $D = 0$, anything can happen: f can have a local maximum, or a local minimum, or a saddle point, or none of these, at (x_0, y_0).

Find the local extrema of the function $f(x, y) = 8y^3 + 12x^2 - 24xy$.

EXAMPLE: $f(x,y) = x^4 + y^4 - 4xy + 1$
Find local extrema & saddle points.

Q $f(x,y) = 3x^2 - y^3 - 6xy$
Find and discriminate all critical points.