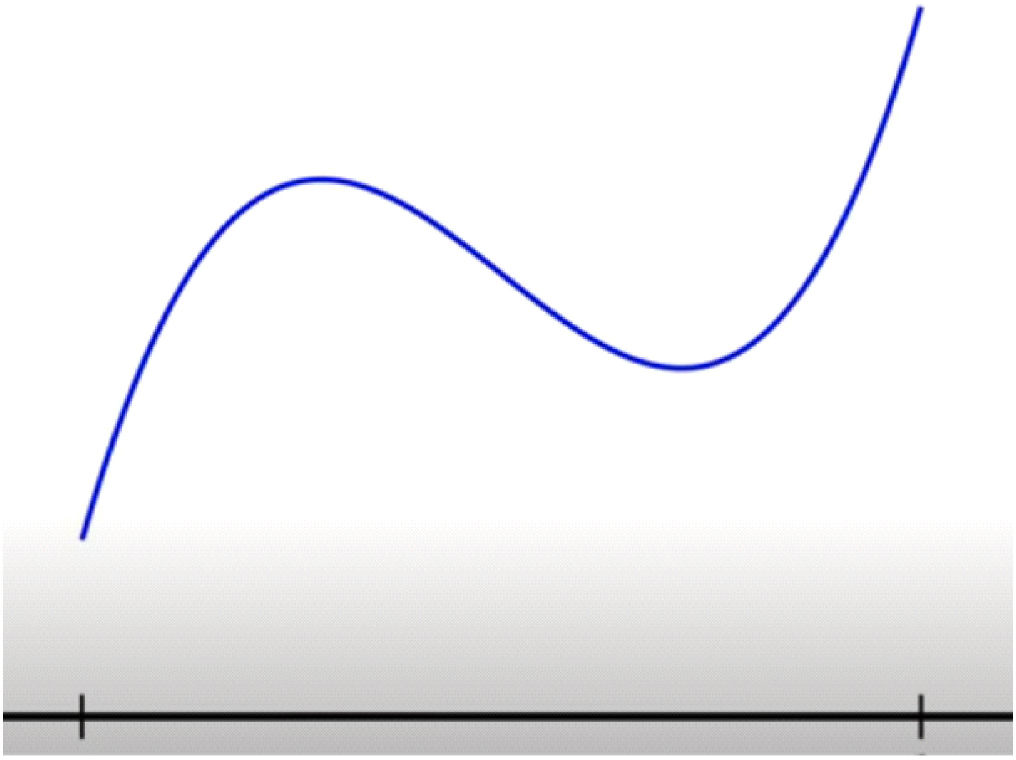
Lagrange Multiplier

Friday, 16 August 2024 7:12 pm



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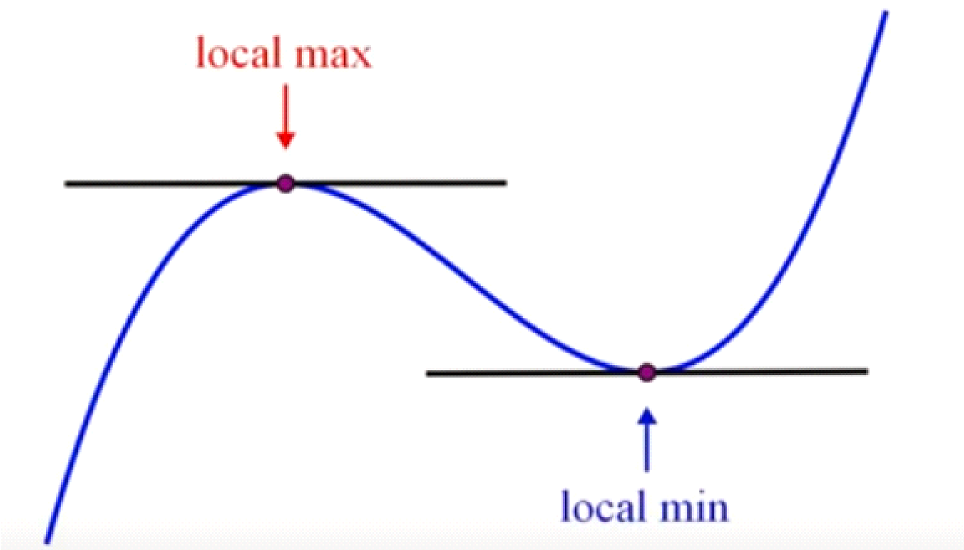
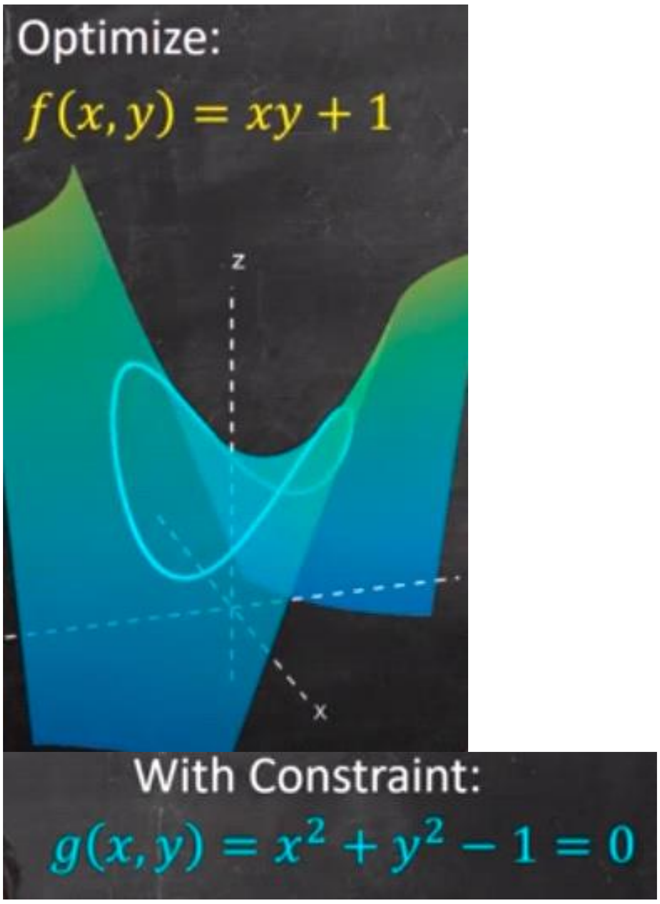
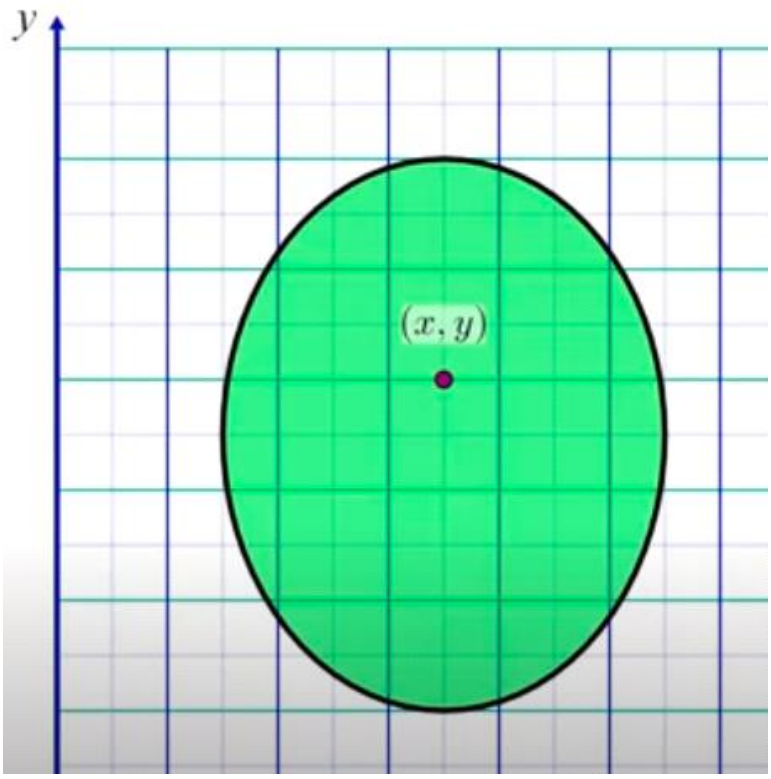
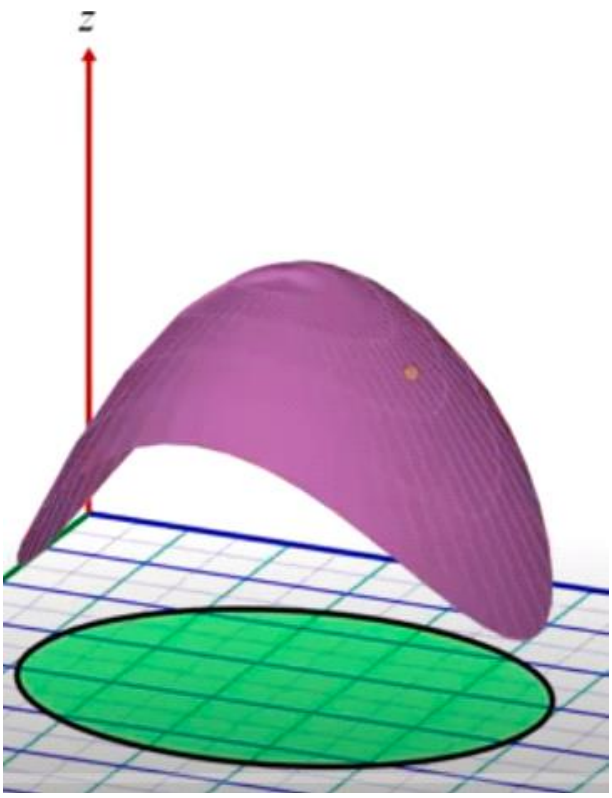




Constraints (Conditions)

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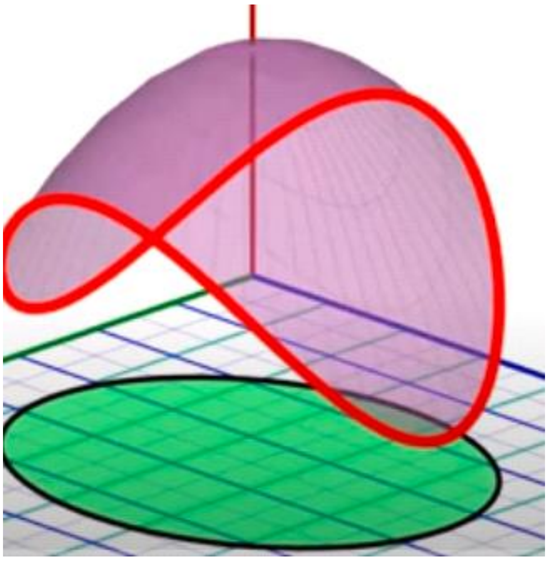
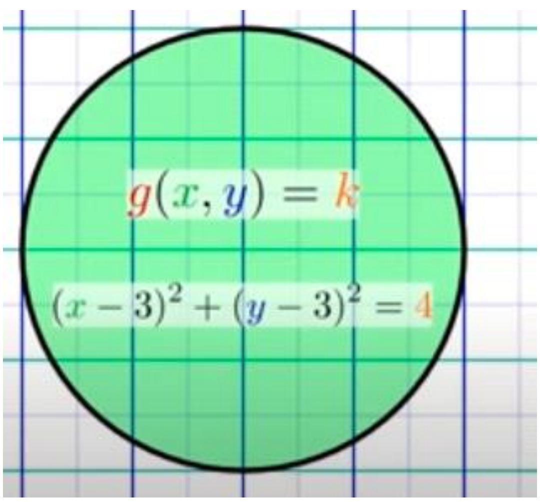
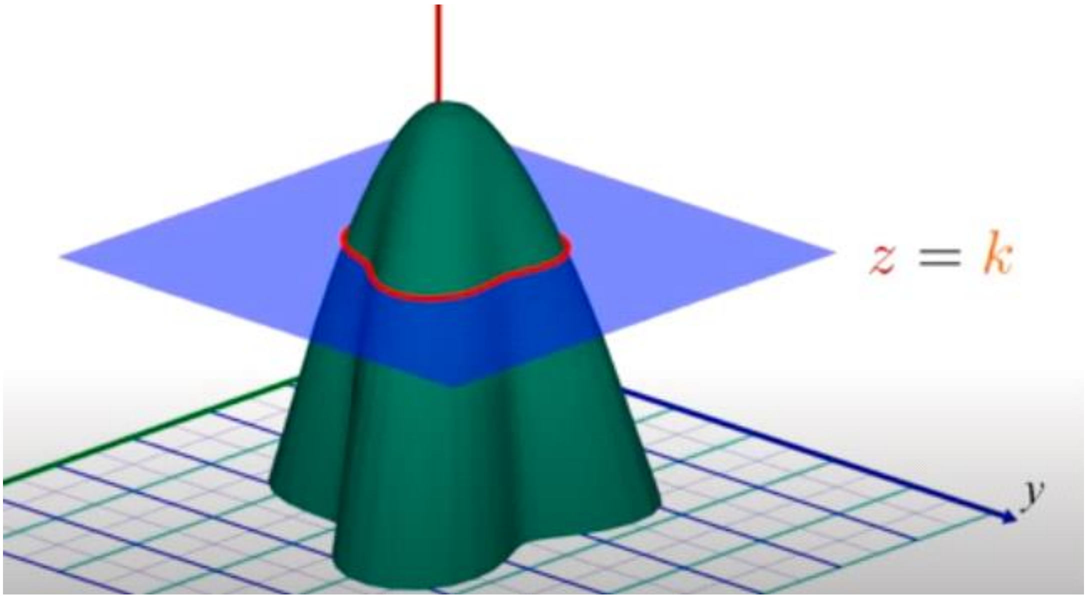



$g(x, y) = k$
$(x-3)^2 + (y-3)^2 = 4$

Example:



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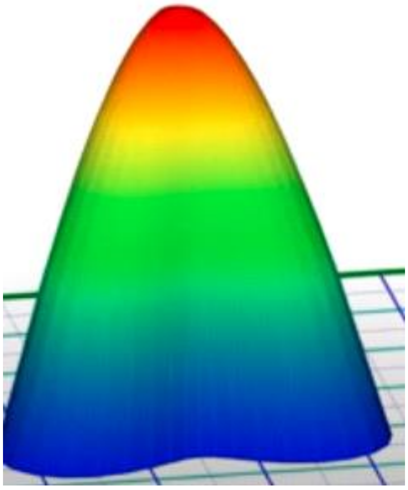

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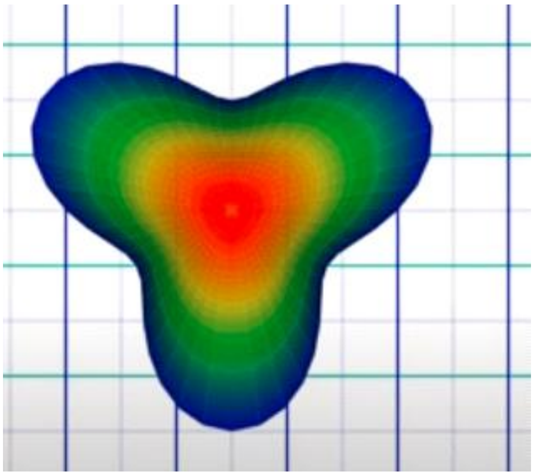

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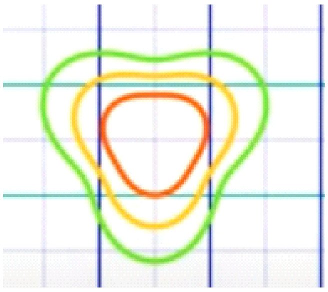

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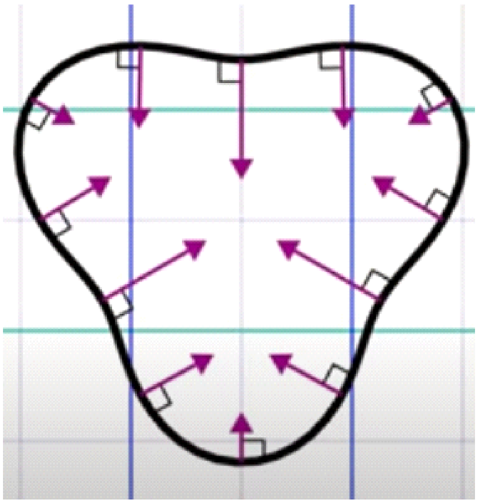

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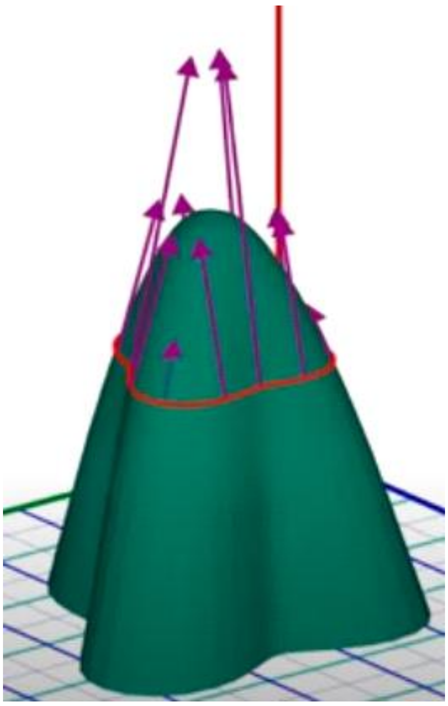

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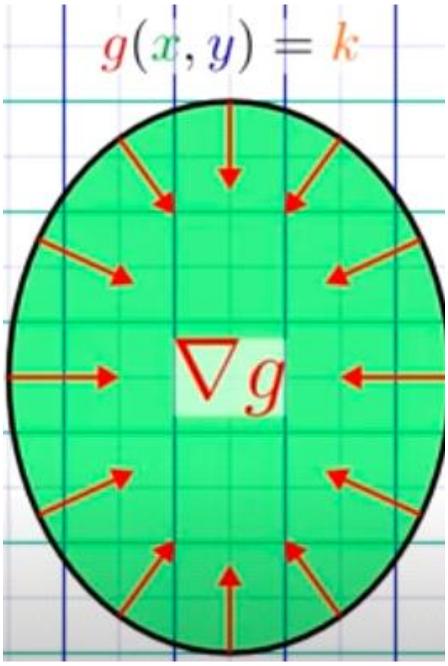

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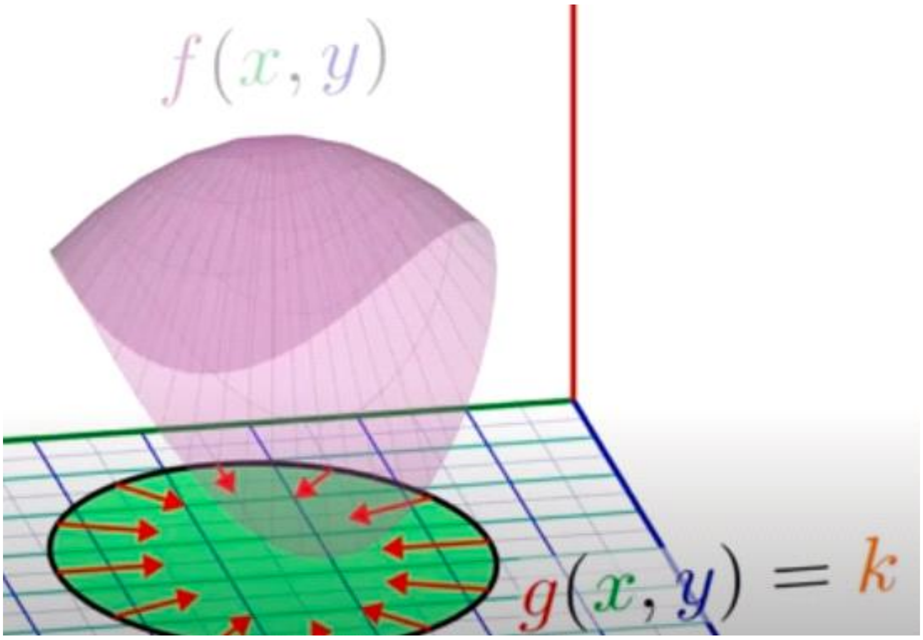

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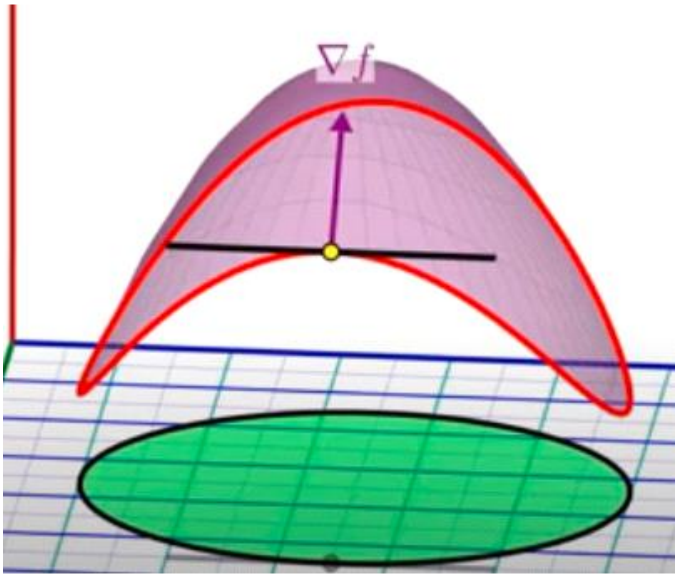

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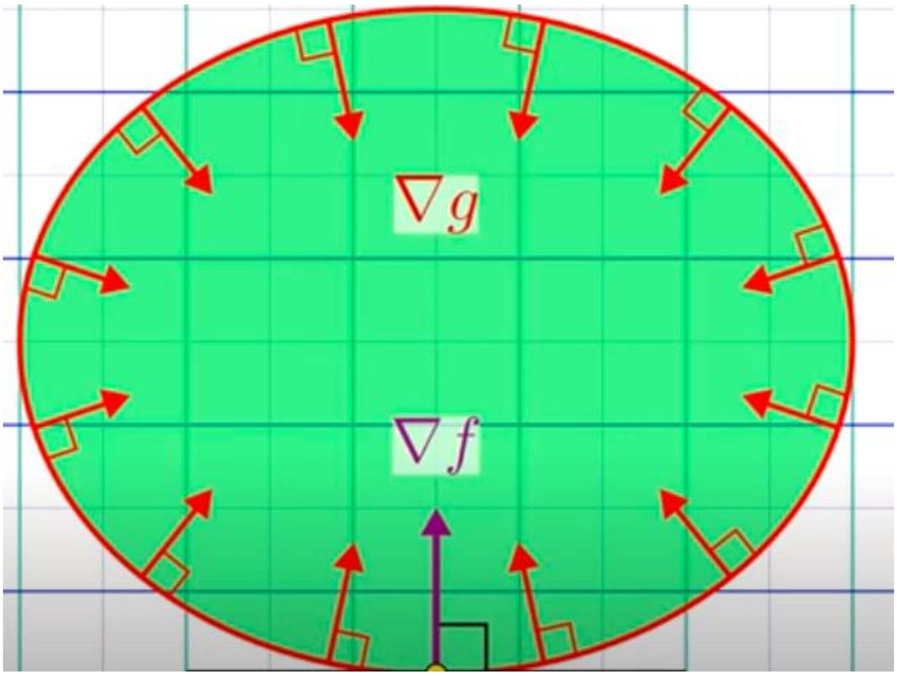

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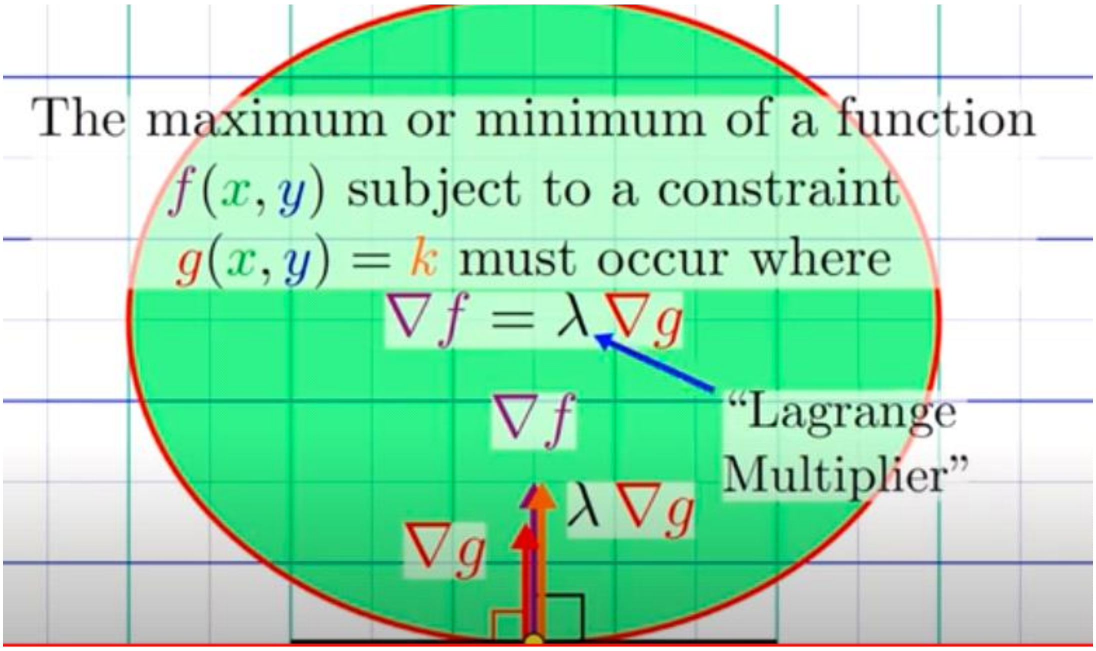

Finding Maxes/Mins Using Lagrange Multipliers

$$\nabla f = \lambda \nabla g$$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \lambda \frac{\partial g}{\partial x}, \lambda \frac{\partial g}{\partial y} \right\rangle$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ g(x, y) = k \end{cases}$$

Example 1 Find the maximum and minimum values of $x + y$ on the circle $x^2 + y^2 = 4$.

The objective function is

$$f(x, y) = x + y,$$

and the constraint is

$$g(x, y) = x^2 + y^2 = 4.$$

Since $\operatorname{grad} f = f_x\vec{i} + f_y\vec{j} = \vec{i} + \vec{j}$ and $\operatorname{grad} g = g_x\vec{i} + g_y\vec{j} = 2x\vec{i} + 2y\vec{j}$, the condition $\operatorname{grad} f = \lambda \operatorname{grad} g$ gives

$$1 = 2\lambda x \quad \text{and} \quad 1 = 2\lambda y,$$

so

$$x = y.$$

We also know that

$$x^2 + y^2 = 4,$$

giving $x = y = \sqrt{2}$ or $x = y = -\sqrt{2}$. The constraint has no endpoints (it's a circle) and $\operatorname{grad} g \neq \vec{0}$ on the circle, so we compare values of f at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Since $f(x, y) = x + y$, the maximum value of f is $f(\sqrt{2}, \sqrt{2}) = 2\sqrt{2}$; the minimum value is $f(-\sqrt{2}, -\sqrt{2}) = -2\sqrt{2}$. (See Figure 15.29.)

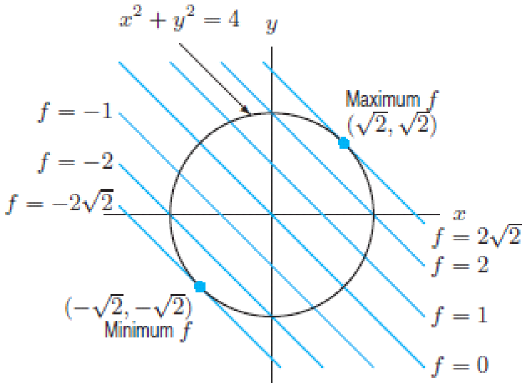

Figure 15.29: Maximum and minimum values of $f(x, y) = x + y$ on the circle $x^2 + y^2 = 4$ are at points where contours of f are tangent to the circle

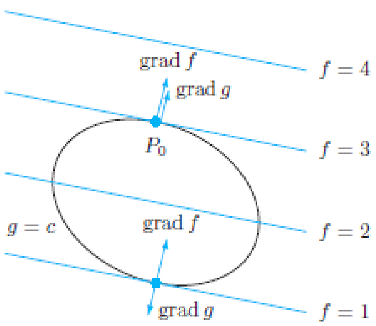

Figure 15.28: Maximum and minimum values of $f(x, y)$ on $g(x, y) = c$ are at points where grad f is parallel to grad g

Example 2 Find the maximum and minimum values of $f(x, y) = (x-1)^2 + (y-2)^2$ subject to the constraint $x^2 + y^2 \leq 45$.

Practice Questions

$$f(x, y) = x^3 + y, \quad x + y \geq 1$$

$$f(x_1, x_2) = {x_1}^2 + {x_2}^2, \quad x_1 + x_2 = 1$$

$f(x, y) = xy, \quad x^2 + 2y^2 \leq 1$

<u>EXAMPLE</u>: Find the extreme values of

$$f(x,y) = x^2 + 2y^2$$

on the circle $x^2+y^2 = 1$.

<u>EXAMPLE</u>: Find the points on the sphere

$$x^2+y^2+z^2 = 4$$

that are closest to and furthest from $(3, 1, -1)$.