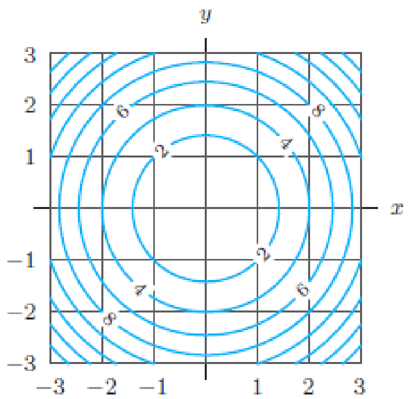
Directional Derivative

Friday, 16 August 2024 7:08 pm

14.4 GRADIENTS AND DIRECTIONAL DERIVATIVES IN THE PLANE

The Rate of Change in an Arbitrary Direction: The Directional Derivative



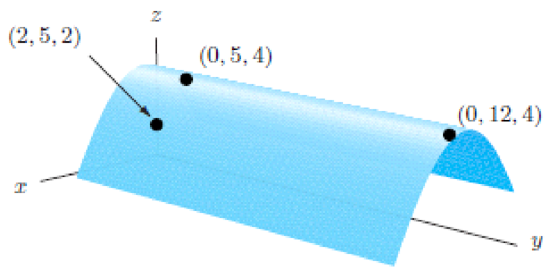


Figure 14.36

Example 2 For each of the functions f, g, and h in Figure 14.30, decide whether the directional derivative at the indicated point is positive, negative, or zero, in the direction of the vector $\vec{v} = \vec{i} + 2\vec{j}$, and in the direction of the vector $\vec{w} = 2\vec{i} + \vec{j}$.

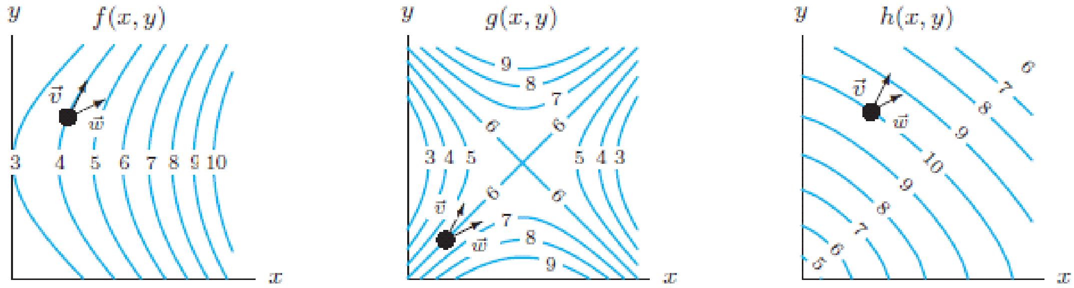


Figure 14.30: Contour diagrams of three functions with direction vectors $\vec{v} = \vec{i} + 2\vec{j}$ and $\vec{w} = 2\vec{i} + \vec{j}$ marked on each

The Gradient Vector of a differentiable function f at the point (a, b) is

$$\text{grad}\, f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$$

The Directional Derivative and the Gradient

If f is differentiable at (a,b) and $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, then

$$f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = \operatorname{grad} f(a,b) \cdot \vec{u}\,.$$

Example 3 Calculate the directional derivative of $f(x,y) = x^2+y^2$ at $(1,0)$ in the direction of the vector $\vec{i}+\vec{j}$.

Solution First we have to find the unit vector in the same direction as the vector $\vec{i}+\vec{j}$. Since this vector has magnitude $\sqrt{2}$, the unit vector is

$$\vec{u} = \frac{1}{\sqrt{2}}(\vec{i}+\vec{j}) = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\,.$$

Thus,

$$\begin{aligned} f_{\vec{u}}(1,0) &= \lim_{h\to 0}\frac{f(1+h/\sqrt{2},h/\sqrt{2})-f(1,0)}{h} = \lim_{h\to 0}\frac{(1+h/\sqrt{2})^2+(h/\sqrt{2})^2-1}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{2}h+h^2}{h} = \lim_{h\to 0}(\sqrt{2}+h) = \sqrt{2}. \end{aligned}$$

Example 5 Find the gradient vector of $f(x,y) = x+e^y$ at the point $(1,1)$.

Solution Using the definition, we have

$$\operatorname{grad} f = f_x\vec{i} + f_y\vec{j} = \vec{i} + e^y\vec{j}\,,$$

so at the point $(1,1)$

$$\operatorname{grad} f(1,1) = \vec{i} + e\vec{j}\,.$$

Geometric Properties of the Gradient Vector in the Plane

If f is a differentiable function at the point (a,b) and $\operatorname{grad} f(a,b) \neq \vec{0}$, then:

- The direction of $\operatorname{grad} f(a,b)$ is
  - Perpendicular[1] to the contour of f through (a,b);
  - In the direction of the maximum rate of increase of f.
- The magnitude of the gradient vector, $\|\operatorname{grad} f\|$, is
  - The maximum rate of change of f at that point;
  - Large when the contours are close together and small when they are far apart.

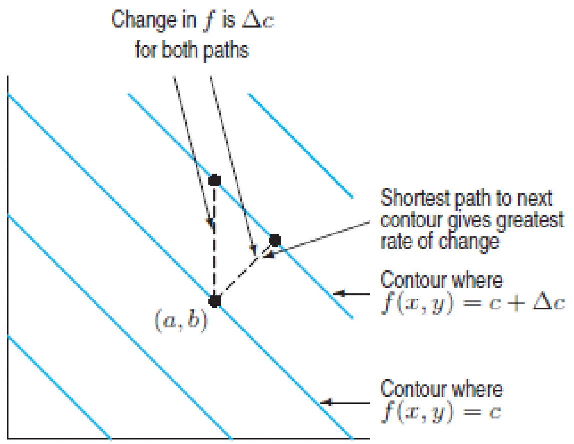


Figure 14.32: Close-up view of the contours around (a, b), showing the gradient is perpendicular to the contours

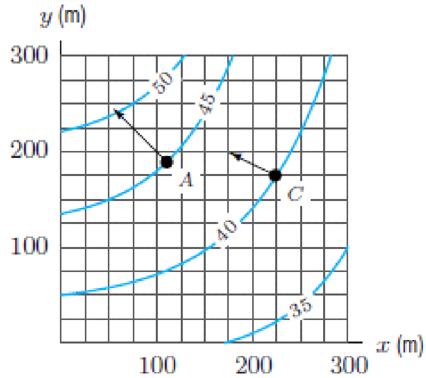


Figure 14.33: A temperature map showing directions and relative magnitudes of two gradient vectors

Let $f(x, y) = x^2y^3$. At the point $(-1, 2)$, find a vector

(a) In the direction of maximum rate of change.
(b) In the direction of minimum rate of change.
(c) In a direction in which the rate of change is zero.

Example 7 Use the gradient to find the directional derivative of $f(x, y) = x + e^y$ at the point $(1, 1)$ in the direction of the vectors $\vec{i} - \vec{j}$, $\vec{i} + 2\vec{j}$, $\vec{i} + 3\vec{j}$.

Solution In Example 5 we found

$$\text{grad}\, f(1, 1) = \vec{i} + e\vec{j}.$$

A unit vector in the direction of $\vec{i} - \vec{j}$ is $\vec{s} = (\vec{i} - \vec{j})/\sqrt{2}$, so

$$f_{\vec{s}}(1, 1) = \text{grad}\, f(1, 1) \cdot \vec{s} = (\vec{i} + e\vec{j}) \cdot \left(\frac{\vec{i} - \vec{j}}{\sqrt{2}}\right) = \frac{1 - e}{\sqrt{2}} \approx -1.215.$$

A unit vector in the direction of $\vec{i} + 2\vec{j}$ is $\vec{v} = (\vec{i} + 2\vec{j})/\sqrt{5}$, so

$$f_{\vec{v}}(1, 1) = \text{grad}\, f(1, 1) \cdot \vec{v} = (\vec{i} + e\vec{j}) \cdot \left(\frac{\vec{i} + 2\vec{j}}{\sqrt{5}}\right) = \frac{1 + 2e}{\sqrt{5}} \approx 2.879.$$

A unit vector in the direction of $\vec{i} + 3\vec{j}$ is $\vec{w} = (\vec{i} + 3\vec{j})/\sqrt{10}$, so

$$f_{\vec{w}}(1, 1) = \text{grad}\, f(1, 1) \cdot \vec{w} = (\vec{i} + e\vec{j}) \cdot \left(\frac{\vec{i} + 3\vec{j}}{\sqrt{10}}\right) = \frac{1 + 3e}{\sqrt{10}} \approx 2.895.$$

<u>EXAMPLE</u>: Find $D_u f(x,y)$ if

$$f(x,y) = x^3 - 3xy + 4y^2$$

and $\hat{u}$ is the unit vector given by angle $\theta = \pi/6$, what is $D_u f(1,2)$?

can

THE GRADIENT

<u>Q</u> For $f(x,y) = x^2 y - 4y^3$, find $D_u f(2,1)$ for $\hat{u}$ in the direction from $(2,1)$ to $(4,0)$

<u>EXAMPLE</u>: $f(x,y,z) = x \sin yz$

a) find ∇f

b) $D_u f(1,3,0)$ in the direction of $v = \hat{i} + 2\hat{j} - \hat{k}$