Triple Integration: Cylindrical & Spherical

Thursday, 20 June 2024 4:46 pm

Section 16.5: Integration in Cylindrical and Spherical Coordinates

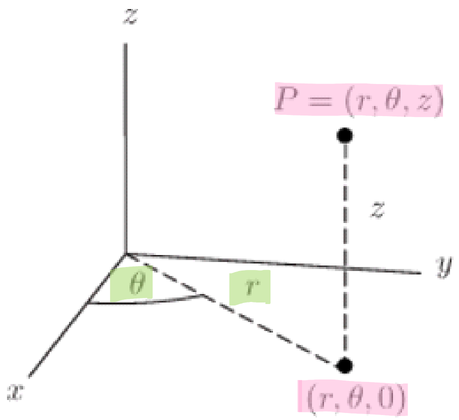
Integration in Cylindrical Coordinates

The cylindrical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are obtained by representing the x and y coordinates using polar coordinates (or potentially the y and z coordinates or x and z coordinates) and letting the third coordinate remain unchanged.

RELATION BETWEEN CARTESIAN AND CYLINDRICAL COORDINATES: Each point in $\mathbb{R}^3$ is represented using $0 \le r < \infty$, $0 \le \theta \le 2\pi$, $-\infty < z < \infty$.

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = z.$$

As with polar coordinates in the plane, note that $x^2 + y^2 = r^2$.



Notice that we can now interpret r as the distance from the point (x, y, z) to the z axis, while the interpretation of θ and z remain unchanged.

Question: What are the surfaces obtained by setting r, θ, and z equal to a constant?

Example 1 Describe in cylindrical coordinates a wedge of cheese cut from a cylinder 4 cm high and 6 cm in radius; this wedge subtends an angle of $\pi/6$ at the center. (See Figure 16.41.)

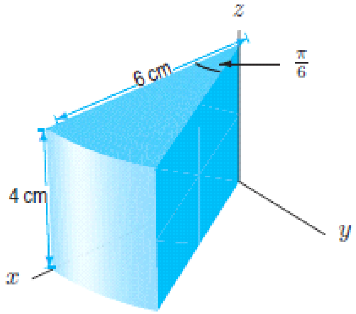


Figure 16.41: A wedge of cheese

Solution The wedge is described by the inequalities $0 \le r \le 6$, and $0 \le z \le 4$, and $0 \le \theta \le \pi/6$.

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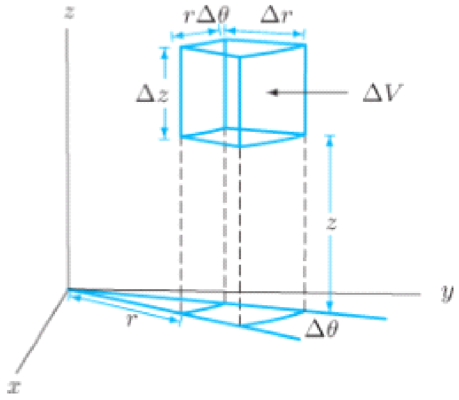
What is dV in Cylindrical Coordinates?

Recall that when integrating in polar coordinates, we set $dA = r\,dr\,d\theta$. When viewing a small piece of volume, ΔV, in cylindrical coordinates, we will see that the correct form for dV is rather intuitive based on this.



It is clear from this image that we should have $\Delta V \approx r\,\Delta r\,\Delta\theta\,\Delta z$. This leads us to the following conclusion:

When computing integrals in cylindrical coordinates, put $dV = r\,dr\,d\theta\,dz$. Other orders of integration are possible.

Examples:

1. Evaluate the triple integral in cylindrical coordinates: $f(x, y, z) = \sin(x^2 + y^2)$, W is the solid cylinder with height 4 with base of radius 1 centered on the z-axis at $z = -1$.

$$\int_W f\,dV = \int_{-1}^{3}\int_0^{2\pi}\int_0^1 (\sin(r^2))\,r dr\,d\theta\,dz$$
$$= \int_{-1}^{3}\int_0^{2\pi} (-\frac{1}{2}\cos r^2)\Big|_0^1\,d\theta\,dz$$

$$= -\frac{1}{2}\int_{-1}^{3}\int_{0}^{2\pi}(\cos 1 - \cos 0)\,d\theta\,dz$$
$$= -\pi\int_{-1}^{3}(\cos 1 - 1)\,dz = -4\pi(\cos 1 - 1) = 4\pi(1 - \cos 1)$$

Example 2 Find the mass of the wedge of cheese in Example 1, if its density is 1.2 grams/cm^3.

Solution If the wedge is W, its mass is

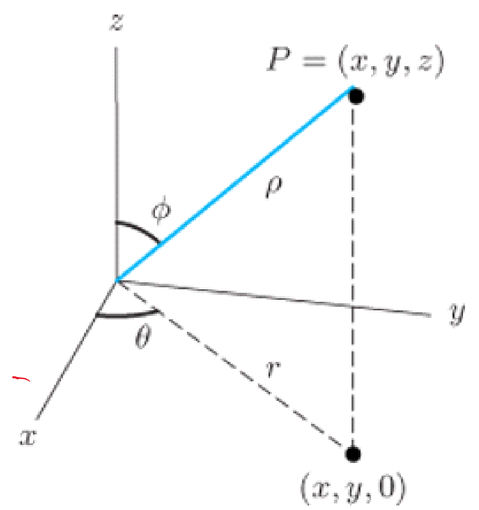
$$\int_W 1.2\,dV.$$

In cylindrical coordinates this integral is

$$\int_0^4\int_0^{\pi/6}\int_0^6 1.2\,r\,dr\,d\theta\,dz = \int_0^4\int_0^{\pi/6} 0.6r^2\Big|_0^6\,d\theta\,dz = 21.6\int_0^4\int_0^{\pi/6}d\theta\,dz$$
$$= 21.6\left(\frac{\pi}{6}\right)4 = 45.239 \text{ grams}.$$

Spherical Coordinates

The spherical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are the analog of polar coordinates in $\mathbb{R}^2$. We define $\rho = \sqrt{x^2 + y^2 + z^2}$ to be the distance from the origin to (x, y, z), θ is defined as it was in polar coordinates, and ϕ is defined as the angle between the positive z-axis and the line connecting the origin to the point (x, y, z).



From the above figure, we can see that $r = \rho\sin\phi$, and $z = \rho\cos\phi$, so using the relationship between Cartesian coordinates (x, y, z) and cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, we arrive at the following:

RELATIONSHIP BETWEEN CARTESIAN AND SPHERICAL COORDINATES: Each point in $\mathbb{R}^3$ is represented using $0 \le \rho < \infty$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

$$x = \rho\sin\phi\cos\theta,$$
$$y = \rho\sin\phi\sin\theta,$$
$$z = \rho\cos\phi.$$

Also, $x^2 + y^2 + z^2 = \rho^2$.

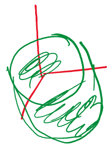
When computing integrals in spherical coordinates, put $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. Other orders of integration are possible.

Examples: Setup

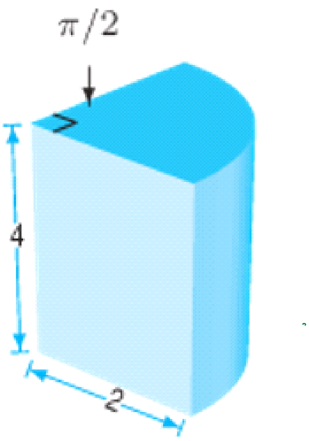
2. ~~Evaluate~~ the triple integral in spherical coordinates. $f(x, y, z) = 1/(\underbrace{x^2 + y^2 + z^2}_{\rho^2})^{1/2}$ over the bottom half of a sphere of radius 5 centered at the origin.

$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{5} \frac{1}{(\rho^2)^{1/2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{5} \rho \sin\phi \, d\rho \, d\phi \, d\theta$$



3. For the following, choose coordinates and set up a triple integral, inlcluding limits of integration, for a density function f over the region.



(a)

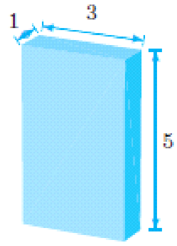
Example 4 Use spherical coordinates to derive the formula for the volume of a ball of radius a.

Solution In spherical coordinates, a ball of radius a is described by the inequalities $0 \le \rho \le a$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$. Note that θ goes from 0 to 2π, whereas ϕ goes from 0 to π. We find the volume by integrating the constant density function 1 over the ball:

$$\text{Volume} = \int_R 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}a^3 \sin\phi\, d\phi\, d\theta$$
$$= \frac{1}{3}a^3 \int_0^{2\pi} -\cos\phi\Big|_0^{\pi}\, d\theta = \frac{2}{3}a^3 \int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}.$$

For Exercises 12–18, choose coordinates and set up a triple integral, including limits of integration, for a density function f over the region.

12.



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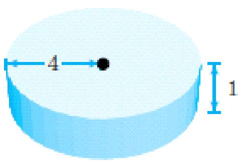

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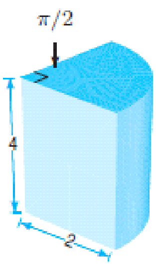

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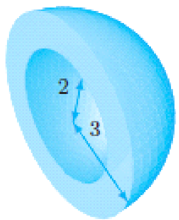

Using Cartesian coordinates, we get:

$$\int_0^3\int_0^1\int_0^5 f\, dz\, dy\, dx$$

Using cylindrical coordinates, we get:

$$\int_0^1\int_0^{2\pi}\int_0^4 f\cdot r dr\, d\theta\, dz$$

Using cylindrical coordinates, we get:

$$\int_0^4\int_0^{\pi/2}\int_0^2 f\cdot r dr\, d\theta\, dz$$

Using spherical coordinates, we get:

$$\int_0^{\pi}\int_0^{\pi}\int_2^3 f\cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$