Assignment 1 with Solution

Sunday, 4 August 2024 12:38 pm

Q1.

The temperature T (in °C) at any point in the region $-10 \leq x \leq 10$, $-10 \leq y \leq 10$ is given by the function

$$T(x, y) = 100 - x^2 - y^2.$$

(a) Sketch isothermal curves (curves of constant temperature) for $T = 100°\text{C}$, $T = 75°\text{C}$, $T = 50°\text{C}$, $T = 25°\text{C}$, and $T = 0°\text{C}$.

(b) A heat-seeking bug is put down at a point on the xy-plane. In which direction should it move to increase its temperature fastest? How is that direction related to the level curve through that point?

(a) To find the level curves, we let T be a constant.

$$T = 100 - x^2 - y^2$$
$$x^2 + y^2 = 100 - T,$$

which is an equation for a circle of radius $\sqrt{100 - T}$ centered at the origin. At $T = 100°$, we have a circle of radius 0 (a point). At $T = 75°$, we have a circle of radius 5. At $T = 50°$, we have a circle of radius $5\sqrt{2}$. At $T = 25°$, we have a circle of radius $5\sqrt{3}$. At $T = 0°$, we have a circle of radius 10.

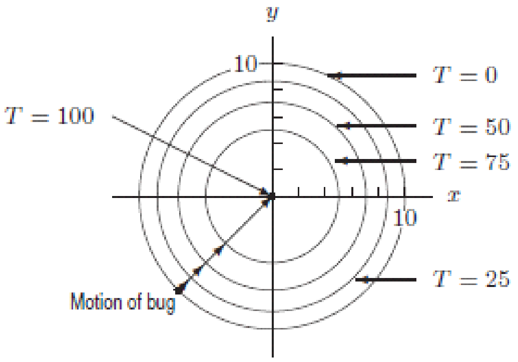

Figure 12.135

(b) No matter where we put the bug, it should go straight toward the origin—the hottest point on the xy-plane. Its direction of motion is perpendicular to the tangent lines of the level curves, as can be seen in Figure 12.135.

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Q2.

Match the functions (a)–(d) with the shapes of their level curves (I)–(IV). Sketch each contour diagram.

(a) $f(x, y) = x^2$ **(b)** $f(x, y) = x^2 + 2y^2$

(c) $f(x, y) = y - x^2$ **(d)** $f(x, y) = x^2 - y^2$

I. Lines II. Parabolas

III. Hyperbolas IV. Ellipses

(a) I
(b) IV
(c) II
(d) III

See Figure 12.96.

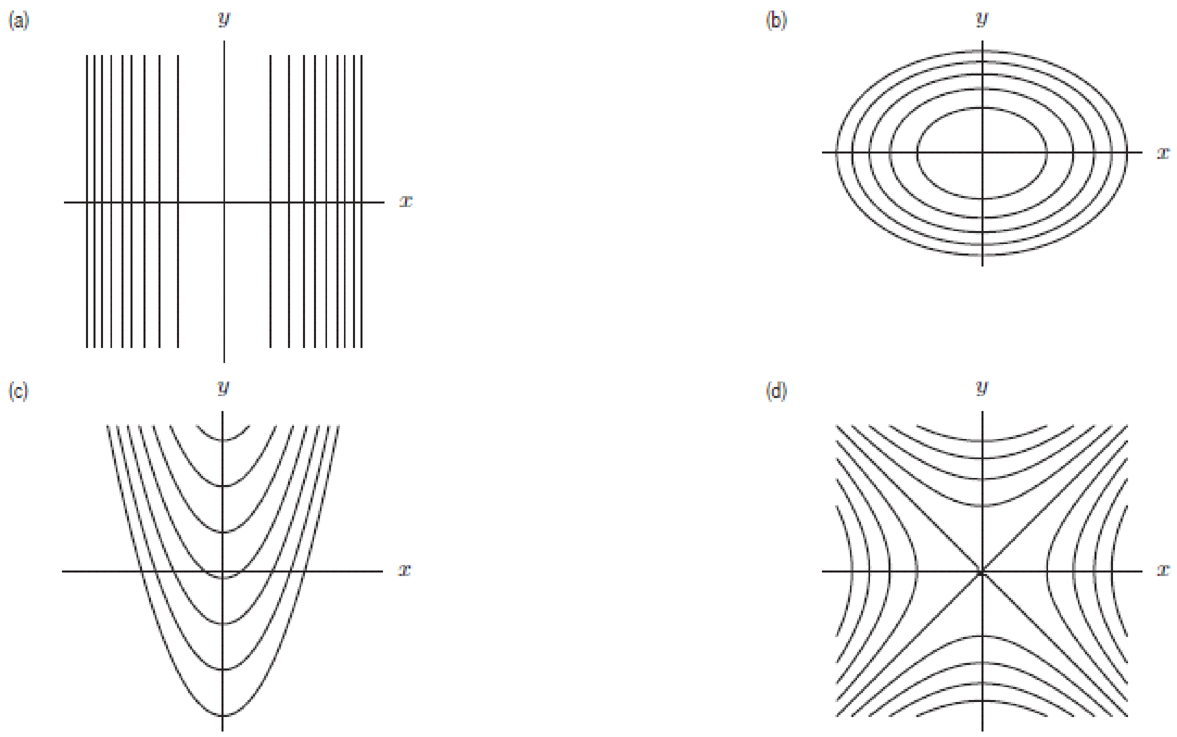

Figure 12.96

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Q3.

A manufacturer sells two goods, one at a price of \$3000 a unit and the other at a price of \$12,000 a unit. A quantity q_1 of the first good and q_2 of the second good are sold at a total cost of \$4000 to the manufacturer.

(a) Express the manufacturer's profit, π, as a function of q_1 and q_2.

(b) Sketch curves of constant profit in the q_1q_2-plane for $\pi = 10{,}000$, $\pi = 20{,}000$, and $\pi = 30{,}000$ and the break-even curve $\pi = 0$.

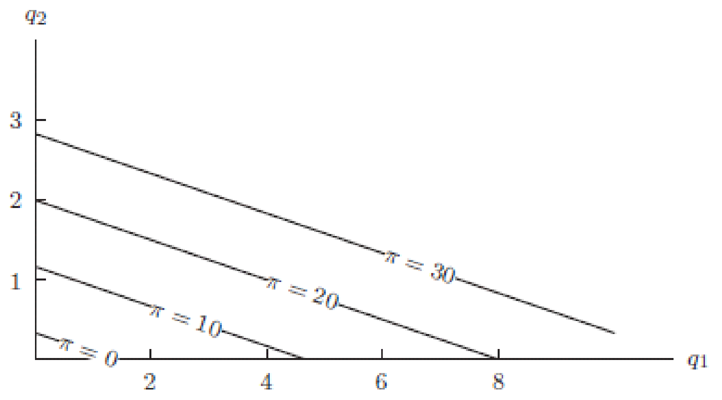
(a) The profit is given by the following:

$$\pi = \text{Revenue from } q_1 + \text{ Revenue from } q_2 - \text{Cost}.$$

Measuring π in thousands, we obtain:

$$\pi = 3q_1 + 12q_2 - 4.$$

(b) A contour diagram of π follows. Note that the units of π are in thousands.



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Q4.

Show that the function f does not have a limit at $(0, 0)$ by examining the limits of f as $(x, y) \to (0, 0)$ along the line $y = x$ and along the parabola $y = x^2$:

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}, \qquad (x,y) \neq (0,0).$$

Let us suppose that (x, y) approaches $(0, 0)$ along the line $y = x$. Then

$$f(x,y) = f(x,x) = \frac{x^3}{x^4 + x^2} = \frac{x}{x^2+1}.$$

Therefore

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = \lim_{x\to 0} \frac{x}{x^2+1} = 0.$$

On the other hand, if (x, y) approaches $(0, 0)$ along the parabola $y = x^2$ we have

$$f(x,y) = f(x,x^2) = \frac{x^4}{2x^4} = \frac{1}{2}$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} f(x,y) = \lim_{x\to 0} f(x,x^2) = \frac{1}{2}.$$

Thus no matter how close they are to the origin, there will be points (x, y) such that $f(x, y)$ is close to 0 and points (x, y) such that $f(x, y)$ is close to $\frac{1}{2}$. So the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.

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Q5.

By approaching the origin along the positive x-axis and the positive y-axis, show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x+y^2}{2x+y}.$$

Points along the positive x-axis are of the form $(x, 0)$; at these points the function looks like $x/2x = 1/2$ everywhere (except at the origin, where it is undefined). On the other hand, along the y-axis, the function looks like $y^2/y = y$, which approaches 0 as we get closer to the origin. Since approaching the origin along two different paths yields numbers that are not the same, the limit does not exist.

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Q6.

Explain why the following function is not continuous along the line $y = 0$:

$$f(x,y) = \begin{cases} 1-x, & y \geq 0, \\ -2, & y < 0. \end{cases}$$

We will study the continuity of f at $(a, 0)$. Now $f(a, 0) = 1 - a$. In addition:

$$\lim_{\substack{(x,y)\to(a,0) \\ y>0}} f(x,y) = \lim_{x\to a}(1-x) = 1-a$$
$$\lim_{\substack{(x,y)\to(a,0) \\ y<0}} f(x,y) = \lim_{x\to a} -2 = -2.$$

If $a = 3$, then

$$\lim_{\substack{(x,y)\to(3,0) \\ y>0}} f(x,y) = 1-3 = -2 = \lim_{\substack{(x,y)\to(3,0) \\ y<0}} f(x,y)$$

and so $\lim_{(x,y)\to(3,0)} f(x,y) = -2 = f(3,0)$. Therefore f is continuous at $(3, 0)$.

On the other hand, if $a \neq 3$, then

$$\lim_{\substack{(x,y)\to(a,0) \\ y>0}} f(x,y) = 1-a \neq -2 = \lim_{\substack{(x,y)\to(a,0) \\ y<0}} f(x,y)$$

so $\lim_{(x,y)\to(a,0)} f(x,y)$ does not exist. Thus f is not continuous at $(a, 0)$ if $a \neq 3$.

Thus, f is not continuous along the line $y = 0$. (In fact the only point on this line where f is continuous is the point $(3, 0)$.)

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Q7.

What value of c makes the following function continuous at $(0, 0)$?

$$f(x,y) = \begin{cases} x^2+y^2+1 & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

The function $f(x, y) = x^2 + y^2 + 1$ gets closer and closer to 1 as (x, y) gets closer to the origin. To make f continuous at the origin, we need to have $f(0, 0) = 1$. Thus $c = 1$ will make the function continuous at the origin.

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Q8.

. Let $f(x,y) = \begin{cases} \dfrac{|x|}{x}y & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$

Is $f(x,y)$ continuous

(a) On the x-axis? **(b)** On the y-axis?

(c) At $(0,0)$?

For $x > 0$, we have

$$f(x,y) = y.$$

Thus, the surface representing f for $x > 0$ is the plane $z = y$.

For $x < 0$, we have

$$f(x,y) = -y.$$

Thus, the surface representing f for $x < 0$ is the plane $z = -y$.

For $x = 0$, we have

$$f(x,y) = 0.$$

Thus, the surface representing f is two half-planes and the y-axis.

(a) The function is continuous at every point on the x-axis.
(b) The function is not continuous at any point on the y-axis, except at the origin, because $f(x,y) = 0$ on the y-axis and not nearby unless $y = 0$.
(c) The function is continuous at the origin.

(a) Yes
(b) No
(c) Yes

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Q9.
find the domain of the given function.

$$f(x,y) = \frac{1}{x} + \sqrt{y+4} - \sqrt{x+1}$$

$x \geq -1$ $\quad$ $x \neq 0$ $\quad$ $y \geq -4$

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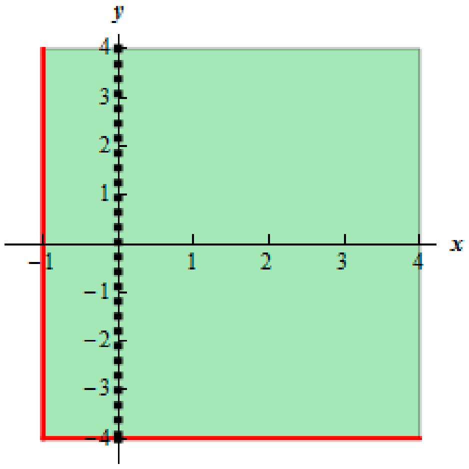

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$$f(x, y) = \ln(2x - 3y + 1)$$

$$2x - 3y + 1 > 0$$

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$$2x + 1 > 3y \qquad \Rightarrow \qquad y < \frac{2}{3}x + \frac{1}{3}$$

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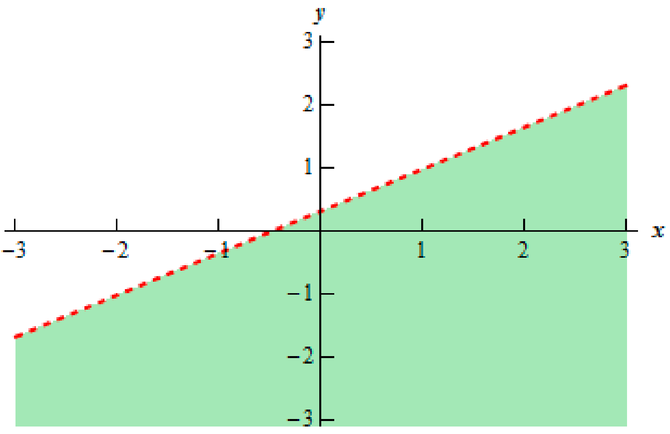

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$$f(x, y, z) = \frac{1}{x^2 + y^2 + 4z}$$

$$x^2 + y^2 + 4z \neq 0$$

$$4z \neq -x^2 - y^2 \qquad \Rightarrow \qquad z \neq -\frac{x^2}{4} - \frac{y^2}{4}$$