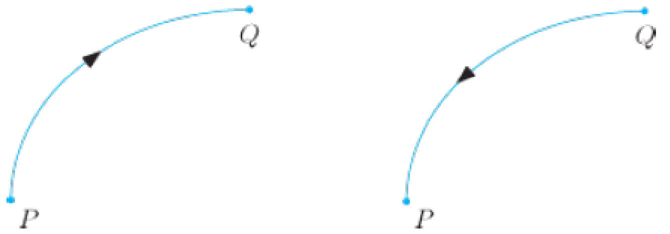
Line Integral, Green' Theorem & Stroke' Theorem

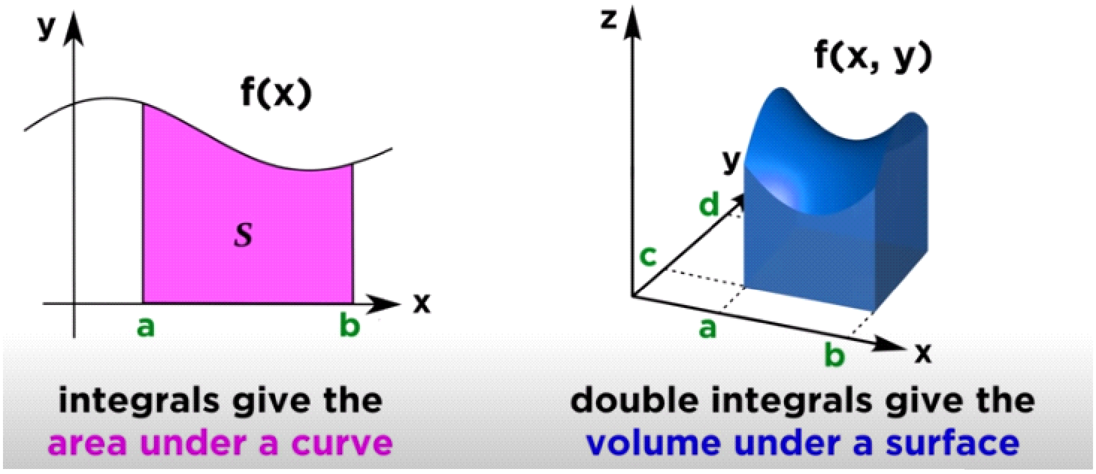
Friday, 28 June 2024 8:31 pm

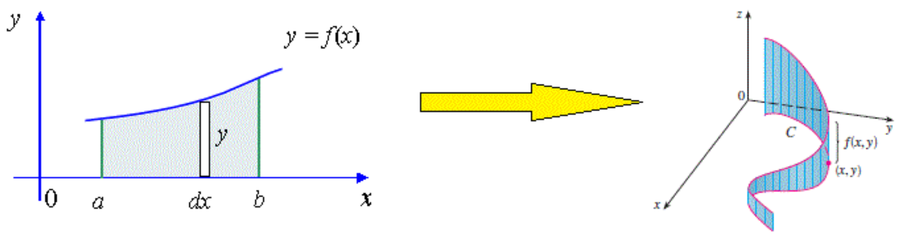
Orientation of a Curve

The concept of orientation of a curve is simple enough to understand. A curve can be traced out in one of two directions. Choosing one of those directions determines an *orientation* of the curve.

A curve is said to be *oriented* if we have chosen a direction of travel on it.



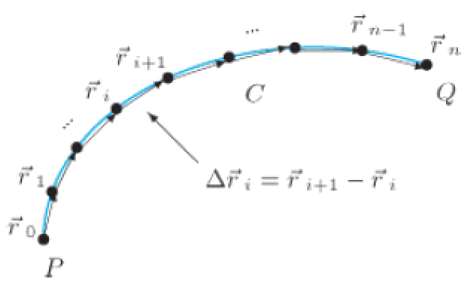


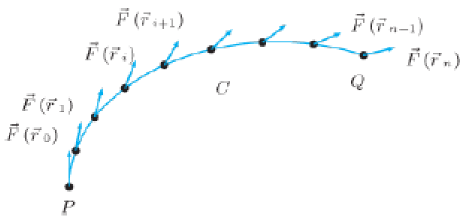


Definite integral - Area of a flat surface

Line integral - Area of a curved surface

Definition of the Line Integral





$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$$

The *line integral* of a vector field $\vec{F}$ along an oriented curve C is

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\vec{r}_i\| \to 0} \sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i.$$

Work Done by $\vec{F} = \int_C \vec{F} \cdot d\vec{r}$.

If C is an oriented, closed curve, the line integral of a vector field F around C is called the *circulation* of $\vec{F}$ around C.

We will often use the notaton $\oint_C \vec{F} \cdot d\vec{r}$ to refer to circulations.

If $\vec{r}(t)$, for $a \le t \le b$, is a smooth parametrization of the oriented curve C and $\vec{F}$ is a vector field that is continuous on C, then

(1) $$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt.$$

In words: To compute the line integral of $\vec{F}$ over C, take the dot product of $\vec{F}$ evaluated on C with the velocity vector, $\vec{r}\,'(t)$, of the parametrization C, then integrate along the curve.

EXAMPLE 2 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ along the curve C given by $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\,\mathbf{k}$, $0 \le t \le 1$.

Solution We have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{t}\,\mathbf{i} + t^3\mathbf{j} - t^2\mathbf{k} \qquad z = \sqrt{t},\ xy = t^3,\ -y^2 = -t^2$$

and

$$\frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + \mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}.$$

Thus,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt \\ &= \int_0^1 \left(2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right)dt \\ &= \left[\left(\frac{3}{2}\right)\left(\frac{2}{5}t^{5/2}\right) + \frac{1}{4}t^4\right]_0^1 = \frac{17}{20}.\end{aligned}$$

EXAMPLE 3 Evaluate the line integral $\int_C -y\,dx + z\,dy + 2x\,dz$, where C is the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$.

Solution We express everything in terms of the parameter t, so $x = \cos t$, $y = \sin t$, $z = t$, and $dx = -\sin t\,dt$, $dy = \cos t\,dt$, $dz = dt$. Then,

$$\begin{aligned}\int_C -y\,dx + z\,dy + 2x\,dz &= \int_0^{2\pi} [(-\sin t)(-\sin t) + t\cos t + 2\cos t]\,dt \\ &= \int_0^{2\pi} [2\cos t + t\cos t + \sin^2 t]\,dt \\ &= \left[2\sin t + (t\sin t + \cos t) + \left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\right]_0^{2\pi} \\ &= [0 + (0+1) + (\pi - 0)] - [0 + (0+1) + (0-0)] \\ &= \pi.\end{aligned}$$

EXAMPLE 4 Find the work done by the force field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$, from $(0, 0, 0)$ to $(1, 1, 1)$ (Figure 16.18).

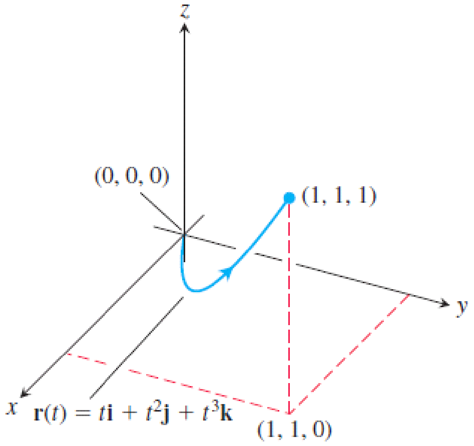


FIGURE 16.18 The curve in Example 4.

Solution First we evaluate $\mathbf{F}$ on the curve $\mathbf{r}(t)$:

$$\begin{aligned}\mathbf{F} &= (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k} \\ &= \underbrace{(t^2 - t^2)}_{0}\mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}.\end{aligned}$$

Substitute $x = t$, $y = t^2$, $z = t^3$.

Then we find $d\mathbf{r}/dt$,

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

Finally, we find $\mathbf{F} \cdot d\mathbf{r}/dt$ and integrate from $t = 0$ to $t = 1$:

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= [(t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}] \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) \\ &= (t^3 - t^4)(2t) + (t - t^6)(3t^2) = 2t^4 - 2t^5 + 3t^3 - 3t^8.\end{aligned}$$

So,

$$\begin{aligned}\text{Work} &= \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\, dt \\ &= \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\right]_0^1 = \frac{29}{60}.\end{aligned}$$

EXAMPLE 7 Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$ (Figure 16.19).

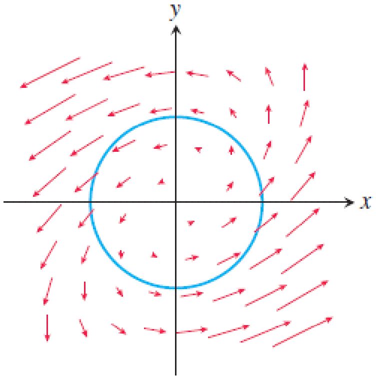


FIGURE 16.19 The vector field $\mathbf{F}$ and curve $\mathbf{r}(t)$ in Example 7.

Solution On the circle, $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Then

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1}$$

gives

$$\text{Circulation} = \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{2\pi} (1 - \sin t \cos t)\, dt$$

$$= \left[t - \frac{\sin^2 t}{2} \right]_0^{2\pi} = 2\pi.$$

As Figure 16.19 suggests, a fluid with this velocity field is circulating *counterclockwise* around the circle, so the circulation is positive. ■

THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over R.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy}_{\text{Curl integral}} \tag{3}$$

EXAMPLE 3 Verify both forms of Green's Theorem for the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region R bounded by the unit circle

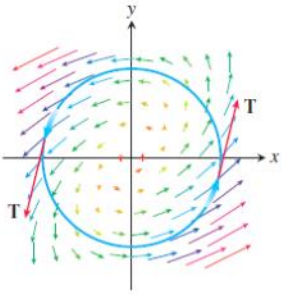
$$C: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Solution Evaluating $\mathbf{F}(\mathbf{r}(t))$ and computing the partial derivatives of the components of $\mathbf{F}$, we have

$$M = \cos t - \sin t, \qquad dx = d(\cos t) = -\sin t\, dt,$$
$$N = \cos t, \qquad dy = d(\sin t) = \cos t\, dt,$$
$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0.$$

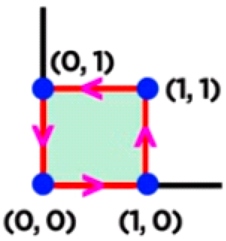
The two sides of Equation (3) are

$$\oint_C M\, dx + N\, dy = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t\, dt) + (\cos t)(\cos t\, dt)$$
$$= \int_0^{2\pi} (-\sin t \cos t + 1)\, dt = 2\pi$$



$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{dy}\right) dx\, dy = \iint_R (1 - (-1))\, dx\, dy$$
$$= 2 \iint_R dx\, dy = 2(\text{area inside the unit circle}) = 2\pi.$$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$$\int_C \vec{F} \cdot d\vec{r}$$

$\vec{F} = \langle xy, x^2 \rangle$

$$\int_C xy\,dx + x^2 dy = \int_0^1 \int_0^1 \left(\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} xy\right) dx\,dy$$

$$= \int_0^1 \int_0^1 x\,dx\,dy$$

$$\int_0^1 \left[(x^2/2)\Big|_0^1\right] dy = \int_0^1 (1/2)\,dy = (y/2)\Big|_0^1 = 1/2$$

THEOREM 6—Stokes' Theorem Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of the curl vector field $\nabla \times \mathbf{F}$ over S:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Curl integral}} \tag{4}$$

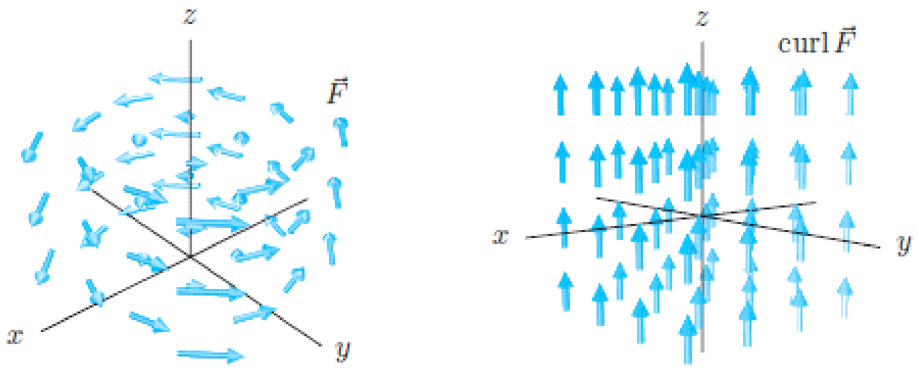


Figure 20.14: The vector fields $\vec{F}$ and curl $\vec{F}$

$$curl\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

EXAMPLE 2 Evaluate Equation (4) for the hemisphere $S: x^2 + y^2 + z^2 = 9, z \geq 0$, its bounding circle $C: x^2 + y^2 = 9, z = 0$, and the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

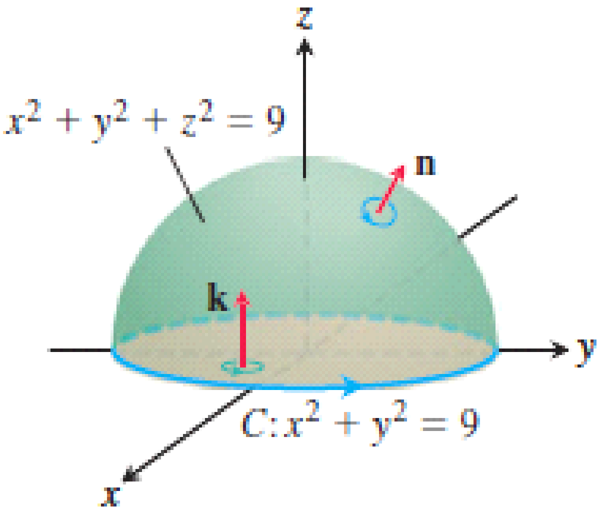


FIGURE 16.58 A hemisphere and a disk, each with boundary C (Examples 2 and 3).

around C (as viewed from above) using the parametrization $\mathbf{r}(\theta) = (3\cos\theta)\mathbf{i} + (3\sin\theta)\mathbf{j}$, $0 \le \theta \le 2\pi$:

$$\begin{aligned} d\mathbf{r} &= (-3\sin\theta\, d\theta)\mathbf{i} + (3\cos\theta\, d\theta)\mathbf{j} \\ \mathbf{F} &= y\mathbf{i} - x\mathbf{j} = (3\sin\theta)\mathbf{i} - (3\cos\theta)\mathbf{j} \\ \mathbf{F}\cdot d\mathbf{r} &= -9\sin^2\theta\, d\theta - 9\cos^2\theta\, d\theta = -9\, d\theta \\ \oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} -9\, d\theta = -18\pi. \end{aligned}$$

For the curl integral of $\mathbf{F}$, we have

$$\begin{aligned} \nabla\times\mathbf{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \\ &= (0-0)\mathbf{i} + (0-0)\mathbf{j} + (-1-1)\mathbf{k} = -2\mathbf{k} \\ \mathbf{n} &= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{3} \qquad \text{Outer unit normal} \\ d\sigma &= \frac{3}{z}\, dA \qquad \text{Section 16.6, Example 7, with } a = 3 \\ \nabla\times\mathbf{F}\cdot\mathbf{n}\, d\sigma &= -\frac{2z}{3}\frac{3}{z}\, dA = -2\, dA \end{aligned}$$

and

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint_{x^2+y^2\le 9} -2\, dA = -18\pi.$$

The circulation around the circle equals the integral of the curl over the hemisphere, as it should from Stokes' Theorem. ■