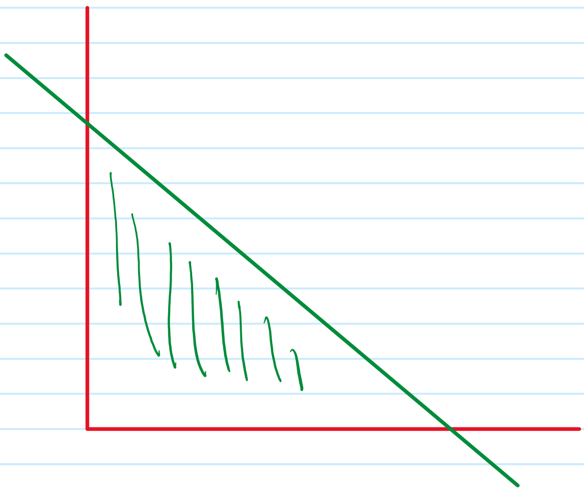
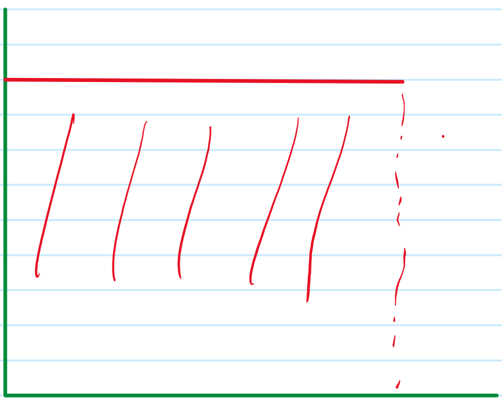
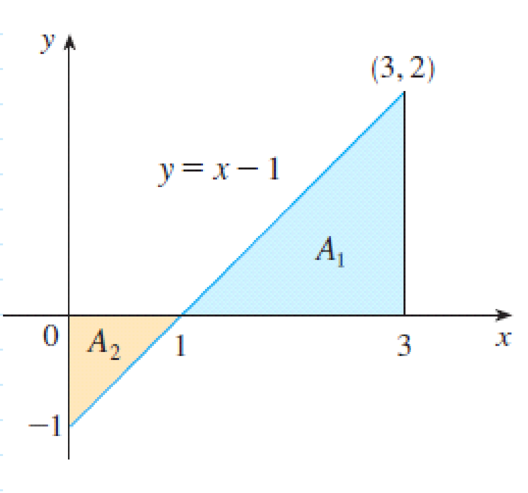
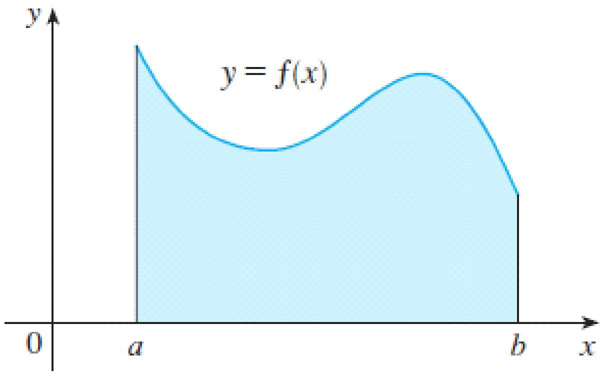
Area under the linear curve

Thursday, 8 May 2025 12:37 pm









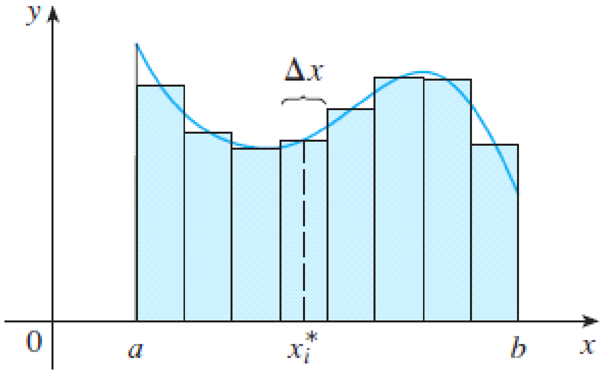


FIGURE 1

If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*) \Delta x$ is the sum of areas of rectangles.

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a), x_1, x_2, \ldots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

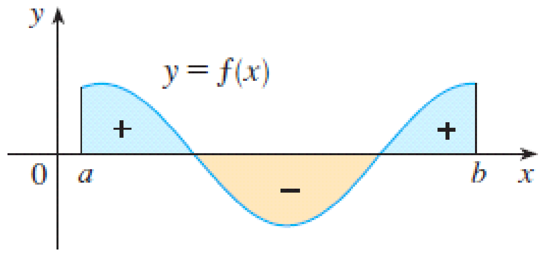


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

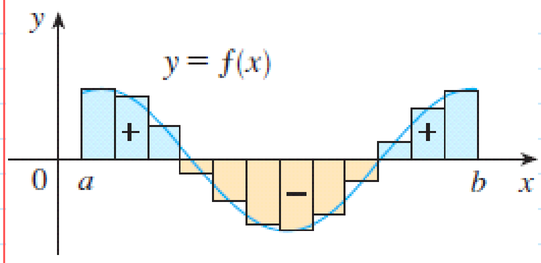
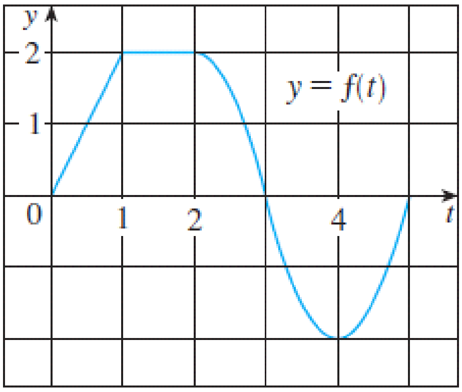
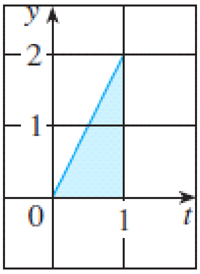
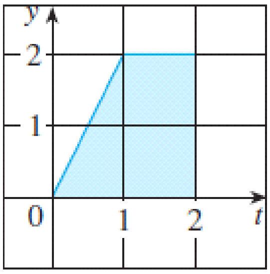


FIGURE 3
$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.

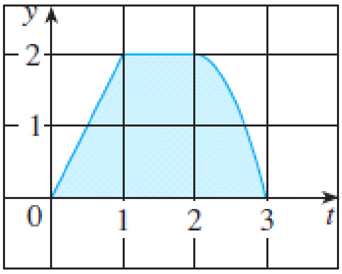




$g(1) = 1$



$g(2) = 3$



$g(3) \approx 4.3$

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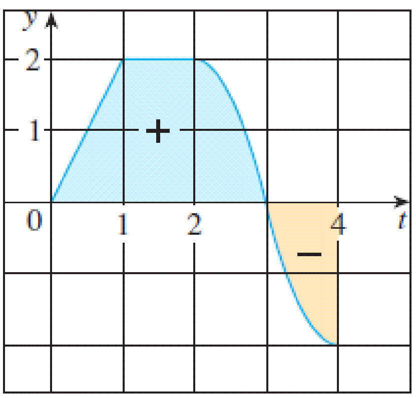
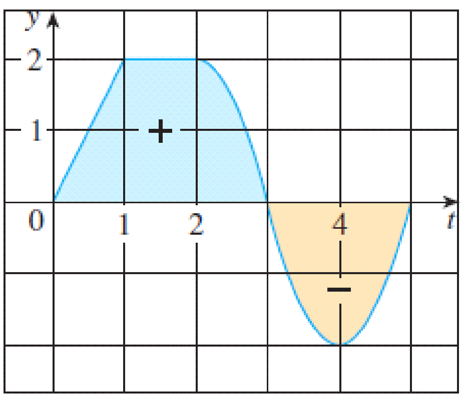


$g(4) \approx 3$



$g(5) \approx 1.7$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

V EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\,dt$.

SOLUTION Since $f(t) = \sqrt{1 + t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + x^2}$$

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

$$\boxed{\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)}$$

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

V EXAMPLE 5 Evaluate the integral $\int_1^3 e^x\,dx$.

SOLUTION The function $f(x) = e^x$ is continuous everywhere and we know that an antiderivative is $F(x) = e^x$, so Part 2 of the Fundamental Theorem gives

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e$$

EXAMPLE 6 Find the area under the parabola $y = x^2$ from 0 to 1.

SOLUTION An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$. The required area A is found using Part 2 of the Fundamental Theorem:

$$A = \int_0^1 x^2\,dx = \frac{x^3}{3}\bigg]_0^1$$

$$= \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

EXAMPLE 7 Evaluate $\int_3^6 \frac{dx}{x}$.

SOLUTION The given integral is an abbreviation for

$$\int_3^6 \frac{1}{x}\,dx$$

An antiderivative of $f(x) = 1/x$ is $F(x) = \ln|x|$ and, because $3 \leqslant x \leqslant 6$, we can write $F(x) = \ln x$. So

$$\int_3^6 \frac{1}{x}\,dx = \ln x\Big]_3^6 = \ln 6 - \ln 3$$

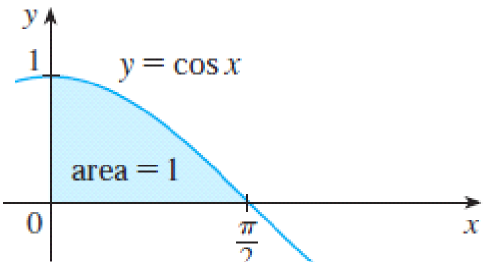
$$= \ln\frac{6}{3} = \ln 2$$

EXAMPLE 8 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

SOLUTION Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, we have

$$A = \int_0^b \cos x\,dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b$$

In particular, taking $b = \pi/2$, we have proved that the area under the cosine curve from 0 to $\pi/2$ is $\sin(\pi/2) = 1$. (See Figure 9.)



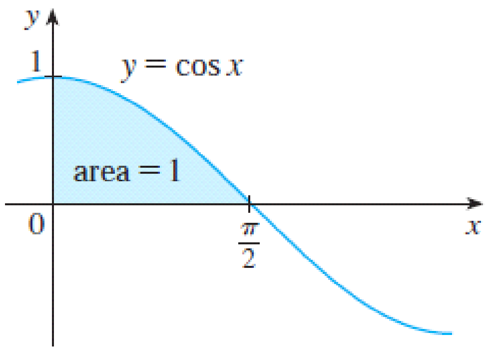


FIGURE 9

EXAMPLE 9 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

Indefinite Integrals

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

SOLUTION Using our convention and Table 1, we have

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\,dx &= 10\int x^4\,dx - 2\int \sec^2 x\,dx \\ &= 10\,\frac{x^5}{5} - 2\tan x + C \\ &= 2x^5 - 2\tan x + C\end{aligned}$$

You should check this answer by differentiating it.

EXAMPLE 3 Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

SOLUTION Using FTC2 and Table 1, we have

$$\int_0^3 (x^3 - 6x)\,dx = \frac{x^4}{4} - 6\frac{x^2}{2}\Big]_0^3$$

$$= \left(\tfrac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\tfrac{1}{4}\cdot 0^4 - 3\cdot 0^2\right)$$

$$= \tfrac{81}{4} - 27 - 0 + 0 = -6.75$$

V EXAMPLE 4 Find $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

SOLUTION The Fundamental Theorem gives

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx = 2\frac{x^4}{4} - 6\frac{x^2}{2} + 3\tan^{-1}x\Big]_0^2$$

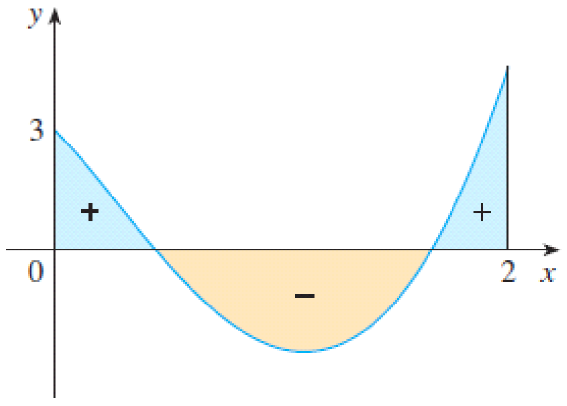
$$= \tfrac{1}{2}x^4 - 3x^2 + 3\tan^{-1}x\Big]_0^2$$

$$= \tfrac{1}{2}(2^4) - 3(2^2) + 3\tan^{-1}2 - 0$$

$$= -4 + 3\tan^{-1}2$$

This is the exact value of the integral. If a decimal approximation is desired, we can use a calculator to approximate $\tan^{-1}2$. Doing so, we get

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx \approx -0.67855$$



EXAMPLE 5 Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$.

SOLUTION First we need to write the integrand in a simpler form by carrying out the division:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Big]_1^9 = 2t + \tfrac{2}{3}t^{3/2} + \frac{1}{t}\Big]_1^9$$

$$= \left(2 \cdot 9 + \tfrac{2}{3} \cdot 9^{3/2} + \tfrac{1}{9}\right) - \left(2 \cdot 1 + \tfrac{2}{3} \cdot 1^{3/2} + \tfrac{1}{1}\right)$$

$$= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1 = 32\tfrac{4}{9}$$

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$