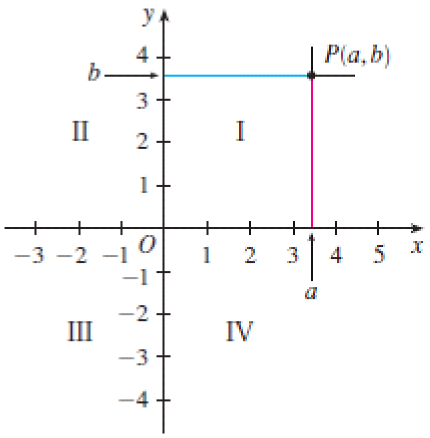
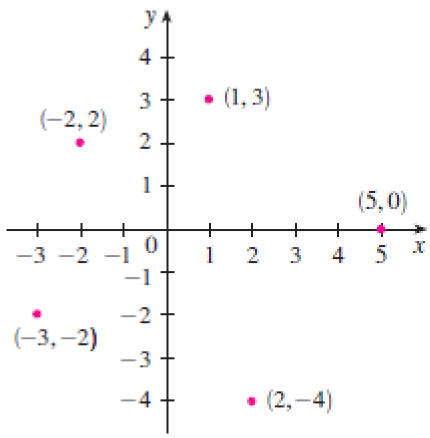
Line & Conic Section

Monday, 28 April 2025 11:35 am

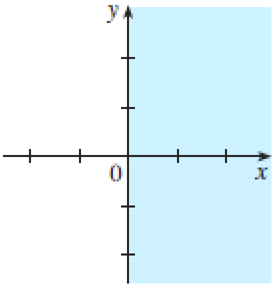
Coordinate Geometry and Lines



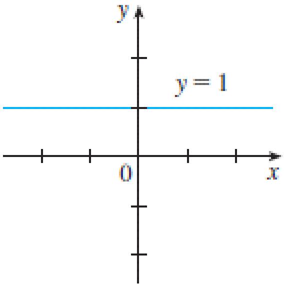


EXAMPLE 1 Describe and sketch the regions given by the following sets.

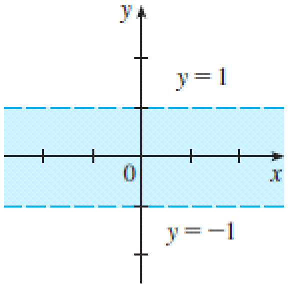
(a) $\{(x, y) \mid x \geqslant 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$



(a) $x \geqslant 0$



(b) $y = 1$



(c) $|y| < 1$

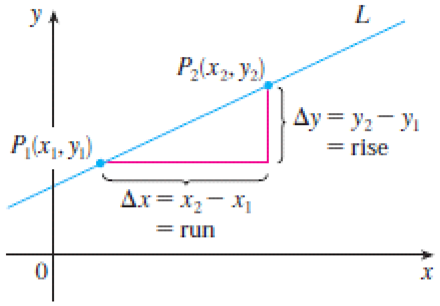
1 Distance Formula The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 2 The distance between $(1, -2)$ and $(5, 3)$ is

$$\sqrt{(5 - 1)^2 + [3 - (-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

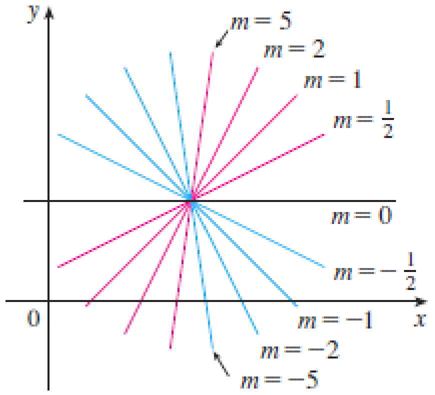
Lines



$\boxed{2}$ **Definition** The **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.



$\boxed{3}$ **Point-Slope Form of the Equation of a Line** An equation of the line passing through the point $P_1(x_1, y_1)$ and having slope m is

$$y - y_1 = m(x - x_1)$$

EXAMPLE 3 Find an equation of the line through $(1, -7)$ with slope $-\frac{1}{2}$.

SOLUTION Using $\boxed{3}$ with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -7$, we obtain an equation of the line as

$$y + 7 = -\tfrac{1}{2}(x - 1)$$

which we can rewrite as

$$2y + 14 = -x + 1 \qquad \text{or} \qquad x + 2y + 13 = 0$$

EXAMPLE 4 Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

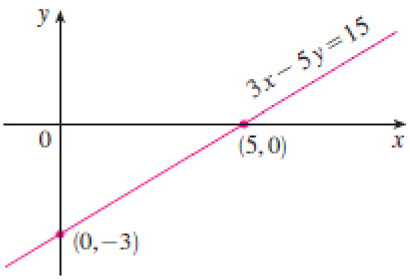
4 **Slope-Intercept Form of the Equation of a Line** An equation of the line with slope m and y-intercept b is

$$y = mx + b$$

$$\boxed{Ax + By + C = 0}$$

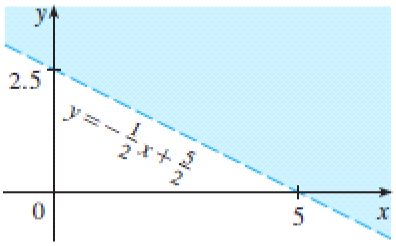
$$y = -\frac{A}{B}x - \frac{C}{B}$$

EXAMPLE 5 Sketch the graph of the equation $3x - 5y = 15$.



EXAMPLE 6 Graph the inequality $x + 2y > 5$.

$$x + 2y > 5$$
$$2y > -x + 5$$
$$y > -\tfrac{1}{2}x + \tfrac{5}{2}$$



Parallel and Perpendicular Lines

6 **Parallel and Perpendicular Lines**

1. Two nonvertical lines are parallel if and only if they have the same slope.
2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

EXAMPLE 7 Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

SOLUTION The given line can be written in the form

$$y = -\tfrac{2}{3}x - \tfrac{5}{6}$$

which is in slope-intercept form with $m = -\frac{2}{3}$. Parallel lines have the same slope, so the required line has slope $-\frac{2}{3}$ and its equation in point-slope form is

$$y - 2 = -\tfrac{2}{3}(x - 5)$$

We can write this equation as $2x + 3y = 16$.

EXAMPLE 8 Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.

SOLUTION The equations can be written as

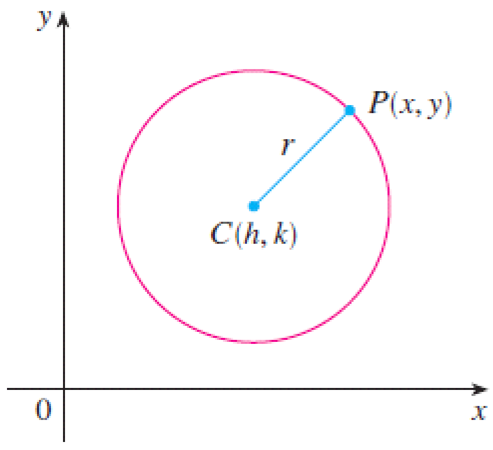
$$y = -\tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{and} \qquad y = \tfrac{3}{2}x - \tfrac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\tfrac{2}{3} \qquad \text{and} \qquad m_2 = \tfrac{3}{2}$$

Since $m_1 m_2 = -1$, the lines are perpendicular.

Circles



1 Equation of a Circle An equation of the circle with center (h, k) and radius r is

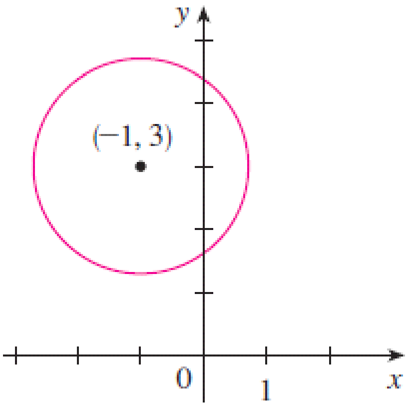
$$(x - h)^2 + (y - k)^2 = r^2$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2$$

EXAMPLE 1 Find an equation of the circle with radius 3 and center $(2, -5)$.

EXAMPLE 2 Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

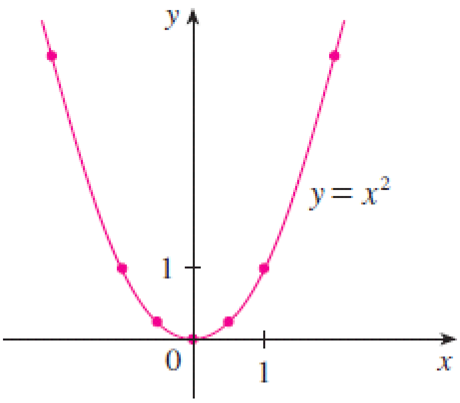


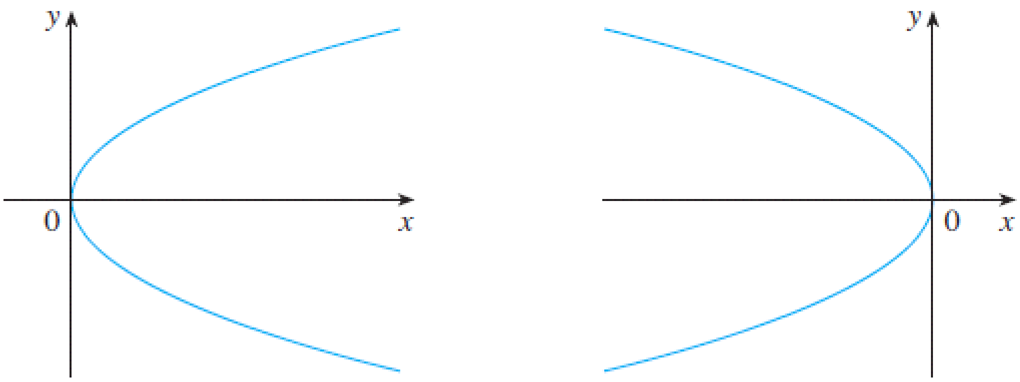
Parabolas

The geometric properties of parabolas are reviewed in Section 10.5. Here we regard a parabola as a graph of an equation of the form $y = ax^2 + bx + c$.

EXAMPLE 3 Draw the graph of the parabola $y = x^2$.

x	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9



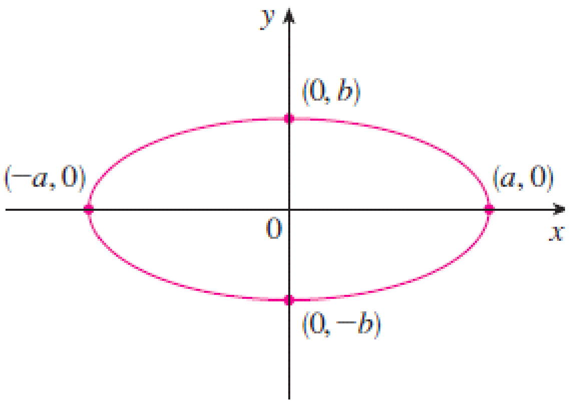


(a) $x = ay^2,\ a > 0$ (b) $x = ay^2,\ a < 0$

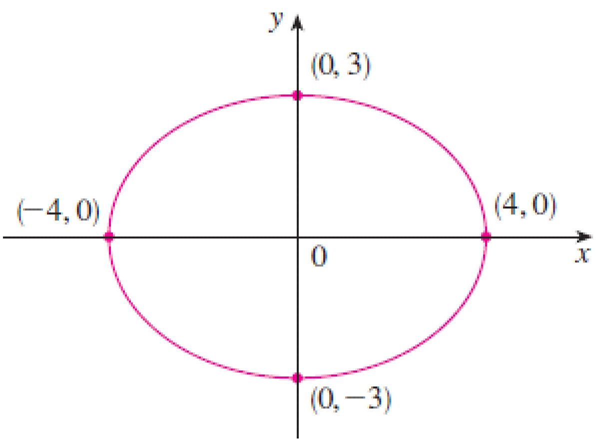
Ellipses

The curve with equation

2 $$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$



EXAMPLE 5 Sketch the graph of $9x^2 + 16y^2 = 144$.



Hyperbolas

The curve with equation

3 $$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

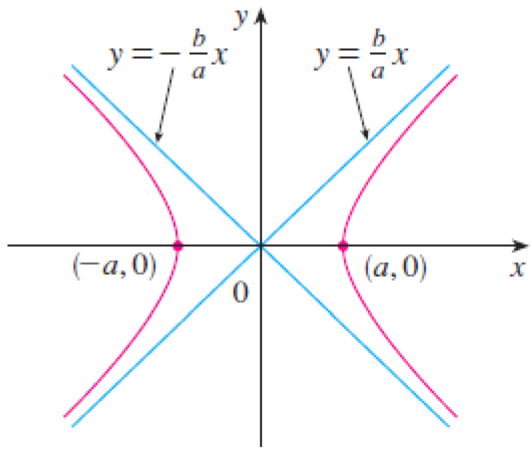


FIGURE 10

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

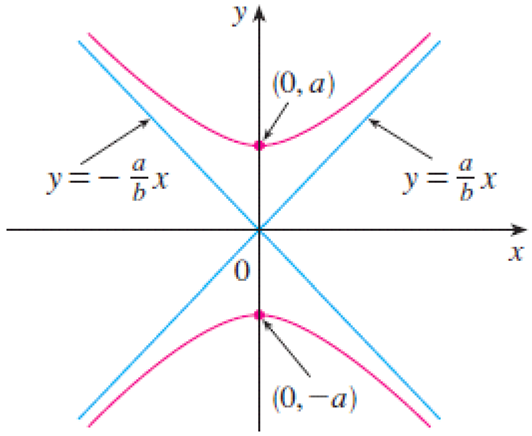
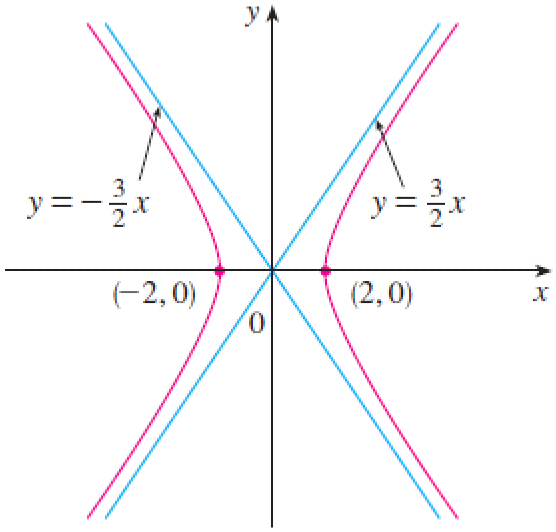
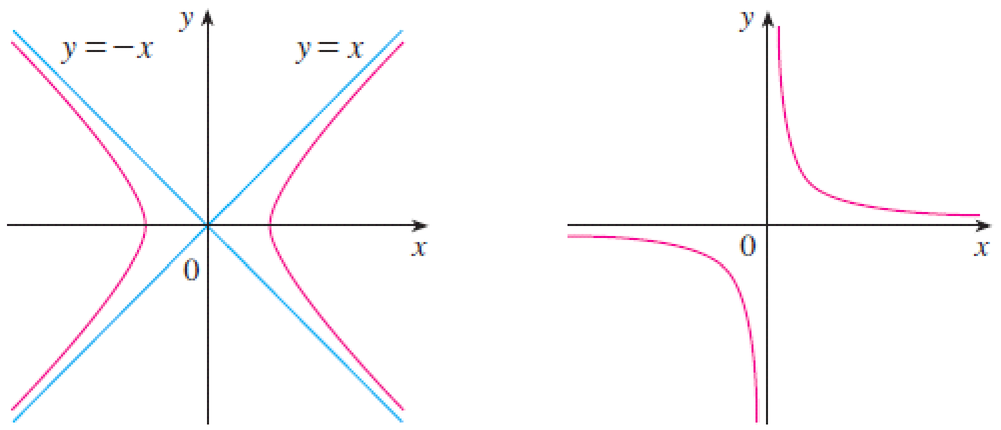


FIGURE 11

The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

EXAMPLE 6 Sketch the curve $9x^2 - 4y^2 = 36$.





(a) $x^2 - y^2 = a^2$ (b) $xy = k$ $(k > 0)$

Shifted Conics

Recall that an equation of the circle with center the origin and radius r is $x^2 + y^2 = r^2$, but if the center is the point (h, k), then the equation of the circle becomes

$$(x - h)^2 + (y - k)^2 = r^2$$

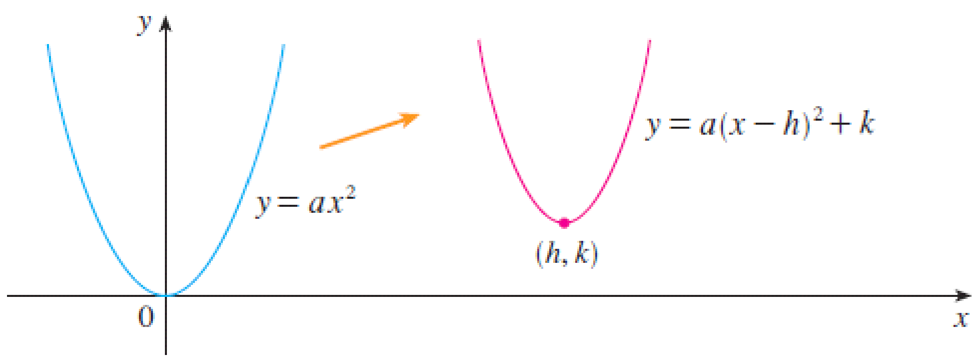
Similarly, if we take the ellipse with equation

4 $$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

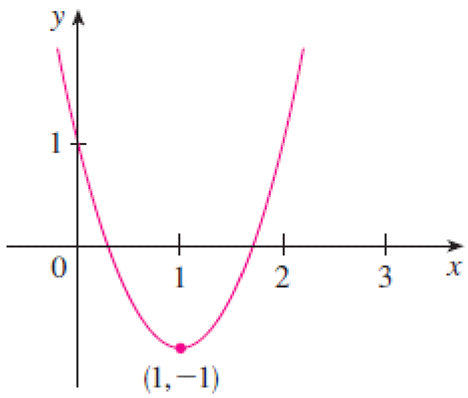
and translate it (shift it) so that its center is the point (h, k), then its equation becomes

5 $$\boxed{\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1}$$

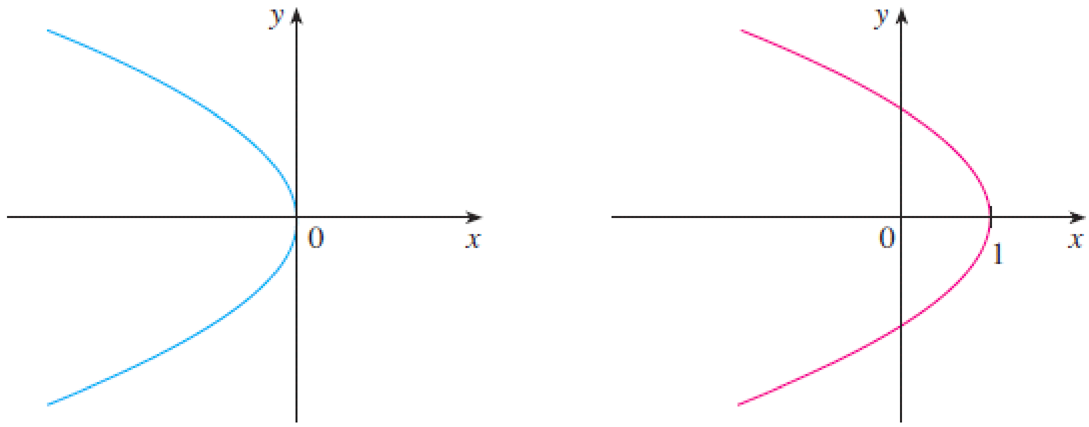
$$y - k = a(x - h)^2 \qquad \text{or} \qquad y = a(x - h)^2 + k$$



EXAMPLE 7 Sketch the graph of the equation $y = 2x^2 - 4x + 1$.



EXAMPLE 8 Sketch the curve $x = 1 - y^2$.



(a) $x = -y^2$ (b) $x = 1 - y^2$