1. Introduction to Functions

- **Definition of a Function:**
  A function is a relation between a set of inputs (domain) and a set of permissible outputs (range) where each input is related to exactly one output.
  - Notation: $f : A \rightarrow B$
  - Example: $f(x) = x^2$
- **Key Components of a Function:**
  - **Domain:** The set of all possible input values (independent variable).
  - **Range:** The set of all possible output values (dependent variable).
  - **Codomain:** The set of all possible values that the function can output (may be larger than the range).
- **Example:**
  For $f(x) = x^2$:
  - Domain: All real numbers ($\mathbb{R}$)
  - Range: Non-negative real numbers ($[0, \infty)$)

2. Types of Functions

- **1. Injective (One-to-One) Functions:**
  - Each element of the domain maps to a unique element in the range.
  - Example: $f(x) = 2x$
- **2. Surjective (Onto) Functions:**
  - Every element in the codomain is mapped to by at least one element in the domain.
  - Example: $f(x) = x^3$
- **3. Bijective Functions:**
  - A function that is both injective and surjective.
  - Example: $f(x) = x$
- **4. Polynomial Functions:**
  - Functions of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$.
  - Example: $f(x) = 3x^2 + 2x + 1$

- **5. Rational Functions:**
  - Functions of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.
  - Example: $f(x) = \frac{x+1}{x-2}$
- **6. Exponential Functions:**
  - Functions of the form $f(x) = a^x$, where $a > 0$.
  - Example: $f(x) = 2^x$
- **7. Logarithmic Functions:**
  - Functions of the form $f(x) = \log_a(x)$, where $a > 0$.
  - Example: $f(x) = \log_2(x)$
- **8. Trigonometric Functions:**
  - Functions like sine, cosine, and tangent.
  - Example: $f(x) = \sin(x)$

3. Domain & Range of Different Types of Functions

- **Polynomial Functions:**
  - Domain: All real numbers ($\mathbb{R}$)
  - Range: Depends on the degree and leading coefficient.

- **Rational Functions:**
  - Domain: All real numbers except where the denominator is zero.
  - Range: Depends on the function.

- **Exponential Functions:**
  - Domain: All real numbers ($\mathbb{R}$)
  - Range: $(0, \infty)$

- **Logarithmic Functions:**
  - Domain: $(0, \infty)$
  - Range: All real numbers ($\mathbb{R}$)

- **Trigonometric Functions:**
  - Example: $f(x) = \sin(x)$
    - Domain: All real numbers ($\mathbb{R}$)
    - Range: $[-1, 1]$

4. Algebraic Functions

- **Definition:**
  Functions that involve only algebraic operations (addition, subtraction, multiplication, division, and roots) on variables.
  - Example: $f(x) = \sqrt{x^2 + 1}$
- **Types of Algebraic Functions:**
  - Polynomial Functions
  - Rational Functions
  - Root Functions

6. Composite Functions

- **Definition:**

  The composition of two functions f and g, denoted $(f \circ g)(x)$, is defined as $f(g(x))$.

- **Example:**
  - $f(x) = x^2$
  - $g(x) = x + 1$
  - $(f \circ g)(x) = f(g(x)) = (x + 1)^2$

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7. Binary Functions

- **Definition:**

  A function that takes two inputs and produces one output.
  - Notation: $f : A \times B \to C$
  - Example: $f(x, y) = x + y$

- **Composition of Binary Functions:**
  - Example: $f(x, y) = x^2 + y^2$ and $g(z) = \sqrt{z}$
  - $(g \circ f)(x, y) = g(f(x, y)) = \sqrt{x^2 + y^2}$

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8. Practice Problems

1. Find the domain and range of $f(x) = \frac{1}{x-3}$.
2. Determine if $f(x) = x^3$ is bijective. If yes, find its inverse.
3. Compute $(f \circ g)(x)$ for $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$.

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9. Summary

- Functions are fundamental in mathematics and computer science.
- Understanding types of functions, their domains, ranges, and compositions is crucial for advanced topics like algorithms, data structures, and machine learning.