Jacobian 3x3 & Taylor Series two variable

Saturday, 19 April 2025 12:02 pm

JACOBIANS

If u and v are functions of the two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the jacobian of u, v with respect to x, y and is written as

$$\frac{\partial (u, v)}{\partial (x, y)} \text{ or } J\left(\frac{u, v}{x, y}\right)$$

Similarly, the jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Screen clipping taken: 19/04/2025 12:03 pm

Example 64. *If* $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

Show that the Jacobian of y_1, y_2, y_3 *with respect to* x_1, x_2, x_3 *is* 4.

(*U.P. I Sem. Jan 2011;* 2004, *Comp.* 2002, *A.M.I.E., Summer* 2002, 2000, *Winter* 2001)

Solution. Here, we have $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$

$$\frac{\partial (y_1, y_2, y_3)}{\partial (x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_1 x_2 \\ x_2 x_3 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_3 x_1 & -x_1 x_2 \end{vmatrix} = \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4$$ **Proved.**

Screen clipping taken: 19/04/2025 12:03 pm

Example 65. *If* $x = r \sin\theta \cos\varphi$,
$y = r \sin\theta \sin\varphi$,
$z = r \cos\theta$,

Show that $\dfrac{\partial (x, y, z)}{\partial (r, \theta, \varphi)} = r^2 \sin\theta$. (*U.P., I Semester, Winter* 2000)

Solution. We have, $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi, \quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\phi, \quad \frac{\partial y}{\partial \varphi} = r\sin\theta\cos\phi, \quad \frac{\partial z}{\partial \varphi} = 0$$

$$\frac{\partial (x, y, z)}{\partial (r, \theta, \varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \sin\theta\, [\sin\theta\cos\phi\,(0 + \sin\theta\cos\phi) - \cos\theta\cos\phi\,(0 - \cos\phi\cos\theta) - \sin\phi\,(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)]$$
$$= r^2 \sin\theta\, [\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi]$$
$$= r^2 \sin\theta\, [(\sin^2\theta + \cos^2\theta)\cos^2\phi + (\sin^2\theta + \cos^2\theta)\sin^2\phi]$$
$$= r^2 \sin\theta\, [\cos^2\phi + \sin^2\phi] = r^2\sin\theta$$

Ans.

1.26 TAYLOR'S SERIES OF TWO VARIABLES

If $f(x, y)$ and all its partial derivatives upto the nth order are finite and continuous for all points (x, y), where

$$a \le x \le a + h, \; b \le y \le b + k$$

Then $f(a + h, b + k) = f(a,b) + \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right) f + \dfrac{1}{2!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f + \dfrac{1}{3!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^3 f + \ldots$

Example 78. *Expand e^x. sin y in powers of x and y, x = 0, y = 0 as far as terms of third degree.*

Solution.

		$x = 0, y = 0$
$f(x, y)$	$e^x \sin y,$	0
$f_x(x, y)$	$e^x \sin y,$	0
$f_y(x, y)$	$e^x \cos y,$	1
$f_{xx}(x, y)$	$e^x \sin y,$	0
$f_{xy}(x, y)$	$e^x \cos y,$	1
$f_{yy}(x, y)$	$-e^x \sin y,$	0
$f_{xxx}(x, y)$	$e^x \sin y,$	0
$f_{xxy}(x, y)$	$e^x \cos y,$	1
$f_{xyy}(x, y)$	$-e^x \sin y,$	0
$f_{yyy}(x, y)$	$-e^x \cos y,$	-1

By Taylor's theorem

$$f(x, y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0) + \ldots$$

$$= f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{x^2}{2!} f_{xx}(0,0) + \frac{2xy}{2!} f_{xy}(0,0) + \frac{y^2}{2!} f_{yy}(0,0) + \frac{1}{3!}x^3 f_{xxx}(0,0) + \frac{3x^2y}{3!} f_{xxy}(0,0) + \frac{3}{3!}xy^2 f_{xyy}(0,0) + \frac{1}{3!}y^3 f_{yyy}(0,0) + \ldots$$

$$e^x \sin y = 0 + x(0) + y(1) + \frac{x^2}{2}(0) + xy(1) + \frac{y^2}{2}(0) + \frac{x^3}{6}(0) + \frac{3x^2y}{6}(1) + \frac{3xy^2}{6}(0) + \frac{y^3}{6}(-1) + \ldots$$

$$= y + xy + \frac{x^2y}{2} - \frac{y^3}{6} + \ldots\ldots$$

Ans.

Example 79. *Find the expansion for cos x cos y in powers of x, y upto fourth order terms.*

Solution.

By Taylor's Series

$$f(x, y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_x^2(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \frac{1}{3!}\left[x^3 f_x^3(0,0) + 3x^2y f_{xy}^2(0,0) + 3xy^2 f_{xy}^{\,2}(0,0) + y^3 f_y^3(0,0)\right]$$

$$+ \frac{1}{4!}\left[x^4 f_x^4(0,0) + 4x^3y f_{xy}^3(0,0) + 6x^2y^2 f_{xy}^{22}(0,0) + 4xy^3 f_{xy}^{\,3}(0,0) + y^4 f_y^4(0,0)\right] + \ldots$$

$$\cos x \cos y = 1 + 0 + 0 + \frac{1}{2}(-x^2 + 0 - y^2) + \frac{1}{6}(0 + 0 + 0 + 0) + \frac{1}{24}(x^4 + 0 + 6x^2y^2 + 0 + y^4)$$

$$= 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^2}{24} + \ldots$$

Ans.

		$x = 0, y = 0$
$f(x, y)$	$\cos x \cos y,$	1
f_x	$-\sin x \cos y,$	0
f_y	$-\cos x \sin y,$	0
f_{xx}	$-\cos x \cos y,$	-1
f_{xy}	$\sin x \sin y,$	0
f_{yy}	$-\cos x \cos y,$	-1
f_{xxx}	$\sin x \cos y,$	0
f_{xxy}	$\cos x \sin y,$	0
f_{xyy}	$\sin x \cos y,$	0
f_{yyy}	$\cos x \sin y,$	0
f_{xxxx}	$\cos x \cos y,$	1
f_{xxxy}	$-\sin x \sin y,$	0
f_{xxyy}	$\cos x \cos y,$	1
f_{xyyy}	$-\sin x \sin y,$	0
f_{yyyy}	$\cos x \cos y,$	1

Example 80. *Find the first six terms of the expansion of the function $e^x \log(1 + y)$ in a Taylor's series in the neighbourhood of the point* (0, 0).

Solution.

Taylor's series is

$$f(x, y) = f(0, 0) + \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right)$$

$$+ \frac{1}{2!}\left(x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2}\right) + \ldots$$

$$\Rightarrow \quad e^x \log(1 + y) = 0 + (x \times 0 + y \times 1)$$

$$+ \frac{1}{2!}[x^2 \times (0) + 2xy \times 1 + y^2 \times (-1)] + ..$$

$$\Rightarrow \quad e^x \log(1 + y) = y + xy - \frac{y^2}{2} \qquad \textbf{Ans.}$$

		$x = 0, y = 0$
$f(x, y)$	$e^x \log(1 + y)$	0
$\frac{\partial f}{\partial x}$	$e^x \log(1 + y)$	0
$\frac{\partial f}{\partial y}$	$\frac{e^x}{1 + y}$	1
$\frac{\partial^2 f}{\partial x^2}$	$e^x \log(1 + y)$	0
$\frac{\partial^2 f}{\partial y^2}$	$-\frac{e^x}{(1 + y)^2}$	-1
$\frac{\partial^2 f}{\partial x \partial y}$	$\frac{e^x}{(1 + y)}$	1