

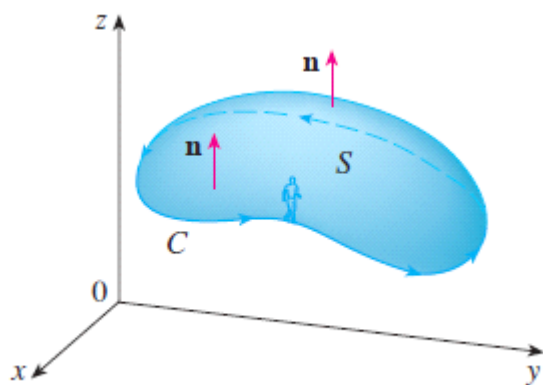
Stokes' Theorem

Monday, 9 June 2025 1:59 pm

Stokes' Theorem

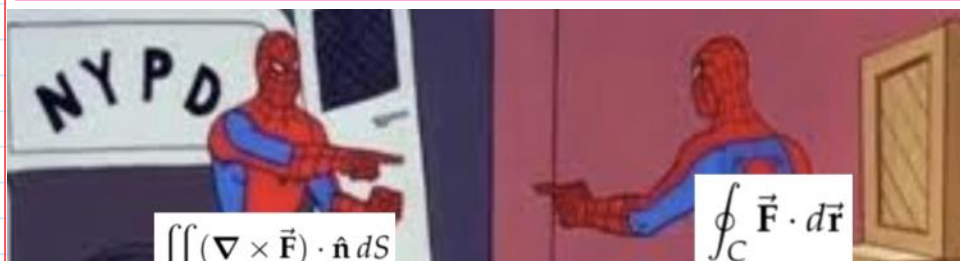


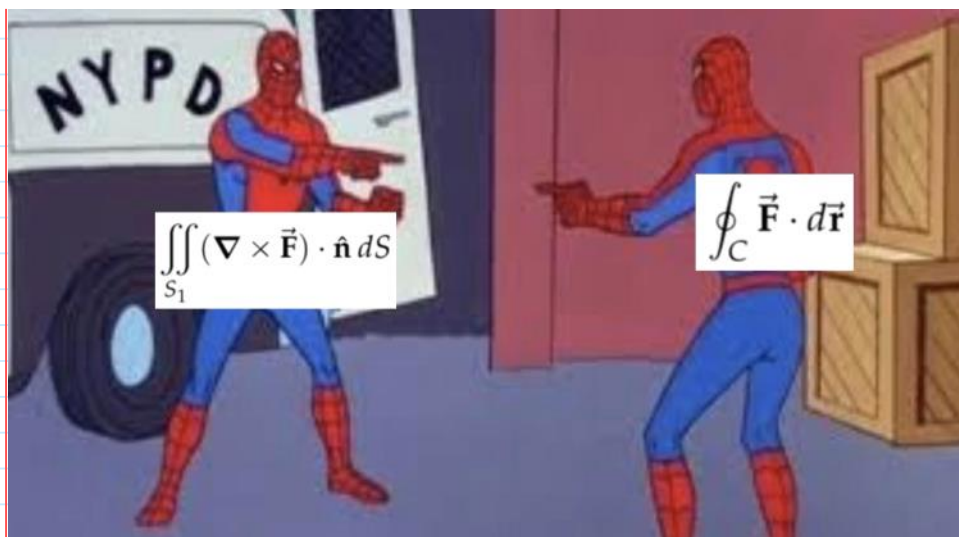
George Gabriel Stokes
(1819-1903)



Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$





$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds \quad \text{and} \quad \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS$$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of \mathbf{F} is equal to the surface integral over S of the normal component of the curl of \mathbf{F} .

The positively oriented boundary curve of the oriented surface S is often written as ∂S , so Stokes' Theorem can be expressed as

$$\boxed{1} \quad \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus. As before, there is an integral involving derivatives on the left side of Equation 1 (recall that $\text{curl } \mathbf{F}$ is a sort of derivative of \mathbf{F}) and the right side involves the values of \mathbf{F} only on the *boundary* of S .

In fact, in the special case where the surface S is flat and lies in the xy -plane with upward orientation, the unit normal is \mathbf{k} , the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA$$

This is precisely the vector form of Green's Theorem given in Equation 16.5.12. Thus we see that Green's Theorem is really a special case of Stokes' Theorem.

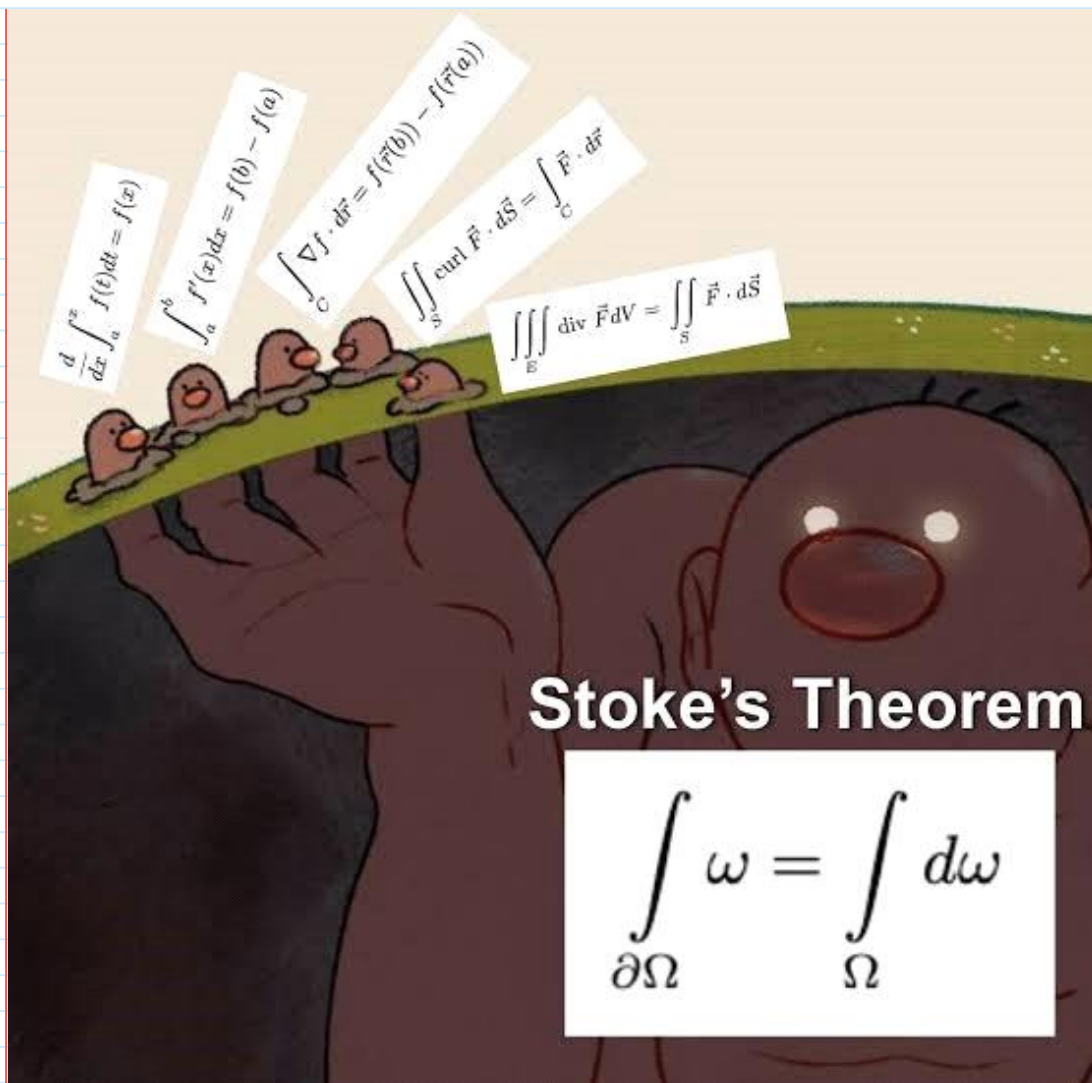
**FUNDAMENTAL
THEOREM OF CALCULUS**



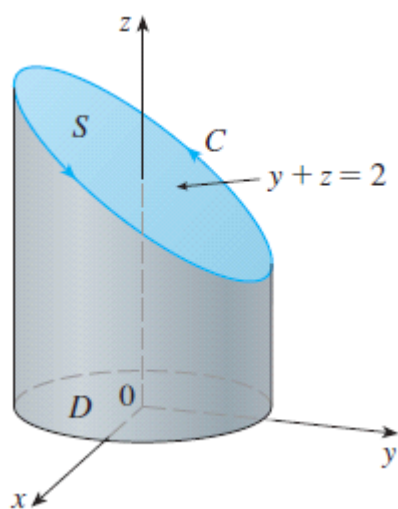
STOKES' THEOREM



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V EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2 \mathbf{i} + x \mathbf{j} + z^2 \mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)



SOLUTION The curve C (an ellipse) is shown in Figure 3. Although $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, it's easier to use Stokes' Theorem. We first compute

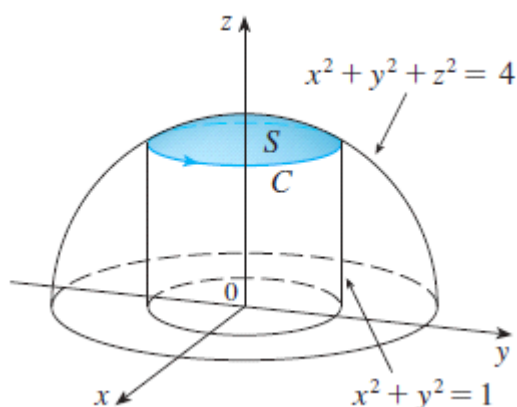
$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y) \mathbf{k}$$

Although there are many surfaces with boundary C , the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C . If we orient S upward, then C has the induced positive orientation. The projection D of S onto the xy -plane is

the disk $x^2 + y^2 \leq 1$ and so using Equation 16.7.10 with $z = g(x, y) = 2 - y$, we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y) dA \\ &= \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta) r dr d\theta \\ &= \int_0^{2\pi} \left[\frac{r^2}{2} + 2 \frac{r^3}{3} \sin \theta \right]_0^1 d\theta = \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin \theta \right) d\theta \\ &= \frac{1}{2}(2\pi) + 0 = \pi \end{aligned}$$

V EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz \mathbf{i} + yz \mathbf{j} + xy \mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy -plane. (See Figure 4.)



SOLUTION To find the boundary curve C we solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Subtracting, we get $z^2 = 3$ and so $z = \sqrt{3}$ (since $z > 0$). Thus C is the circle given by the equations $x^2 + y^2 = 1$, $z = \sqrt{3}$. A vector equation of C is

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \sqrt{3} \mathbf{k} \quad 0 \leq t \leq 2\pi$$

so $\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{3} \cos t \mathbf{i} + \sqrt{3} \sin t \mathbf{j} + \cos t \sin t \mathbf{k}$$

Therefore, by Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} (-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t) dt \\ &= \sqrt{3} \int_0^{2\pi} 0 dt = 0\end{aligned}$$