Parametric Surface & Surface Integral

Monday, 9 June 2025 12:28 pm

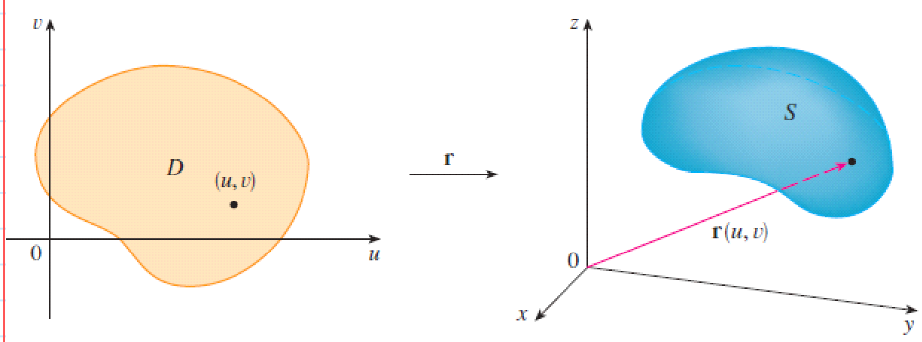
Previous parametric form of the curve:

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

Now, in the case of the surface, it must be two variables

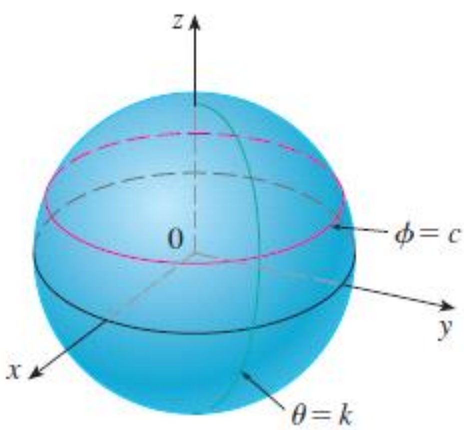
$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$



V **EXAMPLE 4** Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$



$$x = a \sin\phi \cos\theta \qquad y = a \sin\phi \sin\theta \qquad z = a \cos\phi$$

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\, \mathbf{i} + a \sin\phi \sin\theta\, \mathbf{j} + a \cos\phi\, \mathbf{k}$$

EXAMPLE 5 Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \leqslant z \leqslant 1$$

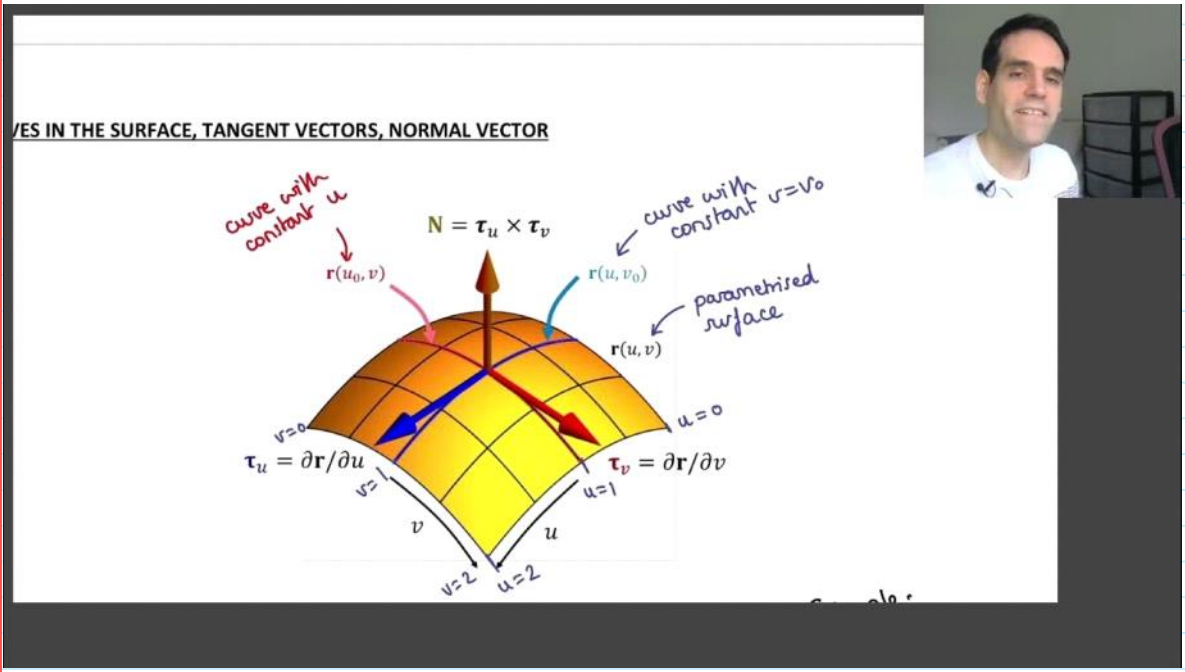
$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = z$$

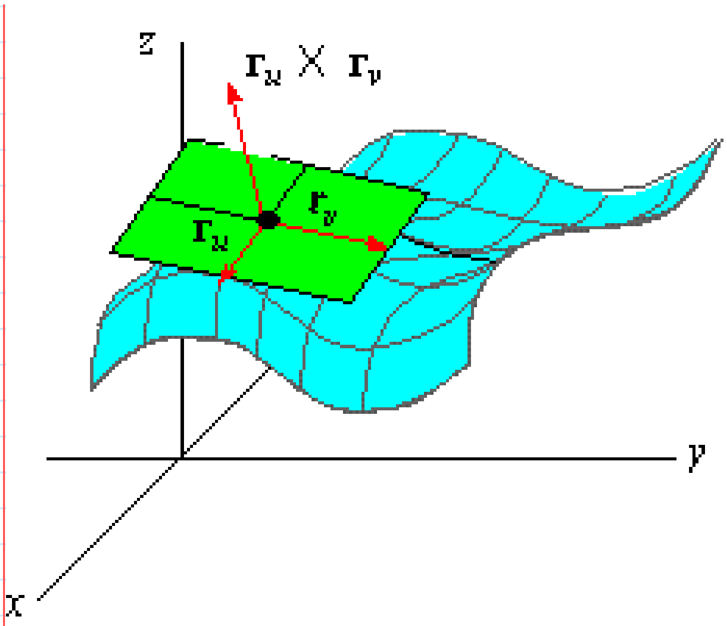
V EXAMPLE 6 Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

$$x = x \qquad y = y \qquad z = x^2 + 2y^2$$

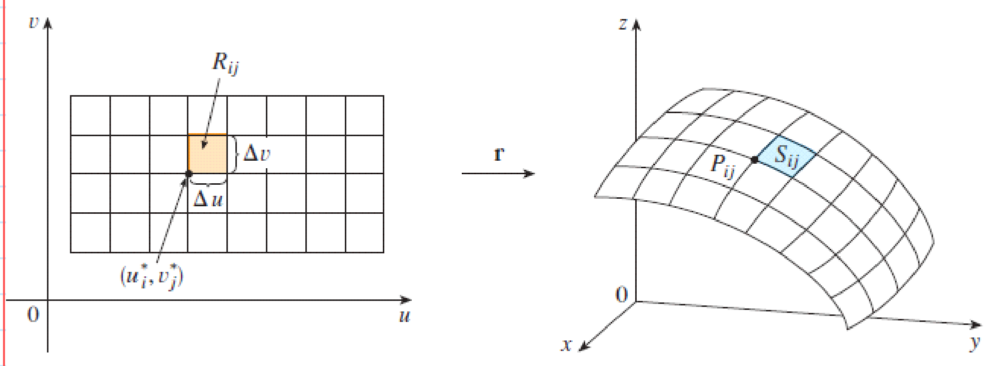
$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + 2y^2)\,\mathbf{k}$$

Normal vector:





Surface Area



6 Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where $\quad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$

EXAMPLE 10 Find the surface area of a sphere of radius a.

SOLUTION In Example 4 we found the parametric representation

$$x = a \sin\phi \, \cos\theta \qquad y = a \sin\phi \, \sin\theta \qquad z = a \cos\phi$$

where the parameter domain is

$$D = \{(\phi, \theta) \mid 0 \leqslant \phi \leqslant \pi, 0 \leqslant \theta \leqslant 2\pi\}$$

We first compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\,\cos\theta & a\cos\phi\,\sin\theta & -a\sin\phi \\ -a\sin\phi\,\sin\theta & a\sin\phi\,\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$$

Thus

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\,\cos^2\theta + a^4\sin^4\phi\,\sin^2\theta + a^4\sin^2\phi\,\cos^2\phi}$$

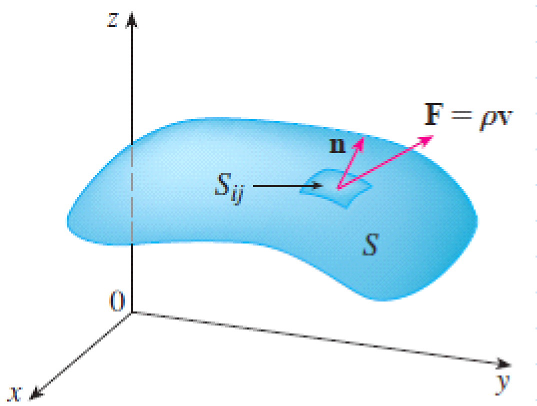
$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\,\cos^2\phi} = a^2\sqrt{\sin^2\phi} = a^2\sin\phi$$

since $\sin\phi \geqslant 0$ for $0 \leqslant \phi \leqslant \pi$. Therefore, by Definition 6, the area of the sphere is

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\,dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\,d\phi\,d\theta$$

$$= a^2\int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\,d\phi = a^2(2\pi)2 = 4\pi a^2$$

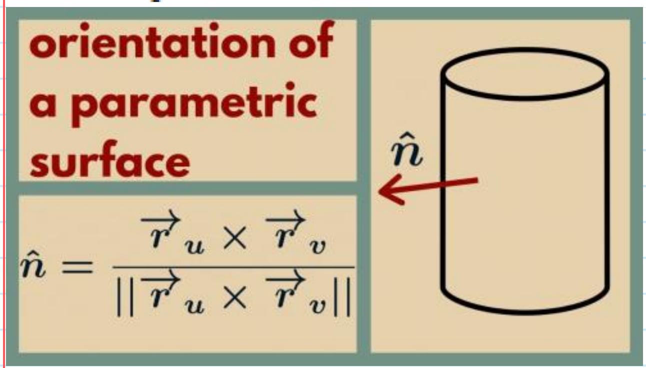
Surface Integrals of Vector Fields



8 Definition If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over** S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.



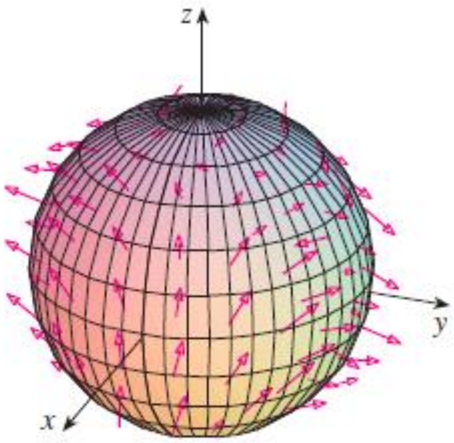
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION As in Example 1, we use the parametric representation

$$\mathbf{r}(\phi, \theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} \qquad 0 \leqslant \phi \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi$$

Then $$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos\phi\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\theta\,\mathbf{k}$$



$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi \cos\theta \, \mathbf{i} + \sin^2\phi \sin\theta \, \mathbf{j} + \sin\phi \cos\phi \, \mathbf{k}$$

Therefore

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos\phi \sin^2\phi \cos\theta + \sin^3\phi \sin^2\theta + \sin^2\phi \cos\phi \cos\theta$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} (2 \sin^2\phi \cos\phi \cos\theta + \sin^3\phi \sin^2\theta) \, d\phi \, d\theta$$

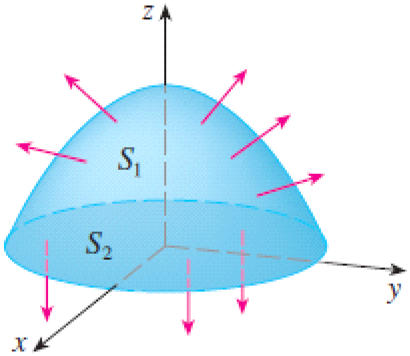
$$= 2 \int_0^{\pi} \sin^2\phi \cos\phi \, d\phi \int_0^{2\pi} \cos\theta \, d\theta + \int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= 0 + \int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} \sin^2\theta \, d\theta \qquad \left(\text{since } \int_0^{2\pi} \cos\theta \, d\theta = 0\right)$$

$$= \frac{4\pi}{3}$$

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA}$$

V EXAMPLE 5 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



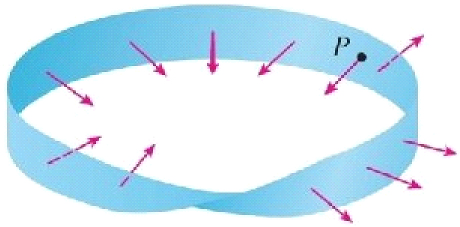
$$P(x, y, z) = y \qquad Q(x, y, z) = x \qquad R(x, y, z) = z = 1 - x^2 - y^2$$

1 $$\frac{\partial g}{\partial x} = -2x \qquad \frac{\partial g}{\partial y} = -2y$$

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \\
&= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\, dA \\
&= \iint_D (1 + 4xy - x^2 - y^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos\theta \sin\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\tfrac{1}{4} + \cos\theta \sin\theta\right) d\theta = \tfrac{1}{4}(2\pi) + 0 = \frac{\pi}{2}
\end{aligned}$$

Oriented Surfaces

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).]



A Möbius strip

Figure 4

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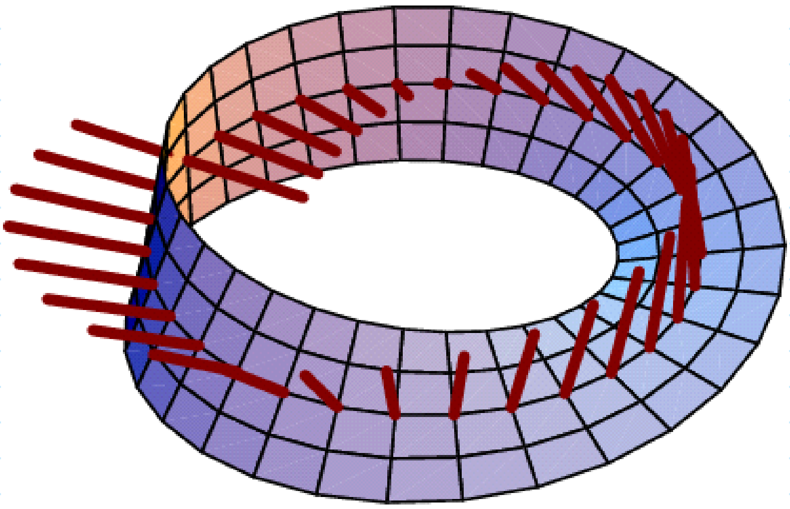

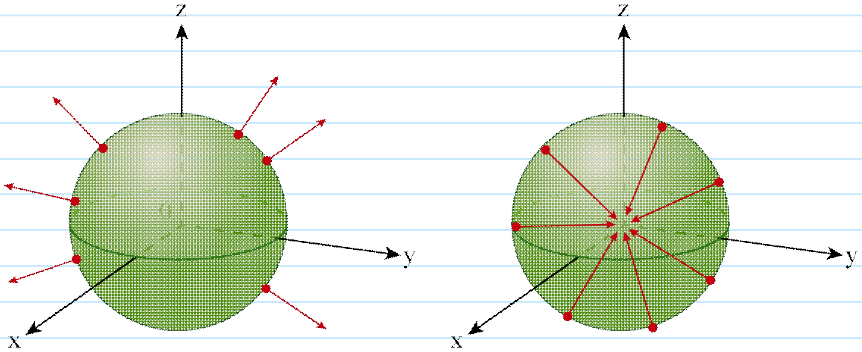

Positive Orientation **Negative Orientation**

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[Mobius Strip Video]