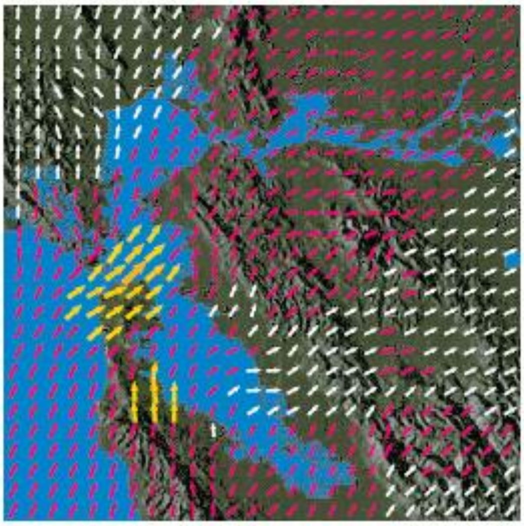
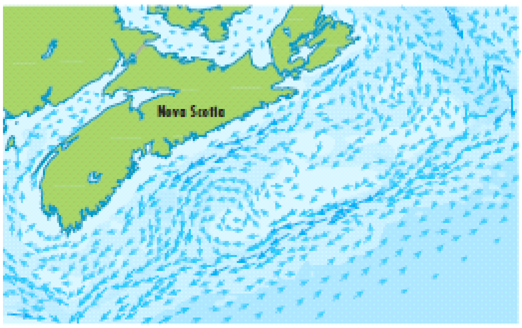
Vector Field & Line Integral

Friday, 16 May 2025 7:03 pm

Vector Fields





(a) Ocean currents off the coast of Nova Scotia

1 Definition Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

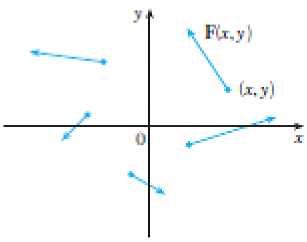


FIGURE 3
Vector field on $\mathbb{R}^2$

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

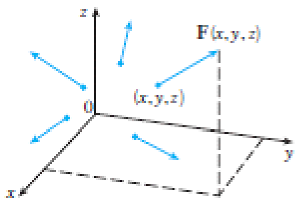


FIGURE 4
Vector field on $\mathbb{R}^3$

V EXAMPLE 1 A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1\rangle$	$(-1, 0)$	$\langle 0, -1\rangle$
$(2, 2)$	$\langle -2, 2\rangle$	$(-2, -2)$	$\langle 2, -2\rangle$
$(3, 0)$	$\langle 0, 3\rangle$	$(-3, 0)$	$\langle 0, -3\rangle$
$(0, 1)$	$\langle -1, 0\rangle$	$(0, -1)$	$\langle 1, 0\rangle$
$(-2, 2)$	$\langle -2, -2\rangle$	$(2, -2)$	$\langle 2, 2\rangle$
$(0, 3)$	$\langle -3, 0\rangle$	$(0, -3)$	$\langle 3, 0\rangle$

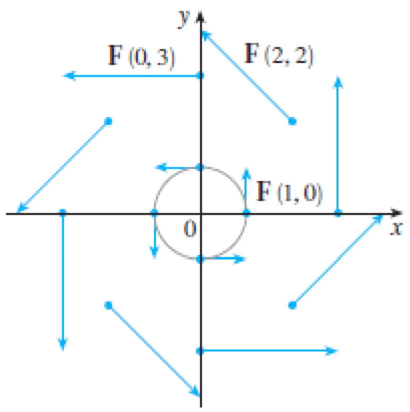


FIGURE 5
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

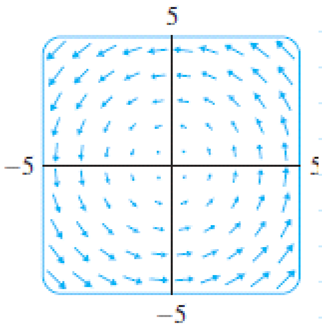


FIGURE 6
$\mathbf{F}(x, y) = \langle -y, x \rangle$

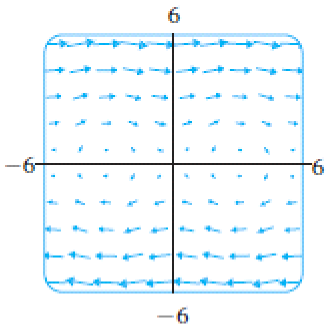


FIGURE 7
$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

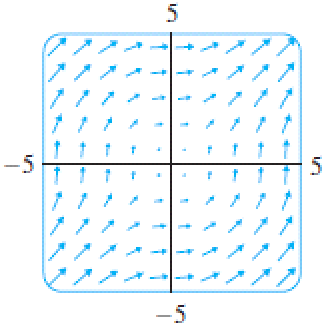
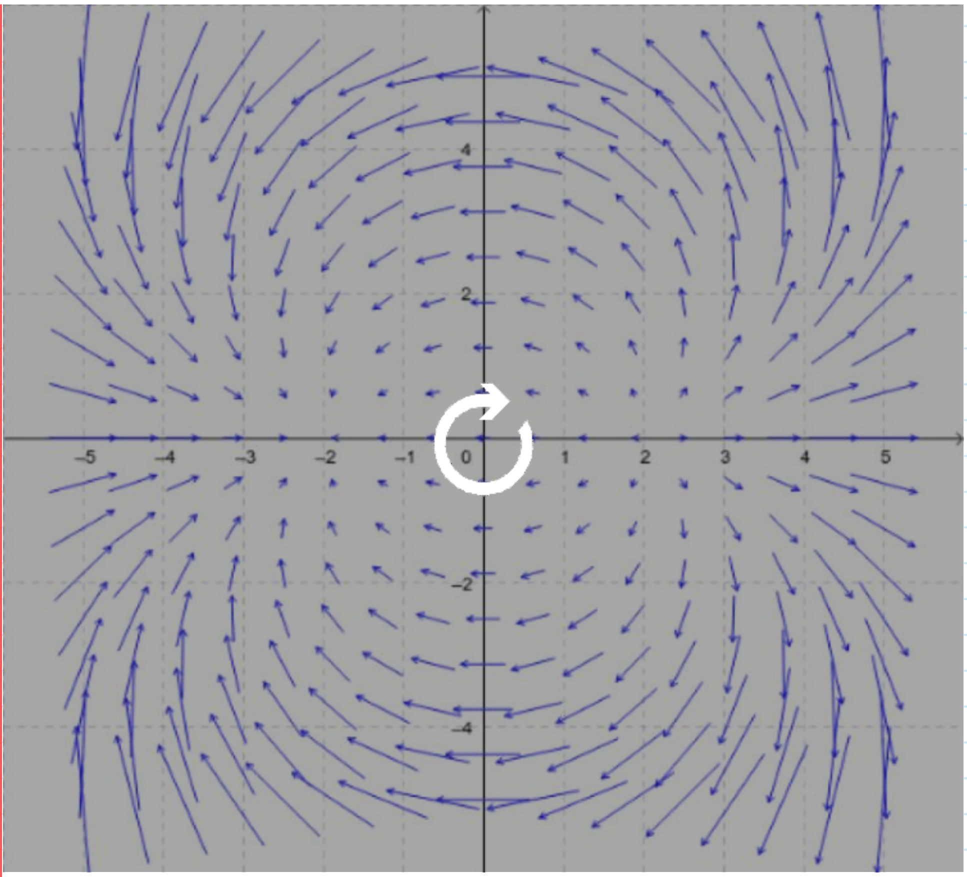
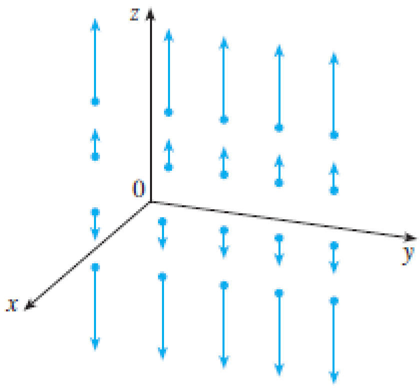


FIGURE 8
$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

[Vector Fields]



V EXAMPLE 2 Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\,\mathbf{k}$.



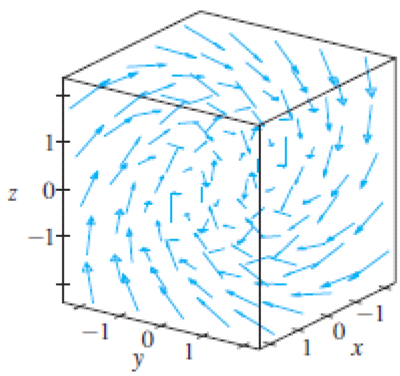


FIGURE 10
$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

Gradient Fields

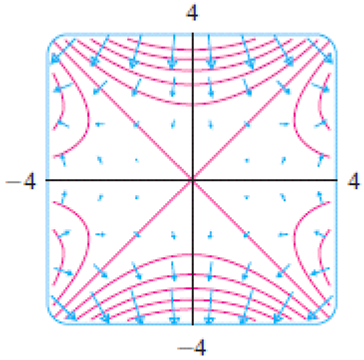
$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

V EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

SOLUTION The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$



1–10 Sketch the vector field **F** by drawing a diagram like Figure 5 or Figure 9.

1. $\mathbf{F}(x, y) = 0.3\,\mathbf{i} - 0.4\,\mathbf{j}$

2. $\mathbf{F}(x, y) = \frac{1}{2}x\,\mathbf{i} + y\,\mathbf{j}$

3. $\mathbf{F}(x, y) = -\frac{1}{2}\,\mathbf{i} + (y - x)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = y\,\mathbf{i} + (x + y)\,\mathbf{j}$

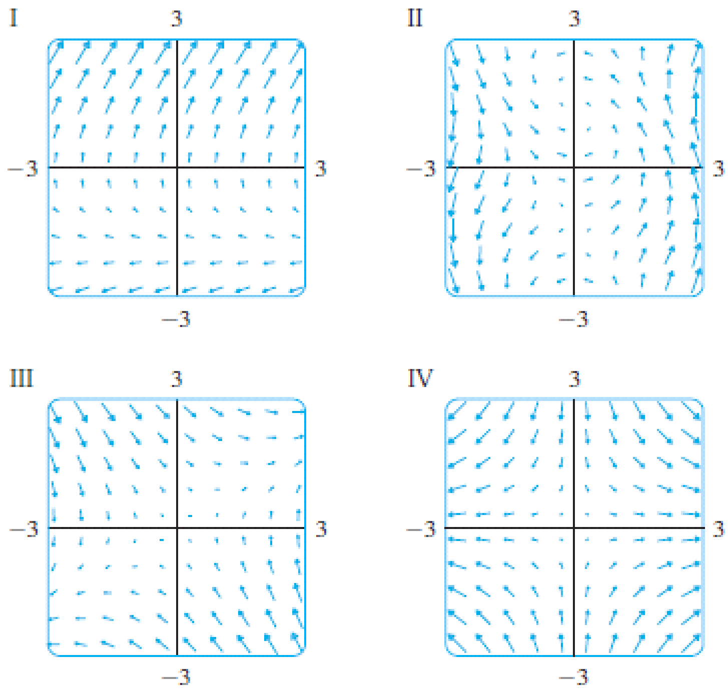
11–14 Match the vector fields **F** with the plots labeled I–IV. Give reasons for your choices.

11. $\mathbf{F}(x, y) = \langle x, -y \rangle$

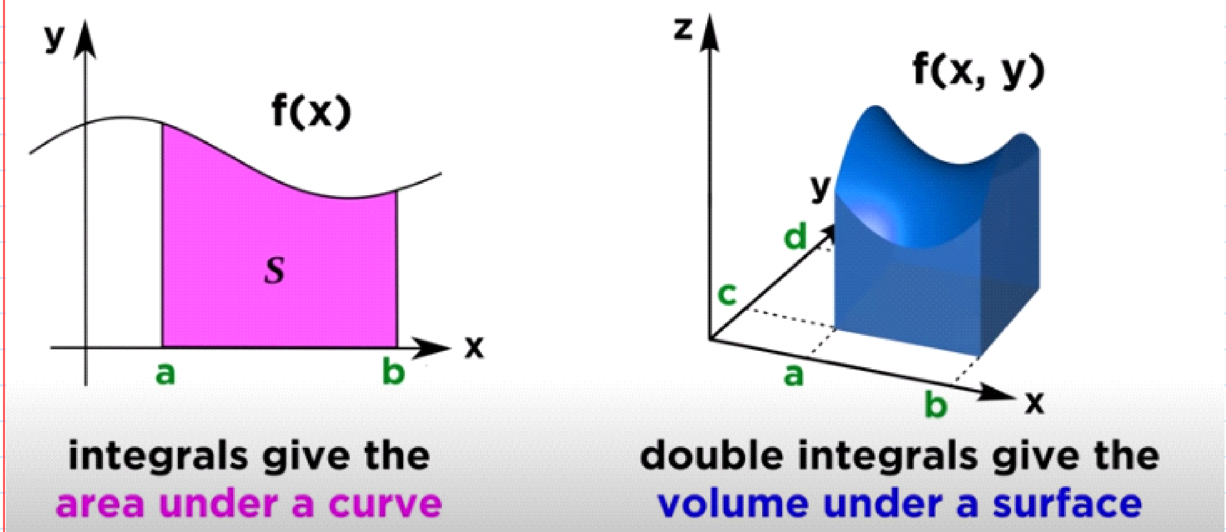
12. $\mathbf{F}(x, y) = \langle y, x - y \rangle$

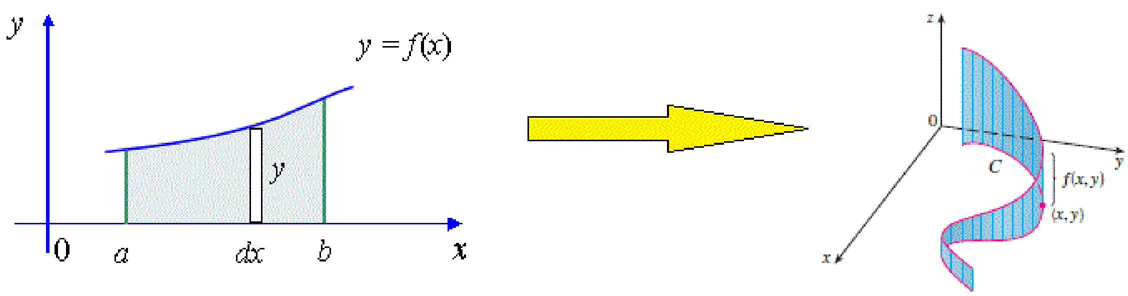
13. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$

14. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$



Line Integrals





Definite integral - Area of a flat surface

Line integral - Area of a curved surface

- **Applications of line Integral:**

- **Calculating work done by a conservative force field:** A line integral can determine the work done in moving an object between two points under the influence of a conservative force field like gravity.

- **Magnetic flux:** In electromagnetism, line integrals are used to compute the magnetic flux passing through a closed loop or surface, which relates to Faraday's law of induction.

- **Electrical potential difference:** The voltage difference between two points in an electric field is given by the line integral of the field along the path connecting them.

- **Momentum/circulation in fluids:** Line integrals quantify the circulation or momentum around closed paths or streamlines in fluid flow applications.

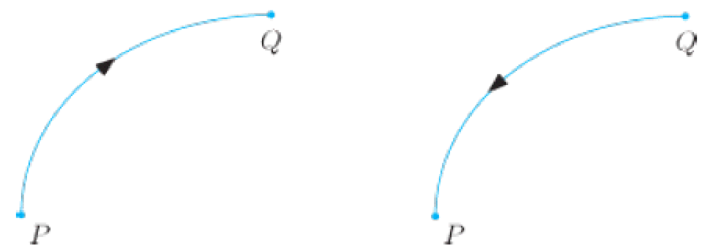
- **Thermodynamic processes:** Calculating work done by pressure-volume changes in thermodynamic systems.

- **Mechanics/orbital motion:** Determining changes in parameters like angular momentum or torque along trajectories.

- **Optimization:** Finding minimum work/time routes between points for logistics, transport etc.

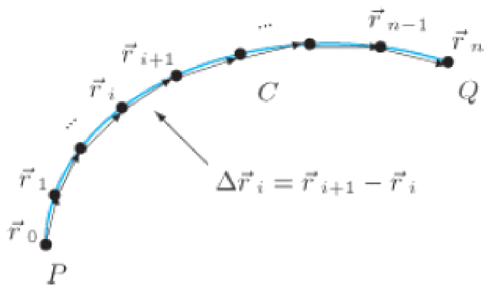
Line Integrals of Vector Fields

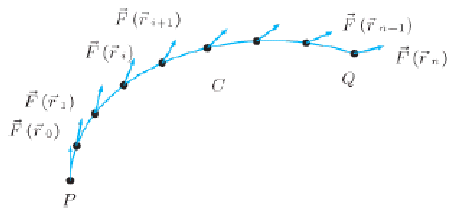
A curve is said to be *oriented* if we have chosen a direction of travel on it.



Definition of the Line Integral

Suppose $\vec{F}$ is a vector field (either in $\mathbb{R}^2$ or $\mathbb{R}^3$), and C is an oriented curve. We develop the line integral the way that we develop all integrals, in this case by first slicing up the curve C into n small, approximately straight pieces along which $\vec{F}$ is approximately constant. Then there is a displacement vector $\Delta\vec{r}_i = \vec{r}_{i+1} - \vec{r}_i$ associated with each piece, and the value of $\vec{F}$ is approximately $\vec{F}(\vec{r}_i)$.





Now, along each displacement vector $\Delta\vec{r}_i$, we can compute the dot product $\vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$ to measure the extent to which $\vec{F}$ points along the curve or against the curve at $\vec{r}_i$.

Summing the dot products over all such pieces, we arrive at a Riemann sum:

$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$$

Taking the limit as $\|\vec{r}_i\| \to 0$, we arrive at the definition of the line integral, provided that the limit exists.

The *line integral* of a vector field $\vec{F}$ along an oriented curve C is

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\vec{r}_i\| \to 0} \sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i.$$

13 Definition Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leq t \leq \pi/2$.

Figure 12 shows the force field and the curve in Example 7. The work done is negative because the field impedes movement along the curve.

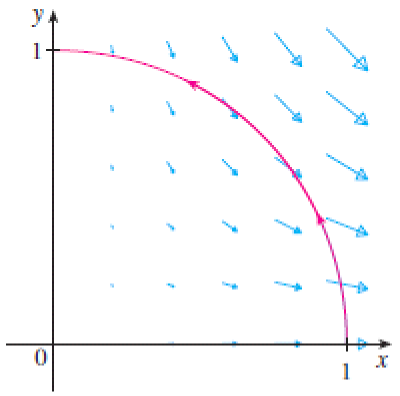


FIGURE 12

SOLUTION Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\,\mathbf{i} - \cos t \sin t\,\mathbf{j}$$

and
$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Therefore the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{\pi/2} (-2\cos^2 t \sin t)\, dt$$

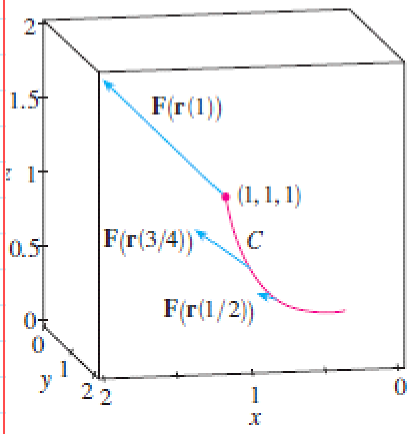
$$= 2\,\frac{\cos^3 t}{3}\Bigg]_0^{\pi/2} = -\frac{2}{3}$$

NOTE Even though $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leq t \leq 1$$



$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\,\mathbf{i} + t^5\,\mathbf{j} + t^4\,\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^1 (t^3 + 5t^6)\, dt = \frac{t^4}{4} + \frac{5t^7}{7}\bigg]_0^1 = \frac{27}{28}$$

Example 1. *If a force* $\vec{F} = 2x^2 y\hat{i} + 3\,xy\hat{j}$ *displaces a particle in the xy-plane from* (0, 0) *to* (1, 4) *along a curve* $y = 4\,x^2$*. Find the work done.*

Solution. Work done $= \int_c \vec{F} . \vec{dr}$

$$= \int_c (2\,x^2 y\,\hat{i} + 3\,xy\,\hat{j}) . (dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_c (2\,x^2 y\, dx + 3\,xy\, dy)$$

$$\begin{bmatrix} \vec{r} = x\hat{i} + y\hat{j} \\ \vec{dr} = dx\,\hat{i} + dy\,\hat{j} \end{bmatrix}$$

$$\begin{pmatrix} y = 4\,x^2 \\ dy = 8\,x\,dx \end{pmatrix}$$

Putting the values of y and dy, we get

$$= \int_0^1 [2x^2 (4x^2)\, dx + 3x\,(4x^2)\, 8x\, dx]$$

$$= 104 \int_0^1 x^4\, dx = 104 \left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5}$$

Example 67. *A vector field is given by*

$\vec{F} = (2y+3)\,\hat{i} + xz\hat{j} + (yz - x)\,\hat{k}$. *Evaluate* $\int_C \vec{F}\cdot d\vec{r}$ *along the path c is* $x = 2t$, $y = t$, $z = t^3$ *from* $t = 0$ *to* $t = 1$. *(Nagpur University, Winter 2003)*

Solution. $\int_C \vec{F}\cdot d\vec{r} = \int_C (2y+3)\, dx + (xz)\, dy + (yz - x)\, dz$

$$\left[\begin{array}{lll} \text{Since } x = 2t & y = t & z = t^3 \\ \therefore \dfrac{dx}{dt} = 2 & \dfrac{dy}{dt} = 1 & \dfrac{dz}{dt} = 3t^2 \end{array}\right]$$

$$= \int_0^1 (2t+3)\,(2\,dt) + (2t)\,(t^3)\, dt + (t^4 - 2t)\,(3t^2\, dt) = \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)\, dt$$

$$= \left[4\,\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1 = \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2} = 7.32857.$$ **Ans.**