Jacobian & Taylor Series

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the transformation $x = 2u + 3v$ and $y = 2u - 3v$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -6 - 6 = -12$$

$$x = \sqrt{2}\,u - \sqrt{\tfrac{2}{3}}\,v,\ y = \sqrt{2}\,u + \sqrt{\tfrac{2}{3}}\,v.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{\tfrac{2}{3}} \\ \sqrt{2} & \sqrt{\tfrac{2}{3}} \end{vmatrix} = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Example 6 Verify that $dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$ when using spherical coordinates.

Hide Solution ▾

Here the transformation is just the standard conversion formulas.

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

The Jacobian is,

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \\ &= -\rho^2\sin^3\varphi\cos^2\theta - \rho^2\sin\varphi\cos^2\varphi\sin^2\theta + 0 \\ &\qquad - \rho^2\sin^3\varphi\sin^2\theta - 0 - \rho^2\sin\varphi\cos^2\varphi\cos^2\theta \\ &= -\rho^2\sin^3\varphi\left(\cos^2\theta + \sin^2\theta\right) - \rho^2\sin\varphi\cos^2\varphi\left(\sin^2\theta + \cos^2\theta\right) \\ &= -\rho^2\sin^3\varphi - \rho^2\sin\varphi\cos^2\varphi \\ &= -\rho^2\sin\varphi\left(\sin^2\varphi + \cos^2\varphi\right) \\ &= -\rho^2\sin\varphi \end{aligned}$$

Finally, dV becomes,

$$dV = \left|-\rho^2\sin\varphi\right| \, d\rho\, d\theta\, d\varphi = \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

1st and 2nd-Degree Taylor Polynomials for Functions of Two Variables

Definition: first-degree Taylor polynomial of a function of two variables, $f(x, y)$

For a function of two variables $f(x, y)$ whose first partials exist at the point (a, b), the 1^{st}**-degree Taylor polynomial** of f for (x, y) near the point (a, b) is:

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

$L(x, y)$ is also called the **linear** (or **tangent plane**) **approximation** of f for (x, y) near the point (a, b).

Definition: Second-degree Taylor Polynomial of a function of two variables, $f(x, y)$

For a function of two variables $f(x, y)$ whose first and second partials exist at the point (a, b), the 2^{nd}**-degree Taylor polynomial** of f for (x, y) near the point (a, b) is:

$$f(x, y) \approx Q(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) + \frac{f_{xx}(a, b)}{2}(x-a)^2 + f_{xy}(a, b)(x-a)(y-b) + \frac{f_{yy}(a, b)}{2}(y-b)^2$$

Determine the 1^{st}- and 2^{nd}-degree Taylor polynomial approximations,

a. $f(x, y) = \sin 2x + \cos y$ for (x, y) near the point $(0, 0)$

b. $f(x, y) = xe^y + 1$ for (x, y) near the point $(1, 0)$

$$f_x(x, y) = 2\cos 2x \quad \text{and} \quad f_y(x, y) = -\sin y$$

$$\begin{aligned} f(0, 0) &= \sin 2(0) + \cos 0 = 1 \\ f_x(0, 0) &= 2\cos 2(0) = 2 \\ f_y(0, 0) &= -\sin 0 = 0 \end{aligned}$$

$$\begin{aligned} L(x, y) &= f(0, 0) + f_x(0, 0)(x-0) + f_y(0, 0)(y-0) \\ &= 1 + 2x \end{aligned}$$

$$\begin{aligned} f_{xx}(x, y) &= -4\sin 2x \\ f_{xy}(x, y) &= 0 \\ f_{yy}(x, y) &= -\cos y \end{aligned}$$

$$\begin{aligned} f_{xx}(0, 0) &= -4\sin 2(0) = 0 \\ f_{xy}(0, 0) &= 0 \\ f_{yy}(0, 0) &= -\cos 0 = -1 \end{aligned}$$

$$\begin{aligned} Q(x, y) &= L(x, y) + \frac{f_{xx}(0, 0)}{2}(x-0)^2 + f_{xy}(0, 0)(x-0)(y-0) + \frac{f_{yy}(0, 0)}{2}(y-0)^2 \\ &= 1 + 2x + \frac{0}{2}x^2 + (0)xy + \frac{-1}{2}y^2 \\ &= 1 + 2x - \frac{y^2}{2} \end{aligned}$$

$f(x, y) = xe^y + 1$.

$f_x(x, y) = e^y \quad \text{and} \quad f_y(x, y) = xe^y$

$$\begin{aligned} f(1,0) &= (1)e^0 + 1 = 2 \\ f_x(1,0) &= e^0 = 1 \\ f_y(1,0) &= (1)e^0 = 1 \end{aligned}$$

$$\begin{aligned} L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= 2 + 1(x-1) + 1y \\ &= 1 + x + y \end{aligned}$$

$$\begin{aligned} f_{xx}(x,y) &= 0 \\ f_{xy}(x,y) &= e^y \\ f_{yy}(x,y) &= xe^y \end{aligned}$$

$$\begin{aligned} f_{xx}(1,0) &= 0 \\ f_{xy}(1,0) &= e^0 = 1 \\ f_{yy}(1,0) &= (1)e^0 = 1 \end{aligned}$$

$$\begin{aligned} Q(x,y) &= L(x,y) + \frac{f_{xx}(1,0)}{2}(x-1)^2 + f_{xy}(1,0)(x-1)(y-0) + \frac{f_{yy}(1,0)}{2}(y-0)^2 \\ &= 1 + x + y + \frac{0}{2}(x-1)^2 + (1)(x-1)y + \frac{1}{2}y^2 \\ &= 1 + x + y + xy - y + \frac{y^2}{2} \\ &= 1 + x + xy + \frac{y^2}{2} \end{aligned}$$