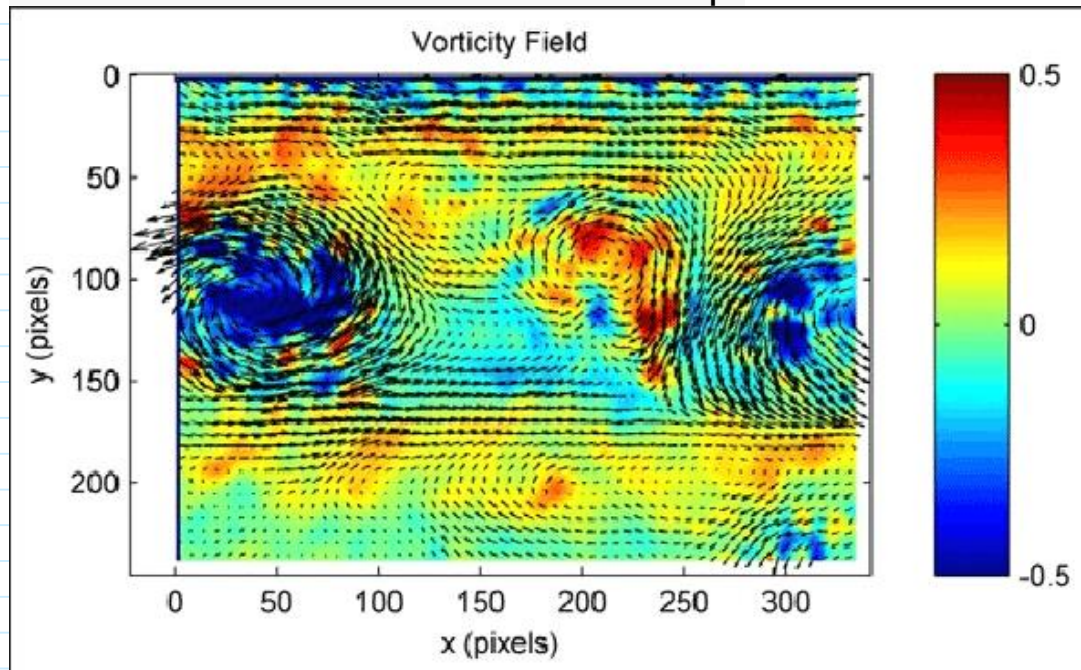


# Curl & Divergence: Motivation & Calculation

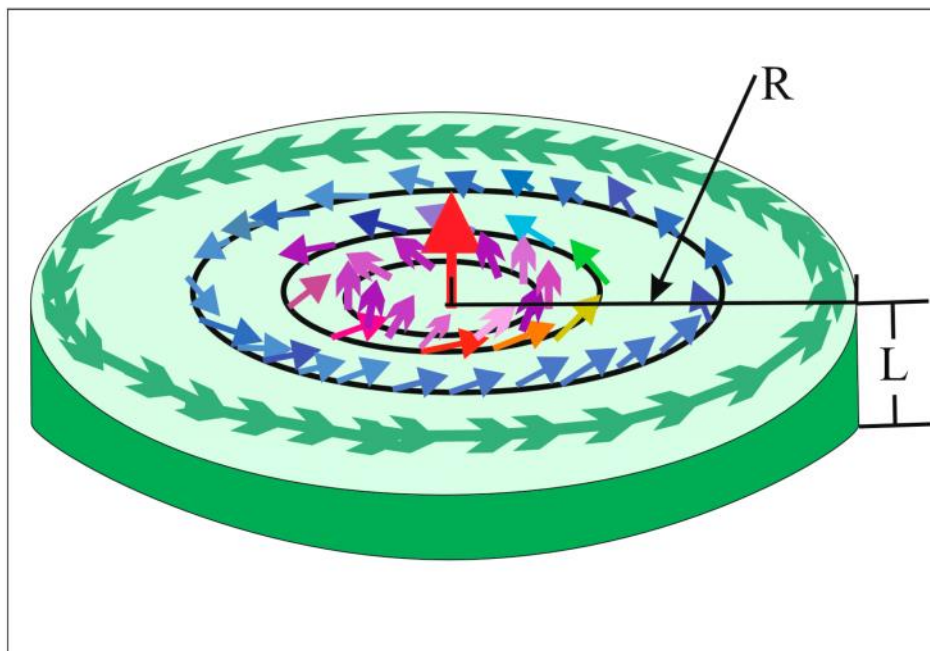
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## Curl:

- Describes rotation/vorticity in fluid flows and measures circulation around an infinitesimal loop.

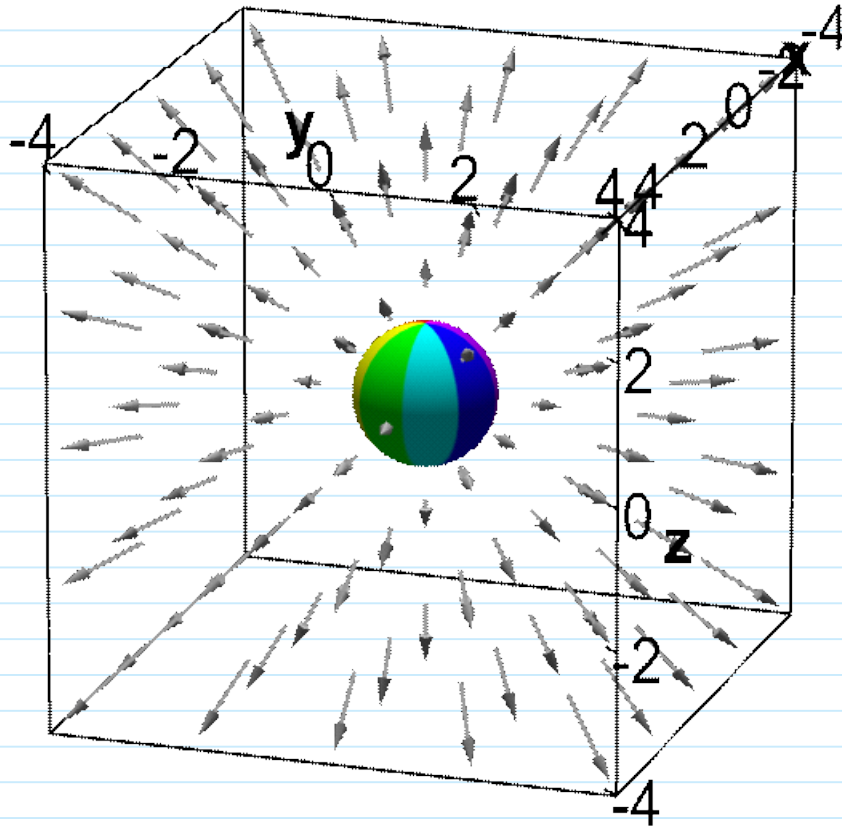


- Characterizes magnetic fields and how field lines twist in space.
- Relates to induced EMF and electromagnetic induction in changing magnetic fields.

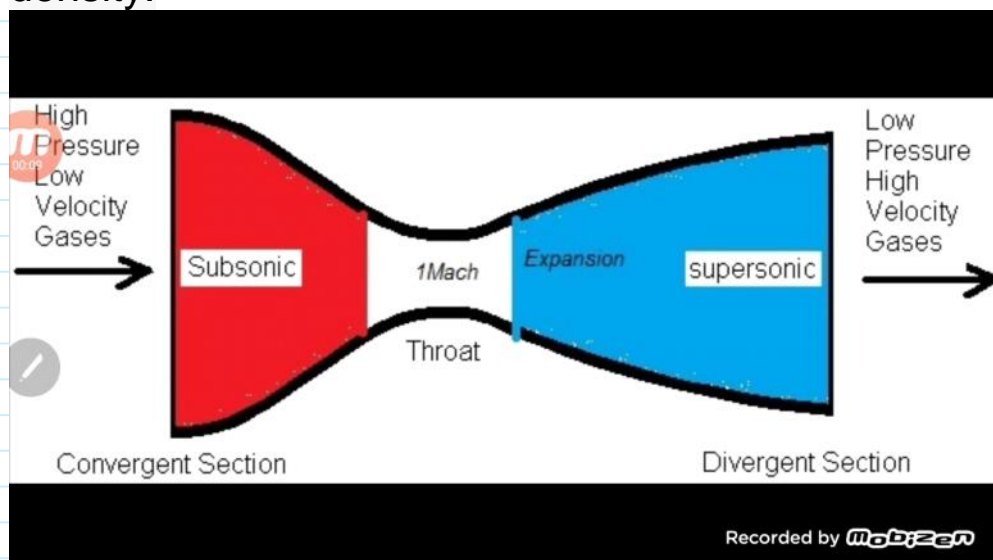


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- Divergence:

- Quantifies how a vector field spreads out or converges at a point.
- In fluid dynamics, divergence relates to sources and sinks of fluid and impacts the continuity equation.



- Describes compressibility in fluids and changes in fluid density.



- Appears in Gauss's law relating electric charge density to the divergence of the electric field.

$$\underbrace{\nabla \cdot}_{\text{Del dot operator means to take the divergence}} \vec{E} = \frac{\rho}{\epsilon_0}$$

Electric field vector

Electric charge volume density (coulombs/m<sup>3</sup>)

Electric permittivity of the free space. Also called dielectric constant

- Characterizes heat, mass, and photon transfer problems through conservation equations using divergence.
- Determines whether a vector field is solenoidal/irrotational for applications in potentials.

Other Applications:

- Optimization and path planning using irrotational/solenoidal conditions.
- Electromagnetism through relationships between curl/div of E, B fields.

## Curl

If  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  is a vector field on  $\mathbb{R}^3$  and the partial derivatives of  $P$ ,  $Q$ , and  $R$  all exist, then the **curl** of  $\mathbf{F}$  is the vector field on  $\mathbb{R}^3$  defined by

$$\boxed{1} \quad \text{curl } \mathbf{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

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$$\nabla f = \mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

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$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\ &= \text{curl } \mathbf{F}\end{aligned}$$

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**EXAMPLE 1** If  $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$ , find  $\text{curl } \mathbf{F}$ .

**SOLUTION** Using Equation 2, we have

$$\begin{aligned}\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} \\ &= \left[ \frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz) \right] \mathbf{i} - \left[ \frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz) \right] \mathbf{j} \\ &\quad + \left[ \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz) \right] \mathbf{k} \\ &= (-2y - xy) \mathbf{i} - (0 - x) \mathbf{j} + (yz - 0) \mathbf{k} \\ &= -y(2 + x) \mathbf{i} + x \mathbf{j} + yz \mathbf{k}\end{aligned}$$

If  $\mathbf{F}$  is conservative, then  $\text{curl } \mathbf{F} = \mathbf{0}$ .

**V EXAMPLE 2** Show that the vector field  $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$  is not conservative.

**SOLUTION** In Example 1 we showed that

$$\text{curl } \mathbf{F} = -y(2 + x) \mathbf{i} + x \mathbf{j} + yz \mathbf{k}$$

This shows that  $\text{curl } \mathbf{F} \neq \mathbf{0}$  and so, by Theorem 3,  $\mathbf{F}$  is not conservative.

**4 Theorem** If  $\mathbf{F}$  is a vector field defined on all of  $\mathbb{R}^3$  whose component functions have continuous partial derivatives and  $\text{curl } \mathbf{F} = \mathbf{0}$ , then  $\mathbf{F}$  is a conservative vector field.

**V EXAMPLE 3**

(a) Show that

$$\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2 z^2 \mathbf{k}$$

is a conservative vector field.

**SOLUTION**(a) We compute the curl of  $\mathbf{F}$ :

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix} \\ &= (6xyz^2 - 6xyz^2)\mathbf{i} - (3y^2 z^2 - 3y^2 z^2)\mathbf{j} + (2yz^3 - 2yz^3)\mathbf{k} \\ &= \mathbf{0} \end{aligned}$$

Since  $\operatorname{curl} \mathbf{F} = \mathbf{0}$  and the domain of  $\mathbf{F}$  is  $\mathbb{R}^3$ ,  $\mathbf{F}$  is a conservative vector field by Theorem 4.

**Divergence**

If  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  is a vector field on  $\mathbb{R}^3$  and  $\partial P/\partial x$ ,  $\partial Q/\partial y$ , and  $\partial R/\partial z$  exist, then the **divergence** of  $\mathbf{F}$  is the function of three variables defined by

**9**

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

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**10**

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

**EXAMPLE 4** If  $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ , find  $\operatorname{div} \mathbf{F}$ .**SOLUTION** By the definition of divergence (Equation 9 or 10) we have

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA$$

God Said:

*Let...*

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

... and there was light!

Maxwell's Equations

Differential form

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations

Integral form

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$



$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \mathbf{J} = \dots \quad \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \dots$$



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## Maxwell

James Clerk Faraday (1831-1879) was an Scottish scientist. He was a gifted mathematician and one of the first scientists to take seriously Faraday's notion of electric and magnetic fields permeating the universe.

