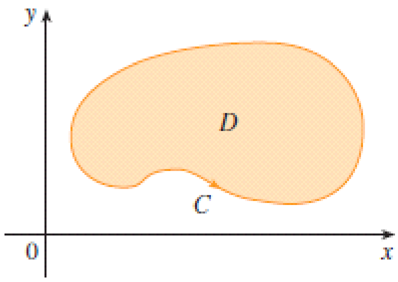
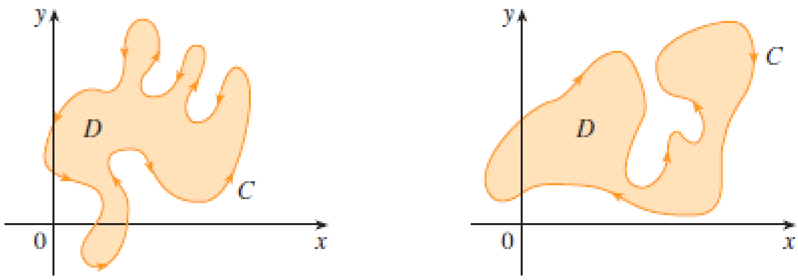
Green's Theorem

Monday, 26 May 2025 11:25 am

Green's Theorem





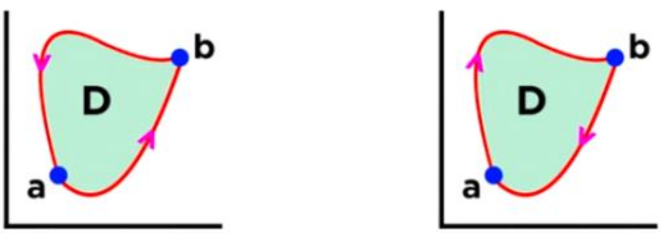
(a) Positive orientation (b) Negative orientation

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_C \mathbf{P}\,\mathbf{dx} + \mathbf{Q}\,\mathbf{dy} = - \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

positive orientation



negative orientation

this curve runs counterclockwise (the theorem applies)

this curve runs clockwise (switch the sign)

EXAMPLE 3 Verify both forms of Green's Theorem for the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region R bounded by the unit circle

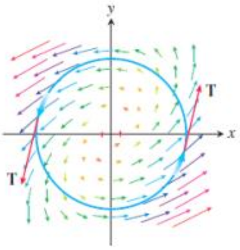
$$C: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Solution Evaluating $\mathbf{F}(\mathbf{r}(t))$ and computing the partial derivatives of the components of $\mathbf{F}$, we have

$$M = \cos t - \sin t, \qquad dx = d(\cos t) = -\sin t\, dt,$$
$$N = \cos t, \qquad dy = d(\sin t) = \cos t\, dt,$$
$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0.$$

The two sides of Equation (3) are

$$\oint_C M\,dx + N\,dy = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t\, dt) + (\cos t)(\cos t\, dt)$$
$$= \int_0^{2\pi} (-\sin t \cos t + 1)\, dt = 2\pi$$



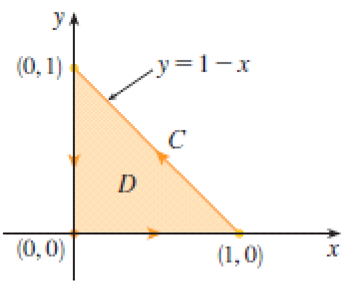
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1 - (-1))\, dx\,dy$$
$$= 2\iint_R dx\,dy = 2(\text{area inside the unit circle}) = 2\pi.$$

V EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

SOLUTION The region D bounded by C is the disk $x^2 + y^2 \le 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$
$$= \iint_D \left[\frac{\partial}{\partial x}\left(7x + \sqrt{y^4 + 1}\right) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$
$$= \int_0^{2\pi}\int_0^3 (7 - 3)\, r\,dr\,d\theta = 4\int_0^{2\pi} d\theta \int_0^3 r\,dr = 36\pi$$

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

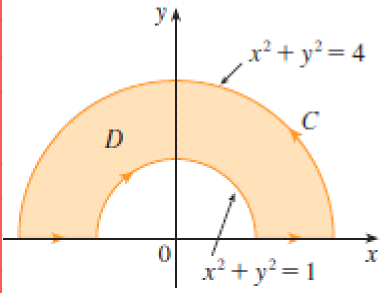


$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \tfrac{1}{2}\int_0^1 (1 - x)^2\,dx$$

$$= -\tfrac{1}{6}(1 - x)^3\Big]_0^1 = \tfrac{1}{6}$$

V EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



$$D = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$$

Therefore Green's Theorem gives

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA$$

$$= \iint_D y\,dA = \int_0^\pi \int_1^2 (r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^\pi \sin\theta\,d\theta \int_1^2 r^2\,dr = \left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{3}r^3\right]_1^2 = \frac{14}{3}$$