Gradient Vector & Directional Derivative

Sunday, 13 April 2025 4:42 pm

The Gradient Vector

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

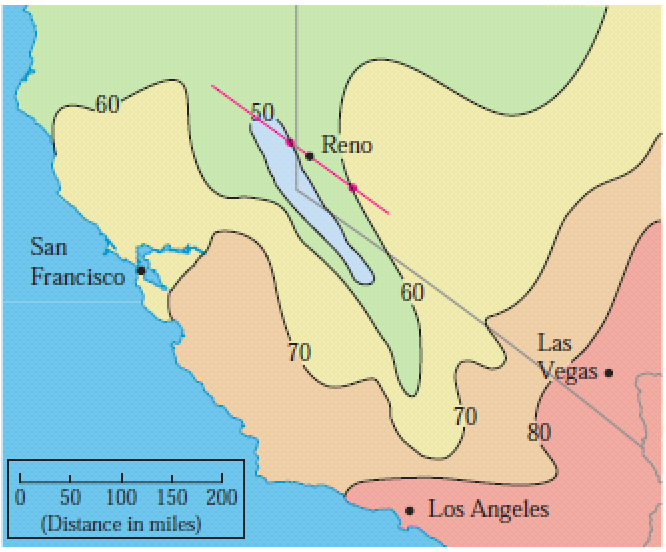
$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

EXAMPLE 3 If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and $$\nabla f(0, 1) = \langle 2, 0 \rangle$$

Directional Derivatives



Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

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$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

$\boxed{2}$ **Definition** The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

$\boxed{3}$ **Theorem** If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

SOLUTION Formula 6 gives

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y) \cos\frac{\pi}{6} + f_y(x, y) \sin\frac{\pi}{6} \\ &= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\tfrac{1}{2} \\ &= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right] \end{aligned}$$

Therefore

$$D_{\mathbf{u}} f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}$$

$$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$$

V EXAMPLE 4 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3\,\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\,\mathbf{i} + 8\,\mathbf{j}$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}$$

Therefore, by Equation 9, we have

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\,\mathbf{i} + 8\,\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

V EXAMPLE 5 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION

(a) The gradient of f is

$$\nabla f(x, y, z) = \langle f_x(x, y, z),\, f_y(x, y, z),\, f_z(x, y, z)\rangle$$

$$= \langle \sin yz,\, xz \cos yz,\, xy \cos yz\rangle$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3\rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\,\mathbf{i} + \frac{2}{\sqrt{6}}\,\mathbf{j} - \frac{1}{\sqrt{6}}\,\mathbf{k}$$

Therefore Equation 14 gives

$$D_{\mathbf{u}} f(1, 3, 0) = \nabla f(1, 3, 0) \cdot \mathbf{u}$$

$$= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\,\mathbf{i} + \frac{2}{\sqrt{6}}\,\mathbf{j} - \frac{1}{\sqrt{6}}\,\mathbf{k}\right)$$

$$= 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}}$$

Maximizing the Directional Derivative

What Does the Gradient Tell Us?

The fact that $f_{\vec{u}} = \operatorname{grad} f \cdot \vec{u}$ enables us to see what the gradient vector represents. Suppose θ is the angle between the vectors $\operatorname{grad} f$ and $\vec{u}$. At the point (a, b), we have

$$f_{\vec{u}} = \operatorname{grad} f \cdot \vec{u} = \|\operatorname{grad} f\| \underbrace{\|\vec{u}\|}_{1} \cos\theta = \|\operatorname{grad} f\| \cos\theta.$$

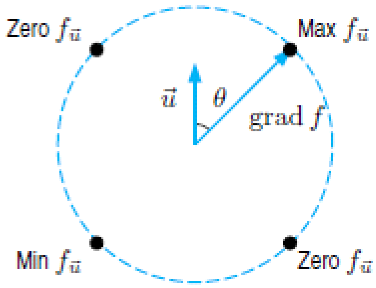

Figure 14.31: Values of the directional derivative at different angles to the gradient

> ### Geometric Properties of the Gradient Vector in the Plane
>
> If f is a differentiable function at the point (a, b) and $\operatorname{grad} f(a, b) \neq \vec{0}$, then:
>
> - The direction of $\operatorname{grad} f(a, b)$ is
>   - Perpendicular[1] to the contour of f through (a, b);
>   - In the direction of the maximum rate of increase of f.
> - The magnitude of the gradient vector, $\|\operatorname{grad} f\|$, is
>   - The maximum rate of change of f at that point;
>   - Large when the contours are close together and small when they are far apart.

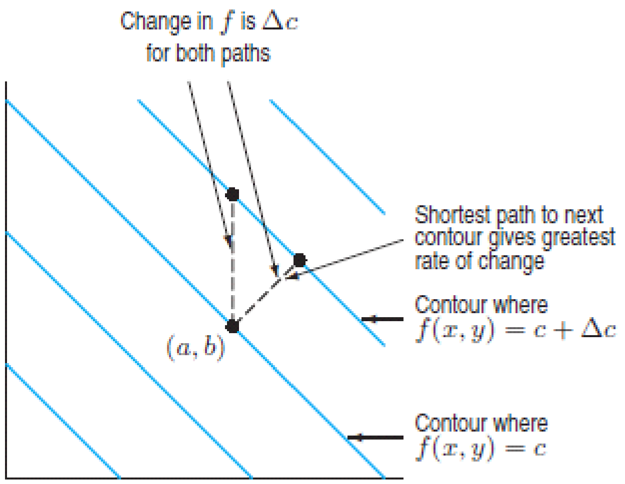

Figure 14.32: Close-up view of the contours around (a, b), showing the gradient is perpendicular to the contours

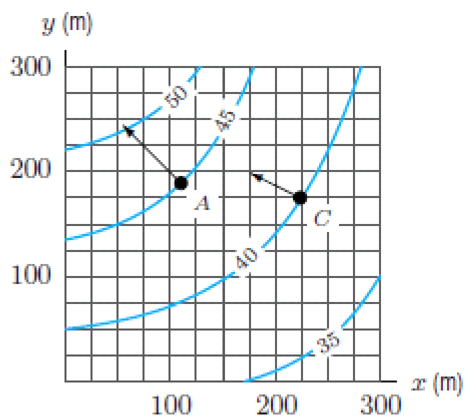

Figure 14.33: A temperature map showing directions and relative magnitudes of two gradient vectors

[1]This assumes that the same scale is used on both axes.