Optimization

Wednesday, 8 January 2025 4:29 pm

Optimization

The dictionary defines *optimum* as "the best or most favorable degree, quantity, number, etc." In mathematics, we optimize by finding the maximum or minimum of a function. Applications in business are to minimize costs or to maximize profits. In this chapter, we describe methods that usually find the point(s) where a function, $f(x, y, z, \ldots)$, has a minimum value. We find maxima by locating the points where the negative of the function is a minimum.

Contents of This Chapter

7.1 Finding the Minimum of $y = f(x)$

Begins by pointing out when getting the minimum from $f'(x) = 0$ has problems. A simple search method can be used, but this is less efficient than methods that narrow the interval that encloses the minimum. Once several values for y at some x-values have been computed, interpolation can locate the minimum with less computational cost. Computer algebra systems and spreadsheets can automate the solution.

7.2 Minimizing a Function of Several Variables

Compares the analytical method of setting partial derivatives to zero and solving the resulting system with numerical procedures. These include graphical techniques and searching procedures. A method called *steepest descent* does the searching along lines on which the function decreases most rapidly, but, for some problems, this is less efficient than another searching procedure called *conjugate gradient*. Newton's method can be adapted to locating a minimum.

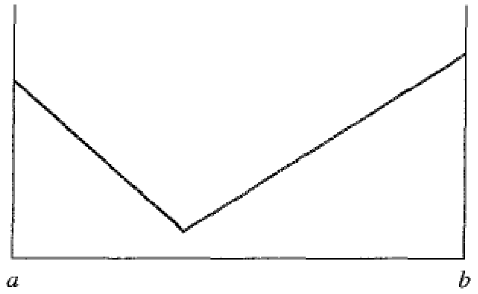
7.3 Linear Programming

Describes a widely used technique in business applications. This applies when the minimum of a linear function is constrained to lie on the boundaries of a region defined by linear relations. The *simplex method* is most often used to solve these problems, and this can determine the effects of changes in the parameters. Again, computer algebra systems and spreadsheets have facilities for doing this.

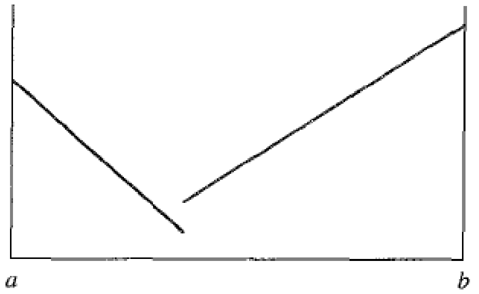
7.4 Nonlinear Programming

Is a more difficult problem than one with a linear function subject to linear constraints. A number of ways to solve such problems are discussed.

The Classical Method $-f'(x) = 0$



(a)



(b)

Searching for the Minimum

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EXAMPLE 7.1 Find the minimum on $[-3, 1]$ of $f(x) = e^x + 2 - \cos(x)$. Use the back-and-forth method. Begin from $x = -3$ and move toward b $(b = 1)$ with $\Delta x = (b - a)/4 = 1$. When the next function value increases, reverse the direction with Δx equal to 1/4 of the previous. Repeat this until $\Delta x < 0.001$.

The successive values are:
At $x = a = -3, f(x)$ is 3.039780, we now begin the search.
With $h = 1$,

$x = -2,\qquad f(x) = 2.551482$
$x = -1,\qquad f(x) = 1.827577$
$x = 0,\qquad f(x) = 2$

We reverse now, $h = -0.25$,

$x = -0.25,\qquad f(x) = 1.809888$
$x = -0.5,\qquad f(x) = 1.728948$
$x = -0.75,\qquad f(x) = 1.740678$

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We reverse now, $h = 0.0625$,

$x = -0.6875, \quad f(x) = 1.729997$
$x = -0.625, \quad f(x) = 1.724298$
$x = -0.5625, \quad f(x) = 1.723858$
$x = -0.5, \quad f(x) = 1.728948$

We reverse now, $h = -0.015625$,

$x = -0.515625, \quad f(x) = 1.727143$
$x = -0.53125, \quad f(x) = 1.725695$
$x = -0.546875, \quad f(x) = 1.724602$
$x = -0.5625, \quad f(x) = 1.723858$
$x = -0.578125, \quad f(x) = 1.723460$
$x = -0.59375, \quad f(x) = 1.723404$
$x = -0.609375, \quad f(x) = 1.723685$

We reverse now, $h = 0.00390625$,

$x = -0.6054688, \quad f(x) = 1.723583$
$x = -0.6015625, \quad f(x) = 1.723502$
$x = -0.5976563, \quad f(x) = 1.723443$
$x = -0.59375, \quad f(x) = 1.723404$
$x = -0.5898438, \quad f(x) = 1.723386$
$x = -0.5859375, \quad f(x) = 1.723390$

We reverse now, $h = -0.0009765625$,

$x = -0.5869141, \quad f(x) = 1.723387$
$x = -0.5864258, \quad f(x) = 1.7233882$

Tolerance of 0.001 is met.

We can see several objections to this crude method. We have to compute an extra function value before we know that the direction is to be reversed. Further, some x-values are duplicated after a reversal but we still recompute $f(x)$. (Keeping track of the function value would be very complicated.) We seek an improvement.

One way to improve the efficiency of this crude method is to use three values that bracket the minimum (at $x = -2, -1, 0$, the first three values in Example 7.1) and fit a quadratic polynomial to them, then find the minimum of that. [When $f(x) = ax^2 + bx + c$, $f'(x)$ is $2ax + b$, and this will be zero at $x = -b/2a$.] The easy way to do this is to form the quadratic polynomial from a difference table and find its minimum point. From these three $x, f(x)$ values, we get an estimate of $x_{\text{min}} = -0.6923658$ and no additional function evaluations are required.

We can continue from here by successively forming quadratics from three points nearest the minimum. We must compute the function value at the new point. Here is the first set:

$$x = -1, \qquad f(x) = 1.827577;$$
$$x = -0.6923658, \qquad f(x) = 1.730653;$$
$$x = 0, \qquad f(x) = 2.$$

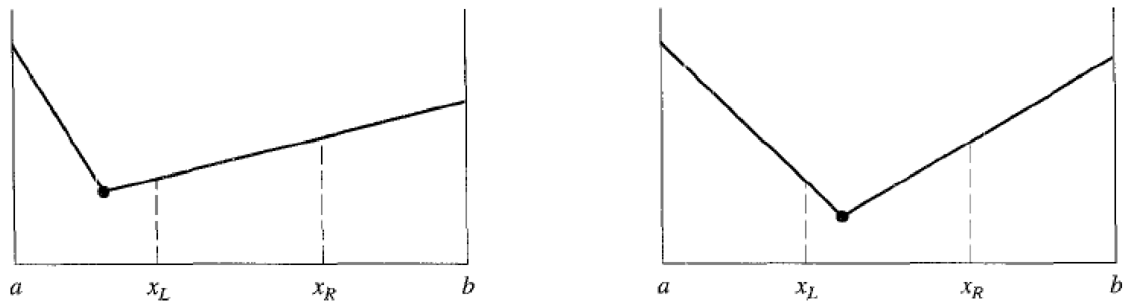
From these points, we find the interpolating quadratic and get its minimum point: $x = -0.6224442$. If we continue, we find the next two estimates of the x-value at the minimum to be

$$-0.5975463 \qquad \text{and} \qquad -0.5878655,$$

which is within 0.0007 of the true $x_{\min}$ of -0.588532744. We have achieved this with only six evaluations of the function rather than the 23 used in the above simple search.

Narrowing the Interval

When we are given a function that has a single minimum point within the interval $[a, b]$, we can say that points a and b bracket or enclose the minimum point. There are ways to narrow that interval and the method known as the *golden section search* is one of the most popular.



(a) **(b)**

Using the Golden Mean to Find a Minimum

Another name for the golden section is the *golden mean.* We begin the search by computing the x-values for the two intermediate points:

$$x_L = a + (1 - r) * (b - a), \quad x_R = a + r * (b - a).$$

(We could have written $x_L = a + r^2 * (b - a)$: $r^2 = 1 - r$.) These points are symmetric about the midpoint of $[a, b]$, $(b - a)/2$. One is 0.381966 times $(b - a)$ from a; the other is 0.618034 times $(b - a)$ from a. Next, we compute the function values at these intermediate points, $F_L = f(x_L)$ and $F_R = f(x_R)$.

We compare the two function values and find a new smaller interval in which the minimum lies:

$$\text{If } F_L < F_R, \text{ then the interval is } [a, x_R] \text{ else it is } [x_L, b].$$

EXAMPLE 7.2. Repeat Example 7.1, but now use the golden section search. The function is $f(x) = e^x + 2 - \cos(x)$ and the minimum is within $[-3, 1]$. Continue the search until the intermediate points are within 0.001 of each other. (The correct answer to nine digits is at $x = -0.588532744$.)

The results from a program are

Starting values

X_L	X_R	F_L	F_R	Interval	Width
−1.4721	−0.5279	2.1309	1.7260	[−3.0000, 1.0000]	4.0000
−0.5279	0.0557	1.7260	2.0589	[−1.4721, 1.0000]	2.4721
−0.8885	−0.5279	1.7807	1.7260	[−1.4721, 0.0557]	1.5279
−0.5279	−0.3050	1.7260	1.7833	[−0.8885, 0.0557]	0.9443
−0.6656	−0.5279	1.7274	1.7260	[−0.8885, −0.3050]	0.5836
−0.5279	−0.4427	1.7260	1.7387	[−0.6656, −0.3050]	0.3607
−0.5805	−0.5279	1.7234	1.7260	[−0.6656, −0.4427]	0.2229
−0.6130	−0.5805	1.7238	1.7234	[−0.6656, −0.5279]	0.1378
−0.5805	−0.5604	1.7234	1.7239	[−0.6130, −0.5279]	0.0851
−0.5929	−0.5805	1.7234	1.7234	[−0.6130, −0.5604]	0.0526
−0.6006	−0.5929	1.7235	1.7234	[−0.6130, −0.5805]	0.0325
−0.5929	−0.5882	1.7234	1.7234	[−0.6006, −0.5805]	0.0201
−0.5882	−0.5852	1.7234	1.7234	[−0.5929, −0.5805]	0.0124
−0.5900	−0.5882	1.7234	1.7234	[−0.5929, −0.5852]	0.0077
−0.5882	−0.5870	1.7234	1.7234	[−0.5900, −0.5852]	0.0047
−0.5889	−0.5882	1.7234	1.7234	[−0.5900, −0.5870]	0.0029
−0.5893	−0.5889	1.7234	1.7234	[−0.5900, −0.5882]	0.0018

Gradient Descent Algorithm — a deep dive | by Robert Kwiatkowski | Towards Data Science