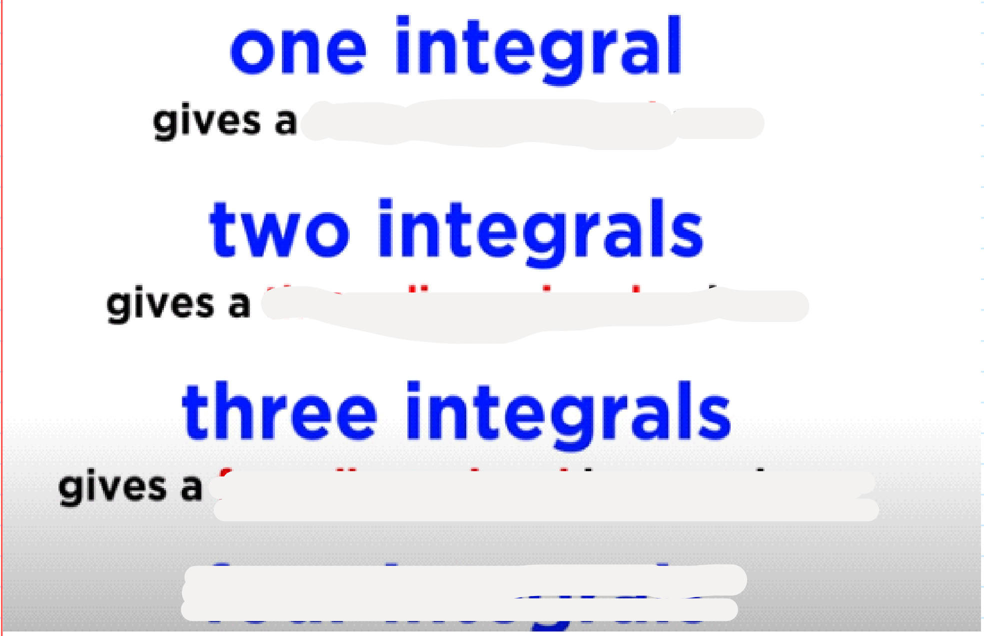
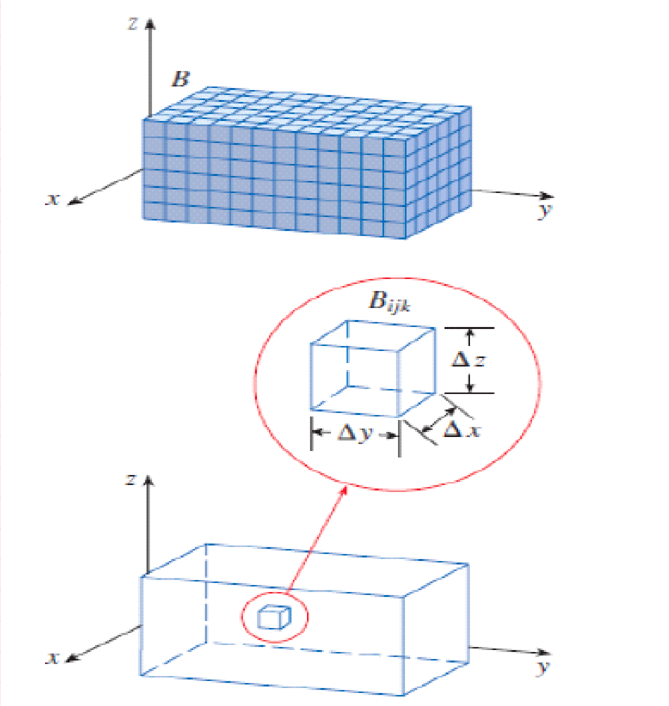
Triple Integral

Thursday, 27 June 2024 10:54 pm





$\Delta V = \Delta x \Delta y \Delta z,$

$$\sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk})\, \Delta V.$$

$$\int_W f\, dV = \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk})\, \Delta x\, \Delta y\, \Delta z.$$

Triple integral as an iterated integral

$$\int_W f\,dV = \int_p^q \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz,$$

where y and z are treated as constants in the innermost (dx) integral, and z is treated as a constant in the middle (dy) integral. Other orders of integration are possible.

2. $h(x,y,z) = ax + by + cz$, W is the rectangular box $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 2$.

Example 1 A cube C has sides of length 4 cm and is made of a material of variable density. If one corner is at the origin and the adjacent corners are on the positive x, y, and z axes, then the density at the point (x,y,z) is $\delta(x,y,z) = 1 + xyz$ gm/cm^3. Find the mass of the cube.

Limits on Triple Integrals

- The limits for the outer integral are constants.
- The limits for the middle integral can involve only one variable (that in the outer integral).
- The limits for the inner integral can involve two variables (those on the two outer integrals).

Example: Set up an iterated integral to compute mass of a solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, if the density is given by $\delta(x, y, z) = z$.

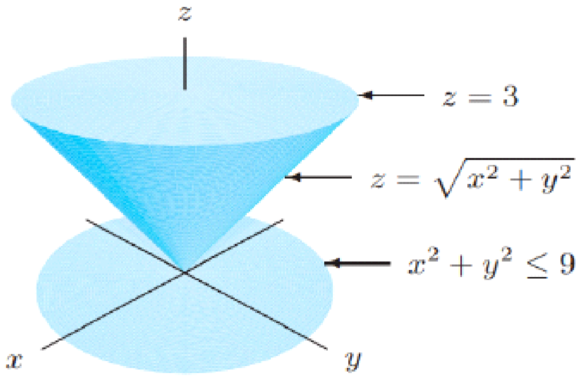


Figure 16.25

Section 16.5: Integration in Cylindrical and Spherical Coordinates

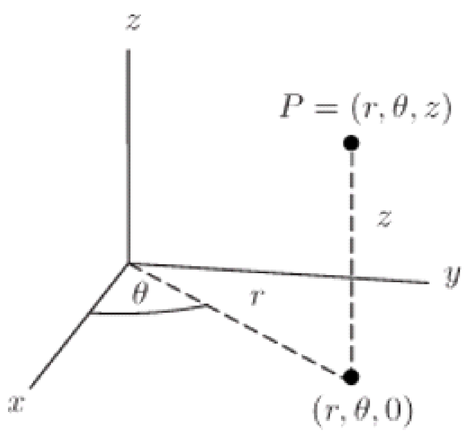
Integration in Cylindrical Coordinates

The cylindrical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are obtained by representing the x and y coordinates using polar coordinates (or potentially the y and z coordinates or x and z coordinates) and letting the third coordinate remain unchanged.

RELATION BETWEEN CARTESIAN AND CYLINDRICAL COORDINATES: Each point in $\mathbb{R}^3$ is represented using $0 \le r < \infty$, $0 \le \theta \le 2\pi$, $-\infty < z < \infty$.

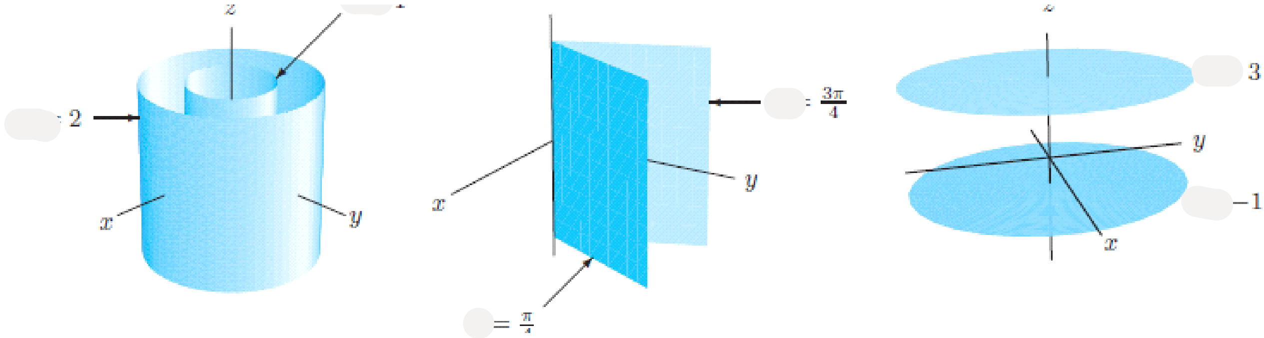
$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = z.$$

As with polar coordinates in the plane, note that $x^2 + y^2 = r^2$.



Notice that we can now interpret r as the distance from the point (x, y, z) to the z axis, while the interpretation of θ and z remain unchanged.

Question: What are the surfaces obtained by setting r, θ, and z equal to a constant?



Example 1 Describe in cylindrical coordinates a wedge of cheese cut from a cylinder 4 cm high and 6 cm in radius; this wedge subtends an angle of $\pi/6$ at the center. (See Figure 16.41.)

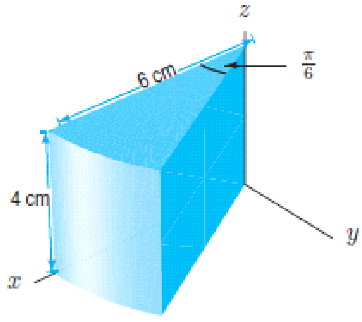
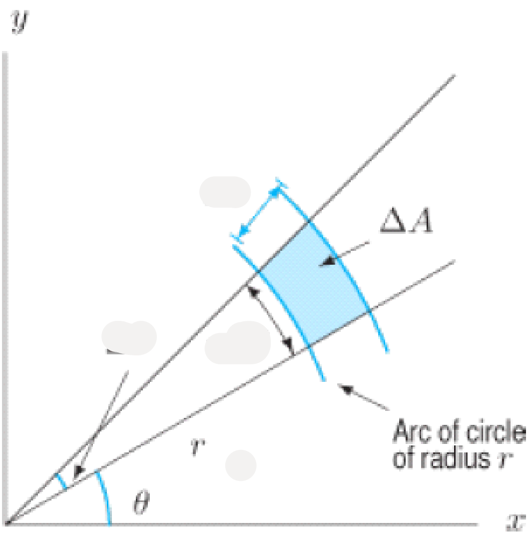


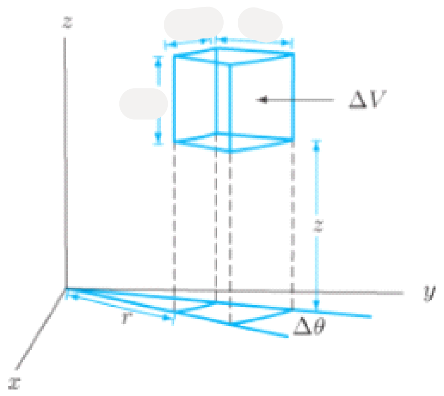
Figure 16.41: A wedge of cheese

Example 2 Find the mass of the wedge of cheese in Example 1, if its density is 1.2 grams/cm^3.



What is dV in Cylindrical Coordinates?

Recall that when integrating in polar coordinates, we set $dA = r\,dr\,d\theta$. When viewing a small piece of volume, ΔV, in cylindrical coordinates, we will see that the correct form for dV is rather intuitive based on this.



It is clear from this image that we should have $\Delta V \approx r\,\Delta r\,\Delta\theta\,\Delta z$. This leads us to the following conclusion:

When computing integrals in cylindrical coordinates, put $dV = r\,dr\,d\theta\,dz$. Other orders of integration are possible.

Examples:

1. Evaluate the triple integral in cylindrical coordinates: $f(x, y, z) = \sin(x^2 + y^2)$, W is the solid cylinder with height 4 with base of radius 1 centered on the z-axis at $z = -1$.

Spherical Coordinates

The spherical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are the analog of polar coordinates in $\mathbb{R}^2$. We define $\rho = \sqrt{x^2 + y^2 + z^2}$ to be the distance from the origin to (x, y, z), θ is defined as it was in polar coordinates, and ϕ is defined as the angle between the positive z-axis and the line connecting the origin to the point (x, y, z).

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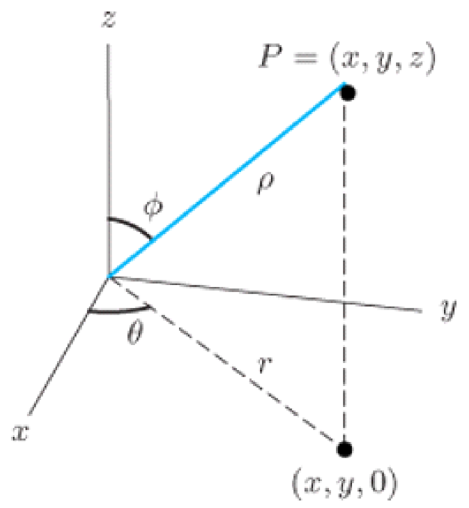

From the above figure, we can see that $r = \rho \sin\phi$, and $z = \rho\cos\phi$, so using the relationship between Cartesian coordinates (x, y, z) and cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, we arrive at the following:

RELATIONSHIP BETWEEN CARTESIAN AND SPHERICAL COORDINATES: Each point in $\mathbb{R}^3$ is represented using $0 \leq \rho < \infty$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$.

$$x = \rho \sin\phi \cos\theta,$$
$$y = \rho \sin\phi \sin\theta,$$
$$z = \rho \cos\phi.$$

Also, $x^2 + y^2 + z^2 = \rho^2$.

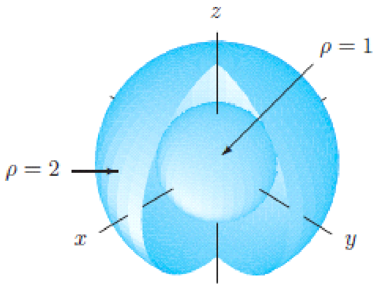

Figure 16.45: The surfaces $\rho = 1$ and $\rho = 2$

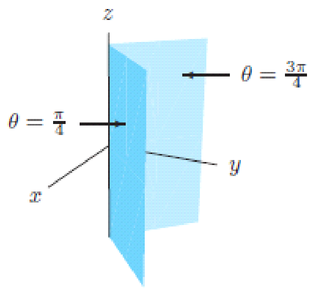

Figure 16.46: The surfaces $\theta = \pi/4$ and $\theta = 3\pi/4$

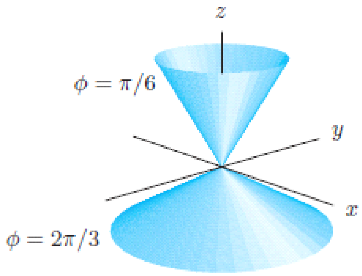

Figure 16.47: The surfaces $\phi = \pi/6$ and $\phi = 2\pi/3$

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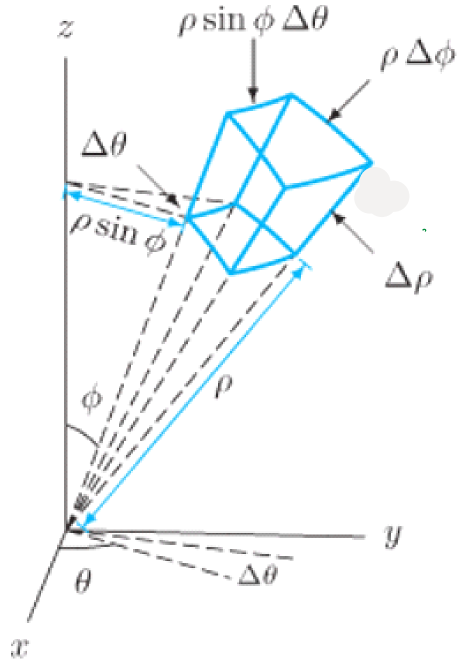
Question: What surfaces are obtained by setting ρ, θ, and ϕ equal to a constant?

What is dV is Spherical Coordinates?

Consider the following diagram:



We can see that the small volume ΔV is approximated by $\Delta V \approx \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$. This brings us to the conclusion about the volume element dV in spherical coordinates:

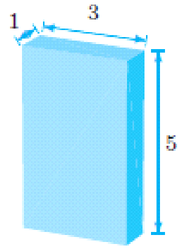
When computing integrals in spherical coordinates, put $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. Other orders of integration are possible.

Example 4 Use spherical coordinates to derive the formula for the volume of a ball of radius a.

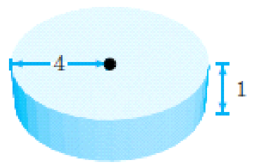
Evaluate the triple integral in spherical coordinates. $f(x, y, z) = 1/(x^2 + y^2 + z^2)^{1/2}$ over the bottom half of a sphere of radius 5 centered at the origin.

For Exercises 12–18, choose coordinates and set up a triple integral, including limits of integration, for a density function f over the region.

12.



13.



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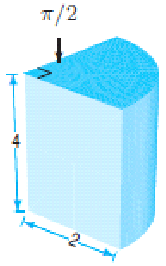

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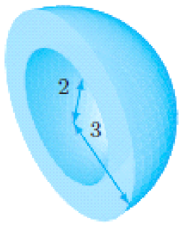
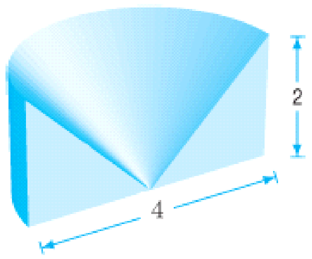
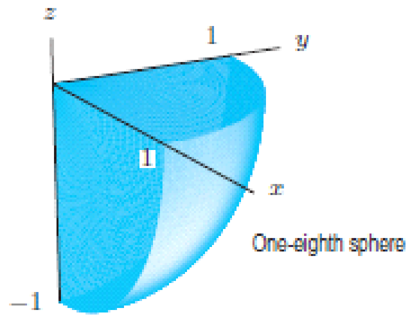




(c)

(a) Cartesian (b) Cylindrical (c) Spherical

24.



1. Match the equations in (a)–(f) with one of the surfaces in (I)–(VII).

(a) $x = 5$ **(b)** $x^2 + z^2 = 7$ **(c)** $\rho = 5$
(d) $z = 1$ **(e)** $r = 3$ **(f)** $\theta = 2\pi$

(I) Cylinder, centered on x-axis.
(II) Cylinder, centered on y-axis.
(III) Cylinder, centered on z-axis.
(IV) Plane, perpendicular to the x-axis.
(V) Plane, perpendicular to the y-axis.
(VI) Plane, perpendicular to the z-axis.
(VII) Sphere.

In Exercises 2–7, find an equation for the surface.

2. The vertical plane $y = x$ in cylindrical coordinates.

3. The top half of the sphere $x^2 + y^2 + z^2 = 1$ in cylindrical coordinates.

4. The cone $z = \sqrt{x^2 + y^2}$ in cylindrical coordinates.

5. The cone $z = \sqrt{x^2 + y^2}$ in spherical coordinates.

6. The plane $z = 10$ in spherical coordinates.

7. The plane $z = 4$ in spherical coordinates.