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Self-contact of a flexible loop under uniform hydrostatic pressure

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ABSTRACT

Critical values of the pressure leading to self-contact in a flexible and inextensible loop subjected to uniform hydrostatic pressure have been predicted using variants and approximations of the harmonic balance method. Moreover, a modified perturbation method has been employed that has shown some advantages over conventional perturbation approach to the problem. A comprehensive comparison is made between the results we obtained using various techniques and the existing reported results for different pressure loadings and rotational symmetries.

1. Introduction

Finding equilibrium shapes of a flexible inextensible elastic ring subjected to a uniform hydrostatic pressure dates back to the Euler's work (Euler, 1744), where he–following the Bernoulli's work–developed a general framework for the calculus of variations. The problem, also known as the Euler's elastica continued to attract researchers' attention for over decades. Generally, the problem is to minimize the bending energy of an elastic ring subject to various constraints. The problem has been extensively studied to find the critical values of the pressure p that lead to buckled shapes of the ring, for various modes of rotational symmetries. More importantly, estimating the critical value of the pressure p at which a closed elastic loop comes to a self-contact has been the subject matter of many research papers. Recently experiments Hazel (Hazel and Mullin, 2016) have been performed to view such equilibrium shapes, where a sudden pressure has been applied on the loops that are confined to stay in polygonal containers.

Much of the motivation for the problem in hand comes from its applications in various biological phenomena such as understanding the shapes of the vesicles and cell membrane, etc. It is explained in Canham (1970) that the minimized bending energy of the red blood cell membrane is responsible for its biconcave shape. Also, Deborah (Deborah et al., 1997) has shown potential application of the underlying problem in illustrating the deformation of the vesicle due to microtubles polymerization within the cell. Usefulness of the problem has also been exhibited in computer vision and image processing as in David (Mumford, 1994). The Euler's Elastica has also contributed to the development of the methods for restoring distorted images, for example see Chan et al. (2002).

Carrier (1974) has explored various related problems including but not limited to the stability of the large buckled pre-stressed elastic ring and dynamic stability of the ring when uniform pressure is applied periodically in time. Following the Carrier's work (Carrier, 1974), Tadjbakhs (Tadjbakhs and Odeh, 1997) derived the governing equations for the curvature of the ring using energy minimum principle and force, moment balance method. Precisely, Tadjbakhs (Tadjbakhs and Odeh, 1997) showed existence of a buckled solution for dimensionless pressure $p > 3$ and solved the problem with a combination of a perturbation method and a numerical approach. Though, Levy. (1884) was the first to express solutions in terms of elliptic functions, Watanabe (Watanabe and Takagi, 2008) provided various ranges on p for the problem to have a unique minimizer and expressed the curvature function explicitly using elliptic functions. Furthermore, he showed that for a certain pressure bifurcation takes place that produces a non circular minimizer of the energy functional. Zhang (1997), minimized the Helfrich energy to find the equilibrium shapes of the axisymmetric vesicles made of bilayers membrane, subject to the volume and area constraints of the vesicle. Following (Zhang, 1997), Shravan (Veerapaneni et al., 2009) determined stability of the equilibrium shapes numerically. However, they did not make a systematic study to find the pressure p that leads to self-contact of the loop. Djondjorov (Djondjorov et al., 2011) provided explicit expressions for the coordinates of the resulting equilibrium curve for a given pressure and reported the contact pressure for various shapes. The analytic study of the 3D version the Euler-Plateau problem and its generalization has been done in Fried (Chen and Fried, 2013; Biria and Fried, 2014). The author Majid (2020) has recently performed stability and bifurcation analysis of the Euler-Plateau problem numerically.

This work is motivated by Baisheng (Wu et al., 2007). In Baisheng

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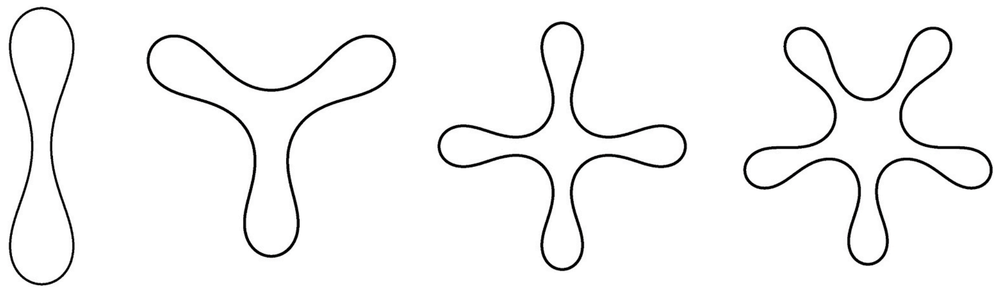

Fig. 1. Plot of curves for $p = 3.06, 3.57, 4.63, 5.247$ using third order HB method for $n = 2, 3, 4, 5$ respectively.

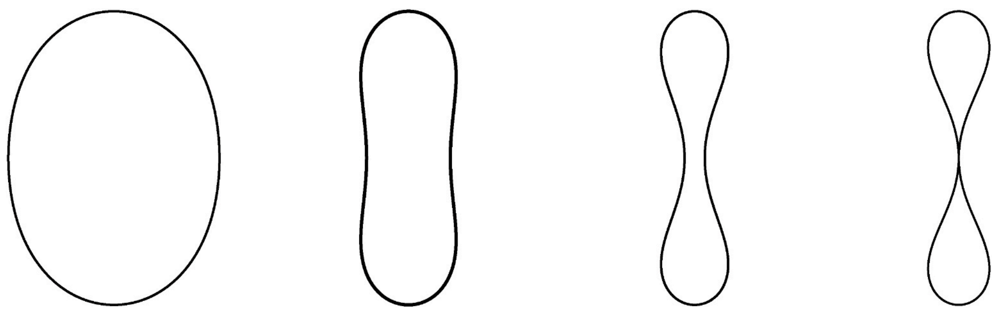

Fig. 2. Equilibrium loops for rotational symmetry $n = 2$ for $p = 3.06, 3.57, 4.63$ and $p_c = 5.247$ using 3rd order HB method.

(Wu et al., 2007), a harmonic balance method is used to find the buckled equilibrium shapes of the elastic ring for various pressure loadings considering only one order of rotational symmetry, namely $n = 2$. In Baisheng (Wu et al., 2007), a comparison was made between the curvatures obtained by the 2nd order harmonic balance method and a numerical approach based on the shooting method. However, they did not show results for $n > 2$ nor did they report if the contact pressure could be predicted accurately even for $n = 2$. In this work, we extend the harmonic balance method for any n mode rotational symmetry and report that the 2nd harmonic balance approximation is not reliable for large post buckling that gives rise to self-contact or self-intersection. Additionally, we use a non-trivial expression of the third order harmonic balance method and report the pressures p corresponding to contact points for various order of rotational symmetries. Our results are in good agreement with already established results in the literature such as Djondjorov (Djondjorov et al., 2011), Flaherty (Flaherty et al., 1972) etc. Contact points reported for $n = 2$ and $n = 3$ in Figs. 1 and 2 of Tadjbakhs (Tadjbakhs and Odeh, 1997) differ significantly from more recent results found in Djondjorov (Djondjorov et al., 2011), Flaherty (Flaherty et al., 1972), etc. We propose a higher order perturbation approach and a modified version of it which provide better estimates of the contact pressure comparable with established results such as Djondjorov (Djondjorov et al., 2011), Flaherty (Flaherty et al., 1972). The series approximation method we use is a natural choice for the periodic nonlinear systems Minoru (Urabe, 1965). Moreover, series solutions are more physical therefore are more appropriate for the bifurcation and stability analysis such as in Tadjbakhs (Tadjbakhs and Odeh, 1997) and Dickey (Dickey and Roseman, 1993). Though the series expressions we work with are long, they are only polynomials and could be handled easily with any symbolic package. By using these series expressions one could avoid to work with intractable elliptic functions used in Djondjorov (Djondjorov et al., 2011). In addition, in the series approximation methods one does not require to construct initial guess, which is a very important concern in purely numerical methods used in Flaherty (Flaherty et al., 1972).

Rest of the paper is organized as follows. In section 2, we discuss mathematical model. Details of the harmonic balance method and perturbation approach are given in section 3. Equilibrium shapes, including self-contact and self-intersection are explained in section 4. Finally, in section 5 we summarize our findings.

2. Model

We skip the derivation of the model as it has been discussed and reproduced in a number of places such as Djondjorov (Djondjorov et al., 2011), Tadjbakhs (Tadjbakhs and Odeh, 1997), Adams (2008) and Ivailo (Mladenov and Hadzhilazova, 2013). Generally, two standard approaches are followed: bending energy minimization subject to the inextensibilty constraint with external loading, force and moment balance method. More details on these methods in the context of elasticity can be found in Antman (1968). Denoting the external pressure by p, Lagrnage multiplier to enforce the inextensibility of the loop by μ and integrating the corresponding Euler Lagrange equation, we arrive at the dimensionless shape equation as in Tadjbakhs (Tadjbakhs and Odeh, 1997),

$$v^{''} + \mu v - \beta + \frac{3}{2}v^2 + \frac{1}{2}v^3 = 0. \tag{1}$$

Where, $\mu = \frac{3}{2} - c$ and $\beta = c + p - \frac{1}{2}$ and $v(s)$ is the difference of the curvatures of the undeformed and the deformed ring for $s \in [0, 2\pi]$, and c is the integral constant. The closure conditions are

$$v'(0) = v'(\pi / n) = 0, \tag{2}$$

and

$$\int_0^{\pi/n} v(s)\, ds = 0. \tag{3}$$

After determining the curvature $v(s)$ for the range $0 \le s \le \frac{\pi}{n}$, the curvature for the entire curve can be produced using the following periodicity conditions.

$$v(-s) = v(s), \quad v\left(s - \frac{2\pi}{n}\right) = v(s). \tag{4}$$

Furthermore, the parametric coordinates $x_1(s)$ and $x_2(s)$ of the curve can be obtained using the quadratures

$$x_1(s) = \int_0^s \cos(\theta(\varepsilon))\, d\varepsilon, x_2(s) = \int_0^s \sin(\theta(\varepsilon))\, d\varepsilon, \qquad 0 \le s \le 2\pi. \tag{5}$$

In equation (5), $\theta(s)$ is the angle between tangent and the x_1-axis which is defined as

$$\theta(s) = s + \int_0^s v(\varepsilon) d\varepsilon, \quad 0 \le s \le 2\pi.$$

3. Methods

In order to find the periodic solution of equations (1)–(3), we use harmonic balance method which is a Glarekin's approximation as in Baisheng (Wu et al., 2007). In the harmonic balance approach, solutions are expressed in the form of Fourier's series and the coefficients are obtained by satisfying (1). It has been shown Minoru (Urabe, 1965) that for a nonlinear periodic system Glarekin's approximation exists and it converges to the exact solution uniformly. Furthermore, based on (Urabe, 1965) many numerical methods have been developed to find the solutions using Glarekin's approximations in Minoru (UrabeAllen, 1966). Solution of equations (1)–(3) is $v(s)$ which is a periodic function of period π and $v(s)$ is also an even function of s, i.e,

$$v(s-\pi) = v(s) \ \text{ and } \ v(-s) = v(s). \tag{6}$$

We express solution $v(s)$ in the Fourier's series

$$v(s) = \sum_{k=0}^{\infty} h_k \cos(nks).$$

The integer n represents mode of the rotational symmetry of the curve whose curvature $v(s)$ is given by (5). Satisfying (3) requires that

$$v(s) = \sum_{k=1}^{\infty} h_k \cos(nks). \tag{7}$$

Considering $A = v(0)$ (difference of the curvatures at $s = 0$ of the undeformed and the deformed ring) and following Baisheng (Wu et al., 2007), the zeroth order harmonic balance approximations to $v(s)$, μ and β for n mode rotational symmetry can be generalized in term of A as

$$v_0^n(s) = A\cos(ns),\ \mu_0^n(A) = \frac{1}{8}\left(8n^2 - 3A^2\right),\ \beta_0^n(A) = \frac{3}{4}A^2. \tag{8}$$

Denoting by m the order of harmonic balance and using Δ as an incremental operator, we can write a general analytic iterative process to find the subsequent approximations to $v(s)$, μ and β as

$$v_m^n(s) = v_{m-1}^n(s) + \Delta v_{m-1}^n(s), \quad \mu_m^n = \mu_{m-1}^n + \Delta\mu_{m-1}^n, \quad \beta_m^n = \beta_{m-1}^n + \Delta\beta_{m-1}^n. \tag{9}$$

After substituting (9) in equations (1)–(3), linearization is performed considering the variables $\Delta v_{m-1}^n(s)$, $\Delta\mu_{m-1}^n$ and $\Delta\beta_{m-1}^n$ to be small. The resulting differential equation is then solved for $\Delta v_{m-1}^n(s)$, $\Delta\mu_{m-1}^n$, $\Delta\beta_{m-1}^n$ using harmonic balance method. Using this procedure we obtain the following first order harmonic balance approximations.

$$v_1^n = \frac{2A^2(\cos(ns) - \cos(2ns))}{A^2 - 4A - 8n^2} + A\cos(ns), \tag{10}$$

$$\mu_1^n = \frac{1}{8}\left(8n^2 - 3A^2\right) - \frac{3(A-2)A^2}{2\left(A^2 - 4A - 8n^2\right)}, \tag{11}$$

$$\beta_1^n = \frac{3A^2}{4} - \frac{3(A-4)A^3}{4\left(A^2 - 4A - 8n^2\right)}. \tag{12}$$

All higher order generalized approximations required extensive calculations producing long expressions. It is impractical to write full expressions here. So, we reproduce only first few terms of the second and third approximations. The second approximation is given by

$$v_2^n = v_1^n + A\cos\left(ns\right) + \frac{6A^2(\cos(ns) - \cos(2ns))}{-12A + 3A^2 - 32n^2 + 8n^n} - \left(\left(\left(\frac{648A^5}{\left(-12A + 3A^2 + 8(-4n^2 + n^n)\right)^3} - \ldots, \tag{13}$$

$$\mu_2^n = \left(24A^{17} - 3A^{18} + 3489660928An^{16} + 1073741824n^{18} + 24A^{16}\left(15n^2 - 13\right) - \ldots, \tag{14}$$

$$\beta_2^n = \left(3A^2\left(-38A^{17} + 3A^{18} - 3489660928An^{16} - 1073741824n^{18} + A^{16}\left(160 - 176n^2\right) + \ldots \tag{15}$$

Finally, we have calculated the third order approximation of the harmonic balance method, which is quite large but easy to handle using Mathematica and it gives much better results. The curvature function using the third order harmonic balance approximation is give by

$$v_3^n = A\cos\left(ns\right) + \frac{2A^2(\cos(ns) - \cos(2ns))}{A^2 - 4A - 8n^2} - \left(\left(\left(\frac{84A^5}{\left(A^2 - 4A - 8n^2\right)^3} - \frac{96A^4}{\left(A^2 - 4A - 8n^2\right)^3} - \ldots, \tag{16}$$

In order to make comparisons with the existing results such as Djondjorov (Djondjorov et al., 2011), Tadjbakhs (Tadjbakhs and Odeh, 1997) and Flaherty (Flaherty et al., 1972), we express pressure p in terms of initial approximation A as

$$p = -1 - \left(-194728877988249\text{60}A^{29} + 11476736730464256A^{28} - 3211742184210432A^{27} +, . \tag{17}$$

All calculations were performed in Mathematica (Wolfram Research Inc, 2018). The Mathematica notebooks containing complete mathematical expressions used for this work are available from the authors on request.

In Tadjbakhs (Tadjbakhs and Odeh, 1997), a regular perturbation method (Murdock, 1999) with the following expressions has been used in combination with Newton's method to obtain the equilibrium shapes.

Table 1
Values of the contact pressure p_c for various order of HB.

n	2	3	4	5
HB zeroth approx	5.05	15.84	32.03	44.351
HB first approx	5.64	24.57	59.12	110.76
HB second approx	5.247	21.44	51.84	95.01
HB third approx	5.247	21.65	51.843	97.455

$$v = C_0\cos(ns) + \varepsilon C_1\cos(2ns) + \varepsilon^2 C_2\cos(3ns) + \ldots = \sum_{k=0}\varepsilon^k C_k\cos((n(k+1))s), \tag{18}$$

$$\mu = \mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \ldots = \sum_{k=0}\varepsilon^k\mu_k, \tag{19}$$

$$\beta = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \ldots = \sum_{k=0}\varepsilon^k\beta_k. \tag{20}$$

Considering $C_0 = v(0) = A$, first we find zeroth order approximation and then use it to find the next approximation. Similarly, the first approximation is used to get the second approximation. We expanded the expansions (18)–(20) to third term unlike Tadjbakhs (Tadjbakhs and Odeh, 1997) which uses only first two terms.

Inspired by some clue from the harmonic balance approach we modify equation (18) to the following form

$$\begin{aligned} v &= \varepsilon^0 C_0\cos(2s) + \varepsilon^1 C_1(\cos(4s) - \cos(2s)) \\ &\quad + \varepsilon^2(C_2(\cos(4s) - \cos(2s)) + C_3(\cos(6s) - \cos(4s))). \end{aligned} \tag{21}$$

and use the regular perturbation method with second order accuracy to approximate $v(s)$. This modified approach has given much better approximation to $v(s)$. Associated results are given in the later section.

4. Equilibrium shapes

In this section, we use versions of the harmonic balance approximation discussed in section 3 to obtain equilibrium shapes for various values of the loading parameter p and n mode rotational symmetries. Accuracy of the higher order harmonic balance method is promising as contact point estimates based on it are very close approximations to the existing available results. Form now on harmonic balance shall be abbreviated as HB.

In Fig. 1, we show one equilibrium shape for each $2 \le n \le 5$ for some sample values of $p = 3.06, 3.57, 4.63, 5.247$ respectively. These solutions are obtained using the third order HB approximation.

4.1. Self-contact

Values of the contact pressure reported by Djondjorov (Djondjorov et al., 2011) seem to be more reliable. They have provided explicit generalized analytic expressions based on transcendental and elliptic functions as solutions to the curvature equation (1). Then, these expressions are used to find the pressure values for the contact points, systematically. Similarly, in the form of equations (16) and (17) we have provided higher order generalized expressions to find the curvature for various values of p and n.

In order to find the contact point for each n, it is pragmatic to notice the fact that at least one contact point lies on

$$x_1(s, A) = 0. \tag{22}$$

The least positive value of A–located on the contour plot (22)–is substituted in equation (17) which yields the corresponding pressure p_c, responsible for the self-contact. Table 1 provides values of the contact pressure for $n = 2, 3, 4, 5$ for zeroth, first, second and third order HB approximations.

In Table 2, we compare the results obtained form the third order HB approximation with Djondjorov (Djondjorov et al., 2011), Flaherty

Table 2
Comparison of the contact pressure p_c between the 3rd order HB method and other existing results.

N	2	3	4	5	6	7	8	9
Tadjbakhs (Tadjbakhs and Odeh, 1997)	5.03	19.44	–	–	–	–	–	–
Flaherty (Flaherty et al., 1972)	5.247	21.65	51.84	–	–	–	–	–
Djondjorov (Djondjorov et al., 2011)	5.247	21.65	51.844	97.834	161.077	242.682	343.517	464.276
HB 3rd approax	5.247	21.65	51.843	97.455	160.157	241.965	343.45	463.616

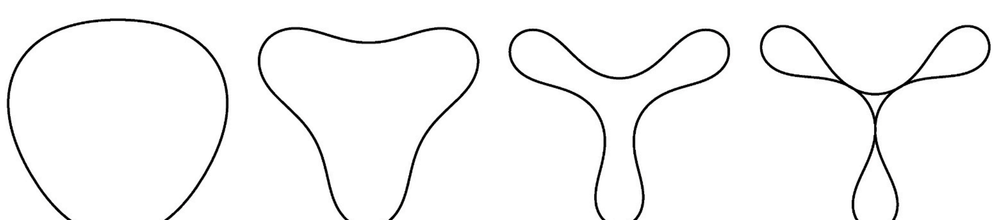

Fig. 3. Equilibrium loops for rotational symmetry $n = 3$ for $p = 8.08, 10.04, 15.78$ and $p_c = 21.65$ using 3rd order HB method.

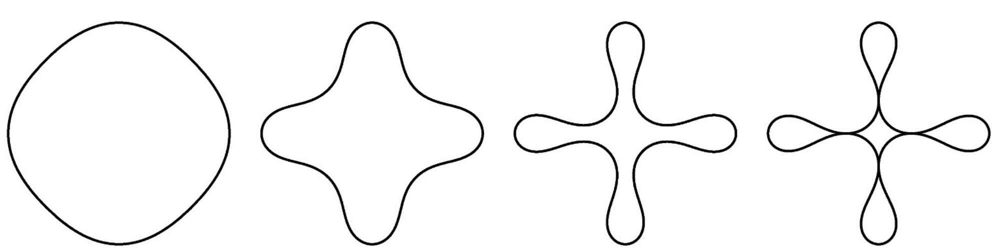

Fig. 4. Equilibrium loops for rotational symmetry $n = 4$ for $p = 15.08, 20.92, 39.49$ and $p_c = 51.84$ using 3rd order HB method.

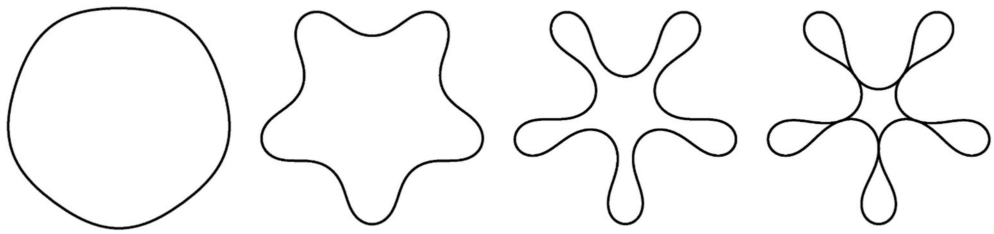

Fig. 5. Equilibrium loops for rotational symmetry $n = 5$ for $p_c = 24.08, 33.76, 72.94$ and $p_c = 97.455$ using 3rd order HB method.

Table 3
Comparison of the contact pressure p_c between 2nd order HBM and perturbation methods.

N	2	3	4	5	6	7	8	9
HB 2nd approx	5.247	21.44	51.84	95.01	154.62	230.742	322.581	438.84
Second order perturbation	5.16	19.60	44.46	81.35	130.3	191.34	266.49	355.65
Modified perturbation	5.247	21.48	50.50	94.39	153.42	229.11	321.53	430

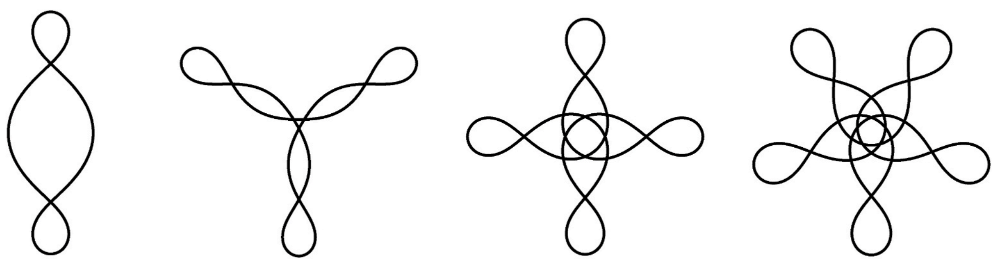

Fig. 6. Self-intersecting loops for $p = 10.34, 70.45, 180, 300$ for $n = 2, 3, 4, 5$ respectively using 3rd order HB method.

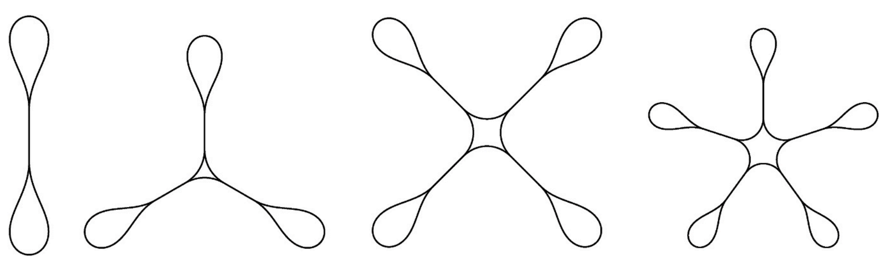

Fig. 7. After restricting self-intersection in Fig. 6, for $p = 10.34, 70.45, 180, 300$ and $n = 2, 3, 4, 5$ respectively using 3rd order HB method.

(Flaherty et al., 1972) and Tadjbakhs (Tadjbakhs and Odeh, 1997). It is evident from Table 2 that the results obtained using 3rd order HB approximation are very close to the results reported by Djondjorov (Djondjorov et al., 2011) and can estimate contact points for any value of n. Figs. 2–5 show equilibrium shapes of the ring for $2 \le n \le 5$ with corresponding values of pressures p. Last figure in each row in Figs. 2–5 is where first contact occurs in the 3rd order HB method. All equilibrium solutions shown in Figs. 2–5 are obtained using the 3rd order HB method.

We also report contact pressure obtained using the second order perturbation method and its modification discussed in section 3. In Table 3, row 2 shows contact pressure for various values of n by extending Tadjbakhs (Tadjbakhs and Odeh, 1997) to the second order as in (18)-(20). In the last row of Table 3, we use modified perturbation approach given by (21) and estimate the contact point pressure. It is clear that the values of the contact pressure p_c based on the modified approach are more close to the 2nd HB approximation, whereas even the extended perturbation approach to Tadjbakhs (Tadjbakhs and Odeh, 1997) fails to match values of the contact pressures obtained from the 2nd order HB method for higher values of n mode rotational symmetry.

4.2. Self-intersection

For the sake of completion, results are also reported for even higher values of the pressure p using 3rd order HB method. Actually, when the pressure p is progressively increased beyond the contact pressure p_c a non-physical phenomenon occurs. Fig. 6 shows self-intersection of the loop when applied pressure $p > p_c$ for $2 \le n \le 5$. Since it is two dimensional problem, such behavior is not possible in experiment. Physically, higher pressure should reduce the area enclosed by the loop instead of giving rise to self-intersection. In order to remove such artifacts we follow Djondjorov (Djondjorov et al., 2011) technique of scaling and transformation and obtain physical results shown in Fig. 7.

5. Discussion

We have used higher order approximation in the harmonic balance method to accurately predict the contact pressure of an elastic ring subjected to uniform hydrostatic pressure. Non-trivial generalized expressions are provided which work for any value of the pressure and rotational symmetry. Expressions are formulated in terms of applied hydrostatics pressure which facilitate physical discussion of the problem and are also suitable for comparison purpose with already available results. Though the expressions are very large and difficult to manage manually, symbolic computational packages such as Mathematica (Wolfram Research Inc, 2018) can easily and efficiently manipulate such expressions. We have also used a higher order perturbation method and a modified version of it to estimate the contact pressure. Modification in the perturbation method suggests that instead of increasing order of the perturbation method a better choice of the initial series can be more effective and accurate for this problem. It is also seen that second order harmonic balance method Baisheng (Wu et al., 2007) is not a good choice to estimate contact pressure p_c for higher values of n. Additionally, the harmonic balance method works well for large buckling, therefore it is also viable to administer self-intersecting situations.

Author agreement statement

We the undersigned declare that this manuscript is original, has not been published before and is not currently being considered for publication elsewhere.

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

We understand that the Corresponding Author is the sole contact for the Editorial process. He/she is responsible for communicating with the other authors about progress, submissions of revisions and final approval of proofs Signed by all authors as follows.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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