HW1 - Electromagnetism

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1. Answer to question 1:

   The relation between the Poisson equation and the Green's function can be deduced through the below identity

$$\{\phi\psi_{x'} - \phi_{x'}\psi\}_0^a = \int_0^a dx'[\phi\psi_{x'x'} - \phi_{x'x'}\psi] \tag{1}$$

   Setting $\phi = \Phi(x')$ and $\psi(x') = G(x, x')$,

$$\phi(x) = \int_0^a dx'\rho(x')G(x, x') - [\Phi(x')G_{x'}(x, x') - \Phi_{x'}(x')G(x, x')]_{x'=0}^{x'=a} \tag{2}$$

   To find the Greens function it suffices to find a symmetric function obeying the homogeneous differential equation, as we prove below. Satisfying the homogeneous boundary condition in the variable y,

$$G(x, x') = \begin{cases} \alpha_1(x)x' + \beta_1(x) & \text{if } x' > x \\ \alpha_2(x)x' + \beta_2(x) & \text{if } x' < x \end{cases}$$

   Now, by requiring the boundary condition we solve the homogeneous BVP, $\alpha_1(x) = -\frac{\beta_1(x)}{a}$, $\beta_2(x) = 0$, explicitly

$$G(x, x') = \begin{cases} \beta_1(x)(1 - \frac{x'}{a}) & \text{if } x' > x \\ \alpha_1(x)x' & \text{if } x' < x \end{cases}$$

   As the dirac-delta distribution is symmetric, which is physically convenient, if $G(x, y)$ is solution, so it is $G(y, x)$. The only possible symmetric function is obtained through

$$G(x, x') = \begin{cases} x(1 - \frac{x'}{a}) & \text{if } x' > x \\ x'(1 - \frac{x}{a}) & \text{if } x' < x \end{cases}$$

   One may indeed easily check that the derivative of the Green's function with respect to x' implies a Heaviside function, i.e, the discontinuity in the derivative of the Green's function is precisely the expected from its definition. The field is written according to (2) and notting the second boundary term to vanish,

$$\Phi = \int dx'\rho(x')G(x, x') - [\Phi(a)x(1 - \frac{1}{a}) + \Phi(0)] \tag{3}$$

   Identifying $\Phi(0) = A$ and $\Phi(a) = B$ the final result is obtained.

2. Answer to question 2

(a) A distribution of charge in spherical coordinates in the 3d space is such that

$$\int_0^\infty drr^2 \int_\Omega \rho(r,\theta,\phi)d\Omega = Q \tag{4}$$

For a shell localized in $r = R$, $R^2 \int d\Omega\rho(\Omega) = Q$. A way to explicitly obtain R^2 in the LHS is to consider $\rho(r,\Omega) = \delta(r-R)\rho(\Omega)$. In particular, for a uniform distribution, $\rho(\Omega) = \rho$, a constant, and $\rho = \frac{Q}{4\pi R^2}$. Thus,

$$\rho(r,\Omega) = \delta(r-R)\frac{Q}{4\pi R^2} \tag{5}$$

(b)

$$\int_0^\infty drr \int_\phi d\phi \int_z \rho(r,\phi,z)dz = Q \tag{6}$$

Now, $\rho(r,\phi,z) = \delta(r-R)\rho$ with ρ a constant, since there is no dependence of the density on either the angle or z position. Therefore, integrating produces $\rho 2\pi RL = Q$, and as $\frac{Q}{L}$ defines a linear density, $\rho = \frac{\lambda}{2\pi R}$.

$$\rho(r,\phi,z) = \delta(r-R)\frac{\lambda}{2\pi R} \tag{7}$$

(c) For a flat circular disk, in cyllindrical coordinates,

$$\rho(z,r,\theta) = \delta(z)\rho(r,\phi) \tag{8}$$

In the case considered, ρ is independent on both r and ϕ, thence $\rho(z,r,\phi) = \delta(z)\rho$ and an integration $\rho\int_0^{2\pi}\int_{r=0}^{R}\int r\sin\phi dr = Q$ yields $\rho = \frac{Q}{\pi R^2}$. Thus,

$$\rho(z,r,\theta) = \frac{Q}{\pi R^2}\delta(z) \tag{9}$$

(d) In spherical coordinates, in an analogous way, $\rho(r,\theta,\phi) = \delta(\theta)\rho(r,\phi) = \delta(\theta)\rho$, with the last identity holding for the uniform case (ρ constant). Thence, an integration leads to $2\rho\pi\frac{R^3}{3} = Q$ and

$$\rho = 3\frac{QR^3}{2\pi}\delta(\theta) \tag{10}$$

3. Answer to question 3

(a) From the Gauss law applied to a small gaussian cyllinder near one of the flat sheets and encompassing a charge of magnitude q,

$$(\vec{E}\cdot\hat{n}_1 - \vec{E}\cdot\hat{n}_2)\Delta a = \frac{q}{\epsilon_0} \tag{11}$$

as $n_2 = -n_1$, and with Δa representing the area of the cyllinder projected onto the plane. $E = \frac{Q}{2\epsilon_0\Delta A}$ The potential difference between the sheets is provided through $\Delta V = Ed = \frac{Qd}{2\epsilon_0\Delta A}$. Finally, the ratio $C = \frac{\Delta V}{Q}$ defines the capacitance

$$C = \frac{d}{2\epsilon_0 A} \tag{12}$$

(b)

In this case, the Gauss law produces the trivial known result $E = \frac{Q}{4\pi r^2}$ through a concentric sphere in between the conductors of radius a and b. The potential difference between the shells is

$$V = \int_a^b E dr = -\frac{Q}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0}(\frac{1}{a} - \frac{1}{b}) \tag{13}$$

Therefore, according to the usual definition for capacitance,

$$C = \frac{1}{4\pi\epsilon_0}(\frac{1}{a} - \frac{1}{b}) \tag{14}$$

(c)

A concentric cyllinder with radius lying in between a and b is used as a gaussian surface. This is suitable since the electric field is constant along this surface. Thence, Gauss law imply $E = \frac{Q}{L}\frac{1}{2\pi r\epsilon_0}$. An integration produces the potential difference between the cyllinders of radii a and b,$V = \frac{Q}{2\pi\epsilon_0 L}\ln\frac{b}{a}$. Hence,

$$C = \frac{1}{2\pi\epsilon_0 L}\ln\frac{b}{a} \tag{15}$$

(d) (Numerical)

4. Answer to question 4

We wish to compute the capacitance density

$$\frac{C}{L} = \frac{Q}{LV} = \frac{\lambda}{V} \tag{16}$$

where λ stands for linear density. Thence, as the cyllinders are infinitely long, not any component for the electric field is expected except the radial one. Let the system of coordinates be placed on the first cyllinder to which we associate the index 1. Thence,

$$V = \frac{\lambda}{2\pi}\int_{a_1}^{d-a_2} dr_1(\frac{1}{r_1} - \frac{1}{d-r_1}) = \frac{\lambda}{2\pi}\ln(\frac{d}{a_1}(1-\frac{a_2}{d})) - \ln(\frac{a_2}{d}\frac{1}{(1-\frac{a_1}{d})}) \tag{17}$$

Due to the logarithm properties and $d >> a_1$ and $d >> a_2$, and denoting $a^2 = a_1 a_2$ the geometric average, it is clear the below relation

$$V = \frac{\lambda}{2\pi}\ln(\frac{d}{a})^2 = \frac{\lambda}{\pi}\ln\frac{d}{a} \tag{18}$$

The capacitance density is, therefore,

$$\frac{C}{L} = \pi(\ln\frac{d}{a})^{-1} \tag{19}$$

5. Answer to question 5

Consider the Green's identity for the Neumman boundary problem

$$\Phi(x) = \langle\Phi\rangle_S + \frac{1}{4\pi}\int_V G(x,x')\rho(x')d^3x' + \oint_S G(x,x')\nabla\Phi(x')\cdot\vec{n}da' \tag{20}$$

with $\langle\Phi\rangle$ denoting the average of the inner function on the surface. In the absence of charge within the volume, it follows

$$\Phi(x) = \langle\Phi\rangle_S - \oint_S G(x,x')E_n(x')da' \tag{21}$$

as $-\nabla\Phi(x')\cdot\vec{n} = E_n$. In the surface, $E_n(x') = \sigma(x')/\epsilon_0$. The absence of charge within the surface imply the mean value theorem, explicitly,

$$\Phi(x) = \langle\Phi\rangle_S \tag{22}$$

6. Answer to questions 6

(a) Let

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{23}$$

$$\nabla^2\Phi' = -\frac{\rho'}{\epsilon_0} \tag{24}$$

Due to the vectorial identity

$$\Phi'\nabla^2\Phi - \Phi\nabla^2\Phi' = \nabla\cdot(\Phi'\nabla\Phi - \Phi\nabla\Phi') \tag{25}$$

the identification $-\nabla\Phi = \vec{E}$, applying the Poisson relations and the Gauss theorem,

$$\int[\Phi\rho' - \Phi'\rho]d^3x = \epsilon_0\int da[\Phi E_n' - \Phi' E_n] \tag{26}$$

In the external surface, $E_n = \frac{\sigma}{\epsilon_0}$, therefore

$$\int[\Phi\rho' - \Phi'\rho]d^3x = \int da[\Phi\sigma' - \Phi'\sigma] \tag{27}$$

which is clearly equivalent to the identity we wish to prove through a simple algebraic rearrangement.

7. Answer to question 7

Consider as gaussian surface as a box containing the sheets at zero potential and far away perpendicular walls (formally, at infinity). Compare to the problem of two capacitor sheets, each with charge density σ, whose physical measurements are labelled with primed functions. Hence, we apply the reciprocal Green's Theorem. In the boundary, $\Phi_a = 0$, therefore (30) is reduced to

$$\int\Phi'\rho d^3x = \int da\sigma\Phi' \tag{28}$$

as $\rho(x) = q\delta(x - x_1)$ for some $x_1 \in (0,d)$, with d the distance between conductors.

$$-q\Phi'(x_1) = \int da\sigma\Phi_a' \tag{29}$$

As Φ' is constant in each conductor surface,

$$-q\Phi'(x_1) = [\Phi'(d) - \Phi'(0)]q_{ind} \tag{30}$$

We have seen that the potential $\Phi'(x) \sim x$ within capacitors, as the electric field is constant. Therefore,

$$q_{ind} = -q\frac{x_1}{d} \tag{31}$$

as required