HW5 - Electromagnetism

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Table of relations :

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l(\cos\theta)\exp[im\phi] \tag{1}$$

$$\frac{1}{|\vec{x}-\vec{x'}|} = \sum_{l,m}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \tag{2}$$

$$\int |Y_{lm}(\theta,\phi)|^2 d\Omega = 1 \tag{3}$$

$$Y^*_{lm}(\theta,\phi) = (-1)^m Y_{l-m}(\theta,\phi) \tag{4}$$

$$\tag{5}$$

Answer to question 3

We compute $\nabla\times\vec{A}$ with

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3} \tag{6}$$

and use the following identities

$$\nabla\times(\Psi\vec{a}) = \nabla\Psi\times\vec{a} + \Psi\nabla\times\vec{a} \tag{7}$$

$$\nabla\times(\vec{a}\times\vec{b}) = \vec{a}\nabla\vec{b} = \vec{b}\nabla\cdot\vec{a} + (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b} \tag{8}$$

with ψ an scalar function. As a preliminary result

$$\nabla\Psi = -\frac{3\vec{n}}{|\vec{x}|^4} \tag{9}$$

and applying (2) and (3),

$$\nabla\times\vec{A} = \frac{\mu_0}{4\pi}[\frac{-3(\hat{n}\cdot\vec{x})\vec{m} - 3(\hat{n}\cdot\vec{m})\vec{x}}{|\vec{x}|^4}] + \frac{1}{|\vec{x}|^3}[\vec{m}\nabla\cdot\vec{x} - \vec{x}\nabla\cdot\vec{m} + (\vec{m}\cdot\nabla)\vec{x} - (\vec{x}\cdot\nabla)\vec{m}] \tag{10}$$

Noticing $(\vec{m}\cdot\nabla)\vec{x} = \vec{m}$, dividing for x the first term and with $\nabla\cdot\vec{x} = 3$.

$$\nabla\times\vec{A} = \frac{\mu_0}{4\pi}\frac{-3\vec{m} - 3(\hat{n}\cdot\vec{m})\vec{n}}{|\vec{x}|^3} + \frac{1}{|\vec{x}|^3}(3\vec{m} - \vec{x}\nabla\cdot\vec{m} + (\vec{m}\cdot\nabla)\vec{x} - (\vec{x}\cdot\nabla)\vec{m}) \tag{11}$$

Thus, summing and noticing $\vec{m} = \frac{1}{2}\int\vec{x'}\times\vec{J}(x')d^3x'$ is a constant vector produces

$$\nabla\times\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m} - 3(\hat{n}\cdot\vec{m})\hat{n}}{|\vec{x}|^3} \tag{12}$$

which is equal to the result up to a change in the sign of $\vec{m}$.

Answer to question 2

Problem 5.3 of Jackson's book

Consider the law for the field (at $\vec{x}$)caused by a current element at the origin of coordinates.

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{x}}{|\vec{x}|^3} \tag{13}$$

Thus,

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{\frac{d\vec{x}'}{dt} \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} dt \tag{14}$$

If we parametrize the curve through $\vec{x}' = (a\cos t, a\sin t, 0)$, the system of coordinates is the center of the spire and is such that $(\vec{x} - \vec{x}') = (-a\cos t, -a\sin t, z)$. Thus, with

$$\frac{d\vec{x}'}{dt} \times (\vec{x} - \vec{x}') = a(z\cos t, z\sin t, 1) \tag{15}$$

$$|\vec{x} - \vec{x}'|^3 = (a^2 + z^2)^{3/2} \tag{16}$$

Therefore, an integration of the first two terms yields zero, as expected from the symmetry. Thus,

$$\vec{B} = \hat{z} \frac{\mu_0 I a}{2} \frac{1}{(a^2 + z^2)^{3/2}} \tag{17}$$

Now, if we have a coil we must sum over different spires placed at different positions x_i

$$\vec{B} = \hat{z} \sum_i \vec{B}_i = \frac{\mu_0 I a^2}{2} \sum_i \frac{1}{(a^2 + (z - z_i)^2)^{\frac{3}{2}}} \frac{\Delta x_i}{\Delta x_i} \tag{18}$$

Making $\Delta x_i = \Delta x = \frac{L}{\text{Number of loops}}$, $\Delta = \frac{1}{N}$, N being the density of spires. Therefore, in the limit $N \to \infty$, the sum is brought to a Riemman sum

$$\vec{B} = \hat{z} \sum_i \vec{B}_i = \frac{\mu_0 I N}{2a} \int \frac{1}{(1 + (\frac{z - z_i}{a})^2)^{\frac{3}{2}}} dx \tag{19}$$

Thence, the integral is transformed for $\phi = \tan^{-1}[\frac{(z-z_i)}{a}]$, and the result is $\sin\phi$. But a quick draw provides the information that $\sin\phi = \cos\theta$ with the angle θ defined as in figure (of the problem). Finally, with the identification $\cos(\pi - \theta_2) = -\cos\theta_2$, it results

$$\vec{B} = \hat{z} \sum_i \vec{B}_i = \hat{z} \frac{\mu_0 I N}{2a} (\cos\theta_1 + \cos\theta_2) \tag{20}$$

Answer to question 5.19 of Jackson's book

$$\Phi_M(x) = -\frac{1}{4\pi} \int \frac{\nabla \cdot \vec{M}}{|\vec{x} - \vec{x}'|} d^3x + \frac{1}{4\pi} \int \frac{\hat{n} \cdot \vec{M}}{|\vec{x} - \vec{x}'|} da' \tag{21}$$

Inside or outside the cylinder, $\nabla \cdot \vec{M} = 0$, since $\vec{M}$ is piecewise constant. The only contribution is due to the second term, which provides

$$\Phi_M(x) = \frac{1}{4\pi} \int da_1' \frac{M_0}{|\vec{x} - \vec{x}_1'|} + \frac{1}{4\pi} \int da_2' \frac{M_0}{|\vec{x} - \vec{x}_2'|} \tag{22}$$

Both da'_1 and da'_2 have the form $\rho' \sin\theta' d\phi' d\rho$, as we integrate for ϕ' only the $m = 0$ term survive. To prove this, notice that $Y_{lm}(\theta', \phi') \sim \exp[im\phi]$ and the integral of last in the interval $[0, 2\pi]$ is only non vanishing for $m = 0$. As $Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta')(2l+1)$, it follows

$$\Phi_{M_{in,out}} = \sum_l P_l(\cos\theta) \frac{2l+1}{2} [\int_0^{\tan^{-1}[\frac{2a}{L}]} + \int_{-\tan^{-1}[\frac{2a}{L}]}^0 P_l(\cos\theta') d\theta'] \int_0^a \{\frac{\rho^l}{\rho'^{l+1}}, \frac{\rho'^l}{\rho^{l+1}}\} \rho' d\rho' \tag{23}$$

where each of the terms in bracket are related respectively to the inner and external region. We can not have the logarithmic term for $l = 0$ (internal case) because the result would not converge (roughly unless we were at $\rho/a \to 0$. Therefore, solving the radial integral

$$\Phi_{M_{in,out}} = \sum_l P_l(\cos\theta) \frac{2l+1}{2} [\int_0^{\tan^{-1}[\frac{2a}{L}]} + \int_{-\tan^{-1}[\frac{2a}{L}]}^0 P_l(\cos\theta') d\theta'] \{\frac{\rho^l}{a^{l-1}(1-l)}, \frac{a^{l+2}}{\rho^{l+1}(l+2)}\} \tag{24}$$

Finally,

$$\vec{H} = -\nabla \Phi_{M_{in,out}} \tag{25}$$

and

$$\vec{B}_{out} = \frac{\vec{H}}{\mu_0} \tag{26}$$
$$\vec{B}_{in} = \frac{\vec{H}}{\mu}$$

in each different region, when the dielectric is present or not. There is a bit of subtlety in this result, since there is the possibility of being within the inner region of expansion but outside of the dielectric. Thus, if one is in this special region, one must take care when applying the result.

Answer to question 5.19 of Jackson's book (Write later)

Answer to question 5.27 of Jackson's book

Answer to Question 3

In the Coulomb Gauge,

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \tag{27}$$

and defining a vector in the circumference as

$$\vec{r} = a(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{28}$$

it follows the derivative

$$\vec{v} = \omega a(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0) \tag{29}$$

As $\sigma 4\pi a^2 = Q$ of the shell, we establish as anzats the following reasonable distribution $\rho = \kappa\delta(r' - a)$, from which $\rho(\vec{x}') = \sigma a^2 \delta(r' - a)$.Thence, from $\vec{J} = \rho(\vec{x}')\vec{v}$, it follows

$$\vec{J} = \sigma\omega a^3 \delta(r' - a)(-\sin\theta'\sin\phi', \sin\theta'\cos\phi', 0) \tag{30}$$

Thence, in component form, after reescaling the coordinates, $x \to \sqrt{\mu_0\sigma\omega a^3} x$ and $y \to \sqrt{\mu_0\sigma\omega a^3} y$,

$$\nabla^2 A_x(\theta, \phi, r) = \sin\theta\sin\phi\delta(r - a) \tag{31}$$
$$\nabla^2 A_y(\theta, \phi, r) = -\sin\theta\cos\phi\delta(r - a) \tag{32}$$
$$\nabla^2 A_z(\theta, \phi, r) = 0 \tag{33}$$

The solutions of these equations is the solution for Poisson (with BVP at infinity)

$$A_x = \int \frac{\sin\theta'\sin\phi'\delta(r'-a)r'^2\sin\theta'}{|\vec{x}-\vec{x'}|}d\theta' d\phi' dr' \tag{34}$$

$$A_y = -\int \frac{\sin\theta'\cos\phi'\delta(r'-a)r'^2\sin\theta'}{|\vec{x}-\vec{x'}|}d\theta' d\phi' dr' \tag{35}$$

Proceeding as usual, let

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{lm}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi') \tag{36}$$

The result for the radial part and θ part is identical, the difference in $\{A_x, A_y\}$ is in ϕ if there is any, for which the ϕ integrals are useful.

$$I_x = \int_0^{2\pi}\sin\phi'\exp[-im\phi']d\phi' = \pi i(-\delta_{m1}+\delta_{m-1}) \tag{37}$$

$$I_y = -\int_0^{2\pi}\cos\phi'\exp[-im\phi']d\phi' = -\pi(\delta_{m1}+\delta_{m-1}) \tag{38}$$

Therefore, we can separate the result in two integrals

$$A_x = \sum_l \sqrt{\frac{2l+1}{4\pi}}\pi i(-\sqrt{\frac{(l-1)!}{(l+1)!}}+\sqrt{\frac{(l+1)!}{(l-1)!}})\int\sin\theta'^2[-P_l(\cos\theta')]d\theta'\int_{r'}\delta(r'-a)r'^2\frac{r_<^l}{r_>^{l+1}}dr'$$

$$A_y = \sum_l \sqrt{\frac{2l+1}{4\pi}}(-\pi)(\sqrt{\frac{(l-1)!}{(l+1)!}}+\sqrt{\frac{(l+1)!}{(l-1)!}})\int\sin\theta'^2[-P_l(\cos\theta')]d\theta'\int_{r'}\delta(r'-a)r'^2\frac{r_<^l}{r_>^{l+1}}dr'$$

in the beforementioned reescaled unities. Now, the field $\vec{B}$ can be expressed as

$$B_x = -\partial_z A_y = \frac{\sin\theta}{r}\partial_\theta A_y - \frac{\cos\theta}{r}\partial_r A_y \tag{39}$$

$$B_y = -\partial_z A_x = \frac{\sin\theta}{r}\partial_\theta A_x - \frac{\cos\theta}{r}\partial_r A_x \tag{40}$$

$$B_z = (\partial_x A_y - \partial_y A_x) = \frac{\cos\theta}{r^2}\partial_\theta(A_y - A_x) - \frac{\cos\theta}{r}(\partial_r A_y - \partial_r A_x) \tag{41}$$

These derivatives can be obtained with any algebraic software for the inner or external part of the sphere. For the inner case, $r_< = r$,$r_> = a$, and for the external region, the relation is the contrary. One must remind that r has been reescaled by a factor of $r \to r\mu_0\sigma\omega a^3$ such that we must return to the original coordinate suitably, i.e, make, $r \to \frac{r}{\mu_0\sigma\omega a^3}$.Concerning the integral of type

$$I = \int_0^\pi \sin^2\theta P_l(\cos\theta) = \int_0^\pi d\theta P_l(\cos\theta) + \int_0^1 xP_l(x)dx \tag{42}$$

The first is known and is provided in Jackson, the second can be trivially obtained using Lagrange polynomial relations.

Question 4

In this case we must substitute μ_0 for μ whenever we are in a dielectric media, this means that the reescaled coordinate is different if we are in the dielectric.