HW4 - Electromagnetism

M.A.Sarmento

January 24, 2020

Table of relations :

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l(\cos\theta)\exp[im\phi] \tag{1}$$

$$\frac{1}{|\vec{x}-\vec{x'}|} = \sum_{l,m}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \tag{2}$$

$$\int |Y_{lm}(\theta,\phi)|^2 d\Omega = 1 \tag{3}$$

$$Y^*_{lm}(\theta,\phi) = (-1)^m Y_{l-m}(\theta,\phi) \tag{4}$$

$$\int_R^\infty r^n \exp[-r] = \Gamma(n+1,R) \tag{5}$$

$$\Gamma(n+1) = \lim_{R\to 0}\Gamma(n+1,R) \tag{6}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}}(\frac{3}{2}\cos^2\theta - \frac{1}{2}) \tag{7}$$

$$Y_{00} = \frac{1}{\sqrt{4}} \tag{8}$$

Answer to question 1 (Problem 4.1 of Jackson's book)
We compute the expansion in spherical harmonics for a set of two charge distribution (refeered through the subindex s2) whose center is placed in the origin of coordinates. We choose it to be aligned in the x-axis and the two-charge system with a characteristic distance of $2a$.

$$\Phi_{s_2}(x,y,z) = \frac{1}{4\pi\epsilon_0}q(\frac{1}{\sqrt{(x-a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+y^2+z^2}}) \tag{9}$$

Thence, as the charge distributions are in the x-y plane $\theta' = \frac{\pi}{2}$ in both expansions, $\phi' = 0$ for the first and $\phi' = \pi$ for the second, yielding (from relation (2) in the table)

$$\Phi_{s_2} = \frac{1}{4\pi\epsilon_0}q\sum_{l,m}\frac{a^l}{r^{l+1}}[Y^*_{lm}(\frac{\pi}{2},0) - Y^*_{lm}(\frac{\pi}{2},\pi)]Y_{lm}(\theta,\phi) \tag{10}$$

i.e,

$$\Phi_{s_2} = \frac{1}{4\pi\epsilon_0}q\sum_{l,m}\frac{a^l}{r^{l+1}}[Y_{l-m}(\frac{\pi}{2},0) - Y_{l-m}(\frac{\pi}{2},\pi)](-1)^m Y_{lm}(\theta,\phi)$$

$$(11)$$

or

$$\Phi_{s2} = \frac{1}{4\pi\epsilon_0}(\sum_{lm} qa^l P_l(0)[1-(-1)^m]\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \tag{12}$$

From which we identify the multipole

$$q_{lm} = \sqrt{\frac{2l+1(l-m)!}{4\pi(l+m)!}} P_l(0)[1-(-1)^m] \tag{13}$$

The factor $\sqrt{\frac{2l+1(l-m)!}{4\pi(l+m)!}}$ comes from the proportionality factor relating the spherical harmonics to Lagrange polynomials as in table. (a) We wish to evaluate the potential due to a four charge distribution as in figure A. For this, consider another distribution which consist of the last charge configuration rotated an angle magnitude of $\frac{\pi}{2}$ in polar coordinates, i.e, such that

$$Y_{l-m}(\frac{\pi}{2},0) \rightarrow Y_{l-m}(\frac{\pi}{2},\frac{\pi}{2}) \tag{14}$$

$$Y_{l-m}(\frac{\pi}{2},\pi) \rightarrow Y_{l-m}(\frac{\pi}{2},\frac{3\pi}{2}) \tag{15}$$

Therefore, the new multipole coefficients are

$$q'_{lm} = \sqrt{\frac{2l+1(l-m)!}{4\pi(l+m)!}} P_l(0)(\exp[-im\frac{\pi}{2}] + \exp[-im\frac{3\pi}{2}]) \tag{16}$$

By the superposition principle, the configuration of four charges as in figure A is obtained by summing the contribution of both rotated and the non-rotated charge configuration. This is equivalent to summing the correspondent multipole expansion terms. For distinction, we represent the multipole coefficients of the system labeling Q_{lm},

$$Q_{lm} = q_{lm} + q'_{lm} \tag{17}$$

(b) Consider

$$\Phi_{s_3}(x,y,z) = \frac{1}{4\pi\epsilon_0}(\frac{-2q}{\sqrt{x^2+y^2+z^2}} + \frac{q}{\sqrt{x^2+y^2+(z-a)^2}} + \frac{q}{\sqrt{x^2+y^2+(z+a)^2}}) \tag{18}$$

and proceed as in the previous example, but now there is symmetry of rotation in the polar angle, i.e, $m = 0$. In this case, we have to change the sign of the charge calculated in the previous context, $q \rightarrow -q$ for one of the charges (the negative one), and carry out the transformation

$$Y_{l-m}(\frac{\pi}{2},0) \rightarrow P_l(1)\sqrt{\frac{2l+1!}{4\pi l!}} \tag{19}$$

$$Y_{l-m}(\frac{\pi}{2},\pi) \rightarrow P_l(-1)\sqrt{\frac{(2l+1)!}{4\pi l!}} \tag{20}$$

Finally we add the contribution of the $-2q$ charge multi poles at the origin. Again rely on the superposition principle of multi pole contributions.

Answer to question 2 (Problem 4.7 of Jackson's book)

(a) The charge distribution is of the form $\rho(r') = R'(r')\sin^2\theta$, with $R'(r') = \frac{1}{64\pi}r'^2\exp[-r']$. It is convenient to write $\sin^2\theta$ as a combination of spherical harmonics. We seek a linear combination with $m = 0$ harmonics in the table of spherical harmonics producing the result. Thence, $\sin^2\theta = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}(Y_{20} - Y_{00}(\theta',\phi')\sqrt{5})$. Explicitly,

$$\rho(\vec{r}') = R'(r')\sqrt{\frac{4\pi}{5}}(Y_{20}(\theta',\phi') - Y_{00}(\theta',\phi')\sqrt{5}) \tag{21}$$

The green's function is also separable in radial and spherical harmonic terms according to (2). Thence, the product of these quantities is separable,

$$\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = -\frac{2}{3}\frac{1}{64\pi}\sum_l \exp[-r']\frac{r'^{l+2}}{r^{l+1}}\sum_m Y_{lm}(\theta,\phi)Y_{lm}(\theta',\phi')\sqrt{\frac{4\pi}{5}}(Y_{20}(\theta',\phi') - Y_{00}(\theta',\phi')\sqrt{5}) \tag{22}$$

with $r' = r_<$ and $r = r_>$ for the multipole expansion. The potential is provided through an integration in the sample coordinates.

$$\Phi(\vec{r})_{r>r'} = -\frac{2}{3}\frac{1}{64\pi}\sum_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\int_0^r r'^{l+2}\exp[-r']r'^2dr' \tag{23}$$
$$\int d\Omega'\sqrt{\frac{4\pi}{5}}(Y_{20}(\theta',\phi') - Y_{00}(\theta',\phi')\sqrt{5})Y_{lm}(\theta',\phi')$$

which due to the orthonormal spherical harmonics and the complete and incomplete Γ definition yields

$$\Phi(\vec{r},\vec{r}') = -\frac{2}{3}\frac{1}{64\pi}\sum_l Y_{l0}(\theta,\phi)\frac{\Gamma(l+5) - \Gamma(l+5,r)}{r^{l+1}}\sqrt{\frac{4\pi}{5}}(\delta_{l2}\delta_{m0} - \delta_{l0}\delta_{m0}\sqrt{5}) \tag{24}$$

Through the first identity in the table and $P_0(\cos\theta) = 0$,

$$\Phi(\vec{r})_{r>r'} = -\frac{2}{3}\frac{1}{64\pi}\sqrt{\frac{4\pi}{5}}(\frac{\Gamma(5) - \Gamma(5,r)}{r} + P_2(\cos\theta)\frac{\Gamma(7) - \Gamma(7,r)}{r^3}) \tag{25}$$

(b) The valid expression for $r < r'$, can be obtained by interchanging r and r' suitably in the Green's function expansion and integrating in the external region,

$$\Phi(\vec{r})_{r<r'} = -\frac{2}{3}\frac{1}{64\pi}\sqrt{\frac{4\pi}{5}}\sum_{lm}Y_{lm}(\theta,\phi)r^l\int_r^\infty r'^{3-l}\exp[-r']dr'\int d\Omega'[Y_{20}(\theta',\phi') - \tag{26}$$
$$Y_{00}(\theta',\phi')\sqrt{5})Y_{lm}(\theta',\phi')]$$

which is reduced to

$$\Phi(\vec{r})_{r<r'} = -\frac{2}{3}\frac{1}{64\pi}\sqrt{\frac{4\pi}{5}}\sum_l \Gamma(4-l)(\delta_{l2} - \delta_{l0}\sqrt{5})r^lP_l(\cos\theta) \tag{27}$$

from which the dependence on r^0 and r^2 is obtained.

Answer to question 3 (Problem 4.8 of Jackson's book - calculation to be completed)

The solution admits an expansion as

$$\Phi_{int} = \sum_{lm} (a_{lm} \cos m\phi + b_{lm} \sin m\phi) r^l \tag{28}$$

$$\Phi_{mid} = \sum_{lm} [\cos m\phi (c_{lm} r^l + d_{lm} r^{-l-1}) + \sin m\phi (e_{lm} r^l + f_{lm} r^{-l-1})] \tag{29}$$

$$\Phi_{ext} = -E_0 r \cos\phi \tag{30}$$

by noticing the extra symmetry $\Phi(\phi) = \Phi(-\phi)$, $b_{lm} = e_{lm} = f_{lm} = h_{lm} = 0$ and last is reduced. We eprform the relabeling $c_{lm} \to b_{lm}$, $d_{lm} \to c_{lm}$, $g_{lm} \to d_{lm}$.

$$\Phi_{int} = \sum_{lm} a_{lm} \cos m\phi r^l \tag{31}$$

$$\Phi_{mid} = \sum_{lm} \cos m\phi (b_{lm} r^l + c_{lm} r^{-l-1}) \tag{32}$$

$$\Phi_{ext} = -E_0 r \cos\phi + \sum_{lm} d_{lm} \cos m\phi r^{-l-1} \tag{33}$$

The boundary conditions put altogether are

$$-\frac{1}{r}\partial_\phi \Phi_{int}|_{r=a} = -\frac{1}{r}\partial_\phi \Phi_{mid}|_{r=a} \tag{34}$$

$$-\epsilon_0 \partial_r \Phi_{int}|_{r=a} = -\epsilon \partial_r \Phi_{mid}|_{r=a} \tag{35}$$

$$-\frac{1}{r}\partial_\phi \Phi_{ext}|_{r=b} = -\frac{1}{r}\partial_\phi \Phi_{mid}|_{r=b} \tag{36}$$

$$-\epsilon_0 \partial_r \Phi_{ext}|_{r=b} = -\epsilon \partial_r \Phi_{mid}|_{r=b} \tag{37}$$

The first and second conditions produces

$$a_{lm} a^l = (b_{lm} a^l + c_{lm} a^{-l-1}) \tag{38}$$

$$\epsilon_0 l a_{lm} a^{l-1} = \epsilon (l b_{lm} a^{l-1} - (l+1) c_{lm} a^{-l-2}) \tag{39}$$

The remaining produce

$$(-E_0 b \delta_{m1} + d_{lm} b^{-l-1}) = (b_{lm} b^l + c_{lm} b^{-l-1}) \tag{40}$$

$$\epsilon_0 (-E_0 \delta_{m1} - (l+1) d_{lm} b^{-l-2}) = \epsilon (l b_{lm} b^{l-1} - (l+1) c_{lm} b^{-l-2}) \tag{41}$$

We write explicitly the b_{lm} dependence on c_{lm} according to the first and second equations set. Solving for b_{lm}, every other constant is determined.

Answer to question 4 (Problem 4.9 of Jackson's book)

Consider the anzats

$$\Phi_{ext} = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{x}|} + \frac{1}{4\pi\epsilon_0} \frac{q'}{|\vec{r} - \vec{x}'|} \tag{42}$$

$$\Phi_{int} = \frac{1}{4\pi\epsilon} \frac{q''}{|\vec{r} - \vec{x}|} \tag{43}$$

Φ_{ext} is solution within the dielectric, since the vector $\vec{x}$ lies outside, and if we choose $\vec{x'}$ to lie within the dielectric, there is only a Dirac delta contribution due to the first term, as expected. We proceed as suggested, with an expansion in spherical harmonics. Therefore, the continuity (equivalent to the rotational condition, due to the orthogonality of the derivative of Lagrange polynomials),

$$\frac{q}{\epsilon_0}(\frac{r^l}{x^{l+1}})+\frac{q'}{\epsilon_0}(\frac{x'^l}{r^{l+1}})|_{r=a}=\frac{q''}{\epsilon}(\frac{x^l}{r^{l+1}})|_{r=a} \tag{44}$$

For the normal derivative condition (gauss law),

$$ql(\frac{r^{l-1}}{x^{l+1}})-(l+1)q'(\frac{x'^l}{r^{l+2}})|_{r=a}=-(l+1)q''(\frac{x^l}{r^{l+2}})|_{r=a} \tag{45}$$

where the orthogonality of the Lagrange polynomials itself is considered. Explicitly,

$$ql(\frac{a^{l-1}}{x^{l+1}})-(l+1)q'(\frac{x'^l}{a^{l+2}})=-(l+1)q''(\frac{x^l}{a^{l+2}}) \tag{46}$$

$$\frac{q}{\epsilon_0}(\frac{a^l}{x^{l+1}})+\frac{q'}{\epsilon_0}(\frac{x'^l}{a^{l+1}})=\frac{q''}{\epsilon}(\frac{x^l}{a^{l+1}}) \tag{47}$$

It is seen that it is not possible to obtain a value of q' independent of l. Establishing q to depend on l is not a problem, since we have seen that outside the sphere there always exist a multipole expansion in the form

$$\Phi(\vec{x})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}}{r^{l+1}} \tag{48}$$

Therefore, after an algebraic procedure, it suffices to let $x'=a^2/d$ and choose $q'=qal(1-\epsilon/\epsilon_0)/(1+l(1+\epsilon/\epsilon_0)$. The complete expansion is therefore, obtained trivially from this suitable choice of dipoles.
Answer to question 5 (Problem 4.10 of Jackson's book)
(a)
The electric field $\vec{E}$ on the left-hand side of the sphere obeys the related potential Laplace equation trivially. Likewise, the electric displacement vector $\vec{D}$ obeys a constant modified Gauss law with null rotational, i.e, there is a potential obeying a Poisson-like equation with the *free* net charge density on the right-hand side. The absence of free net charges in between the dielectrics imply the related potential to be solution to the Laplace equation. Without considerations on the boundary value problem, we known the general solution to the problem is comprised in the shape

$$\Phi_{L,R}=\sum_l[\{B_{Ll},B_{Rl}\}r^l+\{C_{Ll},C_{Rl}\}r^{-(l+1)}]P_l(\cos\theta) \tag{49}$$

such sub index L hold if we are in the left side of the sphere (vacuum), and R in the dielectric medium in the right. We notice that there is a simplification in the usual expansion in terms of spherical harmonics due to the azhimutal symmetry, implying $m=0$. The boundary condition problem consists of providing total charge $-Q$ on the inner sphere, $+Q$ on the external sphere and, of relevant importance, the obedience of Maxwell equations in the dielectric interfaces. The Gauss equation applied for the interface is consistent with the electric field pointing radially. The rotational condition is related to the change of the potential in the radial direction

$$-\frac{\partial\Phi_L}{\partial r}=-\frac{\partial\Phi_R}{\partial r} \tag{50}$$

For this condition to be satisfied everywhere in the interface $r \in [a,b]$, it is necessary the relation to hold

$$\{B_{Ll}, C_{Ll}\} = \{B_{Rl}, C_{Rl}\} \tag{51}$$

Notice thence that $\Phi_L = \Phi_R = \Phi$. We know from the first Maxwell equation the local relation between the potential and charge density.

$$\sigma_{L,R}(a,\theta) = -\{\epsilon_0, \epsilon\} \sum_l (B_{Ll} l a^{l-1} - \frac{(l-1)}{a^{l+2}} C_{Ll}) P_l(\cos\theta) \tag{52}$$

The boundary condition imply

$$QP_0(x) = -\pi(\epsilon + \epsilon_0) \sum_l (B_{Ll} l a^{l-1} - C_{Ll} \frac{(l-1)}{a^{l+2}}) \int_0^\pi P_l(\cos\theta) \sin\theta d\theta \tag{53}$$

Therefore, apart from $l = 0$ the terms vanish.

$$C_{L0} = -\frac{Q}{2\pi(\epsilon + \epsilon_0)} \frac{1}{a^2} \tag{54}$$

and

$$\Phi = -\frac{Q}{2\pi(\epsilon + \epsilon_0)} \frac{1}{a^2 r} \tag{55}$$

$$\vec{E} = \frac{Q}{2\pi(\epsilon + \epsilon_0)} \frac{1}{a^2 r^2} \hat{r} \tag{56}$$

$$\sigma_{L,R} = -\{\epsilon_0, \epsilon\} |\vec{E}| \tag{57}$$

(c) As discussed in theory, $\nabla \cdot \vec{P} = -\sigma_P$ with σ_P the polarization charge. As $\vec{P} = (\epsilon - \epsilon_0)\vec{E}$,

$$\sigma_p = Q \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{1}{a^2 r^3} \tag{58}$$

Answer to question 5

From previous calculations, it is a trivial result the following (due to polar symmetry)

$$\Phi_{s_2} = \frac{1}{4\pi\epsilon_0} q \sum_{l,m} \sqrt{\frac{(2l+1)!}{4\pi l!}} \frac{a^l}{r^{l+1}} [P_l(1) - P_l(-1)] P_l(\cos\theta) \tag{59}$$

We simply used the result of letter (b) of question (1) (but with a charge sign changed) without the extra charge at the origin. Keeping the smaller terms we have one due to the charge and another due to the dipole, defined with $p = qa$.

Answer to question 6

This problem can be solved using techniques similar to previously employed in problem (4), for instance. To be solved soon.