HW7 - Electromagnetism

M.A.Sarmento

January 24, 2020

Answer to question 1

(a) Using eq. (7) and eq. (8) of question 2, we find

$$R = |\frac{1-n}{n+1}| = |\frac{1-\sqrt{\frac{\epsilon}{\epsilon_0}}}{1+\sqrt{\frac{\epsilon}{\epsilon_0}}}| \tag{1}$$

by considering $\mu = \mu_0$, and we proceed to use $\epsilon(\omega)$ in Drude's model, which is

$$\epsilon = \epsilon_b + i\frac{\sigma}{\omega} \tag{2}$$

where ϵ_b takes into account bound electrons, while σ/ω the conduction ones. In the limit of a bad conductor, everything behaves as a dielectric

$$R = |\frac{1-\sqrt{\frac{\epsilon_b(\omega)}{\epsilon_0}}}{1+\sqrt{\frac{\epsilon_b(\omega)}{\epsilon_0}}}| \tag{3}$$

with $\epsilon(\omega)$ provided in the model of oscillators with nonzero hook constant (as discussed in Jackson), with regions of ω where there is absorption (resonance frequencies) or reflection (other frequencies). For the case of a good conductor, $\sigma >> \epsilon_b$, and

$$R = |\frac{1-\sqrt{\frac{i\sigma}{\omega\epsilon_0}}}{1+\sqrt{\frac{i\sigma}{\omega\epsilon_0}}}| \tag{4}$$

With some algebraic manipulation, it can be proved that $R \to 1$, when $\sigma \to \infty$ in accordance with the behavior expected for a conductor: free electrons organize as to cancel the electric field.

(b) The decay of the intensity of the electric field relative to medium 1 is provided through $E_0' = E_0 \exp\big[\mathrm{Re}[i\omega\sqrt{\mu\epsilon}\delta]\big]$. Reflected light is the fraction of light that penetrates only the skin depth δ in material. $R = T(\delta) = \exp\big[\mathrm{Re}[i\omega\sqrt{\mu\epsilon}\delta]\big]$. For a good conductor, $\sqrt{\epsilon(\omega)} = \sqrt{i}\sqrt{\frac{\sigma}{\omega}}$, meaning $R = \exp\big[\cos\frac{3\pi}{4}\sqrt{\omega\mu_0\sigma}\delta\big] = \exp\Big[\sqrt{\frac{\omega\mu_0\sigma}{2}}\delta\Big]$. Dividing and multiplying by c^2 yields

$$R = \exp\left[-\sqrt{\frac{\omega\sigma}{2c^2\epsilon(\omega)}}\delta\right] \tag{5}$$

Substituting again $\epsilon(\omega) = i\frac{\sigma}{\omega}$, imply

$$R = \exp\left(-\sqrt{\frac{\omega^2}{2c^2}(\frac{1}{\sqrt{2}})\delta}\right) = \exp\left[-\frac{1}{2}\frac{\omega}{c}\delta\right] \sim 1 - \frac{1}{2}\frac{\omega}{c}\delta \tag{6}$$

as it is commonly the case that $\delta << \frac{c}{\omega}$. Beyond this penetration depth no light returns to the original medium.

Answer to question 2

(a) The continuity of the parallel electric and perpendicular magnetic field ($\vec{E}$ and $\vec{H}$), as there is no source of current or density. Thence, denoting E_0 the incident field, E_0' the transmitted and E_0'' the reflected field,

$$E_0 + E_0'' = E_0' \tag{7}$$

$$\frac{E_0}{v_{air}} - \frac{E_0''}{v_{air}} = \frac{E_0'}{v_{glass}} \tag{8}$$

As $v_{ar} = c$ and $v_{glass} = c/n$, solving for E_0' and E_0'' yields

$$E_0' = \frac{2}{n+1}E_0 \tag{9}$$

$$E_0'' = \frac{n-1}{n+1}E_0 \tag{10}$$

From which the reflection and transmission coefficients are defined

$$R_{air,glass} \equiv |\frac{E_0''}{E_0}| = |\frac{n-1}{n+1}| \tag{11}$$

$$T_{air,glass} \equiv |\frac{E_0'}{E_0}| = |\frac{2}{n+1}| \tag{12}$$

(b)
To calculate the final transmitted coefficient, we must consider (1) and (2) once more, but with v_{air} switched to v_{glass}, which is the equivalent of $n \to \frac{1}{n}$ in (3) (and (4), but we need only (3)).

$$E_0' = \frac{2n}{n+1}E_0 \tag{13}$$

E_0 in (7) is the incident amplitude in the second interface, and it is the transmitted field in the first interface, E_0', provided through (3). Thence, the final transmitted field in terms of the very first incident amplitude is

$$E_0' = \frac{2n}{n+1}\frac{2}{n+1}E_0 \tag{14}$$

We are half the way to verify that the intensity is maximum at discrete values, and prove that the maximum occurs with the electric field as in (12). This is quiet trivial from applying the boundary condition in the first interface. As $\vec{k}$ points in a direction perpendicular to the interface, say z, the boundary condition in the first interface imply that, inside the glass,

$$\vec{E}' = \vec{E}_0' \exp\left[i\frac{2\pi}{\lambda_{glass}}z\right] \tag{15}$$

We know that the second transmission is proportional to the first transmission (second incidence), and this vector assumes the amplitude value at $z = d$ for

$$\frac{2\pi}{\lambda_{glass} d} = m\pi \tag{16}$$

As ω is constant, $v_{air}/\lambda_{air} = v_{glass}/\lambda_{glass}$, which imply $\lambda_{glass} = \lambda_{air}/n_{glass}$, yielding the result for maximas

$$2nd = m\lambda \tag{17}$$

happening at relative intensity (to that of the very first incident field)

$$I_r = |\frac{2n}{n+1}\frac{2}{n+1}|^2 \tag{18}$$

Answer to question 3

We saw that $\epsilon(\omega)$ is analytic except possibly for isolated points, as $\epsilon^*(\omega^*) = \epsilon(-\omega)$, that is, its real and imaginary parts are independent of ω^*, which is the equivalent of obeying the Cauchy-Riemann equations (holomorphic). From this, it is trivial to derive the Kronig-Kramers relation

$$\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0}] = 1 + \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Im}[\epsilon(z)/\epsilon_0]}{z-\omega} dz \tag{19}$$

which we apply in the following items.
(Case I)
To solve this, consider the integral

$$I_1 = \lambda\mathcal{P}\int_{-\infty}^{\infty} \frac{[\Theta(z-\omega_1) - \Theta(z-\omega_2)]}{z-\omega} dz \tag{20}$$

This integral is zero in the interval of integration such that $z < \omega_1$ or $z > \omega_2$ with ($\omega_1 < \omega_2$), since in either case the integrand vanishes. Thence, the integral is reduced to

$$\lambda\int_{\omega_1+\epsilon}^{\omega_2-\epsilon} \frac{\Theta(z-\omega_1)}{z-\omega} dz = \lambda[\Theta(\omega_2-\omega)\ln|\omega_2-\omega| - \Theta(\omega_1-\omega)\ln|\omega_1-\omega|] \tag{21}$$

Thence, the Kronig relation yield

$$\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0}] = 1 + \frac{\lambda}{\pi}[\Theta(\omega_2-\omega)\ln|\omega_2-\omega| - \Theta(\omega_1-\omega)\ln|\omega_1-\omega|] \tag{22}$$

(Case II)
Consider the integral

$$I_2 = \lambda\gamma\mathcal{P}\int_{-\infty}^{\infty} \frac{1}{[(z-\omega_0)^2+\gamma^2](z-\omega)} =$$

$$\lambda\gamma[\mathcal{P}\int_{-\infty}^{\infty} \frac{1}{2z}\frac{1}{z+i\gamma}\frac{1}{z+\omega_0-\omega} - \mathcal{P}\int_{-\infty}^{\infty} \frac{1}{2z}\frac{1}{z-i\gamma}\frac{1}{z+\omega_0-\omega} dz] \tag{23}$$

$$I_2 = \lambda\gamma i\,\mathrm{Im}[\mathcal{P}\int_{-\infty}^{\infty} [\frac{1}{2z}\frac{1}{z+i\gamma}\frac{1}{z+\omega_0-\omega} dz] \tag{24}$$

There are two real roots $z = 0$ and $z = \omega - \omega_0$ for this integral, and two non real roots, unless γ is a pure imaginary number. The symmetry in the expression is suitable since it is possible to express the result in terms of a real quantity, as it shall be. By considering

$$I_2 = -2\pi i \sum_k \lim_{z \to z_k} [(z - z_k) f(z) - \pi i \sum_j \lim_{z \to z_j} (z - z_j) f(z) \tag{25}$$

(in the LHP - lower half plane) with $f(z)$ the function being integrated and z_j denotes a simple pole in the real line, and z_k a simple pole in the complex region. The integral in the LHP can be taken, and it yields according to (5) and (6).

$$I_2 = -\lambda\gamma \,\mathrm{Im}\{i[2\pi i \frac{1}{2i\gamma} \frac{1}{i\gamma + \omega_0 - \omega} - \pi i (\frac{1}{i\gamma} \frac{1}{\omega_0 - \omega} + \frac{1}{(\omega - \omega_0)} \frac{1}{\omega - \omega_0 + i\gamma})]\} \tag{26}$$

the first root due to $z = 0$ ($j = 1$), the second due to $z = \omega - \omega_0$ ($j = 2$). This quantity of course can be rewritten in a simpler form, but we won't bother, since there is no physical insight gained in this. We merely simplify a bit the result and use the Kronig relation to obtain

$$\mathrm{Re} \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \lambda\gamma \,\mathrm{Re}\{[\frac{1}{(i\gamma + \omega_0 - \omega)} - \frac{1}{\gamma(\omega_0 - \omega)} - \frac{1}{(\omega - \omega_0 + i\gamma)(\omega - \omega_0)}]\} \tag{27}$$

Answer to question 4

This question is just a reminder of elementary calculus in the complex plane. Consider

$$G(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} \frac{\exp[-i\omega\tau]}{\omega_0^2 - \omega^2 - i\gamma\omega} d\omega \tag{28}$$

It is trivial to obtain the roots $\omega_{12} = -\frac{i\gamma}{2} \pm \sqrt{-\frac{\gamma^2}{4} + \omega_0^2}$. Defining $\nu_0^2 = \omega_0^2 - \frac{\gamma^2}{4}$,

$$\omega_{12} = -i\frac{\gamma}{2} \pm \nu_0 \tag{29}$$

The integral in the real line, as usual is brought to an integral in the real line plus a boundary at infinity in the complex plane, in which the integrals vanish (we choose this boundary to be in the UHP for $\tau < 0$ and in the LHP for $\tau > 0$; this assures the vanishing of the integral in the boundary due to the exponential decay. As the integrals are equal we can apply the residue theorem in the form (7) with no roots in the real line (the set j is empty).

$$G(\tau) = \frac{-\omega_p^2 i}{-2\pi i} \int_C \frac{\exp[-i\omega\tau]}{(\omega - \omega_1)(\omega - \omega_2)} d\omega \tag{30}$$

Applying the residue theorem for the lower plane ($\tau > 0$), there is two poles. In the upper half plane ($\tau < 0$), there is no poles. The residue implies that the only nontrivial result occurs for $\tau > 0$, this being provided through

$$G(\tau) = \omega_p^2 i [\frac{\exp[-i\omega_1\tau]}{\omega_1 - \omega_2} - \frac{\exp[-i\omega_2\tau]}{\omega_1 - \omega_2}]\Theta(\tau) \tag{31}$$

$$G(\tau) = -i \exp\left[-\frac{\gamma\tau}{2}\right] \omega_p^2 \frac{1}{2\nu_0} (\exp[i\nu_0] - \exp[-i\nu_0]) = \exp\left[-\frac{\gamma\tau}{2}\right] \omega_p^2 \frac{\sin\nu_0}{\nu_0} \tag{32}$$