HW2 - Electromagnetism

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1. Answer to question 1: Trivial, to be written soon. The principle of superposition applies. We may seek the image charge such that the potential vanishes at the surface for a single particle. If we have another particle, finding the image implying the vanishing of the potential can also be done. The structure of two particles and two image particles produce zero potential by superposition. The relation between image charges and distances is such that there will be a charge in excess for the image configuration.

2. Answer to question 2: This is entirely analogous to a problem solution by Jackson himself, so it is left to be written later.

3. Answer to question 3:

   Consider the center of coordinates as the center of the cylinder held at fixed potential. The symmetry is cylindrical. Through the image method insight, let

$$\Phi(r) = \frac{\tau}{2\pi}\ln[|\vec{r}-\vec{R}|] - \frac{\tau}{2\pi}\ln[|\vec{r}-\vec{R}'|f(R')] \tag{1}$$

   with $\vec{R}$ and $\vec{R}'$ aligned vectors. If the anzats is correct, for $r = b$ the potential is held fixed, say, at V, thence

$$|\frac{\vec{b}-\vec{R}}{(\vec{b}-\vec{R}')f(R')}| = V \tag{2}$$

   Imply that the choice

$$R' = \frac{b^2}{R} \tag{3}$$

$$f(R') = \frac{RV}{b} \tag{4}$$

   satisfies the identity for any angle between the vectors $\vec{b}$ and the co linear charge and image vectors, $\vec{R}$ and $\vec{R}'$. We write the result as

$$\Phi(r) = \frac{\tau}{2\pi}\ln[|\vec{r}-\vec{R}|] - \frac{\tau}{2\pi}\ln[|\vec{r}-\frac{b^2}{R^2}\vec{R}|\frac{RV}{b}] \tag{5}$$

   In the limit $r \to \infty$ the potential converges to

$$\Phi(r\to\infty) = \frac{\tau}{2\pi}\ln[\frac{b}{RV}] \tag{6}$$

For the potential to vanish at infinity we require $V = b/R$. Putting all together,

$$\Phi(r) = \frac{\tau}{2\pi}\{\ln[|\vec{r} - \vec{R}|] - \ln[|\vec{r} - \frac{b^2}{R^2}\vec{R}|]\} \tag{7}$$

Finally, everything can be written in polar coordinates by stabilishing the polar angle ϕ to be defined according to the relative angular displacement with respect to the fixed charge at $\vec{R}$.

$$\Phi(r) = \frac{\tau}{2\pi}\ln[\frac{|\vec{r} - \vec{R}|}{|\vec{r} - \frac{b^2}{R^2}\vec{R}|}] = \frac{\tau}{4\pi}\ln[\frac{r^2 + R^2 - 2rR\cos\phi}{r^2 + \frac{b^4}{R^2} - 2\frac{b^2 r}{R}\cos\phi}] \tag{8}$$

Indeed, at $r \to \infty$, $\Phi(r) = 0$. To check the limit $r = b$, divide the numerator and denominator by R^2,

$$\Phi(b) = \frac{\tau}{4\pi}\ln[\frac{(1 + V^2 - 2V\cos\phi)}{V^2(1 + V^2 - 2V\cos\phi)}] \tag{9}$$

Thence, $V = -\frac{\tau}{2\pi}\ln[V]$ whose inversion is provided through the W-Lambert function, yielding

$$V = -W[\frac{\tau}{2\pi}] = \frac{b}{R} \tag{10}$$

Therefore, once the linear charge density is fixed, both the potential V and radius b are fixed for a given particle placed at R. This is a sufficient condition implying the potential to be 0 at infinity and V in the cyllinder boundary. The potential can also be written in the form

$$\Phi(r) = -2\pi\frac{V}{\ln V}\ln[\frac{\frac{r^2}{R^2} + 1 - \frac{2r}{R}\cos\phi}{V^4(\frac{r^2}{R^2V^4} + 1 - \frac{2r}{RV^2}\cos\phi)}] \tag{11}$$

Finally, the density is provided through

$$\sigma(\phi) = -\epsilon_0\Phi'(b) \tag{12}$$

Calculation is mechanic and details of this are supressed. One may obtain the result through simple algebraic software and display the graphics.

4. Answer to question 4:

   We find the Green's function obeying the boundary condition of constant potential at $r = b$ by letting $\frac{\tau}{2\pi} \to 1$ in eq. (5), and replacing $\vec{R}$ by r' as to agree with the common notation,

$$G_D(r, r') = \frac{\tau}{2\pi}\ln[\frac{|\vec{r} - \vec{r}'|}{|\vec{r} - \frac{b^2}{r'^2}\vec{r}'|}\frac{b}{r'V}] \tag{13}$$

   Thence,

$$G'_D(r, b) = \frac{1}{2\pi b}\frac{b^2 - r^2}{(r^2 + b^2 - 2rb\cos(\phi - \phi'))} \tag{14}$$

   and the density is null inside the cylinder, the Green's function theorem with the Green's function can be applied

$$\Phi(r) = \int da' V(\Omega')G'_D(b, r') = \int d\Omega\frac{V(\Omega)}{2\pi}\frac{b^2 - r^2}{(r^2 + b^2 - 2rb\cos(\phi - \phi'))} \tag{15}$$

such that $\Phi(r'=b,\Omega') = V(\Omega')$. Let V

$$V(\phi') = \begin{cases} \frac{V_1-V_2}{2} \text{ if } -\frac{\pi}{2} < \phi' < \frac{\pi}{2} \\ -\frac{V_1-V_2}{2} \text{ if } \frac{\pi}{2} < \phi' < \pi \end{cases} \tag{16}$$

In which we have placed the gap at $\phi = \pi/2$. Thence,

$$\begin{aligned} \Phi(r) = \frac{1}{\pi}(b^2-r^2)\{\frac{(V_1-V_2)}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{V(\phi')d\phi'}{r^2+b^2-2rb\cos(\phi-\phi')} + \\ -\frac{(V_1-V_2)}{2}\int_{\frac{\pi}{2}}^{\pi} \frac{V(\phi')d\phi'}{r^2+b^2+2rb\cos(\phi-\phi')}\} \end{aligned} \tag{17}$$

Through the parity of the cos(), the displacement $\phi' = \phi + \pi$, the second integral possess the same limits of integration of the first, in a convenient form to write

$$\Phi(r) = \frac{1}{\pi}(b^2-r^2)\frac{(V_1-V_2)}{2}\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi'[\frac{1}{r^2+b^2-2rb\cos(\phi-\phi')} - \frac{1}{r^2+b^2+2rb\cos(\phi-\phi')}] \tag{18}$$

A simple but tedious algebraic simplification can be performed by hand or, in faster way, through Mathematica, yielding

$$\Phi(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(\phi-\phi')}{(b^2-\rho^2)^2+4b^2\rho^2\sin^2(\phi-\phi')}d\phi' = \frac{V_1-V_2}{\pi}\tan^{-1}[\frac{2b\rho}{b^2-\rho^2}\cos\phi] \tag{19}$$

Naturally we have obtained the expression for two cylinders with opposing potentials $\frac{V_1-V_2}{2}$ and $\frac{V_1+V_2}{2}$. The proper way to obtain the result for V_1 and V_2 is through summing the arbitrary potential $\frac{V_1+V_2}{2}$ to the result. In this way we guarantee the proper boundary condition is satisfied in the boundary. Due to the uniqueness theorem, there is only one solution satisfying the boundary condition and we the result is reliable. We consider there is another way to treat the problem by directly imposing V_1 and V_2 in the boundary, but this is very tricky and is to be written later, since it apparently violates the uniqueness of a physical function in a line. It seems simple at a first glance but indeed it need a thorough and careful treatment on its own.

5. Answer to question 5:

(a)

Another way to write Jackson's expression is through explicitly considering the linearly independent terms sin(), cos(), separately. In this way,

$$\Phi(\rho,\phi) = a_0 + b_0\ln\rho + 0 = a_0 + b_0 + \sum_\nu \rho^\nu[A_\nu\sin\nu\phi + A_{*\nu}\cos\nu\phi] + \sum_\nu \rho^{-\nu}[B_\nu\sin\nu\phi + B_{*\nu}\cos\nu\phi] \tag{20}$$

The boundary condition is collected

$$\begin{aligned} \phi(\rho,0) = 0 \\ \phi(\rho,\beta) = 0 \\ \phi(a,\theta) = 0 \end{aligned} \tag{21}$$

From the first,

$$0 = a_0 + b_0 \ln \rho + \sum_{\nu} a_\nu \rho^\nu [A_\nu \sin \nu\phi + A_{*\nu} \cos \nu\phi] + \sum_{\nu} b_\nu \rho^{-\nu} [B_\nu \sin \nu\phi + B_{*\nu} \cos \nu\phi]$$

Thence, $a_0 = 0$, $b_0 = 0$, and $A_{\nu^*=0} = B_{\nu^*=0}$. From the second the value of ν assumes the values $\nu = \frac{n\pi}{\beta}$ with n an integer. The third condition imply

$$0 = \sum_{\nu=1} [a^\nu A_\nu + a^{-\nu} B_\nu] \sin \frac{n\pi\phi}{\beta} \tag{22}$$

From the second the value of ν, it assumes the values $\beta = \frac{\nu\pi}{\beta}$ with n an integer; indeed ν is countable. Thence,

$$B_\nu = a^{2\nu} A_\nu \tag{23}$$

and the solution is retrieved in the form

$$\Phi(\rho, \phi) = \sum_{\nu=1} [a^\nu A_\nu + a^{-\nu} B_\nu] \sin \frac{n\pi\phi}{\beta} \tag{24}$$

Therefore,

$$\Phi(\rho, \phi) = \sum_{\nu} A_\nu [\rho^{\frac{n\pi\phi}{\beta}} + (\frac{a^2}{\rho^\nu})^{\frac{n\pi\phi}{\beta}}] \sin \frac{n\pi\phi}{\beta} \tag{25}$$

The problem is formally solved and what remains is to extract some other measurable physical information.

(b)

To be done, following immediately from (a).

(c)

To be done.