Electrodynamics
Solution to selected problem of Jackson's

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Chapter 1

Answer to question 1.2

Due to the provided line elements,

$$|\partial_u \vec{r}| = \frac{1}{U} \qquad (1)$$
$$|\partial_v \vec{r}| = \frac{1}{V}$$
$$|\partial_w \vec{r}| = \frac{1}{W}$$

where U,V and W are in general functions of (u, v, w) each. Let $\delta(\vec{x}) = f(u, v, w)\delta(u)\delta(v)\delta(w)$. A change of coordinate from $(x, y, z) \to (u, v, w)$ yields

$$1 = \int \delta(\vec{x})d^3x = \int f(u, v, w)\delta(u)\delta(v)\delta(w)|\partial_u \vec{r} \cdot (\partial_v \vec{r} \times \partial_w \vec{w})|dudvdw \qquad (2)$$

Due to the orthogonality of the coordinate system, it follows

$$1 = \int f(u, v, w)\delta(u)\delta(v)\delta(w)\frac{dudvdw}{UVW} \qquad (3)$$

Therefore, it is necessary $f(u, v, w) = U(u, v, w)V(u, v, w)W(u, v, w)$, and

$$\delta(\vec{x}) = UVW\delta(u)\delta(v)\delta(w) \qquad (4)$$

Answer to question 1.3

(a) A distribution of charge in spherical coordinates in the 3d space is such that

$$\int_0^\infty drr^2 \int_\Omega \rho(r, \theta, \phi)d\Omega = Q \qquad (5)$$

For a shell localized in $r = R$, $R^2 \int d\Omega\rho(\Omega) = Q$. A way to explicitly obtain R^2 in the LHS is to consider $\rho(r, \Omega) = \delta(r - R)\rho(\Omega)$. In particular, for a uniform distribution, $\rho(\Omega) = \rho$, a constant, and $\rho = \frac{Q}{4\pi R^2}$. Thus,

$$\rho(r, \Omega) = \delta(r - R)\frac{Q}{4\pi R^2} \qquad (6)$$

(b)

$$\int_0^\infty drr \int_\phi d\phi \int_z \rho(r, \phi, z)dz = Q \qquad (7)$$

Now, $\rho(r,\phi,z) = \delta(r-R)\rho$ with ρ a constant, since there is no dependence of the density on either the angle or z position. Therefore, integrating produces $\rho 2\pi RL = Q$, and as $\frac{Q}{L}$ defines a linear density, $\rho = \frac{\lambda}{2\pi R}$.

$$\rho(r,\phi,z) = \delta(r-R)\frac{\lambda}{2\pi R} \tag{8}$$

(c) For a flat circular disk, in cyllindrical coordinates,

$$\rho(z,r,\theta) = \delta(z)\rho(r,\phi) \tag{9}$$

In the case considered, ρ is independent on both r and ϕ, thence $\rho(z,r,\phi) = \delta(z)\rho$ and an integration $\rho\int_0^{2\pi}\int_{r=0}^{R}\int r\sin\phi dr = Q$ yields $\rho = \frac{Q}{\pi R^2}$. Thus,

$$\rho(z,r,\theta) = \frac{Q}{\pi R^2}\delta(z) \tag{10}$$

(d) In spherical coordinates, in an analogous way, $\rho(r,\theta,\phi) = \delta(\theta)\rho(r,\phi) = \delta(\theta)\rho$, with the last identity holding for the uniform case (ρ constant). Thence, an integration leads to $2\rho\pi\frac{R^3}{3} = Q$ and

$$\rho = 3\frac{QR^3}{2\pi}\delta(\theta) \tag{11}$$

Answer to question 1.5

To extract the charge distribution (in electrostatics) from the potential distribution is a simply matter of applying the Laplacian as

$$-\epsilon_0\nabla^2\phi = \rho \tag{12}$$

Thus,

$$-\epsilon_0\nabla^2\phi = \rho = -\frac{q}{4\pi}[\nabla^2(\frac{\exp[-ar]}{r}) - \frac{a}{2}\nabla^2\exp[-ar])] \tag{13}$$

Expanding,

$$\rho = -\frac{q}{4\pi}\{\exp[-ar]\nabla^2(\frac{1}{r}) + (\frac{1}{r} - \frac{a}{2})\nabla^2\exp[-ar] + 2\nabla(\frac{1}{r})\cdot\nabla(\exp[-ar])\} \tag{14}$$

Thence seeing ρ as a distribution, it is possible to consider an integration by parts as to obtain the form of the divergence due to the term $\nabla(\frac{1}{r})$ in terms of $\nabla^2(\frac{1}{r}) = \delta(r)$.

$$\rho = -\frac{q}{4\pi}\{-\exp[-ar]4\pi\delta(r) + (\frac{1}{r} - \frac{a}{2})\nabla^2\exp[-ar]\} \tag{15}$$

$$\begin{aligned}\rho &= \frac{q}{4\pi}\exp[-ar]\{4\pi\delta(r) - \exp[-ra]\frac{a}{r^2}(\frac{ra}{4} - \frac{1}{2})^2\} \\ \rho &= qa^3\exp[-ar]\delta(r) - \frac{q}{4\pi}\exp[-ra]\frac{a^3}{(ra)^2}(\frac{ra}{4} - \frac{1}{2})^2\}\end{aligned} \tag{16}$$

Where the expression is written for an nondimensional r, i.e, $\delta(r)$ is in fact $\delta(ra) = a^3\delta(r)$ for some length unity a. This makes the expression as a whole consistent. The first term approaches a positive point charge contribution when $r \to 0$, the second term has a mix of exponential and power decay. The involved charges are the same, but opposite. The first term is due to a localized positive charge (nucleus), the other term is due to non-localized negative charge (electronic cloud) for the choice $q > 0$. This is phonologically expected for a symmetric atomic distribution.

Answer to question 1.6

(a) From the Gauss law applied to a small gaussian cyllinder near one of the flat sheets and encompassing a charge of magnitude q,

$$(\vec{E}\cdot\hat{n}_1 - \vec{E}\cdot\hat{n}_2)\Delta a = \frac{q}{\epsilon_0} \tag{17}$$

as $n_2 = -n_1$, and with Δa representing the area of the cylinder projected onto the plane. $E = \frac{Q}{2\epsilon_0 \Delta A}$ The potential difference between the sheets is provided through $\Delta V = Ed = \frac{Qd}{2\epsilon_0 \Delta A}$. Finally, the ratio $C = \frac{\Delta V}{Q}$ defines the capacitance

$$C = \frac{d}{2\epsilon_0 A} \tag{18}$$

(b)

In this case, the Gauss law produces the trivial known result $E = \frac{Q}{4\pi r^2}$ through a concentric sphere in between the conductors of radius a and b. The potential difference between the shells is

$$V = \int_a^b E dr = -\frac{Q}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0}(\frac{1}{a} - \frac{1}{b}) \tag{19}$$

Therefore, according to the usual definition for capacitance,

$$C = \frac{1}{4\pi\epsilon_0}(\frac{1}{a} - \frac{1}{b}) \tag{20}$$

(c)

A concentric cyllinder with radius lying in between a and b is used as a gaussian surface. This is suitable since the electric field is constant along this surface. Thence, Gauss law imply $E = \frac{Q}{L}\frac{1}{2\pi r\epsilon_0}$. An integration produces the potential difference between the cyllinders of radii a and b,$V = \frac{Q}{2\pi\epsilon_0 L}\ln\frac{b}{a}$. Hence,

$$C = \frac{1}{2\pi\epsilon_0 L}\ln\frac{b}{a} \tag{21}$$

Answer to question 1.7

We wish to compute the capacitance density

$$\frac{C}{L} = \frac{Q}{LV} = \frac{\lambda}{V} \tag{22}$$

where λ stands for linear density. Thence, as the cyllinders are infinitely long, not any component for the electric field is expected except the radial one. Let the system of coordinates be placed on the first cyllinder to which we associate the index 1. Thence,

$$V = \frac{\lambda}{2\pi}\int_{a_1}^{d-a_2} dr_1(\frac{1}{r_1} - \frac{1}{d-r_1}) = \frac{\lambda}{2\pi}\ln(\frac{d}{a_1}(1-\frac{a_2}{d})) - \ln(\frac{a_2}{d}\frac{1}{(1-\frac{a_1}{d})}) \tag{23}$$

Due to the logarithm properties and $d >> a_1$ and $d >> a_2$, and denoting $a^2 = a_1 a_2$ the geometric average, it is clear the below relation

$$V = \frac{\lambda}{2\pi}\ln(\frac{d}{a})^2 = \frac{\lambda}{\pi}\ln\frac{d}{a} \tag{24}$$

The capacitance density is, therefore,

$$\frac{C}{L} = \pi(\ln\frac{d}{a})^{-1} \tag{25}$$

Answer to question 1.10

Consider the Green's identity for the Neumman boundary problem

$$\Phi(x) = \langle\Phi\rangle_S + \frac{1}{4\pi}\int_V G(x,x')\rho(x')d^3x' + \oint_S G(x,x')\nabla\Phi(x')\cdot\vec{n}da' \tag{26}$$

with $\langle\Phi\rangle$ denoting the average of the inner function on the surface. In the absence of charge within the volume, it follows

$$\Phi(x) = \langle\Phi\rangle_S - \oint_S G(x,x')E_n(x')da' \tag{27}$$

as $-\nabla\Phi(x')\cdot\vec{n} = E_n(x')$. In the surface, $E_n(x') = \sigma(x')/\epsilon_0$. The absence of charge within the surface imply the mean value theorem, explicitly,

$$\Phi(x) = \langle\Phi\rangle_S \tag{28}$$

Answer to question 1.12

(a) Let

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{29}$$

$$\nabla^2\Phi' = -\frac{\rho'}{\epsilon_0} \tag{30}$$

Due to the vectorial identity

$$\Phi'\nabla^2\Phi - \Phi\nabla^2\Phi' = \nabla\cdot(\Phi'\nabla\Phi - \Phi\nabla\Phi') \tag{31}$$

the identification $-\nabla\Phi = \vec{E}$, applying the Poisson relations and the Gauss theorem,

$$\int[\Phi\rho' - \Phi'\rho]d^3x = \epsilon_0\int da[\Phi E_n' - \Phi' E_n] \tag{32}$$

In the external surface, $E_n = \frac{\sigma}{\epsilon_0}$, therefore

$$\int[\Phi\rho' - \Phi'\rho]d^3x = \int da[\Phi\sigma' - \Phi'\sigma] \tag{33}$$

which is clearly equivalent to the identity we wish to prove through a simple algebraic rearrangement.

Answer to question 1.13

Consider as gaussian surface as a box containing the sheets at zero potential and far away perpendicular walls (formally, at infinity). Compare to the problem of two capacitor sheets, each with charge density σ, whose physical measurements are labelled with primed functions. Hence, we apply the reciprocal Green's Theorem. In the boundary, $\Phi_a = 0$,

$$\int \Phi'\rho d^3x = \int da\sigma\Phi_a' \tag{34}$$

as $\rho(x) = q\delta(x - x_1)$ for some $x_1 \in (0, d)$, with d the distance between conductors.

$$-q\Phi'(x_1) = \int da\sigma\Phi_a' \tag{35}$$

As Φ' is constant in each conductor surface,

$$-q\Phi'(x_1) = [\Phi'(d) - \Phi'(0)]q_{ind} \tag{36}$$

We have seen that the potential $\Phi'(x) \sim x$ within capacitors, as the electric field is constant. Therefore,

$$q_{ind} = -q\frac{x_1}{d} \tag{37}$$

as required.

Chapter 2

Answer to question 2.10

Choose a system of coordinates placed in the center of the boss. The solution to the Laplace equation is provided through the spherical coordinate system.

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta) \tag{38}$$

With boundary conditions

$$\Phi(r=a) = 0 \tag{39}$$

$$\partial_z\Phi|_{z\to\infty} = E_0 \tag{40}$$

The first condition imply

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l[r^l - \frac{a^{2l+1}}{r^{l+1}}]P_l(\cos\theta) \tag{41}$$

The chain rule provides

$$\partial_z\Phi = \partial_r\Phi(r,\cos\theta)\partial_z r + \partial_{\cos\theta}\Phi(r,\cos\theta)\partial_z(\cos\theta) \tag{42}$$

Since $\cos\theta = \frac{z}{r}$, $\partial_z(\cos\theta) = \frac{1}{r} - \frac{z^2}{r^3}$. In the limit of large z, $z \sim r$ and the main contribution is due to the first term. It is possible to compute the entire solution, however we restrict to this approximation for now (to be extended to the general solution later).

$$\partial_z\Phi|_{z\to\infty} \sim \partial_r\Phi(r,\cos\theta)|_{r\to\infty} = E_0 \tag{43}$$

Notice we could have written this expression directly before, however it would not allow for obtaining the most accurate solution we are to derive afterwards. This hold only if $A_1 = E_0$ and $A_2 = 0$, $A_3 = 0$,...,$A_\infty = 0$. A_0 is arbitrary but it is a constant not important to the description of the problem, since it does not matter the value of the potential at infinity, rather its variation. We set $A_0 = 0$. Therefore,

$$\Phi(r,\phi) = E_0[r\cos\theta - \sum_{l=1}^{\infty}\frac{a^{2l+1}}{r^{l+1}}P_l(\cos\theta)] \tag{44}$$

as $P_1(\cos\theta) = \cos\theta$. Thence, on the boss,

$$\sigma(a,\theta) = -\epsilon_0\Phi'(a,\theta) = -\epsilon_0 E_0(\cos\theta + \sum_{l=1}^{\infty}(l+1)a^{l-1}P_l(\cos\theta)) \tag{45}$$

Considering r close to a, the main contribution is that for $l = 1$, which is independent of the size of a, $\sigma(a,\phi) = -\epsilon_0 E_0(3\cos\theta)$.

As $\partial_z r|_{z=0} = 0$ and $\partial_z(\cos\theta)|_{z=0} = \frac{1}{r}$. Therefore, along the plane $z = 0$,

$$\begin{aligned}\sigma(z=0) = \sigma(\cos\theta = \pm 1, r >> a) = -\epsilon_0\partial_z\Phi|_{z=0} = -\frac{\epsilon_0}{r}\partial_{\cos\theta}\Phi(r,\cos\theta)|_{\cos\theta=\pm 1} = \\ -\epsilon_0[E_0 - \sum_{l=1}^{\infty}\frac{a^{2l+1}}{r^{l+2}}]P_l'(\cos\theta)\end{aligned} \tag{46}$$

For $l = 1$, in fact the region where this approach is correct (see dimension in powers of a and r),

$$\sigma(z=0) = -\epsilon_0[E_0 - \frac{a^3}{r^3}] \tag{47}$$

(b)

$$Q_{boss} = 3\epsilon_0 E_0 \int_{\cos\theta=-1}^{\cos\theta=1} \cos\theta d(\cos\theta) \times (2\pi) \times a^2 = 0 \tag{48}$$

However, if the charge in half of the boss is considered, it magnitude is

$$Q_{boss} = 3\epsilon_0 E_0 \int_{\cos\theta=-1}^{\cos\theta=0} \cos\theta d(\cos\theta) \times (2\pi) \times a^2 = 3\pi a^2 \epsilon_0 E_0 \tag{49}$$

Answer to question 2.13

$$G_D(x,x') = \frac{1}{4\pi} \log[\frac{x^2 + x'^2 - 2xx'\cos\gamma}{(\frac{xx'}{a})^2 + a^2 - 2xx'\cos\gamma}] \tag{50}$$

$$\partial_{x'} G_D(x,x')|_{x'=a} = \frac{1}{2\pi a} \frac{a^2 - x^2}{a^2 + x^2 - 2ax\cos(\phi' - \phi)} \tag{51}$$

The superposition principle can be applied.

$$\Phi = \frac{V_1 + V_2}{2} + \frac{a^2 - x^2}{2\pi} \int_{\phi'=0}^{2\pi} d\phi' \frac{f(V_1, V_2)}{a^2 + x^2 - 2ax\cos(\phi' - \phi)} \tag{52}$$

$$f(V_1, V_2) = \begin{cases} \frac{V_1 - V_2}{2} \text{ if } 0 < \phi' < \pi \\ -\frac{V_1 - V_2}{2} \text{ if } \pi < \phi' < 2\pi \end{cases} \tag{53}$$

where the integration is done in the element of line $ad\phi'$, the a cancels. Therefore, it remains to evaluate the integral on (3), coined $\mathcal{I}$. This is divided piecewisely according to (4), with the multiplying factor of $\frac{(V_1 - V_2)}{2}(a^2 - x^2)/2\pi$ emitted.

$$\mathcal{I} = \int_{\phi'=0}^{\pi} \frac{d\phi'}{a^2 + x^2 - 2ax\cos(\phi' - \phi)} - \int_{\pi}^{2\pi} \frac{d\phi'}{a^2 + x^2 - 2ax\cos(\phi' - \phi)} \tag{54}$$

The same limit of integration is met through $\phi' \to \phi' + \pi$ in the second integrand,

$$\mathcal{I} = \int_{\phi'=0}^{\pi} d\phi' [\frac{1}{a^2 + x^2 - 2ax\cos(\phi' - \phi)} - \frac{1}{a^2 + x^2 + 2ax\cos(\phi' - \phi)}] \tag{55}$$

Thence,

$$\mathcal{I} = \int_{\phi'=0}^{\pi} d\phi' \frac{4ax\cos(\phi' - \phi)}{(a^2 + x^2)^2 - 4a^2x^2\cos^2(\phi' - \phi)} = \int_{\phi'=0}^{\pi} d\phi' \frac{4ax\cos(\phi' - \phi)}{(a^2 - x^2)^2 + 4a^2x^2\sin^2(\phi' - \phi)} \tag{56}$$

with the trigonometric substitution $z = \sin(\phi' - \phi)$, and $\gamma = \frac{2axz}{a^2 - x^2}$

$$\mathcal{I} = 4ax \int_{\sin\phi}^{-\sin\phi} \frac{dz}{(a^2 - x^2)^2 + 4a^2x^2z^2} = \frac{4ax}{(a^2 - x^2)^2} \int_{\sin\phi}^{-\sin\phi} \frac{dz}{1 + (\frac{2axz}{a^2 - x^2})^2} \tag{57}$$

$$\mathcal{I} = \frac{4ax}{(a^2 - x^2)^2} \frac{(a^2 - x^2)}{2ax} \int_{-\frac{\sin\phi 2ax}{a^2 - x^2}}^{\frac{\sin\phi 2ax}{a^2 - x^2}} \frac{d\gamma}{1 + \gamma^2} = \frac{4}{a^2 - x^2} \tan^{-1}[\frac{2ax\sin\phi}{a^2 - x^2}] \tag{58}$$

Thence, multiplying $\frac{V_1 - V_2}{2}(a - x)^2/2\pi$ for the integral $\mathcal{I}$ and returning to Φ,

$$\Phi = \frac{V_1 + V_2}{2} + \frac{V_1 + V_2}{2} \tan^{-1}[\frac{2ax\sin\phi}{a^2 - x^2}] \tag{59}$$

The appearance of $\cos\phi$ is trivially obtained through an active rotation of the system.

Answer to question 2.15

(a)

$$G(x, y, x', y') = 2\sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi x')g_n(y, y') = \sum_{n=-\infty}^{\infty} \sin(n\pi x)\sin(n\pi x')g_n(y, y') \tag{60}$$

where observing the parity, we write as to encompass all possible n values. Introducing in the Poison equation with a $-4\pi\delta(\vec{x} - \vec{x}')$ delta source plus boundary conditions specifies the Green's function,

$$-\sum_{n=-\infty}^{\infty} n^2\pi^2 \sin(n\pi x)\sin(n\pi x')g_n(y, y') + \sum_{n=-\infty}^{\infty} \sin(n\pi x)\sin(n\pi x')\partial_y^2 g_n(y, y') = -4\pi\delta(x - x')\delta(y - y') \tag{61}$$

Applying the operator $\mathcal{O} = \int_{-1}^{1}\int_{-1}^{1} dxdx' \sin(n'\pi x)\sin(n'\pi x')()$ to both sides of the equation and through

$$\int_{-1}^{1} \sin(n\pi x)\sin(n'\pi x)dx = \delta_{nn'}$$

$$\{-n^2\pi^2 g_n(y, y') + \partial_y^2 g_n(y, y')\} = -4\pi\delta(y - y') \tag{62}$$

since $\mathcal{O}(\delta(x - x')) = \int_{-1}^{1} dx \sin(n'\pi x)\sin(n'\pi x) = 1$. Thus,

$$(\partial_y^2 - n^2\pi^2)g_n(y, y') = -4\pi\delta(y - y') \tag{63}$$

The homogeneous boundary condition at $x = 0$ and $x = 1$ (with $y \in (0, 1)$) is already satisfied in the representation. For the $y = 0$ and $y = 1$ (with $x \in (0, 1)$) condition

$$\sum_{n=-\infty}^{\infty} \sin(n\pi x)\sin(n\pi x')g_n(0, y') = 0 \tag{64}$$

$$\sum_{n=-\infty}^{\infty} \sin(n\pi x)\sin(n\pi x')g_n(1, y') = 0 \tag{65}$$

For this to hold, the orthogonality condition imply

$$g_n(0, y') = 0 \tag{66}$$

$$g_n(1, y') = 0 \tag{67}$$

Thence, $g(y, y')$ is a 1-D Green's function to be specified.

(b)

For any solution to the homogeneous equation which is a mixture of hiperbolic functions *(apart from the pure hyperbolic cosine solution, which is indeed not possible due to the boundary condition $g_n(0, y') = 0$)* there is the representation

$$g_n(y, y') = \begin{cases} C_{1_n}(y')\sinh(n\pi(y - \phi_1)) \text{ if } y < y' \\ C_{2_n}(y')\sinh(n\pi(y - \phi_2)) \text{ if } y > y' \end{cases} \tag{68}$$

Further it must obey

$$g_n(1, y') = C_{2_n}(y')\sinh(n\pi(1 - \phi_2)) = 0 \tag{69}$$

$$g_n(0, y') = C_{1_n}(y')\sinh(-n\pi\phi_1) = 0 \tag{70}$$

which provides $\phi_2 = 1$ and $\phi_1 = 0$, thence,

$$g_n(y, y') = \begin{cases} C_{1_n}(y')\sinh(n\pi(y)) \text{ if } y < y' \\ C_{2_n}(y')\sinh(n\pi(y - 1)) \text{ if } y > y' \end{cases} \tag{71}$$

Thence,

$$g_n(y',y) = \begin{cases} C_{2_n}(y)\sinh(n\pi(y'-1)) \text{ if } y<y' \\ C_{1_n}(y)\sinh(n\pi(y')) \text{ if } y>y' \end{cases} \tag{72}$$

And the symmetry in the Green's function imply the need of

$$C_{2_n}(y')\sinh(n\pi(y-1)) = C_{1_n}(y)\sinh(n\pi y') \tag{73}$$

which is solved (nontrivial solution) iff $C_{1_n}(y) = \mathcal{C}_n \sinh(n\pi(y-1))$ and $C_{2_n}(y) = \mathcal{C}_n \sinh(n\pi y)$ for some constant $\mathcal{C}_n$. Thence,

$$g_n(y,y') = \begin{cases} \mathcal{C}_n \sinh(n\pi y)\sinh(n\pi(y'-1)) \text{ if } y<y' \\ \mathcal{C}_n \sinh(n\pi y')\sinh(n\pi(y-1)) \text{ if } y>y' \end{cases} \tag{74}$$

or, in Jackson notation $g_n(y,y') = \mathcal{C}_n \sinh(n\pi y_<)\sinh(n\pi(y_>-1))$ and it remains to compute the constant $\mathcal{C}_n$. This is obtained from the contribution of the delta discontinuity. An integration of the equation for g_n in between $y'-\epsilon$ and $y'+\epsilon$ in the limit of $\epsilon \to 0$ for $\epsilon > 0$.

$$\lim_{\epsilon\to 0} \partial_y g_n(y,y')|_{y=y'-\epsilon}^{y=y'+\epsilon} = -4\pi\delta(y-y') \tag{75}$$

Then,

$$\mathcal{C}_n n\pi[\sinh(n\pi y')\cosh(n\pi(y'-1)) - \cosh(n\pi y')\sinh(n\pi(y'-1))] = -4\pi \tag{76}$$

$$\mathcal{C}_n = -\frac{4}{n\sin(n\pi)} \tag{77}$$

Therefore,

$$G(x,y,x',y') = 8\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n\sin(n\pi)} \sin(n\pi x')\sinh(n\pi y_<)\sinh(n\pi(1-y_>)) \tag{78}$$

where the sign minus cancels by the oddness of the sinh function. $\sinh(n\pi(1-y_>)) = -\sinh(n\pi(y_>-1))$. q.e.d.

Answer to question 2.20

(a)

The proof is trivial. Sketched: using cylindrical coordinates ($da = \rho d\phi d\rho$), a region encompassing the charges produces their value. A region not encompassing the charges produces zero.

(b)

$$\phi(\rho,\phi) = \frac{1}{4\pi\epsilon_0}\int G(\rho,\phi,\rho',\phi')\sigma(\rho',\phi')da' \tag{79}$$

$$\Phi(\rho,\phi) = \frac{2\lambda}{4\pi\epsilon_0}[G(\rho,a,\phi,0) - G(\rho,a,\phi,\frac{\pi}{2}) + G(\rho,a,\phi,\pi) - G(\rho,a,\phi,\frac{3\pi}{2})] \tag{80}$$

Thus,

$$\Phi(\rho,\phi) = \frac{\lambda}{\pi\epsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\frac{a^m}{\rho^m}[\cos m\phi - \cos(m(\phi-\pi/2)) + \cos(m(\phi-\pi)) - \cos(m\phi - 3\pi/2)] \tag{81}$$

(the logarithmic term cancels). Concerning the angular term, it is rewritten as

$$\cos m\phi + \cos m\phi(-1)^m - \cos m\phi\cos\frac{m\pi}{2} - \cos(m\phi)\cos(\frac{m\pi}{2}+m\pi)] = \tag{82}$$

$$\cos m\phi[1+(-1)^m - \cos\frac{m\pi}{2} - \cos\frac{m\pi}{2}(-1)^m] = \cos m\phi(1+(-1)^m)[1-\cos\frac{m\pi}{2}] \tag{83}$$

Defining $f(m) = \cos\frac{m\pi}{2}$, $f(1) = 0, f(2) = -1, f(3) = 0, f(4) = 1, f(5) = 0, f(6) = -1, f(7) = 0, f_8 = 1, ...$ Thence, for m odd, the expression vanishes since the first term vanishes. For m even, and multiple of $4n$, the second term vanishes. The only remaining term is the contribution due to $m = 4k + 2$, $k = 0, 1, 2, ...$ Notice a 4 numeric factor 4 when this is the case.

$$\Phi(\rho, \phi) = \frac{2\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{1}{2k+1}\frac{a^{4k+2}}{\rho^{4k+2}}\cos[(4k+2)\phi] \tag{84}$$

An analogous procedure can be performed for the external problem, however, a and ρ interchanges the role.

(c)

$$\Phi(\rho, \phi) = \frac{2\lambda}{\pi\epsilon_0}\Re\{\sum_{k=0}^{\infty}\frac{1}{2k+1}(\frac{a^2}{\rho^2\exp[2\phi i]})^{2k+1}\} \tag{85}$$

Since $\Re[\exp[-i\phi]] = \cos\phi$. Setting $z = \rho\exp[i\phi]$,

$$\Phi(z) = \frac{2\lambda}{\pi\epsilon_0}\Re\{\frac{1}{2}\log[\frac{1+\frac{a^2}{z^2}}{1-\frac{a^2}{z^2}}]\} \tag{86}$$

since, $\sum_{n_{odd}}\frac{Z^n}{n}$, as Jackson suggests in page 75, and it also can be trivially derived. The function mentioned in the problem is rewritten in the form

$$w(z) = \frac{\lambda}{2\pi\epsilon_0}\log[\frac{z^2+a^2}{z^2-a^2}] = \frac{\lambda}{2\pi\epsilon_0}\log[\frac{1+\frac{a^2}{z^2}}{1-\frac{a^2}{z^2}}] \tag{87}$$

Therefore, comparing the solution with this function, $\Phi(z) = 4\Re[\omega(z)]$.

Answer to question 2.25

(a) The general solution series for the 2D Laplace equation is provided through

$$\Phi(\rho, \phi) = \sum_{n=-\infty}^{\infty} a_n\rho^n + b_0\ln\rho \tag{88}$$

Let (ρ', ϕ') be the coordinate of the particle where the field fails to obey the Laplace equation. Using the boundary conditions: potential vanishes at infinity and in the lower boundary ($\phi = 0 < \phi'$) and in the upper boundary $\phi = \beta > \phi'$ is null for every ρ. This is satisfied only if (an extra condition on n is needed, to be seen later on).

$$\Phi(\rho, \phi) = \begin{cases} \sum_{n=1}^{\infty} a_{1n}(\rho', \phi')\rho^n\sin(n\phi+\gamma_n) \text{ if } \rho < \rho' \text{ and } \phi > \phi' \\ \sum_{n=1}^{\infty} a_{2n}(\rho', \phi')\rho^n\sin(n\phi) \text{ if } \rho < \rho' \text{ and } \phi < \phi' \\ \sum_{n=1}^{\infty} b_{1n}(\rho', \phi')\rho^{-n}\sin(n\phi+\gamma_n) \text{ if } \rho > \rho' \text{ and } \phi > \phi' \\ \sum_{n=1}^{\infty} b_{2n}(\rho', \phi')\rho^{-n}\sin(n\phi) \text{ if } \rho > \rho' \text{ and } \phi < \phi' \end{cases} \tag{89}$$

Due to the feature of the Poisson equation for a 2D point charge, a lot of simplification is carried out. It turns out that in fact there are only two branches to be considered. The Poisson equation for a 2D point charge reads

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Phi) + \frac{1}{\rho^2}\partial_\phi^2\Phi = \frac{1}{\epsilon_0\rho}\delta(\phi-\phi')\delta(\rho-\rho') \tag{90}$$

Integrating in a narrow interval centered in ϕ_c and ρ_c each yields, respectively

$$\lim_{\epsilon\to 0}\frac{1}{\rho}\partial_\phi\Phi|_{\phi_c-\epsilon}^{\phi_c+\epsilon} = \begin{cases} \frac{1}{\epsilon_0}\delta(\rho-\rho') \text{ if } \phi' = \phi_c \\ 0 \text{ if } \phi' \neq \phi_c \end{cases} \tag{91}$$

$$\lim_{\epsilon\to 0}\rho\partial_\rho\Phi(\rho,\phi)|_{\rho_c-\epsilon}^{\rho_c+\epsilon} = \begin{cases} \frac{1}{\epsilon_0}\delta(\phi-\phi') \text{ if } \rho'=\rho_c \\ 0 \text{ if } \rho'\neq\rho_c \end{cases} \tag{92}$$

This is the local condition for tangential fields, that of $\nabla\times\vec{E}=0$. The second is the local Gauss law.

The potential is continuous for $\phi_c\neq\phi'$ for either $\rho>\rho'$ or $\rho<\rho'$, i.e,

$$\Phi(\rho,\phi) = \begin{cases} \sum_{n=1}^{\infty} a_{1n}(\rho',\phi')\rho^n\sin(n\phi) \text{ if } \rho<\rho' \\ \sum_{n=1}^{\infty} b_{1n}(\rho',\phi')\rho^{-n}\sin(n\phi) \text{ if } \rho>\rho' \end{cases} \tag{93}$$

The boundary condition on the upper boundary imply $n=\frac{m\pi}{\beta}$ for $m\in\{1,2,...\}$.

Also, the solution is continue at $\rho=\rho'$ if $\phi_c\neq\phi$ (null rotational). This and Gauss law are written

$$\sum_{m=1}^{\infty}[a_{1m}(\rho',\phi')\rho'^{\frac{m\pi}{\beta}}\sin(\frac{m\pi\phi}{\beta}) - b_{1m}(\rho',\phi')\rho'^{-\frac{m\pi\phi}{\beta}}\sin(\frac{m\pi\phi}{\beta})] = 0 \tag{94}$$

$$\sum_{m=1}^{\infty}\{\frac{m\pi}{\beta}[a_{1n}(\rho',\phi')\rho'^n\sin(\frac{m\pi\phi}{\beta}) + b_{1n}(\rho',\phi')\rho'^{-\frac{m\pi}{\beta}}\sin(\frac{m\pi\phi}{\beta})]\} = \frac{1}{\epsilon_0}\delta(\phi-\phi') \tag{95}$$

Applying $\int_0^\beta \sin(\frac{m\pi\phi}{\beta})d\phi$ to both sides,

$$a_{1n} = \frac{2}{m\pi\epsilon_0}b_{1n}\rho'^{-\frac{2m\pi}{\beta}} \tag{96}$$

$$a_{1n}\rho'^n + b_{1n}\rho'^{-n} = \frac{2}{m\pi\epsilon_0}\sin(\frac{m\pi\phi'}{\beta}) \tag{97}$$

whose solution is

$$b_{1m} = \frac{1}{m\pi\epsilon_0}\sin(\frac{m\pi\phi'}{\beta})\rho'^{\frac{m\pi}{\beta}} \tag{98}$$

$$a_{1m} = \frac{1}{m\pi\epsilon_0}\sin(\frac{m\pi\phi'}{\beta})\rho'^{-\frac{m\pi}{\beta}} \tag{99}$$

Therefore, for a particle at (ρ',ϕ')

$$\phi(\rho,\phi) = \begin{cases} \sum_{n=1}^{\infty}\frac{1}{m\pi\epsilon_0}\sin(n\phi')\sin(n\phi)\rho^{\frac{m\pi}{\beta}}\rho'^{-\frac{m\pi}{\beta}} \text{ if } \rho<\rho' \\ \sum_{n=1}^{\infty}\frac{1}{m\pi\epsilon_0}\sin(n\phi')\sin(n\phi)\rho'^{\frac{m\pi}{\beta}}\rho^{-\frac{m\pi}{\beta}} \text{ if } \rho>\rho' \end{cases} \tag{100}$$

Multiplying for $4\pi\epsilon_0$, the Green function is, in a concise form

$$G(\rho,\phi;\rho',\phi') = 4\sum_{n=1}^{\infty}\frac{1}{m}\sin(\frac{m\pi\phi}{\beta})\sin(\frac{m\pi\phi'}{\beta})\rho_<^{\frac{m\pi}{\beta}}\rho_>^{-\frac{m\pi}{\beta}} \tag{101}$$

(b)

From

$$\sin(\frac{m\pi\phi}{\beta})\sin(\frac{m\pi\phi'}{\beta}) = \frac{1}{2}[\cos(\frac{m\pi(\phi-\phi')}{\beta}) - \cos(\frac{m\pi(\phi+\phi')}{\beta})] = \frac{1}{4}(x_-^m + x_-^{*\,m} - x_+^m - x_+^{*m}) \tag{102}$$

$x_- = \exp[\frac{im\pi(\phi-\phi')}{\beta}]$ and $x_+ = \exp[\frac{im\pi(\phi+\phi')}{\beta}]$. Thence, the green function is rewritten as

$$G(\rho,\phi;\rho',\phi') = \sum_{n=1}^{\infty}\frac{1}{m}[(\frac{\rho_<}{\rho_<})^{\frac{\pi}{\beta}}]^m[x_-^m + x_-^{*m} - x_+^m - x_+^{*m}] \tag{103}$$

We need to evaluate a sum of the kind

$$S[g] = \sum_{m=1}^{\infty} \frac{1}{m} g^m \tag{104}$$

Thence,

$$\partial_g S[g] = \sum_{m=1}^{\infty} g^{m-1} = \sum_{m=0}^{\infty} g^m = \frac{1}{1-g} \text{if } |g| < 1 \tag{105}$$

Thence,

$$S[g] = -\log[\alpha_0(1-g)] \tag{106}$$

The interest is on $g = (\frac{\rho_<}{\rho_>})^{\frac{\pi}{\beta}} x_i$, $i \in \{+,-\}$. It is possible to apply the identity since $|g| < 1$ produces $\alpha_0 = 1$ as $S[g=0] = 0$. Then,

$$G(\rho,\phi,\rho',\phi') = -\log[(1-x_-)(1-x_-^*)] + \log[(1-x_+)(1-x_+^*)] = \log[\frac{1-2\Re(x_+)+|x_+|^2}{1-2\Re(x_-)+|x_-|^2}] \tag{107}$$

$$G(\rho,\phi,\rho',\phi') = \log[\frac{1-2(\frac{\rho_<}{\rho_>})^{\frac{\pi}{\beta}}\cos[\frac{\pi(\phi+\phi')}{\beta}] + (\frac{\rho_<}{\rho_>})^{\frac{2\pi}{\beta}}}{1-2(\frac{\rho_<}{\rho_>})^{\frac{\pi}{\beta}}\cos[\frac{\pi(\phi-\phi')}{\beta}] + (\frac{\rho_<}{\rho_>})^{\frac{2\pi}{\beta}}}] \tag{108}$$

Multiplying for $\rho_>^{2\pi}$ above and below yields

$$G(\rho,\phi,\rho',\phi') = \log[\frac{\rho_<^{2\pi/\beta} + \rho_>^{2\pi/\beta} - 2\rho_<^{\pi/\beta}\rho_>^{\pi/\beta}\cos(\frac{\pi}{\beta}(\phi+\phi'))}{\rho_<^{2\pi/\beta} + \rho_>^{2\pi/\beta} - 2\rho_<^{\pi/\beta}\rho_>^{\pi/\beta}\cos(\frac{\pi}{\beta}(\phi-\phi'))}] \tag{109}$$

Chapter 3

Preamble on Bessel

Consider the Poisson equation

$$\nabla^2\Phi = -\frac{q}{\epsilon_0}\frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') \tag{110}$$

Consider solutions to the Laplace equation. Due to the separation of variables of the kind $\Phi = R(\rho)Q(\phi)Z(z)$, the Laplace equation is reduced to one of the cases.

$$\begin{aligned} Z'' &= k^2 Z \\ Q'' &= -m^2 Q \\ R(\rho) &= dJ_m(k\rho) + eN_m(k\rho) \end{aligned} \tag{111}$$

or

$$\begin{aligned} Z'' &= -k^2 Z \\ Q'' &= -m^2 Q \\ R(\rho) &= dI_m(k\rho) + eK_m(k\rho) \end{aligned} \tag{112}$$

and the general solution is therefore provided through superposition as either

$$\Phi = \sum_m a_m \exp[im\phi] \int_k dk[b(k)\sinh(kz) + c(k)\cosh(kz)][d(k)J_m(k\rho) + e(k)N_m(k\rho)] \tag{113}$$

$$\Phi = \sum_m a_m \exp[im\phi] \int_k dk[b(k)\sin(kz) + c(k)\cos(kz)][d(k)I_m(k\rho) + e(k)K_m(k\rho)] \tag{114}$$

respectively, without considerations on boundary conditions (free space).

Particular case: with the consideration of homogeneous boundary conditions at $\rho = a$:

For a zero potential at $\rho = a$, $k = a/x_{mn}$, x_{mn} is the n-th root of the m-Bessel function. In this case, k assumes only discrete values. Explicitly,

$$\Phi = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \exp[im\phi][b_{mn} \sinh(\frac{x_{mn}}{a} z) + c_{mn} \cosh(\frac{x_{mn}}{a} z)] \times$$
$$[d_{mn} J_m(\frac{x_{mn}}{a} \rho) + e_{mn} N_m(\frac{x_{mn}}{a} \rho)]$$
$$\Phi = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \exp[im\phi][b_{mn} \sin(\frac{x_{mn}}{a} z) + c_{mn} \cos(\frac{x_{mn}}{a} z)] \times$$
$$[d_{mn} I_m(\frac{x_{mn}}{a} \rho) + e_{mn} K_m(\frac{x_{mn}}{a} \rho)]$$

Answer to 3.9 and 3.10

We use the expansion of Bessel solutions such that the orthogonality properties can be applied in z and ϕ, since the potential is specified at $\rho = a$. The expansion that suits is the second expansion form of Laplace equation in cylindrical coordinates, as $\exp[im\phi]$ and $\{\sin(kz), \cos(kz)\}$ are orthogonal functions. The boundary conditions on z imply

$$k = \frac{n\pi}{L}$$
$$Z(z) \propto \sin(kz) \tag{115}$$

Thence, the integral is replaced by a sum and

$$\Phi = \sum_m a_{mn} \exp[im\phi] I_m(\frac{n\pi\phi}{L}) \sin(\frac{n\pi z}{L}) \tag{116}$$

Therefore,

$$a_{mn} = \frac{1}{\pi L I_m(\frac{n\pi\phi}{L})} \int_{\phi=0}^{2\pi} \int_{z=0}^{L} V(\phi, z) \sin(\frac{n\pi z}{L}) \exp[-im\phi] dz d\phi \tag{117}$$

In particular for for $+V$ in the inverval $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ and $-V$ in $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$, Thence, the integral on z can be made alone, yielding

$$\int_0^L \sin(\frac{n\pi z}{L}) dz = [1 + (-1)^{n+1}] \tag{118}$$

Therefore, only odd terms contribute in n. The integral in ϕ is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp[-im\phi] d\phi = \frac{-1}{im} \exp[-im\phi]|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{m} \sin(\frac{\pi}{2} m) \tag{119}$$

$$a_{mn} = \frac{2(1 + (-1)^{n+1}) \sin(\frac{\pi}{2} m)}{\pi m L I_m(\frac{n\pi\phi}{L})} \tag{120}$$

Answer to 3.14

(a)

The density is of the form (spherical coordinates)

$$\rho(r', \theta') = \begin{cases} \mathcal{K}(d^2 - r'^2 \cos^2\theta')[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)] \text{ if } r' < d \\ 0 \text{ , otherwise} \end{cases} \tag{121}$$

It is more convenient the coordinate $x' = \cos\theta'$, this is equivalent to

$$\rho(r', x') = \begin{cases} \mathcal{K}(d^2 - r'^2)[\delta(x'-1) + \delta(x'+1)] \text{ if } r' < d \\ 0 \text{ , otherwise} \end{cases} \tag{122}$$

this representation allows to separate in a single product of the kind $\rho = \mathcal{K}X(x')R(r')$. As the sphere is grounded, the only contribution is due to

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')G(x,x')d^3x' \tag{123}$$

The green's function in Legendre polynomial is obtained by setting $a \to 0$ on 3.125 of Jackson, yielding (since $Y_{l0}(\theta',\phi') = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta')$)

$$G(x,x') = \sum_{l=0}^{\infty} P_l(x)P_l(x')r_<^l(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}) \tag{124}$$

Thence,

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}P_l(x) \times (2\pi)\int_{-1}^{1} dx' \int_{r'=0}^{d} P_l(x')(r'^2dr')r_<^l(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}})\rho(x', r) \tag{125}$$

Or more conveniently,

$$\Phi(x) = \frac{\mathcal{K}}{4\pi\epsilon_0}P_l(x) \times (2\pi)\int_{-1}^{1} dx'X(x')P_l(x') \int_{r'=0}^{d} (r'^2dr')r_<^l(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}})R(r') \tag{126}$$

The angular integral of interest:

$$\int_{-1}^{1} P_l(x')X(x') = \int_{-1}^{1} P_l(x')[\delta(x'-1) + \delta(x'+1)]dx = \\ P_l(1) - P_l(-1) = 1 - (-1)^l \tag{127}$$

Consider the integrals on r'. Thence, for an arbitrary function of $(r_>, r_<)$,

$$\int_{r'=0}^{d} r'^2dr'R(r')f(r_>, r_<) = \int_{r'=0}^{r} r'^2R(r')f(r,r') + \int_{r'=r}^{d} r'^2R(r')f(r',r) \tag{128}$$

The first and second radial integrals of interest:

$$\int_{r'=0}^{r} r'^2(d^2 - r'^2)r'^l(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}})dr' = r^{l+3}(\frac{d^2}{l+3} - \frac{r^2}{l+5})(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}) \tag{129}$$

$$\int_{r'=r}^{d} r'^2(d^2 - r'^2)r^l(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}})dr' = r^l\{d^2[\frac{d^{-l+2} - r^{-l+2}}{-l+2} - \frac{d^{l+3} - r^{l+3}}{(l+3)b^{2l+1}}] - \\ [\frac{d^{-l} - r^{-l}}{-l} - \frac{d^{l+5} - r^{l+5}}{(l+5)b^{2l+5}}]\} \tag{130}$$

with the constant $\mathcal{K}$ to be determined. The angular dependence of ρ yields

$$\int_0^{2\pi} d\phi' \int_{-1}^{1} [\delta(x-1) + \delta(x+1)]dx = 2\pi \times 2 \tag{131}$$

Then,

$$Q = 4\pi\mathcal{K}\int_{r'=0}^{d} (d^2 - r'^2)r'^2dr' = 4\pi\mathcal{K}\frac{2d^5}{15} \tag{132}$$

Finally,

$$\mathcal{K} = \frac{15Q}{8\pi d^5} \tag{133}$$

Collecting the results,

$$\Phi(x) = \sum_{l=0}^{\infty} \frac{15Q}{4\pi\epsilon_0 d^5} P_{2l}(\cos\theta) G_{2l}(r) \tag{134}$$

with

$$\begin{aligned} G_l(r) &= r^{l+3}\left(\frac{d^2}{l+3} - \frac{r^2}{l+5}\right)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) \\ &+ r^l\{d^2[\frac{d^{-l+2} - r^{-l+2}}{-l+2} - \frac{d^{l+3} - r^{l+3}}{(l+3)b^{2l+1}}] - [\frac{d^{-l} - r^{-l}}{-l} - \frac{d^{l+5} - r^{l+5}}{(l+5)b^{2l+5}}]\} \end{aligned} \tag{135}$$

Where we neglect possible algebraic simplifications.

(b) We simply take

$$\sigma = -\epsilon_0 \partial_r \Phi|_{r=b}$$

by differentiating the powers or r^l.

Answer to 3.16

As the green's function in free space is $(\frac{1}{|x-x'|})$, in seeking for the Green's function obeying the free boundary condition we are obtaining a representation to $(\frac{1}{|x-x'|})$:

$$\nabla^2 G(x, x') = \frac{-4\pi}{\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z') \tag{136}$$

We seek solutions to the Laplace in cylindrical coordinates based on the first expansion. Since there is no physical divergence at $\rho = 0$, the coefficient attached to Neumann cancels. The z part is treated differently,

$$\frac{1}{|x - x'|} = -4\pi \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk \exp[im\phi] J_m(k\rho) g_{mk}(z; \rho', \phi', z') \tag{137}$$

Then,

$$\begin{aligned} \sum_m \int_0^{\infty} dk\{[\frac{1}{\rho}\partial_\rho(\rho\partial_\rho J_m(k\rho)) - \frac{m^2}{\rho^2} J_m(k\rho)] g_{mk} + J_m(k\rho) g''_{mk}(z)\} \exp[im\phi] = \\ -\frac{1}{4\pi\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z') \end{aligned} \tag{138}$$

From Bessel,

$$\sum_m \int_0^{\infty} dk\{-k^2 g_{mk} + g''_{mk}\} J_m(k\rho) \exp[im\phi] = -\frac{1}{4\pi\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z') \tag{139}$$

Applying the operator,

$$O() = \int \frac{d\phi}{2\pi} \exp[-im\phi] \int_0^{\infty} d\rho\rho J_m(k\rho) d\rho() \tag{140}$$

and since

$$\frac{1}{k}\delta(k - k') = \int_0^{\infty} \rho J_m(k\rho) J_m(k'\rho) d\rho \tag{141}$$

Defining

$$g_{mk}(z;\rho',\phi',z') = -\frac{k}{2}\exp[-im\phi']J_m(k\rho')G_{mk}(z,z') \tag{142}$$

It follows

$$-k^2 G_{mk} + G''_{mk} = \delta(z-z') \tag{143}$$

Thence,

$$G_{mk}(z;z') = \begin{cases} A(z')\exp[kz] + B(z')\exp[-kz], \text{if } z > z' \\ C(z')\exp[kz] + D(z')\exp[-kz] \text{ if } z < z' \end{cases} \tag{144}$$

Thence, as to avoid divergence, $A = 0$, and $D = 0$. Due to symmetry, once more,

$$G_{mk}(z;z') = \mathcal{C}\exp[-k(z_> - z_<)] \tag{145}$$

The constant $\mathcal{C}$ is determined

$$(\partial_{z_>}G_{mk} - \partial_{z_<}G_{mk})|_{z'} = \delta(z-z') \tag{146}$$

Determines

$$\mathcal{C} = -\frac{1}{2k} \tag{147}$$

And hence,

$$g_{mk}(z;\rho',\phi',z') = \frac{1}{4}\exp[-im\phi']J_m(k\rho')\exp[-k(z_> - z_<)] \tag{148}$$

Finally,

$$\frac{1}{|x-x'|} = -\pi\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\exp[im(\phi-\phi')]J_m(k\rho)J_m(k\rho')\exp[-k(z_> - z_<)] \tag{149}$$

There is the mistake of $-\pi$ in the answer (to be checked);

Answer to 3.17 (b)

This is (almost) identical to last up to the application of the Boundary condition for the green's function.

$$G_{mk}(z;z') = \begin{cases} A(z')\exp[kz] + B(z')\exp[-kz], \text{ if } z > z' \\ C(z')\exp[kz] + D(z')\exp[-kz], \text{ if } z < z' \end{cases} \tag{150}$$

In this case, it is clear the Green's function is

$$G_{mk} = \mathcal{C}\sinh(kz_<)\sinh(k(L - z_>)) \tag{151}$$

since it obeys

$$\begin{aligned} G_{mk}(z=0,z') = G_{mk}(z=L,z') = 0 \\ G_{mk}(z,z') = G_{mk}(z',z) \end{aligned} \tag{152}$$

The constant $\mathcal{C}$ is determined,

$$(\partial_{z_>}G_{mk} - \partial_{z_<}G_{mk})|_{z'} = 1 \tag{153}$$

Implying

$$-k\mathcal{C}[\cosh(k(L-z'))\sinh(kz')+\cosh(kz')\sinh(kz')]=1 \tag{154}$$

i.e,

$$\mathcal{C}=-\frac{1}{k\sinh(kL)} \tag{155}$$

To remember:

$$g_{mk}(z;\rho',\phi',z')=-\frac{k}{2}\exp[-im\phi']J_m(k\rho')G_{mk}(z,z') \tag{156}$$

Thence,

$$g_{mk}(z;\rho',\phi',z')=\exp[-im\phi']J_m(k\rho')\frac{\sinh(kz_<)\sinh(k(L-z_>))}{2\sinh(kL)} \tag{157}$$

As in the previous problem,

$$G(x,x')=-4\pi\sum_{m=-\infty}^{\infty}\int_0^{\infty}dk\exp[im\phi]J_m(k\rho)g_{mk}(z;\rho',\phi',z') \tag{158}$$

Then

$$\begin{aligned}G(x,x')=-2\pi\sum_{m=-\infty}^{\infty}\int_0^{\infty}dk\exp[im(\phi-\phi')]J_m(k\rho)J_m(k\rho')\times\\ \frac{\sinh(kz_<)\sinh(k(L-z_>))}{\sinh(kL)}\end{aligned} \tag{159}$$

There is the mistake of $-\pi$ in the answer (to be checked)

Answer to problem 3.23

(a) From the previous examples, the function dependence on z is treated apart, as it is defined differently for either $z>z'$ or $z<z'$. As at $\rho=0$, the solution must not diverge, $e_{mn}=0$, and the special z treatment is provided (based on the first expansion). *A solution based on the first expansion suits because there is not a orthogonality relations for the modified Bessel functions $\{I_m(k\rho),K_m(k\rho)\}$, but these is for the standard Bessel functions $\{J_m(k\rho),N_m(k\rho)\}$.*

$$\Phi=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\exp[im\phi]J_m(\frac{x_{mn}}{a}\rho)g_{mn}(z;z',\rho',\phi') \tag{160}$$

Thence,

$$\begin{aligned}\sum_{mn}\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{d}{d\rho}J_m(\frac{x_{mn}\rho}{a}))g_{mn}-\frac{m^2}{\rho^2}J_{mn}g_{mn}+\frac{d^2g_{mn}}{dz^2}J_m(\frac{k_{mn}\rho}{a}))\exp[im\phi]=\\ -\frac{q}{\epsilon_0}\frac{\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')}{\rho}\end{aligned} \tag{161}$$

Due to the standard Bessel equation

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{d}{d\rho}J_m(\frac{x_{mn}\rho}{a}))-\frac{m^2}{\rho^2}J_{mn}=-(\frac{x_{mn}}{a})^2J_m(\frac{x_{mn}}{a}\rho) \tag{162}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}(-\frac{x_{mn}^2}{a^2}g_{mn}+g''_{mn;zz})\exp[im\phi]J_m(\frac{x_{mn}\rho}{a})=$$
$$-\frac{q}{\epsilon_0}\frac{\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')}{\rho} \tag{163}$$

Applying the operator,

$$O()=\int\frac{\exp[-im\phi]}{2\pi}d\phi\int d\rho\rho\frac{\rho J_m(\frac{x_{mn}\rho}{a})}{\frac{a^2}{2}J_{m'+1}(x'_m n')}() \tag{164}$$

To both sides and through the orthonormalization of the Fourier and Fourier-Bessel yields

$$-\frac{x_{mn}^2}{a^2}g_{mn}+g''_{mn;zz}=-\frac{q}{\epsilon_0}\frac{\exp[-im\phi']}{2\pi}J_m(\frac{x_{mn}\rho'}{a})(\frac{a^2}{2}J_{m+1}^2(x_{mn}))^{-1}\delta(z-z') \tag{165}$$

Thence, as ϕ', ρ' are constants, defining

$$g_{mn}(z;z',\phi',\rho')=-\frac{q}{\epsilon_0}J_m(\frac{x_{mn}\rho'}{a})\exp[-im\phi']\frac{1}{\pi a^2 J_{m+1}^2(x_{mn})}G_{mn}(z;z') \tag{166}$$

Imply

$$G''_{mn}(z;z')-\frac{x_{mn}^2}{a^2}G_{mn}(z;z')=\delta(z-z') \tag{167}$$

Finally, the solutions for Laplace obeying the Boundary condition in each domain reads

$$G_{mn}(z,z')=\begin{cases} A(z')\sinh(\frac{x_{mn}}{a}z) \text{ if } z<z' \\ B(z')\sinh(\frac{x_{mn}}{a}(L-z)) \text{ if } z>z' \end{cases} \tag{168}$$

As the potential (green's function) obeys the Dirichlet boundary condition, the solution is symmetric with respect to the interchange of primed and not primed coordinates. In particular, in the argument z,

$$G_{mn}(\rho,\rho')=\mathcal{C}\sinh(\frac{n\pi\rho_<}{L})\sinh(\frac{n\pi\rho_>}{L}) \tag{169}$$

The dirac contribution is computed

$$\partial_{z_>}G_{mn}-\partial_{z_<}G_{mn}=\delta(z-z') \tag{170}$$

Finally,

$$\mathcal{C}=-\frac{a}{x_{mn}}(\sinh(\frac{x_{mn}L}{a}))^{-1} \tag{171}$$

Returning to the potential, and collecting the terms,

$$\Phi=\frac{q}{\pi\epsilon_0 a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{\exp[im(\phi-\phi')]J_m(\frac{x_{mn}\rho}{a})J_m(\frac{x_{mn}\rho'}{a})}{x_{mn}J_{m+1}^2(x_{mn})\sinh(\frac{x_{mn}}{a}L))}\times$$
$$\sinh(\frac{x_{mn}}{a}z_<)\sinh(\frac{x_{mn}(L-z_>)}{a}) \tag{172}$$

which is symmetric with respect to the interchange of arguments (primed to not primed), as it should be.

(b)

In this case, the ρ component is treated as special. Therefore, in seeking the solution, there is no need for orthogonal relations. *A solution based on the second expansion suits because there is not a orthogonality relation for* $\{\sinh kz, \cosh kz\}$*, though there is for* $\{\sin(kz), \cos(kz)\}$.

Considering the second expansion with a special treatment of the ρ coordinate, the only way for the solution to vanish both at $z = 0$ and $z = a$ is if $c_{mn} = 0$. Thence,

$$\Phi = \sum_{m=-\infty}^{\infty} \sum_{m=1}^{\infty} \exp[im\phi] \sin(\frac{n\pi z}{L}) g_{mn}(\rho; z', \phi', \rho') \tag{173}$$

A immediate $\delta(z - z')$ and $\delta(\phi - \phi')$ representation is

$$\delta(z - z') = \sum_{n=1}^{\infty} \frac{2}{a} \sin(\frac{\pi n z}{a}) \sin(\frac{\pi n z'}{a}) \tag{174}$$

$$\delta(\phi - \phi') = \sum_{m=-\infty}^{\infty} \frac{\exp[im(\phi - \phi')]}{2\pi} \tag{175}$$

Instead the orthogonality relation for these could be used as in the last case, this is essentially the same concept. A substitution into Poisson yields

$$\begin{aligned} &\sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} (\frac{1}{\rho}\partial_\rho(\rho\partial_\rho g_{mn}) - \frac{m^2}{\rho^2} - \frac{n^2\pi^2}{L^2}) \exp[im\phi] \sin(\frac{n\pi z}{L}) = \\ &- \frac{q}{\epsilon_0} \frac{\delta(\rho - \rho')}{\rho} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\exp[im(\phi - \phi')]}{2\pi} (\frac{2}{L}) \sin(\frac{n\pi z}{L}) \sin(\frac{n\pi z'}{L}) \end{aligned} \tag{176}$$

with a comparison;

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho g_{mn}) - (\frac{m^2}{\rho^2} + \frac{n^2\pi^2}{L^2}) g_{mn} = -\frac{2q}{\epsilon_0 L\pi} \exp[-im\phi'] \sin(\frac{n\pi z'}{L}) \frac{\delta(\rho - \rho')}{\rho} \tag{177}$$

Thence, defining G_{mn} with

$$g_{mn}(\rho; z', \phi', \rho') = -\frac{2q}{\epsilon_0 L\pi} G_{mn}(\rho; \rho') \exp[-im\phi'] \sin(\frac{n\pi z'}{L})$$

produces

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho G_{mn}) - (\frac{m^2}{\rho^2} + \frac{n^2\pi^2}{L^2}) G_{mn} = \frac{\delta(\rho - \rho')}{\rho} \tag{178}$$

By defining $\rho = xk$, with $k = (\frac{L}{n\pi})$, it follows the modified Bessel equation.

$$\frac{1}{x}\partial_x(x\partial_x G_{mn}) - (\frac{m^2}{x^2} + 1) = k^2(\frac{\delta(x - x')}{k(kx)}) = \frac{\delta(x - x')}{x} \tag{179}$$

The solutions are those of Laplace for $x \neq x'$.

$$G_{mn}(x; x') = \begin{cases} A_m(x')I_m(x) + B_m(x')K_m(x) \text{ if } x < x' \\ C_m(x')I_m(x) + D_m(x')K_m(x) \text{ if } x > x' \end{cases} \tag{180}$$

For $\rho = 0$, $x = 0$ and K_m diverges; therefore, $B_m = 0$. For $\rho = a$, $x = \frac{a}{k} = \frac{n\pi a}{L}$; $C_m(x') = -D_m(x')\frac{K_m(\frac{n\pi a}{L})}{Im(\frac{n\pi a}{L})}$. Finally, due to symmetry,

$$G_{mn}(x_<, x_>) = \mathcal{C} I_m(x_<)[K_m(x_>) - \frac{K_{mn}(\frac{n\pi a}{L})}{I_m(\frac{n\pi a}{L})} I_m(x_>)] \tag{181}$$

Taking into account the discontinuity,

$$\partial_{x_>} G_{mn} - \partial_{x_<} G_{mn} = \frac{\delta(x - x')}{x} \tag{182}$$

Only mixed Bessel functions contribute,

$$\mathcal{C}[I_m(x)K'_m(x) - K_m(x)I'_m(x)] = \frac{\delta(x-x')}{x} \tag{183}$$

but according to Jackson (many ways to prove)

$$W[I_m(x), K_m(x)] = I_m K'_m(x) - K_m(x)I'_m(x) = -\frac{1}{x} \tag{184}$$

From which follows

$$\mathcal{C} = -1 \tag{185}$$

By collecting the terms and writting for the original coordinate ($x_{\{>,<\}} = \frac{n\pi}{L}\rho_{\{>,<\}}$, since $n > 1$),

$$\Phi = \frac{q}{\pi\epsilon_0 L}\sum_{mn}\frac{I_m(n\pi\rho_>)}{I_m(\frac{n\pi a}{L})}[-K_m(\frac{n\pi a}{L})I_m(\frac{n\pi\rho_>}{L}) + K_m(\frac{n\pi\rho_>}{L})I_m(\frac{n\pi a}{L})] \times$$

$$\exp[im(\phi-\phi')]\sin(\frac{n\pi z}{L})\sin(\frac{n\pi z'}{L}) \tag{186}$$

(c)

(Not detailed calculation in the beginning, straight to the important discussion)

In this last case, not any particular selection is of use, since there are not terms of the kind >,<. We rely on an expansion based on eigenfunctions of the kind (wave equation in the context of quantum theory), and apply its relation to our solution of interest ((3.160) of Jackson).

$$(\nabla^2 + \mathcal{K}^2)\Psi = 0$$

Splitting the variables, as usual, we are led, with the use of boundary conditions on $\rho = a$ and $z = 0$, to

$$\Psi_{\mathcal{K}} = \sum_{mn}\mathcal{N}_{mn;\mathcal{K}}\exp[im\phi]J_m(\frac{x_{mn}\rho}{a})\sin(\sqrt{\mathcal{K}^2 - \frac{a^2}{x_{mn}^2}}z) \tag{187}$$

with $\mathcal{N}_{mn;\mathcal{K}}$ a normalization constant. The boundary condition on $z = L$ imply

$$\mathcal{K}^2 - \frac{x_{mn}^2}{a^2} = (\frac{k\pi}{L})^2 \tag{188}$$

with k an integer. Therefore, to each $\mathcal{K}$ value satisfying the boundary condition there is an integer k.

$$\Psi_k = \sum_{mn}\mathcal{N}_{mn;k}exp[im\phi]J_m(\frac{x_{mn}\rho}{a})\sin(\frac{k\pi z}{L}) \tag{189}$$

In order to apply the result of section 3.12, a normalization on Ψ_k is needed. Thence,

$$\int \Psi_k\Psi_k^*\rho d\rho d\phi dz = 1 \tag{190}$$

imply with the usual normalization,

$$N_{mn;k}^2 = \frac{2}{L}\frac{1}{\frac{a^2}{2}J_{m+1}^2(x_{mn})}\frac{1}{2\pi} \tag{191}$$

Through 3.160,

$$G(\vec{x},\vec{x}') = 4\pi\sum_k \frac{\psi_k^*(\vec{x}')\psi_k(\vec{x})}{\lambda_k - \lambda} \tag{192}$$

with $\lambda = 0$, as we want to solve for Poisson, it follows

$$\Phi = \frac{q}{\epsilon_0} \frac{\sum_k \Psi_k^*(\vec{x})\Psi_k(\vec{x})}{\frac{x_{mn}^2}{a^2} + (\frac{k\pi}{L})^2} \tag{193}$$

and

$$\Phi = \frac{2q}{\pi\epsilon_0 La^2} \sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{\exp[im(\phi-\phi')]\sin(\frac{k\pi z}{L})\sin(\frac{k\pi z'}{L})J_m(\frac{x_{mn}\rho}{a})J_m(\frac{x_{mn}\rho'}{a})}{[\frac{x_{mn}^2}{a^2} + (\frac{k\pi}{L})^2]J_{m+1}^2(x_{mn})} \tag{194}$$

as the factor $\mathcal{N}_{mn;k}^2$ is the one of interest. *It is not clear yet why only the positive k values are considered. An extra sum appear because this method is intrinsically different. The relation between the methods is not clear enough to be written.*

0.0.1 Answer to 3.24

The green's function of interest is obtained through

$$G(x;x') = \frac{\epsilon_0}{q}\Phi(x;x') \tag{195}$$

in the last. Thence, the way to proceed to compute the potential within the cylinder is through

$$\Psi(x,x') = -\frac{1}{4\pi}\oint_S \Phi(x')\partial_{n'}G(x,x')da' \tag{196}$$

The only surface where the integrals fails to vanish is that at $z' = L$. In this case $\partial_{n'} = +\partial_{z'}$. Also, for $z' = L$, ρ' exhibit non null contribution only for $0 < \rho' < b$. Then, as the potential in this interval is constant,

$$\Psi(x,x') = -\frac{V}{4\pi}\frac{\epsilon_0}{q}\int_{\rho'=0}^{b}\int_{\phi'=0}^{2\pi} \rho' d\rho' d\phi' \partial_{z'}\Phi|_{z'=L} \tag{197}$$

Notice that the integration of the ϕ' part yields 0 unless $m = 0$ (This is clear from the azhimutal symmetry, something we would expect). Thence, for the expansion in three sums (last of 3.23):

$$\int_{\rho'=0}^{b}\int_{\phi'=0}^{2\pi} \rho' d\rho' d\phi' \partial_{z'}\Phi|_{z'=L} = -2\pi \times \frac{2q}{\pi\epsilon_0 La^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k\pi}{L}\sin(\frac{k\pi z}{L}) \times \frac{J_0(\frac{x_n\rho}{a})}{[\frac{x_n^2}{a^2} + (\frac{k\pi}{L})^2]J_1^2(x_n)} \times \int_{\rho'=0}^{\rho'=b} \rho' J_0(\frac{x_n\rho'}{a})$$

Thence,

$$\Psi(x,x') = \frac{Vq\pi}{\epsilon_0 L^2 a^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} k\sin(\frac{k\pi z}{L})\frac{J_0(\frac{x_n\rho}{a})}{[\frac{x_n^2}{a^2} + (\frac{k\pi}{L})^2]}\int_{\rho'=0}^{b}\rho' J_0(\frac{x_n\rho'}{a}) \tag{198}$$

The other expansions can be computed analogously. In those cases, as $m = 0$, it remains a single sum over k. As the procedure is clear, and there is no conceptual gain hence on, we move to the next problem.

Chapter 4

Answer to question 4.1

We compute the expansion in spherical harmonics for a set of two charge distribution (refereed through the subindex d) whose center is placed in the origin of coordinates. We choose it to be aligned in the x-axis and the two-charge system with a characteristic distance of $2a$.

$$\Phi_d(x,y,z) = \frac{1}{4\pi\epsilon_0}q(\frac{1}{\sqrt{(x-a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+y^2+z^2}}) \tag{199}$$

Thence, as the charge distributions are in the x-y plane $\theta' = \frac{\pi}{2}$ in both expansions, $\phi' = 0$ for the first and $\phi' = \pi$ for the second, yielding

$$\Phi_d = \frac{1}{4\pi\epsilon_0} q \sum_{l,m} \frac{4\pi}{2l+1} \frac{a^l}{r^{l+1}} [Y^*_{lm}(\frac{\pi}{2}, 0) - Y^*_{lm}(\frac{\pi}{2}, \pi)] Y_{lm}(\theta, \phi) \tag{200}$$

i.e,

$$\Phi_d = \frac{1}{4\pi\epsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} \frac{qa^l}{r^{l+1}} [Y_{l-m}(\frac{\pi}{2}, 0) - Y_{l-m}(\frac{\pi}{2}, \pi)](-1)^m Y_{lm}(\theta, \phi) \tag{201}$$

since $Y^*_{lm}(\theta, \phi) = (-1)^m Y_{l-m}(\theta, \phi)$.

$$\Phi_d = \frac{1}{4\pi\epsilon_0} (\sum_{lm} \frac{4\pi}{2l+1} qa^l \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} P_l^{-m}(0)[(-1)^m - 1] \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \tag{202}$$

since

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} P_l^m(\cos\theta) e^{im\phi} \tag{203}$$

From which we identify the multipole

$$q_{lm} = qa^l \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^{-m}(0)[(-1)^m - 1] \tag{204}$$

and

$$q_{lm} = qa^l \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} P_l^m(0)[1 - (-1)^m] \tag{205}$$

since $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$.

We wish to evaluate the potential due to a four charge distribution as in figure A. For this, consider another distribution which consist of the last charge configuration rotated of an angle $\phi = \frac{\pi}{2}$, i.e,

$$Y_{l-m}(\frac{\pi}{2}, 0) \to Y_{l-m}(\frac{\pi}{2}, \frac{\pi}{2}) \tag{206}$$

$$Y_{l-m}(\frac{\pi}{2}, \pi) \to Y_{l-m}(\frac{\pi}{2}, \frac{3\pi}{2}) \tag{207}$$

Therefore, the multipole coefficients for this distribution are

$$q'_{lm} = qa^l \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^{-m}(0)(\exp[-im\frac{\pi}{2}] - \exp[-im\frac{3\pi}{2}])(-1)^m \tag{208}$$

i.e,

$$q_{lm} = qa^l \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} P_l^m(0)((-i)^m - (i)^m)) \tag{209}$$

since $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$.

The multipole expansion for the union set is equivalent to summing the correspondent multipole expansion terms. For distinction, we represent the multipole coefficients of the system labeling $q_{S;lm}$,

$$q_{S;lm} = q_{lm} + q'_{lm} \tag{210}$$

The multipole coefficients are clearly additive. This is since the superposition principle applies for the potential (and the definition of the multipole coefficients). Explicitly,

$$q_{S;lm} = qa^l\sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}}P_l^m(0)(1-(-1)^m+(-i)^m-(i)^m)) \tag{211}$$

(b) and (c)

Consider

$$\Phi(x,y,z) = \frac{1}{4\pi\epsilon_0}(\frac{-2q}{\sqrt{x^2+y^2+z^2}}+$$
$$\frac{q}{\sqrt{x^2+y^2+(z-a)^2}}+\frac{q}{\sqrt{x^2+y^2+(z+a)^2}}) \tag{212}$$

In this case, the positive charges along the z-axis, then,

$$\Phi_{2,+} = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\frac{qa^l}{r^{l+1}}[Y^*_{lm}(0,\phi')+Y^*_{lm}(\pi,\phi')]Y_{lm}(\theta,\phi) \tag{213}$$

And since

$$Y_{lm}(0,\phi') \to \delta_{m0}P_l(1)\sqrt{\frac{2l+1}{4\pi}} = \sqrt{\frac{2l+1}{4\pi}}\delta_{m0} \tag{214}$$

$$Y_{lm}(\pi,\phi') \to \delta_{m0}P_l(-1)\sqrt{\frac{2l+1}{4\pi}} = (-1)^l\sqrt{\frac{2l+1}{4\pi}}\delta_{m0} \tag{215}$$

The only contribution is that for $m=0$ as expected. It follows

$$\Phi_{2,+} = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\sqrt{\frac{2l+1}{4\pi}}qa^l\delta_{m0}[1+(-1)^l]\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \tag{216}$$

(we could have been less stupid in latter (a), so that we would first transform to generalized Legendre polynomials and exponentials and then conjugate, as in this). Therefore,

$$q'_{lm} = \frac{qa^l}{r^{l+1}}[1+(-1)^l]\delta_{m0} \tag{217}$$

The charge $-2q$ at the origin is placed at $r'=0$, and therefore, it does not contribute to the dipole moment in this coordinate system (the form of the expansion is r'^l/r^{l+1}).

$$q_{S;lm} = \delta_{m0}\sqrt{\frac{2l+1}{4\pi}}\frac{qa^l}{r^{l+1}}[1+(-1)^l] \tag{218}$$

Answer to question 4.5

(a)

$$\vec{F} = \sum_i \hat{x}_i\{\int\rho(x')E_i(0)dV + \sum_j\int\rho(x')x_j\partial_jE_i(0)dV + \sum_{jk}\frac{1}{2}\int\rho(x')x_jx_k\partial_j\partial_kE_i(0)dV\} \tag{219}$$

But the first and last terms are rewritten

$$\sum_i\int\rho(x')E_i(0)dV = \sum_i E_i(0)\int\rho(x')dV = \vec{E}(0)q \tag{220}$$

$$\sum_i\sum_j\hat{x}_i\int\rho(x')x_j\partial_jE_i(0)dV = \sum_i\hat{x}_i\partial_i[\sum_j\int\rho(x')x_jE_j(0)dV] = \nabla[\vec{p}\cdot\vec{E}](0) \tag{221}$$

$$\tag{222}$$

where the last is due to $\partial_i E_j = \partial_j E_i$ ($\nabla \times \vec{E} = 0$). Permuting the indices i and k in the last integral,

$$\frac{1}{2}\sum_i \hat{x}_i \sum_{jk} \int x_j x_k \rho(x')\partial_j\partial_k E_i(0) dV = \frac{1}{2}\hat{x}_i\partial_i \int dV x_j x_k \rho(x')\partial_j E_k(0) =$$
$$\frac{1}{2}\sum_{jk} \nabla \int dV x_j x_k \rho(x')\partial_j E_k(0) \tag{223}$$

We can make $\int dV x_j x_k \rho(x')$ traceless by summing and subtracting

$$\sum_{jk} \int dV \frac{r^2 \delta_{jk}}{3} \partial_j E_k(0) = 0 \tag{224}$$

, since $\nabla \cdot \vec{E}_{out} = 0$. Thus,

$$\vec{F} = \vec{E}(0)q + \nabla[\vec{p}\cdot\vec{E}](0) + \frac{1}{6}\nabla\sum_{jk} Q_{jk}\partial_j E_k(0) \tag{225}$$

with

$$Q_{jk} = 3x_j x_k \rho(x')\partial_j E_k(0) - r^2\delta_{jk}$$

(b)

$$N_1 = \int \rho(x)(x_2 E_3(x) - x_3 E_2(x))dV \tag{226}$$

where $E_2(x)$ and $E_3(x)$ are scalars. We can expand up to first-order

$$N_1 = \int \rho(x)[(x_2E_3(0) - x_3E_2(0)) + (x_2x_j\partial_jE_3(0) - x_3x_j\partial_jE_2(0)) \tag{227}$$

Changing the indices as before ($\nabla \times \vec{E} = 0$)

$$N_1 = \int \rho(x)[(x_2E_3(0) - x_3E_2(0)) + (x_2x_j\partial_3E_j(0) - x_3x_j\partial_2E_j(0)) \tag{228}$$

From which

$$N_1 = \int \rho(x)[(x_2E_3(0) - x_3E_2(0)) + (x_2x_j\partial_3E_j(0) - x_3x_j\partial_2E_j(0)) \tag{229}$$

Thence, we identify

$$N_1 = [\vec{p}\times E^{(0)}(0)]_1 + \frac{1}{3}(Q_{2j}\partial_3E_j(0) - Q_{3j}\partial_2E_j(0)) \tag{230}$$

Since

$$Q_{ij} \propto \frac{(3x_ix_j - r^2\delta_{ij})}{3}$$

with the adition of the divergenceless term.

Problem 4.7 of Jackson's book

(a) The charge distribution is of the form $\rho(r') = \frac{1}{64\pi}\exp[-r']r'^2\sin^2\theta'$. It is convenient to write $\sin^2\theta$ as a combination of spherical harmonics. Thence,

$$\sin^2\theta = \frac{2}{3}\sqrt{\frac{4\pi}{5}}(\sqrt{5}Y_{00}(\theta,\phi) - Y_{20}(\theta,\phi)) \tag{231}$$

since

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}(\frac{3}{2}\cos^2\theta - \frac{1}{2}) = \sqrt{\frac{5}{4\pi}}(1 - \frac{3}{2}\sin^2\theta) \tag{232}$$

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \tag{233}$$

Explicitly,

$$\rho(r') = \frac{1}{64\pi}\exp[-r']r'^2\frac{2}{3}\sqrt{\frac{4\pi}{5}}(\sqrt{5}Y_{00}(\theta',\phi') - Y_{20}(\theta',\phi')) \tag{234}$$

Therefore,

$$4\pi\epsilon_0\phi = \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{1}{96\pi}\sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)Y_{lm}(\theta',\phi')$$
$$\sqrt{\frac{4\pi}{5}}(\sqrt{5}Y_{00}(\theta',\phi') - Y_{20}(\theta',\phi'))\exp[-r']r'^2\frac{r'^l}{r^{l+1}} \tag{235}$$

with $r' = r_<$ and $r = r_>$ for the multipole expansion. The potential is provided through an integration over the whole space.

$$\Phi(\vec{r})_{r>r'} = \frac{1}{4\pi\epsilon_0}\frac{1}{96\pi}\sum_{lm}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\int d\Omega'\sqrt{\frac{4\pi}{5}}(\sqrt{5}Y_{00}(\theta',\phi') -$$
$$Y_{20}(\theta',\phi'))Y_{lm}(\theta',\phi')\int_{r'=0}^{\infty} r'^{l+2}\exp[-r']r'^2 dr'$$

which due to the orthonormal spherical harmonics and the Γ definition yields

$$\Phi(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{96\pi}\sum_{lm}Y_{lm}(\theta,\phi)\sqrt{\frac{4\pi}{5}}(\sqrt{5}\frac{1}{2\times 0+1}\delta_{l0}\delta_{m0} - \frac{1}{2\times 2+1}\delta_{l2}\delta_{m0})\frac{\Gamma(l+5)}{r^{l+1}} \tag{236}$$

Therefore,

$$\Phi(\vec{r})_{r>r'} = \frac{1}{4\pi\epsilon_0}\frac{1}{24}\sqrt{\frac{4\pi}{5}}(\frac{\Gamma(5)}{r}\sqrt{5}Y_{00}(\theta,\phi) - \frac{\Gamma(7)}{5r^3}P_2(\theta,\phi)) \tag{237}$$

$$\Phi(\vec{r})_{r>r'} = \frac{1}{4\pi\epsilon_0}\frac{\Gamma[5]}{24}(\frac{1}{r}P_0(\theta) - \frac{5\times 6}{5r^3}P_2(\theta)) \tag{238}$$

$$\Phi(\vec{r})_{r>r'} = \frac{1}{4\pi\epsilon_0}(\frac{1}{r} - \frac{6}{r^3}P_2(\theta)) \tag{239}$$

(b)

$$\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\epsilon_0}\frac{1}{96\pi}\sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)Y_{lm}(\theta',\phi')$$
$$\sqrt{\frac{4\pi}{5}}(\sqrt{5}Y_{00}(\theta,\phi) - Y_{20}(\theta,\phi))\exp[-r']r'^2\frac{r^l}{r'^{l+1}} \text{ for } r < r' \tag{240}$$

which is the case when $r = 0$. Thence, the integral over the spherical part yields,

$$\Phi(\vec{r})_{r\to 0<r'} = \frac{1}{4\pi\epsilon_0}\frac{1}{24}\sum_{lm}\sqrt{\frac{4\pi}{5}}Y_{lm}(\theta,\phi)(\sqrt{5}\delta_{l0}\delta_{m0} - \frac{1}{5}\delta_{l2}\delta_{m0})r^l\int_{r'=0}^{\infty} dr' r'^2\exp[-r']\frac{1}{r'^{l-1}} \tag{241}$$

$$\Phi(\vec{r})_{r\to 0<r'} = \frac{1}{4\pi\epsilon_0}\frac{1}{24}\sum_{lm} Y_{lm}(\theta,\phi)(\sqrt{5}\delta_{l0}\delta_{m0} - \frac{1}{5}\delta_{l2}\delta_{m0})r^l\Gamma[4-l] \tag{242}$$

$$\Phi(\vec{r})_{r\to 0<r'} = \frac{1}{4\pi\epsilon_0}\frac{1}{24}\sqrt{\frac{4\pi}{5}}(\sqrt{5}\Gamma[4]Y_{00}(\theta,\phi) - \frac{\Gamma[2]}{5}Y_{20}(\theta,\phi)r^2) \tag{243}$$

$$\Phi(\vec{r})_{r\to 0<r'} = \frac{1}{4\pi\epsilon_0}\frac{1}{24}(\Gamma[4]P_0(\theta) - \frac{\Gamma[2]}{5}P_2(\theta)r^2) \tag{244}$$

$$\Phi(\vec{r})_{r\to 0<r'} = \frac{1}{4\pi\epsilon_0}\frac{1}{24}(3\times 2P_0(\theta) - \frac{1}{5}P_2(\theta)r^2) \tag{245}$$

Finally,

$$\Phi(\vec{r})_{r\to 0<r'} = \frac{1}{4\pi\epsilon_0}(\frac{1}{4} - \frac{1}{120}P_2(\theta)r^2) \tag{246}$$

Answer to question 4.8

Notice

$$\nabla\cdot\vec{D} = \rho_L = 0 \tag{247}$$
$$\tag{248}$$

in each i-region, $i\in\{1,2,3\}$,

$$\epsilon_i\nabla\vec{E} = -\epsilon_i\nabla^2\phi_i = 0 \tag{249}$$

In this problem, $\epsilon_1 = \epsilon_3 = \epsilon_0$, $\epsilon_2 = \epsilon$. The physical solutions on each region alone have the shape.

$$\Phi_1 = \sum_{m=1}^{\infty} a_m\rho^m\cos m\phi \tag{250}$$

$$\Phi_2 = b_0 + b_L\log\rho + \sum_{m=1}^{\infty}[b_m\rho^m + c_m\rho^{-m}]\cos m\phi \tag{251}$$

$$\Phi_3 = d_0 - E_0\rho\cos\phi + \sum_{m=1}^{\infty} d_m\rho^{-m}\cos m\phi \tag{252}$$

The solution in the first region avoids a nonphysical divergence of the electric field at the origin (absence of $\log\rho$ and powers of ρ^{-1}. In our case, at infinity the behaviour in powers of ρ shall be as in (3) if the x-axis is placed along the direction of the electric field. The absence of $\sin m\phi$ is clear from the solution parity.

These are to be reduced with considerations on the boundary condition in between different media. From Gauss law for D,

$$D_{1\perp} = D_{2\perp}|_{\rho=a} \to -\epsilon_0\nabla\phi_1 = -\epsilon\nabla\phi_2|_{\rho=a} \tag{253}$$
$$D_{3\perp} = D_{2\perp}|_{\rho=b} \to -\epsilon_0\nabla\phi_3 = -\epsilon\nabla\phi_2|_{\rho=b} \tag{254}$$

Therefore,

$$-\epsilon_0\sum_{m=1}^{\infty} m a_m a^{m-1}\cos m\phi = -\epsilon\frac{b_L}{a} - \epsilon\sum_{m=1}^{\infty}(m b_m a^{m-1} - m c_m a^{-m-1})\cos m\phi \tag{255}$$

$$-\epsilon_0[-E_0\cos\phi - \sum_{m=1}^{\infty} m d_m b^{-m-1}\cos m\phi] = -\epsilon\frac{b_L}{b} - \epsilon\sum_{m=1}^{\infty}[b_m m b^{m-1} - c_m m b^{-m-1}]\cos m\phi \tag{256}$$

Therefore, the only way for (9) to hold is $b_L = 0$. The relations for $m = 1$ and $m > 1$ are therefore

$$b_L = 0 \tag{257}$$
$$\epsilon_0 a_m a^{m-1} = \epsilon(b_m a^{m-1} - c_m a^{-m-1}) \text{ if } m \geq 1 \tag{258}$$
$$\epsilon_0 d_m b^{-m-1} = \epsilon(b_m b^{m-1} - c_m b^{-m-1}) \text{ if } m > 1 \tag{259}$$
$$\epsilon_0(-E_0 - d_1 b^{-2}) = \epsilon(b_1 - c_1 b^{-2}) \ (m = 1) \tag{260}$$

From a circuit, for $\vec{E}$ we obtain,

$$E_{1||} = E_{2||}|_{\rho=a} \to -\frac{1}{a}\partial_\phi \phi_1 = -\frac{1}{a}\partial_\phi \phi_2|_{\rho=a} \tag{261}$$
$$E_{3||} = E_{2||}|_{\rho=b} \to -\frac{1}{b}\partial_\phi \phi_3 = -\frac{1}{b}\partial_\phi \phi_2|_{\rho=b} \tag{262}$$

This condition is almost equivalent to the continuity of the potential of the fields (it does not specify, however, how the terms not dependent on ϕ shall be related, as the continuity does). Therefore,

$$a_m a^{m-1} = b_m a^{m-1} + c_m a^{-m-1} \text{ if } m \geq 1 \tag{263}$$
$$b_m b^{m-1} + c_m b^{-m-1} = d_m b^{-m-1} \text{ if } m > 1 \tag{264}$$
$$b_1 + c_1 b^{-2} = -E_0 + d_1 b^{-2} \ (m = 1) \tag{265}$$

Writting the expressions with validity for $m > 1$ altogether

$$a_m a^{m-1} = b_m a^{m-1} + c_m a^{-m-1} \tag{266}$$
$$\epsilon_0 a_m a^{m-1} = \epsilon(b_m a^{m-1} - c_m a^{-m-1}) \tag{267}$$
$$\epsilon_0 d_m b^{-m-1} = \epsilon(b_m b^{m-1} - c_m b^{-m-1}) \tag{268}$$
$$b_m b^{m-1} + c_m b^{-m-1} = d_m b^{-m-1} \tag{269}$$

This is an homogeneous system with four unknowns and with 0 determinant. Therefore, the solution is only the trivial case $a_m = b_m = c_m = d_m = 0$ for $m > 1$. It remains to consider $m = 1$.

$$\epsilon_0(-E_0 - d_1 b^{-2}) = \epsilon(b_1 - c_1 b^{-2}) \tag{270}$$
$$b_1 + c_1 b^{-2} = -E_0 + d_1 b^{-2} \tag{271}$$
$$\epsilon_0 a_1 = \epsilon(b_1 - c_1 a^{-2}) \tag{272}$$
$$a_1 = b_1 + c_1 a^{-2} \tag{273}$$

To solve this system, consider expressing b_1 in terms of c_1 with the two last equations

$$(\epsilon_0 - \epsilon)b_1 + (\epsilon_0 + \epsilon)c_1 a^{-2} = 0 \tag{274}$$
$$b_1 = \frac{\epsilon_0 + \epsilon}{\epsilon_0 - \epsilon}\frac{c_1}{a^2} \ ; \ a_1 = \frac{c_1}{a^2}\frac{2\epsilon_0}{\epsilon_0 - \epsilon} \tag{275}$$

and rewritting and comparing the first two equations

$$-\epsilon_0 E_0 = \epsilon b_1 - \epsilon c_1 b^{-2} + \epsilon_0 d_1 b^{-2} \tag{276}$$
$$-\epsilon_0 E_0 = \epsilon_0 b_1 + \epsilon_0 c_1 b^{-2} - \epsilon_0 d_1 b^{-2} \tag{277}$$
$$(\epsilon - \epsilon_0)b = (\epsilon + \epsilon_0)c_1 b^{-2} + 2\epsilon_0 d_1 b^{-2} \tag{278}$$
$$d_1 = -\frac{(\epsilon - \epsilon_0)}{2\epsilon_0}\frac{b_1}{b^2} + \frac{(\epsilon + \epsilon_0)}{2\epsilon_0}c_1 \tag{279}$$

Therefore, d_1 can also be written in terms of c_1.

$$d_1 = \frac{(\epsilon + \epsilon_0)}{2\epsilon_0}[-\frac{c_1 a^2}{b^2} + 1] \tag{280}$$

We have the following result in terms of c_1, which can be found with a substitution of the before-mentioned terms in one of the inhomogeneous equations.

$$\Phi_1 = a_1(c_1)\rho\cos m\phi \tag{281}$$
$$\Phi_2 = (b_1(c_1)\rho + c_1\rho^{-1})\cos\phi \tag{282}$$
$$\Phi_3 = -E_0\rho\cos\phi + \frac{d_1(c_1)}{\rho}\cos\phi \tag{283}$$
$$\tag{284}$$

For the limiting cases it suffices to take the limit $a \to b$ for a cylindrical cavity of and $a \to 0$ for a solid cylindrical cavity.

Answer to question 4.9

(a)

We look for solutions obeying Laplace inside and outside (apart from the charge expansion)

$$\phi_{ext} = \sum_{l=0}^{\infty}[\frac{B_l}{r^l} + q'\frac{r_<^l}{r_>^{l+1}}]P_l(\cos\theta) \tag{285}$$
$$\phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{286}$$

with $q' = q/4\pi\epsilon_0$ Avoiding nonphysical divergence at infinity and at the origin. Therefore, Maxwell equations in the frontier yields

$$\epsilon l A_l a^{l-1} = \epsilon_0[q'l\frac{a^{l-1}}{d^{l-1}} - (l+1)\frac{B_l}{a^{l+2}}] \tag{287}$$
$$A_l a^l = \frac{B_l}{a^{l+1}} + q'\frac{a^l}{d^{l+1}} \tag{288}$$

Multiplying the second by $\epsilon l/a$ and comparing both,

$$\epsilon_0[q'l\frac{a^{l-1}}{d^{l+1}} - (l+1)\frac{B_l}{a^{l+2}}] = \epsilon[l\frac{B_l}{a^{l+2}} + q'l\frac{a^{l-1}}{d^{l+1}}] \tag{289}$$

Therefore,

$$B_l(-\frac{\epsilon_0(l+1)}{a^{l+2}} - \frac{l\epsilon}{a^{l+2}}) = (\epsilon-\epsilon_0)q'l\frac{a^{l-1}}{d^{l+1}} \tag{290}$$
$$B_l = \frac{(\epsilon_0-\epsilon)l}{l\epsilon+\epsilon_0(l+1)}\frac{q}{4\pi\epsilon_0}\frac{a^{2l+1}}{d^{l+1}} \tag{291}$$

And

$$A_l = \frac{B_l}{a^{2l+1}} + \frac{q}{4\pi\epsilon_0 d^{l+1}} \tag{292}$$

The solution is fully determined, as we wish.

(b)

To look for the limit $\epsilon/\epsilon_0 \to \infty$, it is equivalent to $\epsilon_0/\epsilon \to 0$,

$$B_{l=0} = 0 \tag{293}$$
$$B_{l\neq 0} = \lim_{\epsilon_0/\epsilon\to 0}\frac{(\frac{\epsilon_0}{\epsilon}-1)l}{l+\frac{\epsilon_0}{\epsilon}(l+1)}\frac{q}{4\pi\epsilon_0}\frac{a^{2l+1}}{d^{l+1}} = -\frac{q}{4\pi\epsilon_0}\frac{a^{2l+1}}{d^{l+1}} \tag{294}$$

In this limit,

$$\Phi_{out} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}[\frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l+1}}{d^{l+1}r^{l+1}}]P_l(\cos\theta) \tag{295}$$

$$\Phi_{in} = \frac{q}{4\pi\epsilon_0 d} + \sum_{l=1}^{\infty}[-\frac{q}{4\pi\epsilon_0}\frac{a^{2l+1}}{d^{l+1}}\frac{1}{a^{2l+1}} + \frac{q}{4\pi\epsilon_0 d^{l+1}}]r^l P_l(\cos\theta) \tag{296}$$

$$\Phi_{in} = \frac{q}{4\pi\epsilon_0 d} + \sum_{l\neq 0}^{\infty}[-\frac{q}{4\pi\epsilon_0} + \frac{q}{4\pi\epsilon_0}]\frac{r^l}{d^{l+1}}P_l(\cos\theta) \tag{297}$$

$$\Phi_{in} = \frac{q}{4\pi\epsilon_0 d} \tag{298}$$

As we expect for a perfect conductor.

Answer to question 4.10

(a)
The general solution to the problem is comprised in the shape

$$\Phi_L = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(\cos\theta) \tag{299}$$

$$\Phi_R = \sum_{l=0}^{\infty}(C_l r^l + D_l r^{-(l+1)})P_l(\cos\theta) \tag{300}$$

(without obedience to boundary condition) except the azhimutal symmetry corresponding to the choice of the z axis chosen suitably. The further specification of the boundary condition consists of providing total charge $-Q$ on the inner sphere, $+Q$ on the external sphere. Connecting both solutions there is the condition. The only condition to be specified is related to the change of the potential in the radial direction as

$$-\frac{\partial\Phi_L}{\partial r} = -\frac{\partial\Phi_R}{\partial r} \tag{301}$$

($\nabla\times\vec{E} = 0$). For this condition to be satisfied everywhere in the interface $r \in [a, b]$,

$$\sum_l(lA_l r^{l-1} - (l+1)B_l r^{-l-2})P_l(0) = \sum_l(lC_l r^{l-1} - (l+1)D_l r^{-l-2})P_l(0) \tag{302}$$

i.e, this holds if $B_0 = D_0$ and A_0 and C_0 arbitrary and $B_i = D_i$ and $A_i = C_i$ for $i \geq 1$.

From the first Maxwell equation the local relation between the potential and charge density.

$$\sigma_L(a,\theta) = -\epsilon_0\sum_{l=1}^{\infty}A_l l a^{l-1}P_l(\cos\theta) + \epsilon_0\sum_{l=0}^{\infty}\frac{(l+1)}{a^{l+2}}B_l P_l(\cos\theta) \tag{303}$$

$$\sigma_R(a,\theta) = -\epsilon\sum_{l=1}^{\infty}C_l l a^{l-1}P_l(\cos\theta) + \epsilon\sum_{l=0}^{\infty}\frac{(l+1)}{a^{l+2}}D_l P_l(\cos\theta) \tag{304}$$

For $l = 0$, there is no contribution of A_0 and C_0. Thefore, it is convenient to define

$$\sigma_L(a,\theta) = -\epsilon_0\sum_{l=0}^{\infty}f_l(r)P_0(\cos\theta) \tag{305}$$

$$\sigma_R(a,\theta) = -\epsilon\sum_{l=0}^{\infty}f_l(r)P_l(\cos\theta) \tag{306}$$

$$\tag{307}$$

with

$$f_l(a) = A_l l a^{l-1} A_l + \frac{(l+1)}{a^{l+2}} B_l \tag{308}$$

The boundary condition for the total charge to be $+q$ imply

$$Q = \sum_{l=0}^{\infty} \{-\epsilon f_l(a) \int_{-\pi/2}^{\pi/2} \int_0^{\pi} P_l(\cos\theta) \sin\theta d\theta d\phi - \\ \epsilon_0 f_l(a) \int_{\pi/2}^{3\pi/2} \int_0^{\pi} \sigma_a a^2 P_l(\cos\theta) \sin\theta d\theta d\phi\} \tag{309}$$

Thus,

$$Q = -\pi(\epsilon + \epsilon_0) \sum_{l=1}^{\infty} f_l(a) a^2 \int_0^{\pi} P_l(\cos\theta) \sin\theta d\theta \tag{310}$$

Thus,

$$Q = -\pi(\epsilon + \epsilon_0) \sum_{l=0}^{\infty} f_l(a) a^2 \frac{P_{l+1}(\cos\theta) - P_{l-1}(\cos\theta)}{2l+1} |_{\pi}^{0} \tag{311}$$

where we have applied $P_l(x) = \frac{P'_{l+1}(x) - P'_{l-1}(x)}{2l+1}$ and since $P_{-1}(x) = \mathcal{C}$ is an arbitrary constant (see Legendre equation), it can be divided into two parts, $l \geq 1$ and $l = 0$.

$$Q = -\pi(\epsilon + \epsilon_0) f_0(a) a^2 (2P_1(1)) - \pi(\epsilon + \epsilon_0) \sum_{l=1}^{\infty} f_l(a) a^2 \frac{[(-1)^{l-1} - (-1)^{l+1}]}{2l+1} \tag{312}$$

The terms cancels out and it remains only the necessity of determining $l = 0$,

$$Q = -2\pi(\epsilon + \epsilon_0) f_0(a) a^2 \tag{313}$$

implying

$$B_0 = -\frac{Q}{2\pi} \tag{314}$$

On the other hand (the sign on the RHS changes because the normal to the surface changes),

$$-Q = 2\pi(\epsilon + \epsilon_0) f_0(b) b^2 \tag{315}$$

which again implies the same result consistently. The choice of continuity forces $A_0 = C_0$ (we may choose 0 as it is not of importance). If we want a nondiverging field at infinity, $A_l = 0$ for $l \geq 1$. The simplest solution is of the form $B_1 = 0$, $B_2 = 0$. This choice is discussed in the end, as it is subtle.

Therefore,

$$\Phi = -\frac{Q}{2\pi(\epsilon + \epsilon_0)} \frac{1}{r} \tag{316}$$

$$\vec{E} = \frac{Q}{2\pi(\epsilon + \epsilon_0)} \frac{1}{r^2} \hat{r} \tag{317}$$

$$\sigma_L = \epsilon_0 E(a) = \frac{\epsilon_0 Q}{2\pi(\epsilon + \epsilon_0)} \frac{1}{a^2} \tag{318}$$

$$\sigma_R = \epsilon E(b) = \frac{\epsilon Q}{2\pi(\epsilon + \epsilon_0)} \frac{1}{a^2} \tag{319}$$

Thus, the electric field is still symmetric, however, the displacement field vary. (c)

As discussed in the theory, $-\nabla \cdot \vec{P} = \rho_P$ and $\vec{P} = (\epsilon - \epsilon_0)\vec{E}$. Therefore, with Gauss law in a pill containing an area element of the surface with radius a, and due to the absence of polarization for $r < a$ ($\epsilon = \epsilon_0$ returns $\vec{P} = \vec{0}$, as expected),

$$\sigma_{pol,a} = -(\vec{P}|_a \cdot \hat{r} - \vec{0}) = -\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{Q}{2\pi a^2} \tag{320}$$

Therefore, the polarization charge has a contrary sign to the free density charge it carries as we would expect from microscopic arguments.

Discussion: $B_2 = B_3 = B_4 ... = B_\infty = 0$ is the only possible solution for the problem satisfying the BC within the region of interest, otherwise there would be a dependence on θ for the potential outside of the cylinder (rotational condition for the electric field in between the conductor and the region of interest). This would imply a field tangential to the conductor, which violates this second boundary condition, resulting in an absurd.

Chapter 5

Problem 5.1

Let

$$d\vec{B}(\vec{x}) = -\frac{\mu_0 I}{4\pi} d\vec{l'} \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \tag{321}$$

In component form

$$dB_i = -\frac{\mu_0 I}{4\pi}\hat{x}_i \cdot (d\vec{l'} \times \nabla'(\frac{1}{|\vec{x} - \vec{x}'|}) = d\vec{l'} \cdot \nabla'(\frac{1}{|\vec{x} - \vec{x}'|} \times \vec{x}_i) \tag{322}$$

The integral with Stokes theorem produces

$$B_i = -\frac{\mu_0 I}{4\pi} \int \nabla' \times (\nabla'(\frac{1}{|\vec{x} - \vec{x}'|}) \times \hat{x}_i)\hat{n}' da' \tag{323}$$

Applying the property for $\nabla \times \vec{A} \times \vec{B}$ for $\vec{B} = \vec{x}_i$ a constant vector (and setting $\vec{A} = \nabla'(\frac{1}{|\vec{x} - \vec{x}'|})$,

$$B_i = -\frac{\mu_0 I}{4\pi} \int [(\hat{x}_i \cdot \nabla')\nabla'(\frac{1}{|\vec{x} - \vec{x}'|}) - \hat{x}_i \delta(x - x')]\hat{n}' da' \tag{324}$$

But $(\hat{x}_i \cdot \nabla') = \partial_{x_i}$ and as we are integrating outside of the loop,

$$B_i = -\frac{\mu_0 I}{4\pi}\partial_{x_i} \int \nabla'(\frac{1}{|\vec{x} - \vec{x}'|})\hat{n}' da' \tag{325}$$

Therefore,

$$B_i = \frac{\mu_0 I}{4\pi}\partial_{x_i} \int d\Omega \tag{326}$$

See the definition of Jackson in page 33. The proof is trivially complete and just rewritten by collecting the components.

$$\vec{B} = \frac{\mu_0 I}{4\pi}\nabla\Omega \tag{327}$$

Problem 5.3

(a)

Consider the law for the field (at $\vec{x}$)caused by a current element at the origin of coordinates.

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{x}}{|\vec{x}|^3} \tag{328}$$

Thus,

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{\frac{d\vec{x'}}{dt} \times (\vec{x} - \vec{x'})}{|\vec{x} - \vec{x'}|^3} dt \tag{329}$$

If we parametrize the curve through $\vec{x}' = (a\cos t, a\sin t, 0)$, the system of coordinates is the center of the spire and is such that $(\vec{x} - \vec{x}') = (-a\cos t, -a\sin t, z)$. Thus, with

$$\frac{d\vec{x}'}{dt} \times (\vec{x} - \vec{x}') = a(z\cos t, z\sin t, 1) \tag{330}$$

$$|\vec{x} - \vec{x}'|^3 = (a^2 + z^2)^{3/2} \tag{331}$$

Therefore, an integration of the first two terms yields zero, as expected from the symmetry. Thus,

$$\vec{B} = \hat{z} \frac{\mu_0 I a}{2} \frac{1}{(a^2 + z^2)^{3/2}} \tag{332}$$

Now, if we have a coil we must sum over different spires placed at different positions x_i

$$\vec{B} = \hat{z} \sum_i \vec{B}_i = \frac{\mu_0 I a^2}{2} \sum_i \frac{1}{(a^2 + (z - z_i)^2)^{\frac{3}{2}}} \frac{\Delta x_i}{\Delta x_i} \tag{333}$$

Making $\Delta x_i = \frac{L}{n} = \frac{1}{N}$, n being the number of loops and N being their density. Therefore, in the limit $n \to \infty$, the sum is brought to a Riemman sum and

$$\vec{B} = \hat{z} \sum_i \vec{B}_i = \frac{\mu_0 I N}{2a} \int \frac{1}{(1 + (\frac{z - z_i}{a})^2)^{\frac{3}{2}}} dx \tag{334}$$

Thence, the integral is transformed for $\phi = \tan^{-1}[\frac{(z-z_i)}{a}]$, and the result is $\sin\phi$. But a quick draw provides the information that $\sin\phi = \cos\theta$ with the angle θ defined as in figure (of the problem). Finally, $\cos(\pi - \theta_2) = -\cos\theta_2$, and

$$\vec{B} = \hat{z} \sum_i \vec{B}_i = \hat{z} \frac{\mu_0 I N}{2a} (\cos\theta_1 + \cos\theta_2) \tag{335}$$

Problem 5.4

$$\vec{B}(\rho, z) = \sum_{n=0}^{\infty} \frac{B_z^{(n)}(0, z)\rho^n}{n!} \hat{z} + \sum_{n=0}^{\infty} \frac{B_\rho^{(n)}(0, z)\rho^n}{n!} \hat{\rho} \tag{336}$$

Thence, in the absence of current,

$$\nabla \cdot \vec{B} = 0 \frac{B_\rho}{\rho} + \partial_\rho B_\rho + \partial_z B_z = 0 \tag{337}$$

$$\nabla \times \vec{B} = 0 \partial_z B_\rho - \partial_\rho B_z = 0 \tag{338}$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{\dot{B}_{\rho}^{(n)}(0,z)}{n!}\rho^{n}-\sum_{n=1}^{\infty}\frac{nB_{z}^{(n)}(0,z)}{n!}\rho^{n-1}=0 \tag{339}$$

$$\sum_{n=0}^{\infty}\frac{B_{\rho}^{(n)}(0,z)}{n!}\rho^{n-1}+\sum_{n=1}^{\infty}\frac{nB_{\rho}^{(n)}(0,z)}{n!}\rho^{n-1}+\sum_{n=0}^{\infty}\frac{\dot{B}_{z}^{(n)}(0,z)\rho^{n}}{n!}\rho^{n}=0 \tag{340}$$

Therefore,

$$\sum_{n=0}^{\infty}\dot{B}_{\rho}^{(n)}(0,z)\frac{\rho^{n}}{n!}-\sum_{n=1}^{\infty}B_{z}^{(n)}(0,z)\frac{\rho^{n-1}}{(n-1)!}=0 \tag{341}$$

which displacing the second index to make the second sum to run from 0 to ∞ yields

$$\dot{B}_{\rho}^{(n)}(0,z)=B_{z}^{(n+1)}(0,z) \text{ for } n\geq 0 \tag{342}$$

By noticing $B_{\rho}^{(0)}(0,z)=0$ due to the cyllindrical symmetry, the term $n=0$ can be supressed in the first sum,i.e,

$$\sum_{n=1}^{\infty}\frac{B_{\rho}^{(n)}(0,z)}{n!}\rho^{n-1}+\sum_{n=1}^{\infty}\frac{nB_{\rho}^{(n)}(0,z)}{n!}\rho^{n-1}+\sum_{n=0}^{\infty}\frac{\dot{B}_{z}^{(n)}(0,z)\rho^{n}}{n!}\rho^{n}=0 \tag{343}$$

Making the first sum to rum from 0 to ∞ with a displacement of n and equating equal powers of ρ,

$$\frac{B_{\rho}^{(n+1)}(0,z)}{(n+1)!}+\frac{B_{\rho}^{(n+1)}(0,z)}{n!}=-\frac{\dot{B}_{\rho}^{(n)}(0,z)}{n!} \tag{344}$$

Rearranging,

$$\dot{B}_{z}^{(n)}(0,z)=-B_{\rho}^{(n+1)}(0,z)(\frac{n+2}{n+1}) \tag{345}$$

Thence, we have the following recursion relations for $n\geq 0$,

$$\dot{B}_{\rho}^{(n)}(0,z)=B_{z}^{(n+1)}(0,z) \tag{346}$$

$$B_{\rho}^{(n+1)}(0,z)=-\dot{B}_{z}^{(n)}(0,z)(\frac{n+1}{n+2}) \tag{347}$$

Thence, displacing the second and applying the relations recurrently three times produces

$$B_{\rho}^{(n)}(0,z)=-\dot{B}_{z}^{(n-1)}(\frac{n}{n+1})=-\ddot{B}_{\rho}^{(n-2)}(\frac{n}{n+1})=\dddot{B}_{z}^{(n-3)}(\frac{n}{n+1})(\frac{n-2}{n-1}) \tag{348}$$

As expected, $B_{\rho}(0)=0$ (initially set as this) and we also have $B_{\rho}^{(1)}=-\frac{1}{2}\dot{B}_{z}^{(0)}(0,z)$, $B_{\rho}^{(3)}=\frac{3}{4}\frac{1}{2}\dot{B}_{z}^{(0)}(0,z)$. Therefore,

$$B_{\rho}(\rho,z)=-\frac{\rho}{2}\dot{B}_{z}^{(0)}(0,z)+\rho^{3}\frac{3}{4}\frac{1}{2}\frac{\dddot{B}_{z}^{(0)}(0,z)}{3!}+... \tag{349}$$

Which is identical to the result presented in the book up to third order. Now we can obtain $B_{z}(\rho,z)$ by integrating the monopole (absence of this) equation

$$B_{z}=\int d\rho\partial_{z}B_{\rho}=B_{z}(0,z)-\frac{\rho^{2}}{4}\ddot{B}_{z}(0,z)+... \tag{350}$$

where we have neglected terms of higher orders.

Problem 5.8

Consider

$$\vec{J}(r',\theta') = J(r',\theta')\hat{\phi}' \tag{351}$$

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\exp[im\phi] \tag{352}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)Y_{lm}(\theta',\phi')^* =$$
$$\sum_{lm}\frac{r_<^l}{r_>^{l+1}}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(\cos\theta')\exp[im(\phi-\phi')] \tag{353}$$

$$I = \int_0^{2\pi}\cos(\phi-\phi')\exp[im(\phi-\phi')]d\phi' = -\int_{\phi}^{\phi-2\pi}\cos\phi'\exp[im\phi']d\phi' =$$
$$-\int_{\phi-2\pi}^{\phi}\cos\phi\exp[im\phi]d\phi = -\int_{\phi-2\pi}^{\phi}\frac{\exp[i\phi']+\exp[-i\phi']}{2}\exp[im\phi']d\phi' =$$
$$\pi(\delta_{m1}+\delta_{m-1})$$

$$A_\phi = \frac{\mu_0}{4\pi}\sum_{lm}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)\{r^l,r^{-l-1}\}\int J(r',\theta')\{\frac{1}{r'^{l+1}},r'^l\}P_l^m(\cos\theta')$$
$$r'^2\sin\theta' dr'd\theta' \times I \tag{354}$$

Then,

$$A_\phi = \frac{\mu_0}{4}\{\sum_l\frac{1}{l(l+1)}P_l^1(\cos\theta)\{r^l,r^{-l-1}\}\int J(r',\theta')\{\frac{1}{r'^{l+1}},r'^l\}P_l^1(\cos\theta')r'^2\sin\theta' dr'd\theta' +$$
$$\sum_l l(l+1)P_l^1(\cos\theta)\{r^l,r^{-l-1}\}\int J(r',\theta')\{\frac{1}{r'^{l+1}},r'^l\}P_l^{-1}(\cos\theta')r'^2\sin\theta' dr'd\theta'\} \tag{355}$$

and since

$$P_l^{-m}(\cos\theta) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta) \tag{356}$$

$$P_l^{-1}(\cos\theta) = -\frac{1}{l(l+1)}P_l^1(\cos\theta) \tag{357}$$

$$P_l^{-1}(\cos\theta)P_l^{-1}(\cos\theta') = \frac{1}{l^2(l+1)^2}P_l^1(\cos\theta)P_l^1(\cos\theta') \tag{358}$$

Thence, the terms of the sum become equal

$$A_\phi = \frac{2\mu_0}{4}\sum_l\frac{1}{l(l+1)}P_l^1(\cos\theta)\{r^l,r^{-l-1}\}\int J(r',\theta')\{\frac{1}{r'^{l+1}},r'^l\}P_l^1(\cos\theta')$$
$$r'^2\sin\theta' dr'd\theta' \tag{359}$$

Finally,

$$A_\phi = \frac{2\mu_0}{4}\sum_l\frac{1}{l(l+1)}P_l^1(\cos\theta)\{r^l,r^{-l-1}\}\frac{1}{2\pi}\int J(r',\theta')\{\frac{1}{r'^{l+1}},r'^l\}P_l^1(\cos\theta')d^3x' \tag{360}$$

and

$$A_\phi = \frac{\mu_0}{4\pi}\sum_l \frac{1}{l(l+1)} P_l^1(\cos\theta)\{r^l, r^{-l-1}\}\int J(r',\theta')\{\frac{1}{r'^{l+1}}, r'^l\}P_l^1(\cos\theta')d^3x' \tag{361}$$

This proves

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_l m_l r^l P_l^1(\cos\theta) \tag{362}$$

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_l \mu_l r^{-l-1} P_l^1(\cos\theta) \tag{363}$$

with

$$m_l = -\frac{1}{l(l+1)}\int d^3x' r'^{-l-1} P_l^1(\cos\theta')J(r',\theta') \tag{364}$$

$$\mu_l = -\frac{1}{l(l+1)}\int d^3x' r'^l P_l^1(\cos\theta)J(r',\theta') \tag{365}$$

Problem 5.9

We can rewrite the current in spherical coordinates as

$$J = \hat{\phi}\frac{I}{a}\delta(r'-a)[\delta(\cos\theta' - \frac{b}{2\sqrt{a^2+\frac{b^2}{4}}}) + \delta(\cos\theta' + \frac{b}{2\sqrt{a^2+\frac{b^2}{4}}})] \tag{366}$$

Therefore,

$$m_l = -\frac{2\pi I}{l(l+1)}a^{-l-2}[P_l^1(\cos\theta') + P_l^1(-\cos\theta')] \tag{367}$$

$$\mu_l = -\frac{2\pi I}{l(l+1)}a^{l-1}[P_l^1(\cos\theta') + P_l^1(-\cos\theta')] \tag{368}$$

But (from see 3.50),

$$P_l^m(-x) = (-1)^{l+m}P_l(x) \tag{369}$$

Therefore,

$$m_l = -\frac{2\pi I}{l(l+1)}a^{-l-2}(1+(-1)^{1+l})P_l^1(\cos\theta') \tag{370}$$

$$\mu_l = -\frac{2\pi I}{l(l+1)}a^{l-1}(1+(-1)^{1+l})P_l^1(\cos\theta') \tag{371}$$

with $\cos\theta' = \frac{b}{2\sqrt{a^2+\frac{b^2}{4}}}$. Therefore, only odd values of l contribute,

$$m_{l_{odd}} = -\frac{4\pi I}{l(l+1)}a^{-l-2}P_l^1(\cos\theta') \tag{372}$$

$$\mu_{l_{odd}} = -\frac{4\pi I}{l(l+1)}a^{l-1}P_l^1(\cos\theta') \tag{373}$$

In the required interval,

$$\{m_1, \mu_1\} = -2\pi I\{a^{-3}, 1\}P_1^1(\cos\theta') \tag{374}$$

$$\{m_3, \mu_3\} = -\frac{1}{3}\pi I\{a^{-5}, a^2\}P_3^1(\cos\theta') \tag{375}$$

$$\{m_5.\mu_5\} = -\frac{2}{15}\pi I\{a^{-7}, a^4\}P_5^1(\cos\theta') \tag{376}$$

(b)

Notice

$$B_z = \frac{1}{\rho}\partial_\rho(\rho A_\phi)|_{\rho=0} = \frac{A_\phi}{\rho} + \partial_\rho A_\phi \tag{377}$$

but when $\theta = \pi/2$, which is the equivalent of the z axis in our coordinate system, $\rho = r$, and

$$B_z = \frac{A_\phi}{r} + \partial_r A_\phi \tag{378}$$

As we are within the distribution,

$$A_\phi(r, \theta \to \pi/2) = -\frac{\mu_0}{4\pi}\sum_l (l+1) m_l r^{l-1} P_l^1(0) \tag{379}$$

The agreement with exercise 5.7 up to fourth order in z can be found by setting $r = z$. As $m_1 \neq 0$, $m_2 \neq 0$ and $m_5 \neq 0$ are the only nontrivial terms, it is clear the appearance of the power orders 0, 2 and 4 as in exercise 5.7.