HW6 - Electromagnetism

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Answer to question 1 (Problem 6.1 of Jackson's)

(a) Consider

$$\Psi(\vec{x},t) = \int d^3x' \frac{f(x',t')_{ret}}{|\vec{x}-\vec{x}'|} \tag{1}$$

Thus, performing the trivial Dirac delta integrals yields

$$\Psi(\vec{x},t) = \int_{-\infty}^{\infty} dz' \frac{\delta(t - \frac{1}{c}\sqrt{x^2+y^2+(z-z')^2})}{\sqrt{x^2+y^2+(z-z')^2}} \tag{2}$$

Defining $g(z') = t - \sqrt{x^2+y^2+(z-z')^2}$, and changing the variable of integration from z' to g, the local expression for the integral above is below defined and rewritten

$$I = \int_{-\epsilon}^{\epsilon} dz' \frac{\delta(t - \frac{1}{c}\sqrt{x^2+y^2+(z-z')^2}}{\sqrt{x^2+y^2+((z-z')^2}} = \int_{-|\epsilon|}^{|\epsilon|} dg |\frac{1}{\frac{dg}{dz'}}|(z'(g)) \frac{\delta(g(z'))}{c(-g(z')+t)} = \int_{-|\epsilon|}^{|\epsilon|} dg \frac{\delta(g)}{c|z-z'|} \tag{3}$$

with ϵ a real small number, and the value of ϵ in the LHS different from that in the RHS, which is its image due to g, but denoted the same without loss or extra info. In the vicinity of any of the roots of $z'(g)$ with $|z-z'| = \sqrt{-(x^2+y^2)+c^2(g(z')-t)^2}$. As we are integrating in the whole space, there are two roots of $g(z')$, meaning that the integral is divided in two regions where $g(z')$ is inversible, each contributing equally.

$$\Psi(\vec{x},t) = \frac{2c\Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}} \tag{4}$$

The presence of $\Theta(c^2t^2-\rho^2) = \Theta(ct-\rho)$ is to take into account the possibility of unphysical imaginary roots $z'(g)$.

(b) Similarly to last, a sheet is such that

$$\Psi(\vec{x},t) = \int \frac{\delta(x')\delta(t - \frac{|\vec{x}-\vec{x}'|}{c})}{|\vec{x}-\vec{x}'|} dx'dy'dz' \tag{5}$$

Defining $(y-y') = \rho\cos\phi$ and $(z-z') = \rho\sin\phi$ and integrating in x' yield

$$\Psi(\vec{x},t) = 2\pi \int_{\rho=0}^{\infty} \rho d\rho \frac{\delta(t - \frac{\sqrt{x^2+\rho^2}}{c})}{\sqrt{x^2+\rho^2}} \tag{6}$$

Due to a local change of variable as before, $g(\rho) = t - \frac{\sqrt{x^2+\rho^2}}{c}$, now $g(\rho)$ is inversible in the interval $\rho \in [0, \infty)$, and there is only one branch, a way to relate ρ and $g(\rho)$ in the considered interval.

$$\Psi(\vec{x}, t) = 2\pi \int_{-\epsilon}^{\epsilon} dg\rho(g) |\frac{1}{\frac{dg}{d\rho}}| (\rho(g)) \frac{\delta(g)}{c(t-g)} \tag{7}$$

Thence, as $dg/d\rho = -\frac{\rho(g)}{c^2(t-g)}$,

$$\Psi(\vec{x}, t) = 2\pi c \Theta(ct - |x|) \tag{8}$$

The Heaviside appear due to the reality of ρ in the vicinity of $g = 0$, as $\rho = \sqrt{c^2t^2 - x^2} = \sqrt{(ct-x)(ct+x)}$, which implies $ct < -x$ or $ct > x$.

Answer to question 2 (Problem 6.8 of Jackson's)

(a) The external total force per unit volume is provided through

$$\vec{f} = \epsilon_0(\vec{E}(\nabla \cdot \vec{E}) + c^2\vec{B}(\nabla \cdot \vec{B}) - \epsilon_0(\vec{E} \times \nabla \times \vec{E} + c^2\vec{B} \times \nabla \times \vec{B}) \tag{9}$$

from an analysis of (6.116) of Jackson's. This is rewritten as

$$\vec{f} = (\epsilon_0 \frac{1}{2} \nabla(\vec{E}^2 + c^2\vec{B}^2) - (\epsilon_0 \vec{E} \times \nabla \times \vec{E} + \frac{1}{\mu_0}\vec{B} \times \nabla \times \vec{B}) \tag{10}$$

with the generalized divergent theorem, and performing an integration, we observe that the total force per unity area is

$$\vec{f}_S = \frac{\epsilon_0}{2}(\vec{E}^2 + c^2\vec{B}^2) \tag{11}$$

while the remaining part of the force in the volume vanishes, as the terms vanish since the Maxwell equations with $\vec{E}$ and $\vec{B}$ pointing to a fixed direction causes these terms to vanish. (one may prove here). Considering solutions to Maxwell with polarization in one dimension,

$$\vec{E} = g(\omega t - \vec{k} \cdot \vec{r})\hat{z} \tag{12}$$
$$\vec{B} = g(\omega t - \vec{k} \cdot \vec{r})\hat{y} \tag{13}$$

it is immediate to prove that the vector product vanish, as $\nabla \times \vec{E} = (k_x g, -k_y g, 0)$, hence, $\vec{E} \times (\nabla \times \vec{E}) = (0, 0, g) \times \nabla \times (k_x g, -k_y g, 0) = \vec{0}$. The same for $\vec{B}$. This total force is exerted by the screen on the particle, and hence, by the particle on the screen. What if the polarization is circular or arbitrary?
(b)
The rate of energy per unity volume transferred through the field in volume is provided through the conservation of energy

$$-\nabla \cdot (\vec{E} \times \vec{H}) = \frac{d}{dt}(E_{mech} + E_{fields}) \tag{14}$$

Thence, in a small volume the divergence theorem causes the quantity $(\vec{E} \times \vec{H}) \cdot \hat{n}$ to represent the quantity of energy per time per area, which is the quantity provided in the problem. Then,

$$\frac{d\vec{P}_{mech}}{dt} + \frac{d\vec{P}_{field}}{dt} = 0 \tag{15}$$

it yields

$$\frac{d\vec{P}_{mech}}{dt} + \frac{d}{dt}\int \frac{\vec{E}\times\vec{H}}{c^2} dV = 0 \tag{16}$$

$(\vec{E}\times\vec{H})$ points in the direction of $\vec{n}$, i.e, $\vec{E}\times\vec{H} = [(\vec{E}\times\vec{H})\cdot\hat{n}]\hat{n}$. The volume comprising momentum that is exchanged with the object momentum is the sector of a sphere, and it changes in time at a rate of $S\times c$ with S the area under consideration and c the velocity of light. The change of momentum can be considered either in the limit of low or great velocities (Minkowsky flat space-time).

$$\sigma S\vec{a} = \frac{[E\times\vec{H}]\cdot\hat{n}}{c^2} cS\hat{n} \tag{17}$$

or simply

$$\vec{a} = \frac{[(\vec{E}\times\vec{H})\cdot\hat{n}]}{\sigma c}\hat{n} = \frac{1.4\times10^3}{10^{-3}\times 3.10^6}\frac{m}{s^2} = 0.46\frac{m}{s^2} \tag{18}$$

Due to relativistic effects, the tendency is for the acceleration to diminish at higher velocities, the reason we are in the correct limit of acceleration if we are to seek its maxima.

Answer to question 3 (Problem 6.11 of Jackson's)
To solve this question, consider

$$\vec{H} = -\frac{1}{\epsilon\mu_0}\int dt \nabla\times\vec{D} \tag{19}$$

where $\vec{D}$ has been determined in chapter 4.

Answer to question 3
(a) Consider Maxwell's equation in the presence of an oscillating electric field in vacuum,

$$\vec{E}(\rho,t) = \mathrm{Re}[E(\rho)\exp[-i\omega t]]\hat{z} \tag{20}$$

Which can be written in the suitable form

$$\vec{E}(\rho,t) = E(\rho)\cos(\omega t + \phi(\rho))\hat{z} \tag{21}$$

with ϕ a phase term dependent on ρ. This is reasonable if the current is provided through $i(t) = I_0\cos(\omega t)$ and the ohms law is valid, for instance. Given the Maxwell's equation in vacuum,

$$\nabla\cdot\vec{E} = 0 \tag{22}$$
$$\nabla\times\vec{E} = -\partial_t\vec{B} \tag{23}$$
$$\nabla\times\vec{B} = \frac{1}{c^2}\partial_t\vec{E} \tag{24}$$

imply

$$\nabla^2\vec{E} = -\frac{\omega^2}{c^2}\vec{E} \tag{25}$$

In the suitable cylindrical coordinate,

$$\rho^2\vec{E}_{\rho\rho} + \rho\vec{E}_{\rho} + \frac{\omega^2}{c^2}\rho^2\vec{E} = 0 \tag{26}$$

where E is the field pointing in the z direction. Defining the dimensionless quantity $r = \rho\omega/c$, last is brought to

$$r^2\vec{E}_{rr} + r\vec{E}_r + r^2\vec{E} = 0 \tag{27}$$

a Bessel equation of order 0, whose solution is the Bessel function of the first kind in avoiding divergences at $\rho = 0$ due to the solution of the second kind. Since

$$\vec{E} = E(\rho)(\cos\omega t\cos\phi(\rho) - \sin\omega t\sin\phi(\rho))\hat{z} \tag{28}$$

the linear independence of sin and cos terms imply that both $E(\rho)\cos\phi(\rho)$ and $E(\rho)\sin\phi(\rho)$ are solution to the Bessel eq. Thence, (27) is condensed into

$$\vec{E} = (\cos\omega tA - \sin\omega tB)J_0(\frac{\rho\omega}{c})\hat{z} = E_0J_0(\frac{\rho\omega}{c})\sin(\omega t+\phi)\hat{z} \tag{29}$$

Through (22) integrated in time,

$$\vec{B} = -\frac{E_0}{\omega}\cos(\omega t+\phi)\nabla\times[J_0(\frac{\rho\omega}{c})\hat{z}] \tag{30}$$

i.e,

$$\vec{B} = \frac{E_0}{c}\cos(\omega t+\phi)J_1(\frac{\rho\omega}{c})\hat{\phi} \tag{31}$$

from the property $J_0'(x) = -J_1(x)$ for the Bessel function. Collecting the result,

$$\vec{E} = E_0J_0(\frac{\rho\omega}{c})\sin(\omega t+\phi)\hat{z} \tag{32}$$

$$\vec{B} = \frac{E_0}{c}J_1(\frac{\rho\omega}{c})\cos(\omega t+\phi)\hat{\phi} \tag{33}$$

The value of E_0 and ϕ can be obtained from the info of the problem due to the limit of low frequencies. In this limit, explicitly $\omega << \frac{a}{c}$, (31) and (32) is reduced to $\vec{E} = E_0\cos\phi\hat{z}$. In this static limit, $\vec{E} \sim \frac{\sigma}{\epsilon_0}\hat{z}$. Further, through $\sigma = \int\frac{I}{\pi a^2}dt = \frac{\int I_0\sin(\omega t+\phi)dt}{\pi a^2}$, in the static limit, $\omega \to 0$, $\sigma = -\frac{I_0\cos\phi}{\omega\pi a^2}$. From (31) and the local Gauss theorem $\vec{E} = \frac{I_0\cos\phi}{\omega\pi a^2\epsilon_0}\hat{z}$, a comparison leads to an arbitrary phase and $E_0 = \frac{I_0}{\omega\pi a^2\epsilon_0\cos\phi}$, leading to

$$\vec{E} = \cos\phi\frac{I_0}{\omega\pi a^2\epsilon_0}J_0(\frac{\rho\omega}{c})\cos(\omega t+\phi)\hat{z} \tag{34}$$

$$\vec{B} = \frac{1}{c}\cos\phi\frac{I_0}{\omega\pi a^2\epsilon_0}J_1(\frac{\rho\omega}{c})\sin(\omega t+\phi)\hat{\phi} \tag{35}$$

The choice $\phi = 0$ leads to

$$\vec{E} = \frac{I_0}{\omega\pi a^2\epsilon_0}J_0(\frac{\rho\omega}{c})\sin(\omega t)\hat{z} \tag{36}$$

$$\vec{B} = \frac{1}{c}\frac{I_0}{\omega\pi a^2\epsilon_0}J_1(\frac{\rho\omega}{c})\cos(\omega t)\hat{\phi} \tag{37}$$

(b) To calculate the total energy, consider the densities

$$\omega_e = \frac{\epsilon_0}{4}|\vec{E}|^2 = \frac{\epsilon_0}{4}\frac{I_0^2}{\omega^2\pi^2a^4\epsilon_0^2}|J_0(\frac{\rho\omega}{c})|^2|\cos^2(\omega t)| \tag{38}$$

$$\omega_m = \frac{1}{4\mu_0}|\vec{B}|^2 = \frac{1}{4\mu_0}\frac{1}{c^2}\frac{I_0^2}{\omega^2\pi^2a^4\epsilon_0^2}|J_1(\frac{\rho\omega}{c})|^2|\sin^2(\omega t)| = \tag{39}$$

$$\frac{\epsilon_0}{4}\frac{I_0^2}{\omega^2\pi^2a^4\epsilon_0^2}|J_1(\frac{\rho\omega}{c})|^2|\sin^2(\omega t)| \tag{40}$$

Therefore, the total energy is provided through an integration within the capacitor volume.

$$U = 2\pi d \int_0^a d\rho\rho[\omega_e(\rho) + \omega_m(\rho)] \tag{41}$$

$$U_E = \frac{1}{4}\frac{I_0^2}{\omega^2\pi^2a^4\epsilon_0}2\pi d \int_0^a d\rho\rho|J_0(\frac{\rho\omega}{c})|^2|\cos^2\omega t| \tag{42}$$

$$U_M = \frac{1}{4}\frac{I_0^2}{\omega^2\pi^2a^4\epsilon_0}2\pi d \int_0^a d\rho\rho|J_1(\frac{\rho\omega}{c})|^2|\sin^2\omega t| \tag{43}$$

$$U_E \sim \frac{\cos^2(\omega t)}{4}\frac{I_0^2}{\omega^2\pi^2a^4\epsilon_0}2\pi d\frac{a^2}{2} = \frac{\cos^2\omega t}{4}\frac{dI_0^2}{\omega^2\pi a^2\epsilon_0} \tag{44}$$

$$U_M \sim \frac{\sin^2(\omega t)}{4}\frac{I_0^2}{\pi^2\epsilon_0}2\pi d\frac{1}{c^2}\frac{1}{16} = \frac{\sin^2(\omega t)}{32}\frac{dI_0^2}{\pi\epsilon_0}\frac{1}{c^2} \tag{45}$$

Where we have used the expansion

$$J_0(x) = 1 - \frac{x^2}{4} + O(>) \tag{46}$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{16} + O(>) \tag{47}$$

and considered the smallest contribution, that from $J_0 = 1$ and $J_1 = \frac{x}{2}$ alone. The Poynting vector at $\rho = a$ is

$$\vec{S}(\rho = a) = \frac{1}{2}\vec{E}\times\vec{H}(\rho = a) = \frac{1}{2\mu_0}\vec{E}\times\vec{B} = \frac{-1}{2\mu_0\epsilon_0^2c}\frac{I_0^2}{\omega^2\pi^2a^4}J_0(\frac{a\omega}{c})J_1(\frac{a\omega}{c})\sin\omega t\cos\omega t\hat{\rho} \sim \frac{-1}{2\mu_0\epsilon_0^2c}\frac{I_0^2}{\omega^2\pi^2a^4}\frac{a\omega}{2c}\sin\omega t\cos\omega t \tag{48}$$

The total flux of the field is

$$\Phi = (\vec{S}(a)\cdot\hat{\rho})2\pi ad \tag{49}$$

since there is no flux along the cylinder circles. The smallest term is considered in the expansion, i.e,

$$\Phi = -\frac{1}{2\mu_0\epsilon_0^2c}\frac{I_0^2}{\omega^2\pi^2a^4}\frac{a\omega}{2c}2\pi ad\sin\omega t\cos\omega t \sim -\frac{dI_0^2}{2\omega\pi\epsilon_0a^2}\sin\omega t\cos\omega t\hat{\rho} \tag{50}$$

The Poyinting theorem is obeyd if we consider the largest contribution, that due to the electric field. In this case, it is indeed very clear clear that

$$\Phi = -\partial_t U \tag{51}$$

obeying the Pointing theorem in the absence of electric current.

Answer to question 4

To show the equivalence with other circuit components in series we consider the reactance expression valid in the low-frequency limit

$$\mathcal{X} = \frac{4\omega}{|I|^2}(U_M - U_E) \tag{52}$$

Thence,

$$\mathcal{X} = \frac{4\omega}{|I|^2}\left(\frac{d}{32\pi\epsilon_0 c^2}|I|^2 - \frac{d}{4\omega^2\pi a^2\epsilon_0}|I|^2\right) \tag{53}$$

In which we identify one term dependent on ω and another dependent on $\frac{1}{\omega}$. Therefore, we have an inductor and a capacitor connected in series, as the reactance has the form $\mathcal{X} = \omega L - \frac{1}{\omega C}$.