Fractional Knapsack Problem

Using Greedy Method

Exam-Ready Notes Based on Jenny's Lectures

Problem Statement & Given Data

We are given a knapsack (bag) with a maximum capacity $W = 15$ kg, and 7 items, each with a specific profit and weight. The objective is to select items to maximize the total profit without exceeding the capacity.
Because this is a **fractional** knapsack problem, if the remaining capacity is less than an item's weight, we can break the item and take a fraction of it.

Items Data Matrix:

Item	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6	Item 7
Profit (P)	5	10	15	7	8	9	8
Weight (W)	1	3	5	4	1	3	4

The Optimal Strategy: Profit-by-Weight Ratio

The Intuition (Why use a ratio?)
Imagine you are filling a small bag, and you have to choose between a heavy rock that sells for $10, or a tiny diamond that also sells for $10. Which one do you pick? The diamond, of course! Even though the total profit is the same, the diamond takes up much less space. It gives you a much higher **value per kilogram**.

In the Fractional Knapsack Problem, we cannot just look at the highest profit (because the item might be too heavy and fill the whole bag) or the lowest weight (because the item might be worthless). We need the best balance. We find this by calculating the **Profit-by-Weight Ratio** for every item.

Step 1: Calculate the Ratio

The formula is very simple:

$$\text{Ratio} = \frac{\text{Profit of the Item}}{\text{Weight of the Item}}$$

This ratio tells us exactly how much profit we earn for **every 1 kg** of that specific item. Let's calculate this for all our items:

Item	Profit (P)	Weight (W)	Calculation (P / W)	Ratio (Value per kg)
Item 1	5	1	$5/1$	**5.00**
Item 2	10	3	$10/3$	**3.33**
Item 3	15	5	$15/5$	**3.00**
Item 4	7	4	$7/4$	**1.75**
Item 5	8	1	$8/1$	**8.00** (Highest!)
Item 6	9	3	$9/3$	**3.00**
Item 7	8	4	$8/4$	**2.00**

Step 2: Sort in Descending Order

Why do we sort in descending order? We want to pack our bag with the most valuable items first before we run out of space. By picking the items with the highest "Value per kg" first, we guarantee the maximum possible profit.

Our Selection Order:
Item 5 (8.00) $\rightarrow$ Item 1 (5.00) $\rightarrow$ Item 2 (3.33) $\rightarrow$ Item 3 (3.00) $\rightarrow$ Item 6 (3.00) $\rightarrow$ Item 7 (2.00) $\rightarrow$ Item 4 (1.75)

Step 3: Step-by-Step Selection

Now, we will pick the items one by one according to our sorted list, keeping an eye on our remaining bag capacity (Starts at 15 kg).

Selection 1: Item 5
Item 5 gives the best value per kg. It weighs 1 kg. Our bag can hold 15 kg, so it easily fits. We take the whole item.

Item	Fraction Taken	Profit Added	Weight Used	Remaining Capacity
Item 5	1 (All)	8	1	$15 - 1 =$ **14**

Selection 2: Item 1
The next best is Item 1. It weighs 1 kg. We have 14 kg of space left, so it easily fits. We take the whole item.

Item	Fraction Taken	Profit Added	Weight Used	Remaining Capacity
Item 5	1 (All)	8	1	14
Item 1	1 (All)	5	1	$14 - 1 =$ **13**

Selection 3: Item 2
Next is Item 2, weighing 3 kg. We have 13 kg of space left, so it fits perfectly.

Item	Fraction Taken	Profit Added	Weight Used	Remaining Capacity
Item 5	1 (All)	8	1	14
Item 1	1 (All)	5	1	13
Item 2	1 (All)	10	3	$13 - 3 =$ **10**

Selection 4: Item 3
Next is Item 3, weighing 5 kg. We have 10 kg of space left, so we drop the whole item in.

Item	Fraction Taken	Profit Added	Weight Used	Remaining Capacity
Item 5	1 (All)	8	1	14
Item 1	1 (All)	5	1	13
Item 2	1 (All)	10	3	10
Item 3	1 (All)	15	5	$10 - 5 =$ **5**

Selection 5: Item 6
Next is Item 6, weighing 3 kg. We have 5 kg space remaining. It fits!

Item	Fraction Taken	Profit Added	Weight Used	Remaining Capacity
Item 5	1 (All)	8	1	14
Item 1	1 (All)	5	1	13
Item 2	1 (All)	10	3	10
Item 3	1 (All)	15	5	5
Item 6	1 (All)	9	3	$5 - 3 =$ **2**

Selection 6: Item 7
Note: Our remaining capacity is exactly 2 kg, but our next best item, Item 7, weighs 4 kg. Because it is too heavy to fit entirely, we must break it and take a fraction. We take exactly what we need: $\frac{2}{4}$ (or half) of the item.

Item	Fraction Taken	Profit Added	Weight Used	Remaining Capacity
Item 5	1 (All)	8	1	14
Item 1	1 (All)	5	1	13
Item 2	1 (All)	10	3	10
Item 3	1 (All)	15	5	5
Item 6	1 (All)	9	3	2
Item 7	$2/4$	$8 \times \frac{2}{4} = 4$	2	$2 - 2 =$ **0**

Since our remaining capacity has reached 0, our knapsack is completely full. We stop here and ignore the remaining Item 4.

Final Result

By summing up the Profit Added column, we get our maximum possible profit:
Total Optimal Profit = $8 + 5 + 10 + 15 + 9 + 4 =$ **51**

Summary: Why is this method Optimal?

The Greedy Approach using the **Profit/Weight Ratio** gives us the absolute highest profit (51) for the Fractional Knapsack Problem.
It works because by picking the items with the highest ”value per unit of weight”, we guarantee that every single inch of space inside our knapsack is filled with the most profitable material available. There is no wasted space and no wasted opportunity.

The Golden Rule of Knapsack Problems:

- **Fractional Knapsack** $\rightarrow$ Items *can* be broken into pieces. The **Greedy Method (Ratio)** works perfectly and gives the optimal answer.
- **0/1 Knapsack** $\rightarrow$ Items *cannot* be broken (you either take it completely or leave it). Here, the Greedy Method **fails**, and we must use a completely different technique called **Dynamic Programming**.