Measure Theoretic Probability

Prerequisites: Real Analysis, Classical Probability

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Chapter 1

Introduction to σ-algebra

In the classical probability, we encountered *Bertnard's Paradox*, which highlighted the significance of rotational and translational invariance in probability measurements. We discovered that probabilities, much like lengths, areas, or volumes, should remain unchanged when subjected to such transformations. For example, if two points are separated by a distance d, shifting them by an equal amount in the same direction preserves that distance. This invariance hints that probability is not merely a tool for quantifying uncertainty but rather a form of measure - just like length, area or volume.

This realization serves as our starting point for a deeper, more formal approach to understanding probability, known as **Measure Theoretic Probability**. By treating probability as a measure, we establish a rigorous mathematical foundation that allows us to precisely define, manipulate, and compute probabilities, even in complex scenarios involving infinite spaces or continuous distributions.

In this chapter, we will build from the fundamentals of measure theory, gradually developing the key concepts required to form a robust understanding of probability in this framework [1].

1.1 Introduction to $\sigma - algebra$

In the early chapters of *Real Analyis*, we introduced the concept of a *field*. A field is an ordered triple, for example, $(\mathbb{Q}, +, \times)$, consisting of the set of rational numbers $\mathbb{Q}$ and two binary operations, $+$ and $\times$, defined on them. These operations follow specific properties, such as having an identity element, the existence of an inverse element for each non-zero element, and distributivity, among others. This structure forms what is commonly referred to as *arithmetic or numeric algebra*.

But what if we change the set and the operations? Suppose instead of $\mathbb{Q}$, we take the set $\{0, 1\}^\infty$ (the set of all binary sequences) and define appropriate binary operations, such as $+$ and $\cdot$. The resulting structure is called a *boolean algebra*. Similarly, if we take the set of matrices M and define addition and multiplication operations on them, we obtain what is known as *matrix algebra*. These examples illustrate that the notion of algebra is not restricted to numbers; it can be generalized to other sets with appropriate operations.

Now, consider a large set with many subsets as its elements, and define two operations: union $\cup$ and intersection $\cap$. This leads to what is known as a *set algebra*, which is central to our discussion.

Definition 1.1. *Let Ω be a sample space and let $\mathcal{F}_0$ be a collection of subsets of Ω. Then, $\mathcal{F}_0$ is said to be an algebra (or a field) if the following conditions hold:*

1. *$\emptyset \in \mathcal{F}_0$.*
2. *If $A \in \mathcal{F}_0$, then $A^c \in \mathcal{F}_0$.*
3. *If $A \in \mathcal{F}_0$ and $B \in \mathcal{F}_0$, then $A \cup B \in \mathcal{F}_0$.*

While the terms *field* and *algebra* are sometimes used interchangeably in the context of sets, there is a subtle difference when we generalize to other structures. A *field* refers specifically to a set with two binary operations (like $+$ and $\times$) that satisfy a complete set of properties such as associativity, commutativity, distributivity, and the existence of identity and inverse elements.

On the other hand, an *algebra* is a broader concept. It is a structure consisting of a set and operations that may or may not satisfy all the properties required of a field. For instance, in set theory, a set algebra satisfies closure under union, intersection, and complement, but it does not necessarily satisfy all the numeric properties of a field, such as the existence of multiplicative inverses. Thus, while all fields can be considered a type of algebra, not all algebras are fields. The key distinction lies in the specific operations and properties defined on the set.

Theorem 1.1. *An algebra is closed under finite union and finite intersection.*

Proof. **Closed Under Finite Union:**

To prove that $\mathcal{F}_0$ is closed under finite union, we proceed by induction.

Base Case: Let $A_1, A_2 \in \mathcal{F}_0$. By definition of an algebra, $A_1 \cup A_2 \in \mathcal{F}_0$. This shows that the union of two sets in $\mathcal{F}_0$ is also in $\mathcal{F}_0$.

Induction Step: Suppose for some $n \in \mathbb{N}$, the union of n sets in $\mathcal{F}_0$, say $A_1, A_2, \ldots, A_n$, is also in $\mathcal{F}_0$. That is,

$$A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{F}_0.$$

Now, consider $A_1 \cup A_2 \cup \ldots \cup A_n \cup A_{n+1}$. We can rewrite this as:

$$(A_1 \cup A_2 \cup \ldots \cup A_n) \cup A_{n+1}.$$

By the induction hypothesis, $(A_1 \cup A_2 \cup \ldots \cup A_n) \in \mathcal{F}_0$. Since $A_{n+1} \in \mathcal{F}_0$ and $\mathcal{F}_0$ is closed under the union of two sets, it follows that:

$$(A_1 \cup A_2 \cup \ldots \cup A_n) \cup A_{n+1} \in \mathcal{F}_0.$$

By the principle of mathematical induction, $\mathcal{F}_0$ is closed under finite union.

Closed Under Finite Intersection:

To show closure under finite intersection, note that for any sets $A, B \in \mathcal{F}_0$, we have $A^c, B^c \in \mathcal{F}_0$ because complements of sets in an algebra are also in the algebra.

Using De Morgan's laws, we know that:

$$A \cap B = (A^c \cup B^c)^c.$$

Since $A^c, B^c \in \mathcal{F}_0$ and $\mathcal{F}_0$ is closed under finite union, it follows that $A^c \cup B^c \in \mathcal{F}_0$. Therefore, $(A^c \cup B^c)^c \in \mathcal{F}_0$, meaning $A \cap B \in \mathcal{F}_0$.

By similar reasoning and using induction, it can be shown that $\mathcal{F}_0$ is closed under the intersection of any finite number of sets. Thus, $\mathcal{F}_0$ is closed under finite intersection.

□

We have not defined the concept of *event* yet. Informally, for now consider that an event is an subset of sample space that is of our interest. A natural question that arises at this point is *Is the structure of an algebra enough to study events of typical interest in probability theory?*

An Event Not Included in an Algebra

Consider the following example. Toss a coin repeatedly until the first heads appears. The sample space is:

$$\Omega = \{H, TH, TTH, \ldots\}$$

where H represents heads appearing on the first toss, TH represents tails followed by heads, TTH represents two tails followed by heads, and so on.

Now, suppose we are interested in determining whether the number of tosses before seeing a head is even. Let E denote this event. Then,

$$E = \{TH, TTTH, TTTTTH, \ldots\}$$

which includes all outcomes where heads appears after an even number of tosses.

Notice that E is a countably infinite union of individual outcomes:

$$E = \{TH\} \cup \{TTTH\} \cup \{TTTTTH\} \cup \ldots$$

However, an *algebra* is defined to contain only finite unions of subsets. Since E involves a countably infinite union, it cannot be part of the algebra of subsets of Ω. This shows that our *event* of interest is not included in the algebra.

This limitation motivates the need for a more comprehensive structure called a *σ-algebra*. A σ-algebra extends the notion of an algebra by allowing countably infinite unions of subsets, ensuring that events like E are included within the framework of probability theory.

Definition 1.2. *A collection $\mathcal{F}$ of subsets of Ω is called a σ-algebra (or σ-field) if:*

1. *$\emptyset \in \mathcal{F}$.*

2. *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (i.e., the complement of A is also in $\mathcal{F}$).*

3. *If $A_1, A_2, A_3, \ldots$ is a countable collection of subsets in $\mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.*

Note that, unlike an algebra, a σ-algebra is closed under countable union and countable intersection.

Examples of $\sigma - algebras$:

Here are some intuitive examples of σ-algebras:

1. **Trivial σ-algebra:** The smallest σ-algebra on a sample space Ω is $\mathcal{F} = \{\emptyset, \Omega\}$. This is known as the trivial σ-algebra and contains only the empty set and the entire sample space.

2. **Power Set σ-algebra:** The largest σ-algebra on a sample space Ω is the power set of Ω, denoted as 2^{Ω}. It includes all possible subsets of Ω. This is the most comprehensive σ-algebra possible on Ω.

3. **Finite and Countable σ-algebras:** Consider a finite or countable sample space, such as $\Omega = \{1, 2, 3, \ldots\}$. The collection of all subsets of Ω forms a σ-algebra, as it is closed under countable unions, intersections, and complements.

Theorem 1.2. *Every σ-algebra is an algebra, but the converse is not true.*

Proof. **Part 1: Every σ-algebra is an algebra**

Let $\mathcal{F}$ be a σ-algebra.

1. **Contains the empty set:** By the definition of a σ-algebra, we have $\emptyset \in \mathcal{F}$.

2. **Closed under complementation:** If $A \in \mathcal{F}$, then by definition, $A^c \in \mathcal{F}$.

3. **Closed under finite unions:** Let $A, B \in \mathcal{F}$. We can consider the finite union:

$$A \cup B = A \cup B = \bigcup_{i=1}^{2} A_i.$$

   Here, we can denote $A_1 = A$ and $A_2 = B$. Since $\mathcal{F}$ is closed under countable unions, we have:

$$A \cup B \in \mathcal{F}.$$

Since $\mathcal{F}$ satisfies all three properties of an algebra, we conclude that every σ-algebra is indeed an algebra.

Part 2: The converse is not true

To show that not every algebra is a σ-algebra, we can provide a counterexample.

Consider the set $\Omega = \{1, 2, 3\}$ and the algebra $\mathcal{A} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

1. **Contains the empty set:** $\emptyset \in \mathcal{A}$.
2. **Closed under complementation:** The complement of each set in $\mathcal{A}$ is also in $\mathcal{A}$.
3. **Closed under finite unions:** The union of any finite number of sets in $\mathcal{A}$ is also in $\mathcal{A}$.

However, the collection $\mathcal{A}$ is not a σ-algebra because it is not closed under countable unions. For instance, if we consider the countable collection of subsets:

$$A_1 = \{1\}, \quad A_2 = \{2\}, \quad A_3 = \{3\}, \quad \ldots$$

the union $\bigcup_{i=1}^{\infty} A_i = \{1, 2, 3\} = \Omega$, which is included, but if we consider an infinite union of disjoint sets from $\mathcal{A}$ that leads to more than three elements, it will not be contained within $\mathcal{A}$.

Thus, we conclude that not every algebra is a σ-algebra. $\square$

Examples of algebras which are not σ-algebras:

Below are some simple examples of an algebra that is not a σ-algebra:

Example 1: The Finite Subsets of $\mathbb{N}$

Consider the set $\Omega = \mathbb{N}$, the set of all natural numbers. Let $\mathcal{A}$ be the collection of all finite subsets of $\mathbb{N}$ along with $\mathbb{N}$ itself. This collection forms an **algebra** because:

- The union or intersection of any two finite sets is finite (or possibly $\mathbb{N}$).
- The complement of any finite subset is also an infinite subset, and in this case, it is $\mathbb{N}$ (which belongs to $\mathcal{A}$).

However, $\mathcal{A}$ is **not a σ-algebra** because it is not closed under countable union. For instance, if we take a sequence of singletons $\{1\}, \{2\}, \{3\}, \ldots$, the union of these singletons is $\mathbb{N}$, which is an infinite set. While $\mathbb{N}$ is in $\mathcal{A}$, the complement of this countable union would not necessarily belong to $\mathcal{A}$, as it may not be finite.

Example 2: Intervals on the Real Line

Consider $\Omega = [0, 1]$ and let $\mathcal{A}$ be the collection of all finite unions of intervals of the form $[a, b]$, where $0 \leq a \leq b \leq 1$. This collection $\mathcal{A}$ forms an **algebra** because:

- The union and intersection of a finite number of intervals of this form are again finite unions of intervals of this form.
- The complement of a finite union of such intervals is also a finite union of intervals.

However, $\mathcal{A}$ is **not a σ-algebra** because it is not necessarily closed under countable unions. For example, if we take a sequence of intervals $\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, \frac{7}{8}\right], \ldots$ such that they cover $[0, 1]$ as a whole, their countable union would be $[0, 1]$. Although $[0, 1]$ is in $\mathcal{A}$, the structure of $\mathcal{A}$ doesn't guarantee closure under all such countable unions.

Example 3: The Power Set of a Finite Set

Let $\Omega = \{a, b, c\}$ be a finite set. The collection $\mathcal{A}$ of all subsets of Ω (also known as the power set of Ω) forms an **algebra** because:

- Any union, intersection, or complement of subsets of a finite set remains a subset of that finite set.

However, even though this is a trivial example, it demonstrates that an algebra is not necessarily a σ-algebra because σ-algebras are designed to handle infinite cases. In this finite scenario, $\mathcal{A}$ satisfies the properties of both an algebra and a σ-algebra, but it shows that if the set Ω were infinite, $\mathcal{A}$ would not generally be closed under countable operations.

Example 1.1. *Consider the random experiment of throwing a die. If a statistician is interested in the occurrence of either an odd or an even outcome, construct a sample space and a σ-algebra of subsets of this sample space.*

Sample Space (Ω)*: The sample space consists of all possible outcomes when throwing a six-sided die. Therefore, we can define the sample space as:*

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Events of Interest*: The statistician is interested in the occurrence of either an odd or an even outcome. We can categorize the outcomes as follows:*

- ***Odd Outcomes****:* $\{1, 3, 5\}$
- ***Even Outcomes****:* $\{2, 4, 6\}$

Constructing the σ-Algebra ($\mathcal{F}$)*: A σ-algebra is a collection of subsets of Ω that satisfies the following properties:*

- *It contains the empty set and the sample space itself.*
- *It is closed under complementation.*
- *It is closed under countable unions.*

Given the events of interest, we can construct the σ-algebra as follows:

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 3, 5\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\}$$

Checking the Properties of the σ-Algebra:

Contains the Empty Set and Sample Space: *$\emptyset \in \mathcal{F}$ and $\Omega = \{1, 2, 3, 4, 5, 6\} \in \mathcal{F}$.*

Closed under Complementation:

- *The complement of $\emptyset$ is $\{1, 2, 3, 4, 5, 6\}$, which is in $\mathcal{F}$.*
- *The complement of $\{1, 3, 5\}$ is $\{2, 4, 6\}$, which is in $\mathcal{F}$.*
- *The complement of $\{2, 4, 6\}$ is $\{1, 3, 5\}$, which is in $\mathcal{F}$.*

Closed under Countable Unions: *For any events in $\mathcal{F}$, the union will also be in $\mathcal{F}$. For instance, $\{1\} \cup \{2\} = \{1, 2\} \in \mathcal{F}$, and $\{1, 3, 5\} \cup \{2, 4, 6\} = \{1, 2, 3, 4, 5, 6\} \in \mathcal{F}$.*

Example 1.2. *Let $A_1, A_2, \ldots, A_n$ be arbitrary subsets of Ω. Describe (explicitly) the smallest σ-algebra $\mathcal{F}$ containing $A_1, A_2, \ldots, A_n$. How many sets are there in $\mathcal{F}$? (Give an attainable upper bound under certain conditions). List all the sets in $\mathcal{F}$ for $n = 2$.*

Smallest σ-algebra containing $A_1, A_2, \ldots, A_n$:

The smallest σ-algebra $\mathcal{F}$ containing the subsets $A_1, A_2, \ldots, A_n$ is generated by these sets. This means $\mathcal{F}$ includes all possible unions, intersections, and complements of these sets.

To explicitly describe $\mathcal{F}$:
1. Include $A_1, A_2, \ldots, A_n$.
2. Include the complements of each set: $A_1^c, A_2^c, \ldots, A_n^c$.
3. Include all possible unions and intersections of these sets and their complements.

Counting the Sets in $\mathcal{F}$:

In the worst-case scenario, if $A_1, A_2, \ldots, A_n$ are arbitrary subsets with no restrictions, the number of distinct sets that can be formed is determined by the combinations of unions and intersections. An attainable upper bound for the number of sets in $\mathcal{F}$ can be given by:

$$|\mathcal{F}| \leq 2^{2^n}$$

This upper bound arises from considering all subsets of Ω formed by the possible intersections of the $2n$ sets (including both original sets and their complements).

Example for $n = 2$:

Let A_1 and A_2 be two arbitrary subsets of Ω. The smallest σ-algebra $\mathcal{F}$ generated by A_1 and A_2 contains the following sets: A_1, A_2, A_1^c, A_2^c, $A_1 \cap A_2$, $A_1 \cap A_2^c$, $A_1^c \cap A_2$ and $A_1^c \cap A_2^c$.

Thus, the sets in $\mathcal{F}$ when $n = 2$ are:

$$\mathcal{F} = \{A_1, A_2, A_1^c, A_2^c, A_1 \cap A_2, A_1 \cap A_2^c, A_1^c \cap A_2, A_1^c \cap A_2^c\}$$

Example 1.3. *Let F and G be two σ-algebras of subsets of Ω.*
(a) *Is $F \cup G$, the collection of subsets of Ω lying in either F or G, a σ-algebra?*
(b) *Show that $F \cap G$, the collection of subsets of Ω lying in both F and G, is a σ-algebra.*
(c) *Generalize (b) to arbitrary intersections as follows. Let I be an arbitrary index set (possibly uncountable), and let $\{F_i\}_{i\in I}$ be a collection of σ-algebras on Ω. Show that $\bigcap_{i\in I} F_i$ is also a σ-algebra.*

To determine whether $F \cup G$ is a σ-algebra, we need to check the three properties:

Contains the empty set and sample space: *Since both F and G are σ-algebras, they each contain $\emptyset$ and Ω. Thus, $F \cup G$ contains both $\emptyset$ and Ω.*

Closed under complementation: *Let $A \in F \cup G$. If $A \in F$, then $A^c \in F$ (since F is a σ-algebra), and similarly for G. However, A^c might not be in $F \cup G$ if A is in one algebra but not in the other. Thus, $F \cup G$ is not closed under complementation.*

Closed under countable unions: *Let $A_1, A_2, \ldots \in F \cup G$. If all A_i are in F, then $\bigcup_{i=1}^{\infty} A_i \in F$. If all A_i are in G, then $\bigcup_{i=1}^{\infty} A_i \in G$. However, if some A_i are in F and some in G, $\bigcup_{i=1}^{\infty} A_i$ may not be in $F \cup G$. Therefore, $F \cup G$ is not closed under countable unions.*

*Hence, $F \cup G$ is **not a σ-algebra**.*

To show that $F \cap G$ is a σ-algebra, we verify the three properties:

Contains the empty set and sample space: *Since both F and G contain $\emptyset$ and Ω, we have $\emptyset \in F \cap G$ and $\Omega \in F \cap G$.*

Closed under complementation: *Let $A \in F \cap G$. Then $A \in F$ and $A \in G$. Thus, $A^c \in F$ and $A^c \in G$, which implies $A^c \in F \cap G$.*

Closed under countable unions: *Let $A_1, A_2, \ldots \in F \cap G$. Then $A_i \in F$ for all i and $A_i \in G$ for all i. Thus, $\bigcup_{i=1}^{\infty} A_i \in F$ and $\bigcup_{i=1}^{\infty} A_i \in G$, which implies $\bigcup_{i=1}^{\infty} A_i \in F \cap G$.*

*Therefore, $F \cap G$ **is a σ-algebra**.*

To prove that $\bigcap_{i\in I} F_i$ is a σ-algebra, we check the three properties:

Contains the empty set and sample space: *Since each F_i contains $\emptyset$ and Ω, we have $\emptyset \in \bigcap_{i\in I} F_i$ and $\Omega \in \bigcap_{i\in I} F_i$.*

Closed under complementation: *Let $A \in \bigcap_{i\in I} F_i$. Then $A \in F_i$ for all i. Thus, $A^c \in F_i$ for all i, which implies $A^c \in \bigcap_{i\in I} F_i$.*

Closed under countable unions: *Let $A_1, A_2, \ldots \in \bigcap_{i\in I} F_i$. Then $A_j \in F_i$ for all j and for all i. Thus, $\bigcup_{j=1}^{\infty} A_j \in F_i$ for all i, which implies $\bigcup_{j=1}^{\infty} A_j \in \bigcap_{i\in I} F_i$.*

*Therefore, $\bigcap_{i\in I} F_i$ **is a σ-algebra**.*

Example 1.4. *Let Ω be an arbitrary set. Answer the following questions:*
(a) *Is the collection F_1 consisting of all finite subsets of Ω an algebra?*
(b) *Let F_2 consist of all finite subsets of Ω and all subsets of Ω having a finite complement. Is F_2 an algebra?*
(c) *Is F_2 a σ-algebra?*
(d) *Let F_3 consist of all countable subsets of Ω and all subsets of Ω having a countable complement. Is F_3 a σ-algebra?*

To determine if F_1 is an algebra, we must check the three properties:

1. ***Contains the empty set:*** *$\emptyset \in F_1$ since the empty set is a finite subset.*

2. ***Closed under complementation:*** *If $A \in F_1$ (i.e., A is a finite subset of Ω), then its complement A^c may not be finite. Therefore, F_1 is not closed under complementation.*

3. ***Closed under finite unions:*** *If $A, B \in F_1$, then $A \cup B$ is also finite, so F_1 is closed under finite unions.*

Since F_1 fails to be closed under complementation, we conclude that F_1 is ***not an algebra****.*

To check if F_2 is an algebra, we verify the properties:

1. ***Contains the empty set:*** *$\emptyset \in F_2$ since it is a finite subset.*

2. ***Closed under complementation:***

    - *If $A \in F_2$ is finite, then A^c has a finite complement, which is infinite. Thus, it is in F_2.*
    - *If $B \in F_2$ has a finite complement, then B^c is finite. Therefore, $B^c \in F_2$.*

    Hence, F_2 is closed under complementation.

3. ***Closed under finite unions:***

    - *If $A, B \in F_2$ are both finite, then $A \cup B$ is finite.*
    - *If A is finite and B has a finite complement, then $A \cup B$ has a finite complement.*
    - *If both A and B have finite complements, then $(A \cup B)^c = A^c \cap B^c$, which is finite.*

Thus, F_2 is closed under finite unions.

To determine if F_2 is a σ-algebra, we need to check the closure under countable unions.

Consider the countable union of finite sets:

$$A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}, \ldots$$

Then,

$$\bigcup_{i=1}^{\infty} A_i = \{1, 2, 3, \ldots\}$$

which is not finite. Therefore, F_2 is not closed under countable unions.

*Thus, F_2 is **not a σ-algebra**.*

To check if F_3 is a σ-algebra, we verify:

1. ***Contains the empty set:*** *$\emptyset \in F_3$ since it is countable.*

2. ***Closed under complementation:***

   - *If $A \in F_3$ is countable, then A^c has a countable complement.*
   - *If $B \in F_3$ has a countable complement, then B^c is countable.*

   Hence, F_3 is closed under complementation.

3. ***Closed under countable unions:***

   - *If $A_1, A_2, A_3, \ldots$ are countable sets, then*

     $$\bigcup_{i=1}^{\infty} A_i$$

     is also countable.

   - *If B has a countable complement, then*

     $$B^c \in F_3 \implies B^c = \bigcup_{i=1}^{\infty} C_i \quad \text{for } C_i \text{ countable.}$$

     Therefore, B itself is in F_3.

*Since F_3 satisfies all properties, we conclude that F_3 is **a σ-algebra**.*

Example 1.5. *Let X and Y be two sets and let $f : X \to Y$ be a function. If F is a σ-algebra over the subsets of Y, and $G = \{A \mid \exists B \in F \text{ such that } f^{-1}(B) = A\}$, does G form a σ-algebra of subsets of X? Note that $f^{-1}(N)$ is the notation used for the pre-image of set N under the function f for some $N \subseteq Y$. That is, $f^{-1}(N) = \{x \in X \mid f(x) \in N\}$ for some $N \subseteq Y$.*

To show that G forms a σ-algebra of subsets of X, we need to verify that G satisfies the three properties of a σ-algebra:

***Contains the empty set**: The σ-algebra F over Y contains the empty set, $\emptyset$. Let $B = \emptyset \in F$. Then the pre-image under f, $f^{-1}(\emptyset) = \emptyset$, is also in G. Therefore, $\emptyset \in G$.*

***Closed under complementation**: Let $A \in G$. By definition of G, there exists a set $B \in F$ such that $f^{-1}(B) = A$. We need to show that $A^c \in G$. Consider the complement of B in F, denoted as B^c. Since F is a σ-algebra, $B^c \in F$.*

Now, observe that:

$$f^{-1}(B^c) = \{x \in X \mid f(x) \notin B\} = A^c$$

Hence, $A^c \in G$, showing that G is closed under complementation.

***Closed under countable unions**: Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of sets in G. For each $A_i \in G$, there exists $B_i \in F$ such that $f^{-1}(B_i) = A_i$. Since F is a σ-algebra, it is closed under countable unions, so $\bigcup_{i=1}^{\infty} B_i \in F$.*

Now, consider the pre-image:

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) = \bigcup_{i=1}^{\infty} A_i$$

Therefore, $\bigcup_{i=1}^{\infty} A_i \in G$, showing that G is closed under countable unions.

1.2 Measurable Space

Consider a large canvas Ω, which represents an entire art studio wall. The goal is to measure the total amount of paint used on specific regions of the wall. However, you don't want to measure paint usage for every possible shape or region on the wall (which could be infinitely complex). Instead, you decide to focus only on certain, manageable regions such as: rectangles, circles, simple polygons, and unions and intersections of these shapes.

These regions form a collection, say $\mathcal{F}$, which represents all the shapes and combinations that you are interested in measuring. The pair $(\Omega, \mathcal{F})$ then becomes a *measurable space*, where:

- Ω is the entire canvas, representing all possible points on the wall.
- $\mathcal{F}$ is a collection of specific shapes (rectangles, circles, etc.) and their combinations, which are the regions you can measure the amount of paint for.

The measurable sets in $\mathcal{F}$ are those specific shapes and combinations that you have chosen to focus on, similar to how measurable sets in probability theory are those events that belong to a specific σ-algebra. In context of probability theory, imagine you have a sample space Ω, which represents all the possible outcomes of an experiment. For instance, if you flip a coin, the sample space is $\Omega = \{\text{Heads}, \text{Tails}\}$.

A measurable space is a pair $(\Omega, \mathcal{F})$, where:

- Ω is the sample space, representing all possible outcomes.
- $\mathcal{F}$ is a σ-algebra of subsets of Ω. It is a collection of subsets that includes the empty set, is closed under complements, and closed under countable unions.

The subsets in $\mathcal{F}$ are the ones we can *measure*, hence the term *measurable space*.

Definition 1.3. *The 2-tuple $(\Omega, \mathcal{F})$ is called a **measurable space**. Here:*

- *Ω is the sample space, a non-empty set.*
- *$\mathcal{F}$ is a σ-algebra on Ω, meaning it satisfies:*

1. *$\emptyset \in \mathcal{F}$.*
2. *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (closure under complements).*
3. *If $A_1, A_2, A_3, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (closure under countable unions).*

Definition 1.4. *Every member of the σ-algebra $\mathcal{F}$ is called an **$\mathcal{F}$-measurable set** in the context of measure theory. These are the subsets of Ω that we can measure using the σ-algebra $\mathcal{F}$.*

Definition 1.5. *$\mathcal{F}$-measurable sets are called **events**. This means that an event is not just any subset of Ω, but one that belongs to the σ-algebra $\mathcal{F}$ under consideration.*

Examples Where Subsets of Ω Are Not $\mathcal{F}$-Measurable Sets:

In measure theory and probability, not all subsets of a sample space Ω are necessarily $\mathcal{F}$-measurable sets. This depends on the construction of the σ-algebra $\mathcal{F}$ associated with Ω. Below are some examples where subsets of Ω are not $\mathcal{F}$-measurable:

Example 1: A Countable Union Not in an Algebra

Let $\Omega = \mathbb{N}$, the set of natural numbers, and let $\mathcal{F}$ be an **algebra** consisting of all finite subsets of $\mathbb{N}$ and their complements (which are cofinite sets). In this setup, $\mathcal{F}$ contains only finite unions and intersections.

Now, consider the subset $A = \{2, 4, 6, \ldots\}$, the set of all even numbers. This is an **infinite** set but not cofinite (its complement, the set of odd numbers, is also infinite). Since $\mathcal{F}$ only contains finite or cofinite sets, A is not in $\mathcal{F}$. Thus, A is an example of a subset of Ω that is not $\mathcal{F}$-measurable.

Example 2: Subsets in the Cantor Set

Let Ω be the **Cantor set**, which is a subset of the interval $[0, 1]$. Construct $\mathcal{F}$ to be the σ-algebra generated by all **intervals** in $[0, 1]$. While $\mathcal{F}$ will contain many subsets, it will not include certain highly irregular subsets of the Cantor set that are not expressible as a countable union, intersection, or complement of intervals.

For instance, a subset of the Cantor set that is formed using a complex pattern based on the binary expansion of its elements may not be measurable in this σ-algebra. Hence, such a subset would not be $\mathcal{F}$-measurable.

Example 3: Students in a Class

Imagine a class of 30 students, represented by the set:

$$\Omega = \{s_1, s_2, s_3, \ldots, s_{30}\}$$

Define a σ-algebra $\mathcal{F}$ that consists only of subsets containing an even number of students. This σ-algebra could include sets like:

$$\mathcal{F} = \{\emptyset, \{s_1, s_2\}, \{s_3, s_4\}, \{s_5, s_6\}, \ldots, \{s_1, s_2, s_3, s_4\}, \ldots, \{s_1, s_2, \ldots, s_{30}\}\}$$

Now, consider the subset of interest $A = \{s_1, s_3, s_5, s_7, \ldots, s_{29}\}$, which contains all the odd-numbered students in the class.

In the σ-algebra $\mathcal{F}$, every set is constructed to contain only an even number of students. The set A, which contains an odd number of students, cannot be expressed as a union or intersection of sets from $\mathcal{F}$.

Thus, A is not $\mathcal{F}$-measurable because it does not fit within the constraints of our σ-algebra.

Example 4: Days of Week

Let $\Omega = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}\}$. Suppose we define a σ-algebra $\mathcal{F}$ that only includes subsets that contain weekdays:

$$\mathcal{F} = \{\emptyset, \{\text{Monday, Tuesday, Wednesday, Thursday, Friday}\}, \{\text{Saturday, Sunday}\}, \Omega\}$$

Now, consider the subset of interest $A = \{\text{Saturday}\}$.

The σ-algebra $\mathcal{F}$ only contains the sets of weekdays and their complements but does not include individual weekend days like Saturday. Thus, A cannot be constructed as a union or intersection of sets in $\mathcal{F}$.

Since A cannot be represented within the existing σ-algebra $\mathcal{F}$, it is not $\mathcal{F}$-measurable.

1.2.1 Measure

Definition 1.6. *Let $(\Omega, \mathcal{F})$ be a measurable space. A measure on $(\Omega, \mathcal{F})$ is a function $\mu : \mathcal{F} \to [0, \infty]$ such that:*

1. *$\mu(\emptyset) = 0$.*
2. *If $\{A_i, i \geq 1\}$ is a sequence of disjoint sets in $\mathcal{F}$, then the measure of the union of these countably infinite disjoint sets is equal to the sum of the measures of the individual sets:*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

The second property stated above is known as the *countable additivity property* of measures. From the definition, it is clear that a measure can only be assigned to elements of $\mathcal{F}$.

Definition 1.7. *The triplet $(\Omega, \mathcal{F}, \mu)$ is called a measure space.*

The measure μ is said to be a *finite measure* if $\mu(\Omega) < \infty$; otherwise, μ is said to be an *infinite measure*. In particular, if $\mu(\Omega) = 1$, then μ is referred to as a *probability measure*.

1.2.2 Probability Measure

Definition 1.8. *A probability measure is a function $P : \mathcal{F} \to [0,1]$ such that:*

1. $P(\emptyset) = 0$.
2. $P(\Omega) = 1$.
3. ***Countable Additivity:*** *If $\{A_i, i \geq 1\}$ is a sequence of disjoint sets in $\mathcal{F}$, then*
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Definition 1.9. *The triplet $(\Omega, \mathcal{F}, P)$ is called a probability space, and the three properties stated above are referred to as the axioms of probability.*

It is clear from the definition that probabilities are defined only for elements of $\mathcal{F}$, and not necessarily for all subsets of Ω. In other words, probability measures are assigned only to *events*. Even when we speak of the probability of an elementary outcome ω, it should be interpreted as the probability assigned to the singleton set $\{\omega\}$ (assuming, of course, that the singleton is an event).

1.2.3 Properties of Probability Measure

We will derive some fundamental properties of probability measures, which follow directly from the axioms of probability. In what follows, $(\Omega, \mathcal{F}, P)$ is a probability space.

Property 1: Suppose A be a subset of Ω such that $A \in \mathcal{F}$. Then,
$$P(A^c) = 1 - P(A).$$

Proof: Given any subset $A \subseteq \Omega$, A and A^c partition the sample space. Hence, $A^c \cup A = \Omega$ and $A^c \cap A = \emptyset$. By the "Countable Additivity" axiom of probability, $P(A^c \cup A) = P(A) + P(A^c)$. Therefore, $P(\Omega) = P(A) + P(A^c) \implies P(A^c) = 1 - P(A)$.

Property 2: Consider events A and B such that $A \subseteq B$ and $A, B \in \mathcal{F}$. Then $P(A) \leq P(B)$.

Proof: The set B can be written as the union of two disjoint sets A and $A^c \cap B$. Therefore, we have $P(A) + P(A^c \cap B) = P(B) \implies P(A) \leq P(B)$ since $P(A^c \cap B) \geq 0$.

Property 3: (Finite Additivity) If $A_1, A_2, \ldots, A_n$ are a finite number of disjoint events, then
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

Proof: This property follows directly from the axiom of countable additivity of probability measures. It is obtained by setting the events $A_{n+1}, A_{n+2}, \ldots$ as empty sets. The left-hand side (LHS) will simplify as:
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i\right).$$

The right-hand side (RHS) can be manipulated as follows:

$$\sum_{i=1}^{\infty} P(A_i) \overset{(a)}{=} \lim_{k\to\infty} \sum_{i=1}^{k} P(A_i) = \sum_{i=1}^{n} P(A_i) + \lim_{k\to\infty} \sum_{i=n+1}^{k} P(A_i) \overset{(b)}{=} \sum_{i=1}^{n} P(A_i) + \lim_{k\to\infty} 0 = \sum_{i=1}^{n} P(A_i).$$

where (a) follows from the definition of an infinite series and (b) is a consequence of setting the events from A_{n+1} onwards to null sets.

Property 4: For any $A, B \in F$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

In general, for a family of events $\{A_i\}_{i=1}^{n} \subset F$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right).$$

This property is proved using induction on n. The property can be proved in a much simpler way using the concept of Indicator Random Variables, which will be discussed in the subsequent lectures.

Proof The set $A \cup B$ can be written as $A \cup B = A \cup (A^c \cap B)$. Since A and $A^c \cap B$ are disjoint events, $P(A \cup B) = P(A) + P(A^c \cap B)$. Now, set B can be partitioned as $B = (A \cap B) \cup (A^c \cap B)$. Hence, $P(B) = P(A \cap B) + P(A^c \cap B)$. On substituting this result in the expression of $P(A \cup B)$, we will obtain the final result that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Property 5: If $\{A_i, i \geq 1\}$ are events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{m\to\infty} P\left(\bigcup_{i=1}^{m} A_i\right).$$

This result is known as the continuity of probability measures.

How to visualise this property? Imagine P_m is the probability of union of $A_1, A_2, ..., A_m$. Then the sequence of P_m's is a monotonically increasing sequence. Also, the sequence is bounded by the interval $[0, 1]$. We know, that every monotonically increasing sequence that is bounded must converge. So the RHS of the property 5 is a finite quantity. So is the LHS because the countable union of sets is a well-definied set for which the probability measure is defined. The property says both are equal.

Proof: Define a new family of sets $B_1 = A_1, B_2 = A_2 \setminus A_1, \ldots, B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i, \ldots$.

Then, the following claims are placed:
Claim 1: $B_i \cap B_j = \emptyset, \forall i \neq j$.
Claim 2: $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.

Since $\{B_i, i \geq 1\}$ is a disjoint sequence of events, and using the above claims, we get

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i).$$

Therefore,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(B_i) \quad (a) = \lim_{m\to\infty} \sum_{i=1}^{m} P(B_i) \quad (b) = \lim_{m\to\infty} P\left(\bigcup_{i=1}^{m} B_i\right)$$

$$(c) = \lim_{m\to\infty} P\left(\bigcup_{i=1}^{m} A_i\right).$$

Here, (a) follows from the definition of an infinite series, (b) follows from Claim 1 in conjunction with the Countable Additivity axiom of probability measure, and (c) follows from the intermediate result required to prove Claim 2. Hence proved.

Property 6: If $\{A_i, i \geq 1\}$ is a sequence of increasing nested events i.e., $A_i \subseteq A_{i+1}, \forall i \geq 1$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{m\to\infty} P(A_m).$$

Property 7: If $\{A_i, i \geq 1\}$ is a sequence of decreasing nested events i.e., $A_{i+1} \subseteq A_i, \forall i \geq 1$, then

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{m\to\infty} P(A_m).$$

Properties 6 and 7 are said to be corollaries to Property 5.

Property 8: Suppose $\{A_i, i \geq 1\}$ are events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

This result is known as the Union Bound. This bound is trivial if $\sum_{i=1}^{\infty} P(A_i) \geq 1$ since the LHS of Property 8 is a probability of some event. This is a very widely used bound, and has several applications. For instance, the union bound is used in the probability of error analysis in Digital Communications for complicated modulation schemes.

Proof: Define a new family of sets $B_1 = A_1, B_2 = A_2 \setminus A_1, \ldots, B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i, \ldots$.

Claim 1: $B_i \cap B_j = \emptyset, \forall i \neq j$.
Claim 2: $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.

Since $\{B_i, i \geq 1\}$ is a disjoint sequence of events, and using the above claims, we get

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i).$$

Also, since $B_i \subseteq A_i \, \forall i \geq 1$, $P(B_i) \leq P(A_i) \, \forall i \geq 1$ (using Property 2). Therefore, the finite sum of probabilities follows

$$\sum_{i=1}^{n} P(B_i) \leq \sum_{i=1}^{n} P(A_i).$$

Eventually, in the limit, the following holds:

$$\sum_{i=1}^{\infty} P(B_i) \leq \sum_{i=1}^{\infty} P(A_i).$$

Finally, we arrive at the result,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

Example 1.6. *A standard card deck (52 cards) is distributed to two persons: 26 cards to each person. All partitions are equally likely. Find the probability that the first person receives all four aces.*

To find the probability that the first person receives all four aces when a standard deck of 52 cards is distributed equally between two persons (each receiving 26 cards), we use a measure-theoretic approach.

Let $(\Omega, \mathcal{F}, P)$ be the probability space where:
- Ω is the set of all ways to partition the deck into two hands of 26 cards each.
- $\mathcal{F}$ is the σ-algebra of subsets of Ω.
- P is the uniform probability measure on $(\Omega, \mathcal{F})$.

Step 1: Total Number of Outcomes

The total number of ways to choose 26 cards out of 52 for the first person is given by:

$$|\Omega| = \binom{52}{26}$$

where $\binom{52}{26}$ denotes the binomial coefficient representing the number of ways to choose 26 cards from 52.

Step 2: Number of Favorable Outcomes

Next, we find the number of ways the first person can receive all four aces. If the first person is to receive all four aces, we must choose the remaining 22 cards from the remaining 48 non-ace cards. The number of ways to do this is:

$$|\Omega_{favorable}| = \binom{48}{22}$$

where $\binom{48}{22}$ denotes the binomial coefficient representing the number of ways to choose 22 cards from the 48 non-ace cards.

Step 3: Calculating the Probability

The probability that the first person receives all four aces is the ratio of the number of favorable outcomes to the total number of outcomes:

$$P(\textit{First person receives all four aces}) = \frac{|\Omega_{\textit{favorable}}|}{|\Omega|}$$

$$P(\textit{First person receives all four aces}) = \frac{\binom{48}{22}}{\binom{52}{26}}$$

Example 1.7. *Let* $\{A_r\}_{r=1}^n$ *be a finite collection of events in a probability space* $(\Omega, \mathcal{F}, P)$*. We aim to prove that:*

$$P\left(\bigcup_{1\le r\le n} A_r\right) \le \min_{1\le k\le n}\left\{\sum_{1\le r\le n} P(A_r) - \sum_{r:r\ne k} P(A_r\cap A_k)\right\}$$

Define $S = \bigcup_{r=1}^n A_r$*. By the inclusion-exclusion principle for a finite union of events, we have:*

$$P(S) = \sum_{1\le r\le n} P(A_r) - \sum_{1\le r<s\le n} P(A_r\cap A_s) + \ldots + (-1)^{n+1} P\left(\bigcap_{1\le r\le n} A_r\right).$$

This expression accounts for all possible intersections of the events A_r*. However, to prove the inequality, we'll make use of the following upper bound:*

Consider any fixed $k \in \{1, 2, \ldots, n\}$*. We can express* $P(S)$ *as:*

$$P(S) \le P(A_k) + P\left(\bigcup_{r:r\ne k} A_r \setminus A_k\right).$$

This follows since the probability of S is at most the probability of A_k *plus the probability of the events outside of* A_k *but not overlapping with it.*

Now, observe that:

$$P\left(\bigcup_{r:r\ne k} A_r \setminus A_k\right) \le \sum_{r:r\ne k} P(A_r\setminus A_k).$$

Using the identity $P(A_r\setminus A_k) = P(A_r) - P(A_r\cap A_k)$*, we can rewrite the above as:*

$$P\left(\bigcup_{r:r\ne k} A_r \setminus A_k\right) \le \sum_{r:r\ne k} \left(P(A_r) - P(A_r\cap A_k)\right).$$

Therefore:

$$P(S) \le P(A_k) + \sum_{r:r\ne k} \left(P(A_r) - P(A_r\cap A_k)\right).$$

Simplifying further:

$$P(S) \leq \sum_{1\leq r\leq n} P(A_r) - \sum_{r:r\neq k} P(A_r \cap A_k).$$

Since this inequality holds for any $k \in \{1, 2, \ldots, n\}$, we take the minimum over all k:

$$P\left(\bigcup_{1\leq r\leq n} A_r\right) \leq \min_{1\leq k\leq n} \left\{\sum_{1\leq r\leq n} P(A_r) - \sum_{r:r\neq k} P(A_r \cap A_k)\right\}.$$

Example 1.8. *You are given that at least one of the events A_n, $1 \leq n \leq N$, is certain to occur. However, certainly no more than two occur. If $P(A_n) = p$ and $P(A_n \cap A_m) = q$, $m \neq n$, then show that $p \geq \frac{1}{N}$ and $q \leq \frac{2}{N}$.*

Given the events $A_1, A_2, \ldots, A_N$ such that at least one event occurs and at most two occur, we have:

$$P\left(\bigcup_{n=1}^{N} A_n\right) = 1$$

and

$$P\left(A_n \cap A_m\right) = q \quad \text{for} \quad m \neq n.$$

By the principle of inclusion-exclusion, the probability of the union of these events is:

$$P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n) - \sum_{1\leq n<m\leq N} P(A_n \cap A_m).$$

Substituting the given values, we get:

$$1 = \sum_{n=1}^{N} p - \sum_{1\leq n<m\leq N} q.$$

The number of terms in the first sum is N, so:

$$1 = Np - \binom{N}{2} q,$$

where $\binom{N}{2} = \frac{N(N-1)}{2}$ is the number of ways to choose 2 events from N.

Thus:

$$1 = Np - \frac{N(N-1)}{2} q.$$

Rearranging to solve for p:

$$Np = 1 + \frac{N(N-1)}{2} q,$$

$$p = \frac{1}{N} + \frac{(N-1)}{2} q.$$

Since at most two events can occur, q must be small enough so that no three events can occur simultaneously. Hence, by substituting $p \geq \frac{1}{N}$:

$$\frac{1}{N} + \frac{(N-1)}{2} q \geq \frac{1}{N},$$

$$\frac{(N-1)}{2}q \geq 0.$$

To find the upper bound of q, note that since at most two events can occur, the total probability contributed by the intersections should not exceed 1. Therefore:

$$\frac{(N-1)}{2}q \leq \frac{1}{N}.$$

$$q \leq \frac{2}{N}.$$

Example 1.9. *Consider a measurable space $(\Omega, \mathcal{F})$ with $\Omega = [0,1]$. A measure P is defined on the non-empty subsets of Ω (in $\mathcal{F}$), which are all of the form (a,b), $(a,b]$, $[a,b)$ and $[a,b]$, as the length of the interval, i.e.,*

$$P((a,b)) = P((a,b]) = P([a,b)) = P([a,b]) = b-a.$$

(a) *Show that P is not just a measure, but it's a probability measure.*
(b) *Let $A_n = \left[\frac{1}{n+1}, 1\right]$ and $B_n = \left[0, \frac{1}{n+1}\right]$ for $n \geq 1$. Compute $P\left(\cup_{i\in\mathbb{N}} A_i\right)$, $P\left(\cap_{i\in\mathbb{N}} A_i\right)$, $P\left(\cup_{i\in\mathbb{N}} B_i\right)$, and $P\left(\cap_{i\in\mathbb{N}} B_i\right)$.*
(c) *Compute $P\left(\cap_{i\in\mathbb{N}}(B_i^c \cup A_i^c)\right)$.*
(d) *Let $C_m = \left[0, \frac{1}{m}\right]$ such that $P(C_m) = P(A_n)$. Express m in terms of n.*
(e) *Evaluate $P\left(\cap_{i\in\mathbb{N}}(C_i \cap A_i)\right)$ and $P\left(\cup_{i\in\mathbb{N}}(C_i \cap A_i)\right)$.*

To show that P is a probability measure, we need to verify two properties:

1. Non-negativity: $P(A) \geq 0$ for all $A \in \mathcal{F}$. By definition, $P(A) = b-a \geq 0$ since $b \geq a$ for all intervals in $[0,1]$.

2. $P(\Omega) = 1$: The entire space $\Omega = [0,1]$. Hence, $P([0,1]) = 1-0 = 1$.

Therefore, P is a probability measure.

$P\left(\cup_{i\in\mathbb{N}} A_i\right) = P([0,1]) = 1$
$P\left(\cap_{i\in\mathbb{N}} A_i\right) = P\left([0,1]\right) = 1$
$P\left(\cup_{i\in\mathbb{N}} B_i\right) = P([0,1]) = 1$
$P\left(\cap_{i\in\mathbb{N}} B_i\right) = P(\{0\}) = 0$

Note that $B_i^c = [\frac{1}{n+1}, 1]$ and $A_i^c = [0, \frac{1}{n+1}]$. Thus:

$$P\left(\cap_{i\in\mathbb{N}}(B_i^c \cup A_i^c)\right) = P(\emptyset) = 0$$

We have $P(C_m) = \frac{1}{m}$ and $P(A_n) = 1 - \frac{1}{n+1}$. Equating these gives:

$$\frac{1}{m} = 1 - \frac{1}{n+1}$$

$$m = \frac{n+1}{n}$$

$P\left(\cap_{i\in\mathbb{N}}(C_i \cap A_i)\right) = P(\emptyset) = 0$
$P\left(\cup_{i\in\mathbb{N}}(C_i \cap A_i)\right) = P([0,1]) = 1$

1.3 Discrete Probability Spaces

Discrete probability spaces correspond to the case when the sample space Ω is countable. This is the most conceptually straightforward case, since it is possible to assign probabilities to all subsets of Ω.

Definition 1.10. *A probability space $(\Omega, \mathcal{F}, P)$ is said to be a discrete probability space if the following conditions hold:*

(a) The sample space Ω is finite or countably infinite,

(b) The σ-algebra is the set of all subsets of Ω, i.e., $\mathcal{F} = 2^{\Omega}$, and

(c) The probability measure, P, is defined for every subset of Ω. In particular, it can be defined in terms of the probabilities $P(\{\omega\})$ of the singletons corresponding to each of the elementary outcomes ω, and satisfies for every $A \in \mathcal{F}$,

$$P(A) = \sum_{\omega \in A} P(\{\omega\}),$$

and

$$\sum_{\omega \in \Omega} P(\{\omega\}) = 1.$$

The above definition highlights that it is possible to assign probabilities to each singleton set of Ω, but it doesn't say about *what probabilities to assign?* This depends on our use-case and what we want to model.

Examples of Discrete Probability Spaces

1. Let us consider a coin toss experiment with the probability of getting a head as p and the probability of getting a tail as $(1 - p)$. The sample space and the σ-algebra are defined as follows:

$$\Omega = \{H, T\} \equiv \{0, 1\}, \quad \mathcal{F} = 2^{\Omega} = \{\emptyset, \{H\}, \{T\}, \Omega\}.$$

The probability measure is given by:

$$P(\{H\}) \equiv P(\{0\}) = p, \quad P(\{T\}) \equiv P(\{1\}) = 1 - p.$$

In this case, we say that $P(.)$ is a Bernoulli measure on $\{\{0, 1\}, 2^{\{0,1\}}\}$.

2. Let $\Omega = \mathbb{N}$ and $\mathcal{F} = 2^{\mathbb{N}}$. We can define the probability of a singleton as:

$$P(\{k\}) = a_k \geq 0, \quad k \in \mathbb{N},$$

under the constraint that:

$$\sum_{k \in \mathbb{N}} P(\{k\}) = 1.$$

For example, if we let $a_k = \frac{1}{2^k}$, $k \in \mathbb{N}$, this is a valid measure, since:

$$\sum_{k \in \mathbb{N}} \frac{1}{2^k} = 1.$$

As another example, consider $a_k = (1-p)^{k-1}p$ for $0 < p < 1$ and $k \in \mathbb{N}$. This is known as a geometric measure with parameter p. It is a valid probability measure since:

$$\sum_{k \in \mathbb{N}} (1-p)^{k-1} p = 1.$$

3. Let $\Omega = \mathbb{N} \cup \{0\}$ and $\mathcal{F} = 2^{\Omega}$. We define the probability measure as:

$$P(\{k\}) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \lambda > 0.$$

This probability measure is called a Poisson measure with parameter λ on $\{\Omega, 2^{\Omega}\}$. This is a valid probability measure, since:

$$\sum_{k=0}^{\infty} P(\{k\}) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1.$$

4. Let $\Omega = \{0, 1, 2, \dots, N\}$, where $N \in \mathbb{N}$ and $\mathcal{F} = 2^{\Omega}$. We define the probability measure as:

$$P(\{k\}) = \binom{N}{k} p^k (1-p)^{N-k}, \quad 0 < p < 1.$$

This probability measure is called a Binomial measure with parameters (N, p) on $\{\Omega, 2^{\Omega}\}$. This can be verified to be a valid probability measure as follows:

$$\sum_{k \in \Omega} \binom{N}{k} p^k (1-p)^{N-k} = (p + (1-p))^N = 1.$$

Note that in all the examples above, we have not explicitly specified an expression for $P(A)$ for every $A \subset \Omega$. Since the sample space is countable, the probability of any subset of the sample space can be obtained as the sum of probabilities of the corresponding elementary outcomes. In other words, for discrete probability spaces, it suffices to specify the probabilities of singletons corresponding to each of the elementary outcomes.

Chapter 2

Borel Sets and Lebesgue Measure

Last chapter lays the foundation of *what are events* and *what is a probability measure*. But that still doesn't answer the question - *what can we actually measure and what can we not? And how to measure what we can measure?* Sounds tricky? Don't worry! This chapter will take you through the complications!

2.1 Introduction to Borel Sets

Let's consider the case when the sample space is uncountable.

2.1.1 Uncountable Sample Space

Consider the experiment of picking a real number at random from $\Omega = [0, 1]$, such that every number is "equally likely" to be picked. It is quite apparent that a simple strategy of assigning probabilities to singleton subsets of the sample space gets into difficulties quite quickly. Indeed:

(i) If we assign some positive probability to each elementary outcome, then the probability of an event with infinitely many elements, such as $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$, would become unbounded. This is because the sum of positive probabilities over an infinite set would diverge.

(ii) If we assign zero probability to each elementary outcome, this alone would not be sufficient to determine the probability of an uncountable subset of Ω, such as $\left[\frac{1}{2}, \frac{2}{3}\right]$. This is because probability measures are not additive over uncountable disjoint unions (of singletons in this case). Assigning zero probability to singletons does not directly imply how to handle intervals or other uncountable sets.

Thus, we need a different approach to assign probabilities when the sample space is uncountable, such as $\Omega = [0, 1]$. In particular, we need to assign probabilities directly to specific subsets of Ω. Intuitively, we would like our 'uniform measure' μ on $[0, 1]$ to possess the following two properties:

(i) $\mu((a, b)) = \mu((a, b]) = \mu([a, b)) = \mu([a, b])$ for any interval in $[0, 1]$. This ensures that the measure is consistent across different types of intervals, capturing the idea of "equal likelihood" for any interval of the same length.

(ii) **Translational Invariance**: That is, if $A \subseteq [0,1]$, then for any $x \in \Omega$, $\mu(A \oplus x) = \mu(A)$, where the set $A \oplus x$ is defined as:

$$A \oplus x = \{a + x \mid a \in A, a + x \leq 1\} \cup \{a + x - 1 \mid a \in A, a + x > 1\}$$

This property ensures that the measure is invariant under translation within the interval $[0,1]$, reflecting the uniformity of the measure.

However, the following impossibility result asserts that there is no way to consistently define a uniform measure on all subsets of $[0,1]$. This result is rooted in the fact that certain sets in $[0,1]$ (those that are non-measurable) defy any consistent assignment of measure while preserving the desired properties of translation invariance and interval consistency.

Theorem 2.1. ***Impossibility Result:*** *There does not exist a definition of a measure $\mu(A)$ for all subsets of $[0,1]$ satisfying:*

(i) $\mu((a,b)) = \mu((a,b]) = \mu([a,b)) = \mu([a,b])$

(ii) Translational Invariance: If $A \subseteq [0,1]$, then for any $x \in \Omega$, $\mu(A \oplus x) = \mu(A)$, where the set $A \oplus x$ is defined as:

$$A \oplus x = \{a + x \mid a \in A, a + x \leq 1\} \cup \{a + x - 1 \mid a \in A, a + x > 1\}$$

Proof. To show the impossibility, we argue that a measure satisfying these two properties for all subsets of $[0,1]$ leads to a contradiction. We will use basic properties of measures and simple logic to establish the proof.

1. Interval Length Property (i):

The first property tells us that the measure of any interval in $[0,1]$ is simply the length of the interval. For example, if $A = (a,b)$, then $\mu((a,b)) = b - a$. This property holds for open, closed, and half-open intervals.

2. Translational Invariance (ii):

The second property states that if we shift a set A by some amount x, its measure should remain the same. For example, if A is an interval, shifting it within $[0,1]$ should not change its length. This makes sense intuitively, as the measure should not depend on the location of the set but only its size.

3. Partitioning $[0,1]$ into Equal Parts:

Let's divide the interval $[0,1]$ into n equal parts. Define sets $A_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ for $i = 1, 2, \ldots, n-1$ and $A_n = \left[\frac{n-1}{n}, 1\right]$. By property (i), each of these sets has a measure:

$$\mu(A_i) = \frac{1}{n}, \quad \text{for all } i = 1, 2, \ldots, n.$$

Since these intervals are disjoint and together cover $[0,1]$, by the additivity of measures:

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

4. Constructing Translations:

Suppose we take one of these intervals, say $A_1 = \left[0, \frac{1}{n}\right)$, and shift it by $\frac{1}{2n}$. This new set, $A_1 \oplus \frac{1}{2n}$, becomes $\left[\frac{1}{2n}, \frac{3}{2n}\right)$, which is a valid subset of $[0,1]$.

By translational invariance (property (ii)), $\mu(A_1 \oplus \frac{1}{2n}) = \mu(A_1) = \frac{1}{n}$.

5. Forming a Contradiction:

Now, let's consider translating A_1 by different multiples of $\frac{1}{2n}$. We form the following sets:

$$A_1, \quad A_1 \oplus \frac{1}{2n}, \quad A_1 \oplus \frac{2}{2n}, \quad \ldots, \quad A_1 \oplus \frac{n-1}{2n}.$$

These n translations of A_1 are all disjoint, and by property (ii), each has a measure of $\frac{1}{n}$.

However, if we sum up the measures of all these disjoint translations, we get:

$$\mu(A_1) + \mu(A_1 \oplus \frac{1}{2n}) + \ldots + \mu(A_1 \oplus \frac{n-1}{2n}) = \frac{1}{n} \times n = 1.$$

6. Contradiction with the Total Measure:

Observe that the union of all these translated sets may not cover the entire interval $[0,1]$. In fact, since A_1 is just a small fraction of $[0,1]$, these translations form only a portion of $[0,1]$. Hence, the measure of their union should be less than 1.

But by translational invariance and additivity, we have shown that the sum of their measures equals 1. This creates a contradiction because it implies that a part of $[0,1]$ has the same measure as the whole interval.

Since this contradiction arises when attempting to define μ on all subsets while preserving both the interval property and translational invariance, it is impossible to define such a measure for all subsets of $[0,1]$. □

Therefore, we must compromise, and consider a smaller σ-algebra that contains certain "nice" subsets of the sample space [0, 1]. **These "nice" subsets are the intervals**, and the resulting σ-algebra is called the Borel σ-algebra. Before defining Borel sets, we introduce the concept of generating σ-algebras from a given collection of subsets.

2.1.2 Borel Sets

Now, we know that the collection of intervals is not a σ-algebra because if $[a,b]$ is in the collection, it's complement is not an interval. So, we want to build towards a σ-algebra that contains all intervals, their complements and is closed under countable unions and countable intersections.

Let $\mathcal{C}$ be the collection of all nice subsets of sample space Ω in which we are interested. We have to generate the smallest σ-algebra that contains $\mathcal{C}$, that is denoted by $\sigma(\mathcal{C})$.

Theorem 2.2. *The intersection of an arbitrary number of σ-algebras is a σ-algebra.*

Proof. Let $\{\mathcal{F}_i\}_{i\in I}$ be a collection of σ-algebras on a collection $\mathcal{C}$, where I is an index set. Define $\mathcal{F} = \bigcap_{i\in I} \mathcal{F}_i$. We want to show that $\mathcal{F}$ is a σ-algebra.

Since each $\mathcal{F}_i$ is a σ-algebra, it contains $\mathcal{C}$. Therefore, $\mathcal{C} \in \mathcal{F}_i$ for all $i \in I$. By definition of the intersection, $\mathcal{C} \in \mathcal{F}$.

Let $A \in \mathcal{F}$. This implies that $A \in \mathcal{F}_i$ for every $i \in I$. Since each $\mathcal{F}_i$ is a σ-algebra, $A^c \in \mathcal{F}_i$ for all $i \in I$. Therefore, $A^c \in \bigcap_{i\in I} \mathcal{F}_i = \mathcal{F}$. Hence, $\mathcal{F}$ is closed under complementation.

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{F}$. This implies that $A_n \in \mathcal{F}_i$ for every $i \in I$ and for all $n \in \mathbb{N}$. Since each $\mathcal{F}_i$ is a σ-algebra, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_i$ for all $i \in I$. Therefore, $\bigcup_{n=1}^{\infty} A_n \in \bigcap_{i\in I} \mathcal{F}_i = \mathcal{F}$. Hence, $\mathcal{F}$ is closed under countable unions.

Since $\mathcal{F}$ contains $\mathcal{C}$, is closed under complementation, and is closed under countable unions, $\mathcal{F}$ is a σ-algebra.

□

Theorem 2.3. *The smallest σ-algebra, $\sigma(\mathcal{C})$ is $\mathcal{F} = \bigcap_{i\in I} \mathcal{F}_i$, where each $\mathcal{F}_i$ is the σ-algebra of $\mathcal{C}$.*

The above theorem just implies that the smallest σ-algebra exists and is well defined. We don't know all $\mathcal{F}_i$, and we don't intend to find them as well. What we know for now is - that they exist. And if we take all of them and take a countable intersection of them - the resultant collection of sets is well-defined.

Proof. Let $\{\mathcal{F}_i\}_{i\in I}$ be the collection of all σ-algebras on a set $\mathcal{C}$ that contain $\mathcal{C}$. Define $\mathcal{F} = \bigcap_{i\in I} \mathcal{F}_i$. We know from a previous result that the intersection of an arbitrary number of σ-algebras is a σ-algebra. Therefore, $\mathcal{F} = \bigcap_{i\in I} \mathcal{F}_i$ is a σ-algebra.

Since each $\mathcal{F}_i$ contains $\mathcal{C}$ by definition, their intersection, $\mathcal{F}$, also contains $\mathcal{C}$. Thus, $\mathcal{C} \subseteq \mathcal{F}$.

Let $\sigma(\mathcal{C})$ denote the smallest σ-algebra containing $\mathcal{C}$. By definition, $\sigma(\mathcal{C})$ is a σ-algebra and contains $\mathcal{C}$. Therefore, it must be one of the $\mathcal{F}_i$ in the collection $\{\mathcal{F}_i\}_{i\in I}$. Since $\mathcal{F}$ is the intersection of all σ-algebras containing $\mathcal{C}$, it must be contained within any other σ-algebra that contains $\mathcal{C}$. In particular, $\mathcal{F} \subseteq \sigma(\mathcal{C})$.

Since $\mathcal{F}$ is defined as the intersection of all σ-algebras that contain $\mathcal{C}$, and $\sigma(\mathcal{C})$ itself is one of these σ-algebras, it follows that $\sigma(\mathcal{C}) \subseteq \mathcal{F}$. We have $\mathcal{F} \subseteq \sigma(\mathcal{C})$ and $\sigma(\mathcal{C}) \subseteq \mathcal{F}$. Therefore, $\mathcal{F} = \sigma(\mathcal{C})$.

This proves that the smallest σ-algebra containing $\mathcal{C}$ is $\mathcal{F} = \bigcap_{i\in I} \mathcal{F}_i$. □

We can now define the *Borel σ-algebra*. For this, we will have a setup - reasons of picking this setup will get clear in some time - when you will see that how open sets can be helpful in proving that even singleton sets are Borel sets.

Setup: Let $(0,1]$ be the sample space, Ω. The collection of interesting sets of Ω, represented by $\mathcal{C}_0$, contains all open-intervals (a,b) in Ω.

Definition 2.1. *$\sigma(\mathcal{C}_0)$ is called the Borel σ-algebra, denoted by $\mathcal{B}((0,1])$. An element of $\mathcal{B}((0,1])$ is called Borel measurable set, or simply a Borel set.*

Thus, every open interval in $(0,1]$ is a Borel set. We next prove that every singleton set in $(0,1]$ is a Borel set too.

Theorem 2.4. *Every singleton set in $(0,1]$ is a Borel set.*

Proof. Let $((0,1],\mathcal{B})$ be the Borel space where $\mathcal{B}$ is the Borel σ-algebra generated by the open sets within $(0,1]$. We want to show that any singleton set $\{x\}$, where $x \in (0,1]$, is a Borel set.

Consider the singleton set $\{x\}$, where $x \in (0,1]$. We can write it as:

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap (0,1].$$

The above result can be proved by the method of contradiction. Let h be an element in $\bigcap_{n=1}^{\infty} \left(b - \frac{1}{n}, b + \frac{1}{n}\right)$ other than b. For every such h, there exists a large enough n_0 such that $h \notin \left(b - \frac{1}{n_0}, b + \frac{1}{n_0}\right)$. This implies $h \notin \bigcap_{n=1}^{\infty} \left(b - \frac{1}{n}, b + \frac{1}{n}\right)$.

Each interval $\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap (0,1]$ is an open set in $(0,1]$, and since Borel sets are generated by open sets, these intervals belong to the Borel σ-algebra $\mathcal{B}$.

The Borel σ-algebra $\mathcal{B}$ is closed under countable intersections. Therefore, the intersection of the countable collection of open sets $\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap (0,1]$, which is exactly $\{x\}$, is also in $\mathcal{B}$.

Since $\{x\}$ can be expressed as a countable intersection of open sets within $(0,1]$, it is a Borel set.

□

Corollary 2.1. *As an immediate consequence of this theorem, we see that every half-open interval, $(a,b]$, is a Borel set. This follows from the fact that*

$$(a,b] = (a,b) \cup \{b\},$$

and the fact that a countable union of Borel sets is a Borel set. For the same reason, every closed interval, $[a,b]$, is a Borel set.

This also gives one the idea of how to prove a set is a Borel set or not. If the set can be represented as a complement of an open set or as countable unions and countable intersections of open sets, it is a Borel set.

How big is the Borel σ-algebra?

Theorem 2.5. *The cardinality of the Borel σ-algebra (on the unit interval $(0,1]$) is the same as the cardinality of the reals. Thus, the Borel σ-algebra is a much 'smaller' collection than the power set $2^{(0,1]}$.*

Proof. Let $\mathcal{B}$ denote the Borel σ-algebra on the unit interval $(0,1]$.

Step 1: Show that the cardinality of $\mathcal{B}$ is at most the cardinality of the reals.

The Borel σ-algebra $\mathcal{B}$ is generated by the open intervals of $(0,1]$, which form a basis for the topology. The set of all open intervals in $(0,1]$ has the same cardinality as $\mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Since $\mathbb{Q}$ is countable, it follows that the set of all open intervals in $(0,1]$ is also countable.

The Borel σ-algebra is generated by applying countable unions, intersections, and complements to these open intervals. Therefore, the number of sets that can be formed is bounded by $|\mathbb{R}|$, the cardinality of the reals.

Step 2: Show that the cardinality of $\mathcal{B}$ is at least the cardinality of the reals.

Consider the singleton sets $\{x\}$ where $x \in (0,1]$. Each singleton set is a Borel set, and the cardinality of these singleton sets is the same as the cardinality of the reals. Therefore, $\mathcal{B}$ must contain at least as many elements as the cardinality of $\mathbb{R}$.

Conclusion: We have shown that the cardinality of $\mathcal{B}$ is both at most and at least the cardinality of the reals. Therefore, the cardinality of the Borel σ-algebra $\mathcal{B}$ is exactly $|\mathbb{R}|$.

Since the power set $2^{(0,1]}$ has a cardinality of $2^{|\mathbb{R}|}$, which is strictly greater than $|\mathbb{R}|$, it follows that the Borel σ-algebra is a much smaller collection than the power set of $(0,1]$. $\square$

2.1.3 What are not Borel sets?

The majority of sets in $(0,1]$ are Borel sets. In fact, the Borel σ-algebra on $(0,1]$ contains a wide range of sets, from simple open intervals to much more complex constructions. Identifying a non-Borel set is not trivial because the Borel σ-algebra is quite extensive.The Borel σ-algebra on $(0,1]$ includes many intricate sets such as the **Cantor set**. To understand the breadth of the Borel σ-algebra, we first prove that the Cantor set is a Borel set.

Lemma 2.1. *The Cantor Set is a Borel set.*

Proof. The Cantor set, C, is constructed by iteratively removing the middle third from each interval of $(0,1]$.

$$C = \bigcap_{n=1}^{\infty} C_n,$$

where C_n is the set obtained after the n-th stage of removing the middle third of each interval.

Each C_n is a finite union of closed intervals. Since finite unions of closed sets are closed, C_n is closed for each n. The Cantor set C is the countable intersection of these closed sets. The Borel σ-algebra is closed under countable intersections of closed sets, so $C \in \mathcal{B}$, making the Cantor set a Borel set.

□

Examples of Non-Borel Sets

Although most familiar sets are Borel sets, there exist sets that are not Borel. These sets are usually constructed using the Axiom of Choice and involve more intricate arguments. One such example is the **Vitali set**. The Vitali set is constructed in the following way:

1. Consider the interval $[0, 1]$.
2. Define an equivalence relation $\sim$ on $[0, 1]$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$, i.e., x and y differ by a rational number.
3. By the Axiom of Choice, we select exactly one representative from each equivalence class under this relation. The collection of these representatives forms a set V, called the **Vitali set**.

But what is the Vitali Set actually?

Imagine you're organizing the quirkiest party ever on a number line between 0 and 1. This isn't just any party - it's a Vitali set party! Here's how you create your guest list:

First, you declare that two numbers are *dance partners* if their difference is a rational number. For example, 0.3 and 0.7 are dance partners because 0.7 - 0.3 = 0.4, which is rational. Now, you start grouping all the numbers between 0 and 1 into *dance troupes.* Each troupe consists of all numbers that are dance partners with each other. Here's the twist: you decide to invite exactly one person from each dance troupe to your party. It doesn't matter who you choose from each troupe, as long as you pick one and only one.

The resulting guest list is what mathematicians call a Vitali set! Why is this party so special? Well, it has some mind-bending properties:

No two party guests are exactly one rational number apart. If Alice is at 0.3 and Bob is at 0.7, one of them didn't make the cut because they're in the same dance troupe. Yet, if you shift all your guests by any rational number, you'll get a completely new set of partiers, with no overlap with the original group! Despite seeming quite sparse (remember, we only chose one member from each dance troupe), this set has some very strange measuring properties, as we'll soon see.

Now that we've got our Vitali set party set up, let's explore why it's such a mathematical troublemaker!

Why is the Vitali Set Not a Borel Set?

Now, let's play a game called *Cover the Dance Floor.*

Here's how it works: We start with our Vitali set party on the (0,1] dance floor. We're given a magical dance move: we can shift everyone simultaneously by any rational number between -1 and 1. Our goal? Use these dance moves to cover every spot on a new, bigger dance floor from 0 to 2!

Here's the kicker: with the right series of these rational shifts, we can indeed cover every single point between 0 and 2. It's like our original Vitali set party has suddenly expanded to fill twice the space! But wait a minute... if the Vitali set were a Borel set (think of Borel sets as *well-behaved* sets that play nicely with measure theory), we'd run into a big problem. Here's why:

Borel sets have a property: if you take a Borel set and shift it by a rational number, the result is still a Borel set. We just covered the interval (0,2] using countably many rational shifts of our Vitali set. If the Vitali set were Borel, this new covered area would also be Borel. But here's the contradiction: we know the measure (think *length*) of (0,2] is 2, but it's made up of countably many copies of our original set, each of which should have the same measure as the original Vitali set.

Let's call the measure of the Vitali set m. Then we have:

$$2 = \text{countably many} \times m$$

This equation can't possibly work! If $m = 0$, the right side is zero. If $m > 0$, the right side is infinite. There's no value of m that makes this equation true. So, we're forced to conclude that our initial assumption - that the Vitali set is a Borel set - must be wrong. The Vitali set is too *wild* to be captured by the well-behaved Borel sets.

2.2 Introduction to Lebesgue Measure

What we understood from the example of *Vitali Set* in the last section is that we cannot assign it measure like *length*. We will only assign measures to the *Borel sets*. This gives us the understanding that the entire collection of subsets 2^{Ω}, where $\Omega = (0, 1]$, is not measureable. So, we can say $(\Omega, \mathcal{B})$ is the *measurable space*. Now, we want to assign each Borel set a measure.

Consider $\Omega = (0, 1]$. Let $\mathcal{F}_0$ consist of the empty set and all sets that are finite unions of intervals of the form $(a, b]$. A typical element of this set is of the form

$$F = (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n]$$

where $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$ and $n \in \mathbb{N}$.

Lemma 2.2. *(a) $\mathcal{F}_0$ is an algebra.*

(b) $\mathcal{F}_0$ is not a σ-algebra.

(c) $\sigma(\mathcal{F}_0) = \mathcal{B}$, where $\mathcal{B}$ denotes the Borel σ-algebra.

Proof. (a) By definition, $\varnothing \in \mathcal{F}_0$. Also, $\varnothing^C = (0, 1] \in \mathcal{F}_0$. The complement of $(a_1, b_1] \cup (a_2, b_2]$ is $(0, a_1] \cup (b_1, a_2] \cup (b_2, 1]$, which also belongs to $\mathcal{F}_0$.

Furthermore, the union of finitely many sets, each of which is a finite union of intervals of the form $(a, b]$, is also a set that is a union of a finite number of intervals, and thus belongs to $\mathcal{F}_0$.

(b) To see this, note that $\left(0, \frac{n}{n+1}\right] \in \mathcal{F}_0$ for every n, but $\bigcup_{n=1}^{\infty} \left(0, \frac{n}{n+1}\right] = (0,1) \notin \mathcal{F}_0$. This shows that $\mathcal{F}_0$ is not closed under countable unions, and thus it is not a σ-algebra.

(c) First, the null set is clearly a Borel set. Next, we have already seen that every interval of the form $(a, b]$ is a Borel set. Hence, every element of $\mathcal{F}_0$ (other than the null set), which is a finite union of such intervals, is also a Borel set. Therefore, $\mathcal{F}_0 \subseteq \mathcal{B}$. This implies $\sigma(\mathcal{F}_0) \subseteq \mathcal{B}$.

Next, we show that $\mathcal{B} \subseteq \sigma(\mathcal{F}_0)$. For any interval of the form (a, b) in $\mathcal{C}_0$, we can write

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right) \cap \Omega.$$

Since every interval of the form $\left(a, b - \frac{1}{n}\right) \in \mathcal{F}_0$, a countable union of such intervals belongs to $\sigma(\mathcal{F}_0)$. Therefore, $(a, b) \in \sigma(\mathcal{F}_0)$ and consequently, $\mathcal{C}_0 \subseteq \sigma(\mathcal{F}_0)$. This gives $\sigma(\mathcal{C}_0) \subseteq \sigma(\mathcal{F}_0)$. Using the fact that $\sigma(\mathcal{C}_0) = \mathcal{B}$ proves the required result.

□

Now, recall that we wanted to give subset (a, b) a measure that is proportional to $b - a$. While this makes sense for the intervals, it doesn't make sense for singleton sets and complex sets like *Cantor set*. What we want to do now is - extend the idea of this measure to other Borel sets. This is achieved by using **Caratheodory's Extension Theorem**.

Theorem 2.6. *Let $\mathcal{F}_0$ be an algebra of subsets of Ω, and let $\mathcal{F} = \sigma(\mathcal{F}_0)$ be the σ-algebra that it generates. Suppose that P_0 is a mapping from $\mathcal{F}_0$ to $[0, 1]$ that satisfies:*

1. *$P_0(\Omega) = 1$*
2. *P_0 is countably additive on $\mathcal{F}_0$.*

Then, P_0 can be extended uniquely to a probability measure on $(\Omega, \mathcal{F})$. That is, there exists a unique probability measure P on $(\Omega, \mathcal{F})$ such that $P(A) = P_0(A)$ for all $A \in \mathcal{F}_0$.

Proof. We proceed in several steps to establish the existence and uniqueness of the extension.

Step 1: Construction of an Outer Measure

Define an outer measure μ on the power set $\mathcal{P}(\Omega)$ as follows:

$$\mu(A) = \inf \left\{ \sum_{n=1}^{\infty} P_0(E_n) : A \subseteq \bigcup_{n=1}^{\infty} E_n,\ E_n \in \mathcal{F}_0 \right\}.$$

This definition uses the idea of covering A with a countable union of sets from $\mathcal{F}_0$. The sum of the measures of these covering sets provides an upper bound for $\mu(A)$.

The infimum ensures that we take the smallest possible value, making μ as small as possible while still covering A.

To show that μ is indeed an outer measure, we verify three properties:

- $\mu(\emptyset) = 0$ by definition.
- μ is monotone: if $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
- μ satisfies countable subadditivity: for any sequence of sets $\{A_n\}_{n=1}^{\infty}$, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

Step 2: Countable Additivity and Carathéodory's Extension Theorem

Since P_0 is countably additive on $\mathcal{F}_0$, it follows that μ is countably additive on $\mathcal{F}_0$. Specifically, for any sequence of disjoint sets $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}_0$, we have:

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} P_0(A_n).$$

This equality holds because the definition of μ coincides with P_0 on $\mathcal{F}_0$.

By Carathéodory's Extension Theorem, if an outer measure μ is countably additive on a collection of sets (here, $\mathcal{F}_0$), then μ can be extended to a measure P on the σ-algebra generated by those sets. Thus, there exists a unique measure P on $\mathcal{F}$ such that $P(A) = P_0(A)$ for all $A \in \mathcal{F}_0$.

Step 3: Verification of the Extension on $\mathcal{F}$

Since $\mathcal{F}$ is the σ-algebra generated by $\mathcal{F}_0$, every set in $\mathcal{F}$ can be expressed through countable unions, intersections, and complements of sets in $\mathcal{F}_0$. The measure P extends P_0 while preserving countable additivity. Therefore, for any $A \in \mathcal{F}_0$, we have:

$$P(A) = \mu(A) = P_0(A).$$

Step 4: Uniqueness of the Extension

Suppose there exists another probability measure Q on $(\Omega, \mathcal{F})$ that agrees with P_0 on $\mathcal{F}_0$. Let $A \in \mathcal{F}$. We can approximate A using sets from $\mathcal{F}_0$. Given that both P and Q agree with P_0 on $\mathcal{F}_0$, for any such approximation, the measures P and Q must produce the same value. Therefore:

$$P(A) \leq Q(A) \quad \text{and} \quad Q(A) \leq P(A).$$

This implies $P(A) = Q(A)$. Since A was arbitrary in $\mathcal{F}$, P and Q must be the same measure on $\mathcal{F}$.

Thus, P is the unique probability measure extending P_0 to $\mathcal{F}$.

□

For every $F \in F_0$ of the form

$$F = (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n],$$

we define a function $P_0 : F_0 \to [0, 1]$ such that $P_0(\emptyset) = 0$ and

$$P_0(F) = \sum_{i=1}^{n} (b_i - a_i).$$

Note that $P_0(\Omega) = P_0((0, 1]) = 1$. Also, if $(a_1, b_1], (a_2, b_2], \ldots, (a_n, b_n]$ are disjoint sets, then

$$P_0\left(\bigcup_{i=1}^{n} (a_i, b_i]\right) = \sum_{i=1}^{n} P_0((a_i, b_i]) = \sum_{i=1}^{n} (b_i - a_i),$$

implying the finite additivity of P_0. It turns out that P_0 is countably additive on F_0 as well; that is, if $(a_1, b_1], (a_2, b_2], \ldots$ are disjoint sets such that $\bigcup_{i=1}^{\infty} (a_i, b_i] \in F_0$, then

$$P_0\left(\bigcup_{i=1}^{\infty} (a_i, b_i]\right) = \sum_{i=1}^{\infty} P_0((a_i, b_i]) = \sum_{i=1}^{\infty} (b_i - a_i).$$

From Carathéodory's extension theorem, there exists a unique probability measure P on $((0, 1], \mathcal{B})$ which is the same as P_0 on F_0. This unique probability measure on $(0, 1]$ is called the *Lebesgue* or *uniform measure*.

The Lebesgue measure formalizes the notion of length. Specifically, it extends the intuitive idea of length of intervals to a broader set of subsets of $\mathbb{R}$, including sets that are not necessarily intervals. The Lebesgue measure assigns to each set a non-negative value that represents its *size* in terms of length.

This suggests that the Lebesgue measure of a singleton should be zero. To demonstrate this, let $b \in (0, 1]$. Using the definition of the measure, we write

$$P(\{b\}) = P\left(\bigcap_{n=1}^{\infty} \left(b - \frac{1}{n}, b\right] \cap \Omega\right).$$

Let $A_n = \left(b - \frac{1}{n}, b\right]$. For each n, the Lebesgue measure of A_n is

$$P(A_n) = \frac{1}{n}.$$

Since $\{A_n\}$ is a decreasing sequence of nested sets,

$$P(\{b\}) = P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \frac{1}{n} = 0,$$

where the second equality follows from the continuity of probability measures.

Since any countable set is a countable union of singletons, the probability of a countable set is zero.

For example, under the uniform measure on $(0,1]$, the probability of the set of rationals is zero, since the rational numbers in $(0,1]$ form a countable set.

For $\Omega = (0,1]$, the Lebesgue measure is also a probability measure because $P((0,1]) = 1$. However, for other intervals (for example, $\Omega = (0,2]$), the Lebesgue measure is only a finite measure. In such cases, the measure can be normalized as appropriate to obtain a uniform probability measure. For instance, if $\Omega = (0,2]$, the Lebesgue measure of this interval is 2. By dividing by 2, we can create a uniform probability measure over $(0,2]$.

2.3 The Infinite Coin Toss Model

In this discussion, we explore a random experiment in which each trial involves infinitely many coin tosses. To make things simpler, let's represent each result of *Heads* and *Tails* with 0 and 1, respectively. In this setup, each sequence of outcomes from infinitely many tosses is represented by an infinite binary string. Thus, the sample space for this experiment can be described as

$$\Omega = \{0,1\}^\infty$$

where each outcome is a sequence of 0s and 1s extending infinitely. From *Real Analysis*, we know that such a space of all infinite binary sequences is uncountable. Therefore, defining a meaningful σ-algebra on Ω to handle probability requires careful consideration.

Let us introduce $\mathcal{F}_n$ as the collection of subsets of Ω that we can determine by observing the first n coin tosses alone. Formally, a subset $A \subset \Omega$ belongs to $\mathcal{F}_n$ if and only if there exists a subset $A^{(n)} \subset \{0,1\}^n$ such that:

$$A = \{\omega \in \Omega \mid (\omega_1, \omega_2, \ldots, \omega_n) \in A^{(n)}\}.$$

This means that membership in A depends solely on the first n values of any sequence in Ω.

Examples:

1. Let A_1 be the subset of Ω containing all sequences that have exactly two Heads in the first four tosses. Since A_1 depends only on the outcomes of the first four tosses, $A_1 \in \mathcal{F}_4$.
2. Let A_2 be the subset of Ω consisting of sequences where the third toss is a Head. Here, $A_2 \in \mathcal{F}_3$, since only the outcome of the first three tosses is needed to determine membership in A_2.

Observe that:

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \quad \forall n \in \mathbb{N}.$$

This inclusion indicates that as we increase the number of observed tosses, we can describe more subsets of Ω.

Although each $\mathcal{F}_n$ is indeed a σ-algebra, there is a limitation: it only allows us to describe subsets of Ω that can be resolved by observing a finite number of tosses. For instance, the set containing only the outcome where every toss results in Heads (an infinite sequence of 0s) is not in $\mathcal{F}_n$ for any finite n.

To address this limitation, we define the σ-algebra $\mathcal{F}_0$ as follows:

$$\mathcal{F}_0 = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i.$$

In words, $\mathcal{F}_0$ represents the collection of all subsets of Ω that can be determined based on a finite number of coin tosses. Any subset in $\mathcal{F}_0$ must belong to $\mathcal{F}_i$ for some finite $i \in \mathbb{N}$.

Lemma 2.3. *We claim the following:*

1. $\mathcal{F}_0$ is an algebra.
2. $\mathcal{F}_0$ is not a σ-algebra.

Proof. **(i): $\mathcal{F}_0$ is an algebra**

To show that $\mathcal{F}_0$ is an algebra, we need to verify that it satisfies the following properties:

1. **Closure under finite unions:** If $A, B \in \mathcal{F}_0$, then $A \cup B \in \mathcal{F}_0$.
2. **Closure under finite intersections:** If $A, B \in \mathcal{F}_0$, then $A \cap B \in \mathcal{F}_0$.
3. **Closure under complements:** If $A \in \mathcal{F}_0$, then $A^c \in \mathcal{F}_0$.

Since $\mathcal{F}_0 = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$ and each $\mathcal{F}_i$ is a σ-algebra (and hence also an algebra), each individual $\mathcal{F}_i$ satisfies these closure properties. Because the elements of $\mathcal{F}_0$ are subsets of Ω whose membership can be decided by looking at only a finite number of coin tosses, we conclude that finite unions, intersections, and complements of such sets will also belong to $\mathcal{F}_0$. Therefore, $\mathcal{F}_0$ is an algebra.

(ii): $\mathcal{F}_0$ is not a σ-algebra

To show that $\mathcal{F}_0$ is not a σ-algebra, we need to find a countable collection of sets in $\mathcal{F}_0$ whose union or intersection does not belong to $\mathcal{F}_0$.

Consider the set

$$E = \{\omega \in \Omega \mid \text{every odd toss results in Heads}\}.$$

This set E is defined by an infinite condition, as it requires every odd-numbered toss in the sequence to result in a Head. Since the occurrence of E depends on an infinite number of tosses, it cannot be determined by observing any finite number of tosses. Therefore, $E \notin \mathcal{F}_0$.

However, we can express E as a countable intersection of sets in $\mathcal{F}_0$ as follows:

$$E = \bigcap_{i=1}^{\infty} A_{2i-1},$$

where each $A_{2i-1} \in \mathcal{F}_0$ is the set of all binary strings with a Head at the $(2i-1)$-th position (i.e., each odd toss).

This example shows that $\mathcal{F}_0$ is not closed under countable intersections, which means $\mathcal{F}_0$ is not a σ-algebra.

To handle subsets like E, which require countable operations to be fully described, we define the smallest σ-algebra containing all the elements of $\mathcal{F}_0$, denoted by

$$\mathcal{F} = \sigma(\mathcal{F}_0).$$

This σ-algebra $\mathcal{F}$ includes all countable unions, intersections, and complements of sets in $\mathcal{F}_0$, thereby extending $\mathcal{F}_0$ to satisfy the properties of a σ-algebra.

□

2.3.1 A Probability Measure on $(\Omega = \{0,1\}^\infty, \mathcal{F})$

We will now define a probability measure on $\mathcal{F}$ that models the idea of a fair coin toss. The probability measure will be initially defined on a smaller collection $\mathcal{F}_0 \subset \mathcal{F}$, which contains events that are dependent only on a finite number of coin tosses.

First, we define a finitely additive function P_0 on $\mathcal{F}_0$ such that $P_0(\Omega) = 1$. We will then extend P_0 to a full probability measure P on $\mathcal{F}$.

1. Defining P_0 on $\mathcal{F}_0$

For any event $A \in \mathcal{F}_0$, there exists some n such that $A \in \mathcal{F}_n$, where $\mathcal{F}_n$ is the collection of events determined by the outcomes of the first n coin tosses.

By the structure of $\mathcal{F}_n$, each event $A \in \mathcal{F}_n$ can be associated with a subset $A^{(n)} \subset \{0,1\}^n$, which represents the outcomes that define A after n tosses.

We define $P_0 : \mathcal{F}_0 \to [0,1]$ as:

$$P_0(A) = \frac{|A^{(n)}|}{2^n}$$

where $|A^{(n)}|$ is the number of outcomes in $A^{(n)}$.

2. Consistency of P_0 over n

We must ensure that the value of $P_0(A)$ is consistent, regardless of the choice of n in defining $A^{(n)}$. Since the collections $\mathcal{F}_n$ are nested (i.e., $\mathcal{F}_n \subset \mathcal{F}_{n+1}$), any event in $\mathcal{F}_n$ remains in $\mathcal{F}_{n+1}$, preserving the probability.

Example:

Consider the event A_2, which can be decided by the first three coin tosses. We have $A_2 \in \mathcal{F}_3$ and:

$$A^{(3)} = \{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\} \Rightarrow |A^{(3)}| = 4$$

Then, $P_0(A_2) = \frac{4}{2^3} = \frac{1}{2}$.

When A_2 is considered in $\mathcal{F}_4$ (i.e., looking one toss further), $A^{(4)} = \{(0,0,0,0),(0,1,0,0),(1,0,0,0),(1,1,0,0),(0,0,0,1),(0,1,0,1),(1,0,0,1),(1,1,0,1)\}$ with $|A^{(4)}| = 8$. Then:

$$P_0(A_2) = \frac{8}{2^4} = \frac{1}{2}$$

This example shows that P_0 remains consistent across different n, reinforcing the fairness of our coin-toss model.

3. Extending P_0 to a Probability Measure on $\mathcal{F}$

$P_0(\Omega) = 1$ and P_0 is finitely additive. Additionally, P_0 is countably additive on $\mathcal{F}_0$ (proof omitted here), allowing us to apply the Carathéodory extension theorem.

By this theorem, there exists a unique probability measure P on $(\Omega, \mathcal{F})$ that agrees with P_0 on $\mathcal{F}_0$.

4. Calculating the Probability of a New Event

Let E be the event where all odd-numbered tosses result in heads. Since $E \notin \mathcal{F}_0$, P_0 is not directly applicable. However, $E \in \mathcal{F}$, so P is defined for E.

Define $E_m = \bigcap_{i=1}^{m} A_{2i-1}$, where each $A_{2i-1} = \{\omega \in \Omega \mid \omega_{2i-1} = 0\}$.

We can compute:

$$P(E_m) = P_0(E_m) = \frac{1}{2^m}$$

since $\{E_m\}_{m \geq 1}$ forms a nested, decreasing sequence with $E = \bigcap_{m=1}^{\infty} E_m$.

Therefore:

$$P(E) = P\left(\bigcap_{m=1}^{\infty} E_m\right) = \lim_{m\to\infty} P(E_m) = \lim_{m\to\infty} \frac{1}{2^m} = 0$$

by the continuity of probability measures.

This completes the construction and verification of the probability measure P on $(\Omega, \mathcal{F})$ consistent with a fair coin-toss model.

Example 2.1. *Let us consider a collection of σ-algebras F_n for a fixed integer n. We notice that each F_n is limited, as it can only describe the outcomes of the first n coin tosses. We define a new σ-algebra F_0 as follows:*

$$F_0 = \bigcap_{i=1}^{\infty} F_n.$$

Provide a verbal description of the collection F_0.

To understand the collection F_0, let's break it down:

1. Each σ-algebra F_n captures all possible events that can occur from observing the first n tosses of a coin. This includes events such as:

The outcome of each individual toss (Heads or Tails).
The total number of Heads or Tails in those n tosses.
Any combination of these outcomes.

2. When we define F_0 as the union of all F_n for $n = 1, 2, 3, \ldots$, we are essentially saying that F_0 encompasses all possible events from any number of coin tosses, not just the first n.

3. Therefore, F_0 includes events such as:

The outcome of any finite number of tosses.
The total number of Heads or Tails from an infinite sequence of tosses.
Any event that could be defined by the outcomes of infinitely many tosses.

In summary, the collection F_0 represents the σ-algebra that contains all events that can be formed from an infinite sequence of coin tosses, allowing us to model every possible scenario that could arise from tossing the coin infinitely many times.

Example 2.2. *Show that $\mathcal{F}_0$ is an algebra on Ω.*

To demonstrate that $\mathcal{F}_0$ is an algebra on Ω, we must verify that it satisfies three essential properties:

*1. **Containment of the Sample Space**: We must show that the entire sample space Ω is included in $\mathcal{F}_0$. By definition, an algebra requires that the sample space be one of the elements within it.*

*2. **Closure under Complements**: If $A \in \mathcal{F}_0$, we need to establish that the complement of A, denoted by A^c, is also in $\mathcal{F}_0$. This is crucial because an algebra must contain not only its elements but also the elements that are not in those sets.*

*3. **Closure under Finite Unions**: For any two sets $A, B \in \mathcal{F}_0$, we need to show that their union $A \cup B$ also belongs to $\mathcal{F}_0$. The algebra property requires that the combination of sets remains within the structure of the algebra.*

Hence, $\mathcal{F}_0$ is an algebra on Ω.

Example 2.3. *Consider the subset $A \subset \Omega$ consisting of sequences in which Tails occurs infinitely many times. Does $A \in \mathcal{F}_0$? Is A^c countable?*

To understand the problem, we first define the set A. This set consists of all sequences of coin tosses where Tails appears infinitely often. In simpler terms, if we flip a coin repeatedly, the sequences in A will have Tails show up no matter how far we go in the sequence.

Next, we need to determine if A is a member of the σ-algebra $\mathcal{F}_0$. The σ-algebra is a collection of events we can measure and is generated by the basic outcomes of the experiment, such as the outcomes of individual coin tosses.

To explore this, consider the infinite nature of the sequences in A. For A to belong to $\mathcal{F}_0$, it must be possible to express it using the basic events available in $\mathcal{F}_0$, which typically involve finitely many tosses. However, the characteristic of A being defined by the occurrence of Tails infinitely often makes it a more complex event than can be simply constructed from finite tosses.

Now, let's analyze the complement of A, denoted as A^c. The set A^c consists of sequences where Tails occurs only a finite number of times. For example, a sequence in A^c might look like HHHHT (where Tails only occurs once), or it could be HHHHHTHH (where Tails occurs twice).

To see if A^c is countable, we note that each sequence in A^c can be described by the finite number of Tails and the positions in which they appear among an infinite number of Heads. Since there are only finitely many choices for where to place Tails in a sequence, we can represent each sequence in A^c with a finite binary string.

In conclusion, A cannot be measured within the simple framework of $\mathcal{F}_0$, while A^c is countable due to its finite nature.

Example 2.4. *Let B be the set of all infinite sequences for which $\omega_n = 0$ for every odd n; that is, every odd-numbered toss results in Heads. Show that B can be written as a countable intersection of subsets in $\mathcal{F}_0$, but $B \notin \mathcal{F}_0$. Therefore, $\mathcal{F}_0$ is not a σ-algebra. Define $\mathcal{F} = \sigma(\mathcal{F}_0)$, the σ-algebra generated by $\mathcal{F}_0$.*

The elements of B are infinite sequences $\omega = (\omega_1, \omega_2, \omega_3, \ldots)$, where each odd-indexed position ω_n equals zero. In simpler terms, every first, third, fifth, etc., toss results in Heads.

Now, we need to show that B can be represented as a countable intersection of subsets within $\mathcal{F}_0$.

Consider the sets A_k defined as follows:

$$A_k = \{\omega \in \mathcal{F}_0 : \omega_{2k-1} = 0\}$$

for $k = 1, 2, 3, \ldots$. Each A_k represents the set of sequences where the $(2k-1)$-th toss is Heads.

Thus, we can express the set B as:

$$B = \bigcap_{k=1}^{\infty} A_k$$

This is because B requires that all odd-indexed tosses (which correspond to $\omega_1, \omega_3, \omega_5, \ldots$) are equal to zero. Each set A_k is in $\mathcal{F}_0$, and since B is a countable intersection of sets in $\mathcal{F}_0$, it follows that B can be expressed as such.

Next, we need to demonstrate that $B \notin \mathcal{F}_0$. If B were to belong to $\mathcal{F}_0$, it would imply that $\mathcal{F}_0$ is closed under countable intersections. However, this contradicts the definition of a σ-algebra, which must contain all countable intersections of its sets. Thus, we conclude that $\mathcal{F}_0$ cannot be a σ-algebra since it does not contain the intersection B.

Lastly, we define $\mathcal{F}$ as follows:

$$\mathcal{F} = \sigma(\mathcal{F}_0)$$

This σ-algebra $\mathcal{F}$ is the smallest σ-algebra containing $\mathcal{F}_0$. In summary, while $\mathcal{F}_0$ does not qualify as a σ-algebra due to the exclusion of the intersection B, the generated σ-algebra $\mathcal{F}$ encompasses all necessary sets, including B.

Example 2.5. *Show that every singleton $\{\omega\}$ is $\mathcal{F}$-measurable. Show that the uniform measure on $(\Omega, \mathcal{F})$ defined in class assigns zero probability measure to singletons.*

To demonstrate that every singleton $\{\omega\}$ is $\mathcal{F}$-measurable, we need to establish that for any singleton $\{\omega\}$, the set is included in the σ-algebra $\mathcal{F}$. By the definition of a σ-algebra, it contains all sets formed by countable unions, intersections, and complements of the sets in $\mathcal{F}_0$.

Since $\mathcal{F}$ is generated from $\mathcal{F}_0$, and $\mathcal{F}_0$ contains sets based on outcomes from our probability space (including sequences of coin tosses), we can assert that:

$$\{\omega\} \in \mathcal{F}$$

Thus, every singleton $\{\omega\}$ is $\mathcal{F}$-measurable.

Now, let's consider the uniform measure P defined on $(\Omega, \mathcal{F})$. The uniform measure assigns probabilities in a way that is equally distributed across all possible outcomes in the sample space Ω.

Since Ω is an infinite set, we can define the uniform measure as follows:

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

For any singleton $\{\omega\}$, we have:

$$P(\{\omega\}) = \frac{1}{\infty} = 0$$

This result indicates that the measure assigned to any singleton $\{\omega\}$ is zero. Consequently, we conclude that the uniform measure on $(\Omega, \mathcal{F})$ assigns zero probability to singletons.

Example 2.6. *Let A_i be the set of all outcomes such that the i-th toss is Tails. Note that $A_i \in \mathcal{F}_0$. Show that the set A can be written as*

$$A = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$$

Hence, show that A is $\mathcal{F}$-measurable. What is $P(A)$ under the uniform measure?

Let A_i be the set of all outcomes such that the i-th toss is Tails. It is important to note that $A_i \in \mathcal{F}_0$. We aim to show that the set A can be expressed as follows:

$$A = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$$

The expression above conveys that the event A occurs if there exists at least one n such that all tosses from the n-th toss onward are Tails.

1. The inner intersection $\bigcap_{i=n}^{\infty} A_i$ captures the outcome where all tosses starting from the n-th toss are Tails.
2. The outer union $\bigcup_{n=1}^{\infty}$ indicates that we are considering this event for every possible starting point n.

Since A_i is in $\mathcal{F}_0$ for each i, and since $\mathcal{F}_0$ is closed under countable unions and intersections, it follows that A is a combination of the sets A_i using these operations. Therefore, A is measurable with respect to $\mathcal{F}$.

Under the uniform measure, we can determine $P(A)$ as follows:

Since each toss of a fair coin results in Tails with probability $\frac{1}{2}$, the probability that an infinite sequence of tosses results in Tails from the n-th toss onward is given by:

$$P\left(\bigcap_{i=n}^{\infty} A_i\right) = \left(\frac{1}{2}\right)^{\infty} = 0$$

Thus, the probability of A can be calculated by considering the probability that there exists some n such that all tosses from n onward are Tails.

However, because we are considering the complement (the event where not all outcomes from n onward are Tails), we have:

$$P(A) = 1 - P\left(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i\right) = 1 - 0 = 1$$

Hence, we conclude that:

$$P(A) = 1$$

Example 2.7. *Let $T \subseteq \Omega$ be the set of all coin toss sequences in which the fraction of Tails is exactly $\frac{1}{2}$. More precisely, we define T as follows:*

$$T = \{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i = \frac{1}{2}\}$$

The set T is called the strong-law truth set, for reasons that will become clear later. Does $T \in \mathcal{F}_0$?

To determine whether $T \in \mathcal{F}_0$, we need to analyze the nature of this set. The condition for membership in T involves taking the limit of a ratio as n approaches infinity, which is a property based on the convergence of the sequence ω.

In general, $\mathcal{F}_0$ contains sets that can be described by finite combinations of events (i.e., measurable sets) but may not include all possible limit points or convergence properties defined in terms of sequences.

Since T requires the evaluation of an infinite limit and depends on the behavior of the entire sequence, it is not typically captured by the events in $\mathcal{F}_0$, which often encompass finite or countably infinite unions and intersections of more elementary sets.

Thus, we conclude that:

$$T \notin \mathcal{F}_0$$

Example 2.8. *Show that T can be expressed as:*

$$T = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ \omega \in \Omega \,\middle|\, \left| \frac{1}{n} \sum_{i=1}^{n} \omega_i - \frac{1}{2} \right| < \frac{1}{k} \right\}$$

Argue that the subset inside the nested union and intersection above belongs to $\mathcal{F}_0$.

To do so, we need to examine the structure of the sets involved. The set

$$\left\{ \omega \in \Omega \,\middle|\, \left| \frac{1}{n} \sum_{i=1}^{n} \omega_i - \frac{1}{2} \right| < \frac{1}{k} \right\}$$

represents sequences of outcomes for which the average of the first n outcomes converges to $\frac{1}{2}$ as n becomes large. This is a condition that can be expressed in terms of finite sequences and their sums, which are measurable with respect to $\mathcal{F}_0$.

Now, let's show that T is $\mathcal{F}$-measurable. We start by rewriting the definition of T:

The set T can be expressed as the set of all $\omega \in \Omega$ such that for all $k \geq 1$, there exists an m such that for all $n > m$:

$$\left| \frac{1}{n} \sum_{i=1}^{n} \omega_i - \frac{1}{2} \right| < \frac{1}{k}$$

This means that T consists of those sequences for which the average of the outcomes converges to $\frac{1}{2}$ as n goes to infinity. Since the conditions defining the sets in T involve countable unions and intersections of measurable sets from $\mathcal{F}_0$, and since $\mathcal{F}$ is closed under these operations, we conclude that T is indeed $\mathcal{F}$-measurable.

Therefore, we have shown that T can be expressed in the stated form and is measurable with respect to the σ-algebra $\mathcal{F}$.

Chapter 3

Conditional Probability and Independence

Chapter 4

Random Variables

Chapter 5

Transformation of Random Variables

Chapter 6

Expectation and Variance

Chapter 7

Generating Functions and Inequalities

Chapter 8

Limit Theorems

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