Towards more efficient infection and fire fighting

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Abstract

The firefighter problem models the situation where an infection, a computer virus, an idea or fire *etc.* is spreading through a network and the goal is to save as many as possible nodes of the network through targeted vaccinations. The number of nodes that can be vaccinated at a single time-step is typically one, or more generally $O(1)$. In a non-standard model, the so called *spreading model*, the vaccinations also spread in contrast to the standard model.

Our main results are concerned with general graphs in the spreading model.

We provide a very simple exact $2^{O(\sqrt{n}\log n)}$-time algorithm. In the special case of trees, where the standard and spreading model are equivalent, our algorithm is substantially simpler than that exact subexponential algorithm for trees presented in (Cai et al. 2008). On the other hand, we show that the firefighter problem on weighted directed graphs in the spreading model cannot be approximated within a constant factor better than $1 - 1/e$ unless NP $\subseteq$ DTIME($n^{O(\log\log n)}$).

We also present several results in the standard model. We provide approximation algorithms for planar graphs in case when at least two vaccinations can be performed at a time-step. We also derive trade-offs between approximation factors for polynomial-time solutions and the time complexity of exact or nearly exact solutions for instances of the firefighter problem for the so called directed layered graphs.

Keywords: approximation algorithms, subexponential algorithms

1 Introduction

The *firefighter problem* has received significant attention recently (Anshelevich et al. 2009, Cai et al. 2008, Finbow et al. 2007, Hartnell 1995). We model the underlying network by a directed or undirected graph $G = (V, E)$ with a distinguished vertex s called the source node (or the root), and nonnegative vertex weights. Each vertex $v \in V$ is either *on fire*, *protected*, or *vulnerable*, where the latter implies that v is neither protected nor on fire. At time 0 a fire breaks out at the source node, at each subsequent time step a firefighter may be placed on a vulnerable vertex to protect it. At each subsequent time step the fire also spreads to all vulnerable vertices that are adjacent to a burning vertex. At some time, when the fire can no longer spread, the process ends and all the vertices which are not on fire are considered to be *saved*.

The firefighter problem was first considered by Hartnell (Hartnell 1995). The objective was to determine a deployment of firefighters which maximizes the sum of weights of saved vertices. Later, several other variants of the firefighter problem have also been studied.

Anshelevich et al. (Anshelevich et al. 2009) generalized the classical firefighter problem to include the possibility of placing up to B firefighters at a single time step. They also considered the dual problem where the objective is to minimize the budget constraint B in order to save a given set of nodes $T \subseteq V$. Furthermore, they also introduced the so called *spreading model* (Anshelevich et al. 2009) for the firefighter problem and its dual. In the spreading model, if a node u is protected and v is a vulnerable neighbor of u at time step t, then at the next time step $t+1$, the node v also becomes protected. Note that in this model a firefighter prevails over possible neighboring nodes on fire and the adjacent vulnerable node is protected in the subsequent step.

The introduction of the spreading model stems from the fact that the firefighter problem can also model a diffusive process, such as an infection, which spreads through a network. The objective is to stop this infection by using targeted vaccinations (see e.g. (Anshelevich et al. 2009)). Hence, the firefighting problem in the spreading model is highly relevant to health care efficiency.

1.1 Related work

The firefighter problem appeared to be NP-hard even when restricted to trees (Finbow et al. 2007). It is hard to approximate within n^{α} in polynomial-time for general undirected graphs, for any $\alpha < 1$ while it admits an $(1-e^{-1})$-approximation in polynomial-time for trees (Cai et al. 2008). The greedy heuristic on trees which places a firefighter on a vulnerable vertex that saves the largest number of vertices is known to achieve $\frac{1}{2}$-approximation factor (Hartnell & Li 2000). Even a subexponential, $2^{O(\sqrt{n}\log n)}$-time, exact algorithm has been designed for trees on n vertices (Cai et al. 2008).

In the spreading model, the firefighter problem is much more feasible. For general graphs it can be approximated within $1 - e^{-1}$ in polynomial-time (Anshelevich et al. 2009).

2 Our contributions

To begin with, we observe that essentially all the known approximability results for the firefighter problem on trees (Cai et al. 2008, Hartnell & Li 2000) as well as those on

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general graphs in the spreading model (Anshelevich et al. 2009) immediately follow from the known corresponding results for the so called maximum coverage problem with groups (Chekuri & Kumar 2004).

Our main results are concerned with general graphs in the spreading model.

We provide a very simple exact $2^{O(\sqrt{n}\log n)}$-time algorithm. In the special case of trees, where the standard and spreading model are equivalent, our algorithm is substantially simpler than that exact subexponential algorithm for trees presented in (Cai et al. 2008).

On the other hand, we show that the firefighter problem on weighted directed graphs in the spreading model cannot be approximated within a constant factor better than $1 - 1/e$ unless NP $\subseteq$ DTIME($n^{O(\log\log n)}$).

We also present several results in the standard model.

Firstly, we obtain two approximation results in terms of the degree of the source vertex for planar graphs, assuming that at least two firefighters can be placed in a single step.

Secondly, we derive two trade-offs between approximation factors for polynomial-time solutions and the time complexity of exact or nearly exact solutions for instances of the firefighter problem for the so called directed layered graphs studied in (Anshelevich et al. 2009).

3 Preliminaries: A reduction to a variant of Maximum Coverage Problem

A restriction of the cardinality variant of the problem of **Maximum Coverage with Group Budgets (MCG)** has been defined in (Chekuri & Kumar 2004) as follows. There are given subsets $S_1, S_2, ..., S_m$ of a ground set X, disjoint subsets $G_1, ..., G_l$ of $\{S_1, ..., S_m\}$ called groups and a positive integer k. The objective is to find $H \subseteq \{S_1, ..., S_m\}$ such that $|H| \le k$, $|H \cap G_i| \le 1$ for $i = 1, ..., l$ and the number of elements in X covered by sets in H is maximized.

Chekuri and Kumar proved the following fact in (Chekuri & Kumar 2004).

Fact 1 *The standard greedy heuristic for minimum set cover (stopped when k sets are already included in the cover) yields $\frac{1}{2}$ approximation for the restriction of the cardinality variant of MCG. This problem can be also approximated within $1 - e^{-1}$ by linear programming techniques.*

Corollary 1 The firefighter problem with budget B on trees as well as the firefighter problem with budget B for general (directed or undirected) graphs in the spreading model can be approximated within $1 - e^{-1}$ in polynomial time. Also, the standard greedy heuristic that iterates picking a vertex that saves the largest number of not yet saved vertices yields $\frac{1}{2}$ approximation in both cases.

Proof: Given an instance of the firefighter problem with budget B on trees, we define a corresponding instance of the cardinality variant of MCG as follows.

We root the input tree T at the distinguished vertex and for each other vertex v define S_v as the set of all descendants of v, including v, in the tree. Thus, the ground set is the set of all vertices of T different from the source vertex, and the family of sets consists of the aforementioned sets S_v. We partition the family into groups by accounting into the same group all sets S_v where v share the same level of T. Finally, we set the parameter k to B.

Any feasible solutions to the resulting instance of MCG which covers q vertices is in one-to-one correspondence with a placement of firefighters in T which saves q vertices (place the firefighters on those v for which S_v are in the set cover) and *vice versa* (account to the cover all S_v where a firefighter is placed on v).

This proves the corollary for the firefighter problem on trees.

The proof for general (directed or undirected) graphs G in the spreading model is analogous. It relies on the observation that the set of vertices saved by a placement of firefighters in the spreading model is a union of the sets saved by single firefighter placements included in the placement.

Let d be the maximum distance of a vertex from the fire source in the graph. For $1 \le r \le d$, and each vertex v of G, we define S_v^r as the set of all vertices of G that become directly or indirectly saved if we place a firefighter on the vertex v in the r-th step. Note that v is included into S_v^r and this set can be easily computed in time polynomial in the size of G. For each r, the sets S_v^r form a separate group. □

4 Firefighting in the spreading model

4.1 A subexponential-time algorithm

Our subexponential-time algorithm for general graphs in the spreading model relies on the two following lemmata.

Lemma 1 *After the j-th step, all vertices within distance at most j from the source vertex in the spreading model are either burnt or (directly or indirectly) saved.*

Proof: Let v be a vertex at distance of j from the source. If the fire has not reached v during the j steps then for any shortest path from the source to v, there is a step $1 \le t \le j$ at which the $t+1$st vertex on the path has been directly or indirectly saved. It follows from the assumed model that v must be saved too. □

Lemma 2 *After the $2\sqrt{n}+1$-st step in any optimal solution all vertices at distance of $2\sqrt{n}+1$ from the source are saved.*

Proof: Suppose that there is a vertex v at distance of $2\sqrt{n} + 1$ from the source that is not saved after the $2\sqrt{n} + 1$-st step. It follows from Lemma 1 that it is burnt after this step. Thus, there must be a shortest path P of length $2\sqrt{n} + 1$ from the source to v that is totally burnt after the $2\sqrt{n} + 1$-st step.

On the other hand, among the $\sqrt{n}$ firefighters placed during the first $\sqrt{n}$ steps, there exists at least one, say placed at t-th step that saves uniquely at most $\sqrt{n}$ of the vertices in the optimal solution.

Now, if we move the firefighter placed in the t-th step to the $t+1$-st vertex on P (counting from the source) then we save at least $2\sqrt{n} + 1 - t \ge \sqrt{n} + 1$ new vertices and let to burn at most those $\sqrt{n}$ vertices previously uniquely saved by this firefighter. We obtain a contradiction with the optimality of the solution. □

Corollary 2 Any optimal solution in the spreading model for a graph with n vertices and a distinguished source vertex places at most $2\sqrt{n} + 1$ firefighters.

Proof: By Lemma 2, directly after the $2\sqrt{n} + 1$-st step, all the vertices at distance of $2\sqrt{n}+1$ from the source are saved. Hence, the more remote vertices will be saved too. This implies that that there is no need to place firefighters in the next steps. □

Now, we are ready to derive our main result in this section.

Theorem 1 *An optimal solution in the spreading model for a graph with n vertices and a distinguished source vertex can be found in time $O(n^{2\sqrt{n}+3}) = 2^{O(\sqrt{n}\log n)}$.*

Proof: By Corollary 2, it is sufficient to enumerate all valid placements of at most $2\sqrt{n}+1$ firefighters, for each of them compute the number of saved vertices, and chose the placement maximizing the number of saved vertices. There are $O(n^{2\sqrt{n}+1})$ such placements and the computation of the number of saved vertices for a given placement takes time $O(n^2)$. $\square$

4.2 A lower bound

The *budgeted maximum coverage* problem (BMC for short) is as follows. For a budget k and a family S of sets defined over a domain of n weighted elements, each set having an associated cost, find a subset S' of S such that the total cost of sets in S' does not exceed k and the total weight of elements covered by S' is maximized.

Khuller et al. proved the following approximability hardness result on BMC (Khuller et al. 1999).

Fact 2 *The unit cost version of the budgeted maximum coverage problem cannot be approximated within a constant factor better than $1-1/e$ unless NP $\subseteq$ DTIME($n^{O(\log\log n)}$).*

Lemma 3 *There is a polynomial-time many-one reduction ϕ of the unit cost version of the budgeted maximum coverage problem to the firefighter problem on weighted directed graphs in the spreading model such that for an instance I of the maximum coverage problem the maximum number of the elements covered under budget k is equal to the maximum weight of saved vertices in $\phi(I)$ minus 1 (or even equal in case all elements are covered).*

Proof: Let S be the family of sets in I. Form a layered directed graph $G(I)$ with a distinguished source vertex s as follows. In the bottom layer put vertices in one-to-one correspondence with the elements in the domain. In the next layer put vertices in one-to-one correspondence with the sets in S. Direct from each of them edges to all vertices in the bottom layer corresponding to elements covered by the associated set in S. Now, connect the source s with each of the vertices v on the next to the bottom layer by a unique directed path of length k from s to v. Set the weights of the vertices on the bottom level to one and the weights of all remaining vertices to zero.

Consider a solution to I covering q elements with k sets. Suppose first that it does not cover all elements. Place a firefighter on each of the unique paths connecting the source with a vertex on the next to the bottom level corresponding to a set in the solution to I so no two firefighters are placed at the same distance from s and no firefighter is placed on s. Such a placement is possible since the paths have length k and it saves $q+\frac{k(k+1)}{2}$ vertices of total weight q. Finally, we can save one more vertex of weight 1 corresponding to an uncovered element by placing a firefighter on it in the last $k+1$st step.

On the other hand, since placement of a firefighter on an ancestor of a vertex is never worse than such a placement on the vertex, we may easily transform an optimal solution to $G(I)$ to a normalized one which places k firefighters on the unique paths connecting the source s with the next to the bottom layer during the first k steps. Such a normalized placement saves at most $\frac{k(k+1)}{2}$ vertices on these paths and, say t, vertices on the bottom layer. In the last, $k+1$st step, it places a firefighter on a not yet saved vertex at the bottom level. It follows that the total weight of saved vertices is $t+1$. By picking the k sets corresponding to the vertices on the next to the bottom level saved by the k firefighters placed in the first k steps, we obtain a family of k sets covering t elements.

In case, a (optimal) solution to I covers all n elements we do not need to place any firefighter in the $k+1$st step. Then, on the other hand, the firefighters placed in the first k steps of any normalized optimal solution to $G(I)$ induce a family of k sets covering $t \geq n-1$ elements while the total weight of saved vertices is n. $\square$

Fact 2 combined with the proof of Lemma 3 yields the following lower bound.

Theorem 2 *The firefighter problem on weighted directed graphs in the spreading model cannot be approximated within a constant factor better than $1-1/e$ unless NP $\subseteq$ DTIME($n^{O(\log\log n)}$).*

5 Firefighting in the standard model

5.1 A simple subexponential-time algorithm for trees

Since in the case of trees, there is no difference between the standard model and the spreading model, our simple subexponential-time algorithm for general graphs in the spreading model also works for trees in the standard model. This yields both a much simpler subexponential algorithm as well as analysis for the firefighter problem on trees than those presented in (Cai et al. 2008).

Furthermore, if we adopt the derivation of the subexponential algorithm in the spreading model to the standard firefighter model constrained to trees then we can decrease the constant in the exponent of the upper bound derived in the spreading model substantially.

The following counterpart of Lemma 1 is obvious.

Lemma 4 *In the standard model for trees with a distinguished source vertex, after the j-th step, all vertices within distance at most j from the source vertex are either burnt or (directly or indirectly) saved.*

The following counterpart of Lemma 2 can be simply obtained by replacing Lemma 1 with Lemma 4 in the body of the proof.

Lemma 5 *After the $2\sqrt{n}+1$-st step in any optimal solution in the standard model for a tree with a distinguished source vertex, all vertices at distance of $2\sqrt{n}+1$ from the source are saved.*

Similarly, the proof of the following counterpart of Corollary 2 can be obtained by replacing Lemma 2 with Lemma 5 in the body of the proof.

Corollary 3 Any optimal solution in the standard model for trees with n vertices and a distinguished source vertex places at most $2\sqrt{n}+1$ firefighters.

Finally, the proof of the following counterpart of Theorem 1 can be obtained by replacing Corollary 2 with Corollary 3 in the body of the proof of Theorem 1 and observing that computing the number of saved vertices for a given placement of firefighters requires time linear in n in case of trees.

Theorem 3 *An optimal solution in the standard model for a tree with n vertices and a distinguished source vertex can be found in time $O((\frac{n}{4})^{\sqrt{n}}n^{3/2})$.*

Proof: By Corollary 3, it is sufficient to enumerate all valid placements of at most $2\sqrt{n}+1$ firefighters, for each of them compute the number of saved vertices and chose the placement maximizing the number of saved vertices. We may w.l.o.g consider only the placements that for $i=1,...,2\sqrt{n}+1$ place at most one firefighter at distance of exactly i from the source. The number of the latter placements is at most $(n/(2\sqrt{n}+1))^{2\sqrt{n}+1} \leq n^{\sqrt{n}+\frac{1}{2}}2^{-2\sqrt{n}-1}$. It remains to observe that the computation of the number of saved vertices for a given placement takes time $O(n)$. $\square$

5.2 Approximate firefighting on planar graphs

Planar graphs and their planar embeddings termed as *plane graphs* seem to be a very natural model of a network for the applications of the firefighter problem. If we allow for placement of more than one firefighter in a single step, we can obtain non-trivial approximation results based on the good separator properties of planar graphs.

We can rephrase Lemma 2 from (Lipton & Tarjan 1979) for our purposes in terms of plane graphs as follows.

Lemma 6 *Let G be a plane graph with nonnegative vertex costs summing to W. Suppose G has a rooted spanning tree T of radius r. Then the vertices of G can be partitioned into three sets $A,\ B,\ C$ such that no edge joins a vertex in A with a vertex in B, neither A nor B has total cost exceeding $2W/3$, and C consists of vertices on two paths towards the root of T starting from vertices v and u on a common face and ending at their lowest common ancestor. Furthermore, C can be completed to a simple cycle by the diagonal joining v with u, and the vertices in A lie outside the cycle while those in B inside the cycle.*

Theorem 4 *The firefighter problem on planar graphs with a source vertex r of degree $deg(r)$ and budget $B \geq \max\{2,\ \lceil deg(r)/2 \rceil\} + 1$ can be approximated within $\frac{1}{3}$ in polynomial-time.*

Proof: Let G be a planar graph on n vertices with a source vertex r in which fire starts and let G' be its planar embedding.

Construct the breadth-first search tree BT of G' with the root at r. By Lemma 6, there are two vertices $v,\ u$ in BT such that the set C of vertices on the paths $P_v,\ P_u$ from v and u to the lowest common ancestor of v and u in BT splits G' into two parts each having at least $1/3$ of the total weight of G'.

Now if the aforementioned lowest common ancestor is different from r, for $j = 1, 2, ...$ iterate the following step: Place firefighters on the at most two vertices of P_v and P_u that are at distance of j from r.
It follows from Lemma 6 that the part of G' between these two paths of total weight at least $1/3$ of that of G is saved in this way.

Suppose in turn that the lowest common ancestor is at r. In case the number children of r between P_v and P_u is smaller than $deg(r)/2$ then in the first step, i.e., for $j = 1$, we place at most $\lceil deg(r)/2 \rceil + 1$ firefighters on the aforementioned children and the two children on P_v and P_u, and then proceed analogously as in the previous case. Finally, in case the aforementioned number of children is greater than $deg(r)/2$ we proceed as follows. We place at most $\lceil deg(r)/2 \rceil + 1$ firefighters on all the children of r outside the cycle induced by the paths P_v and P_u and the children of r on these two paths and proceed analogously as in the consecutive steps. In this way, we save all the vertices outside the cycle whose total weight is at least $\frac{1}{3}$ of that of G. □

Theorem 5 *The firefighter problem on a planar graph G with a source vertex r of degree $deg(r)$ and budget $B \geq 2$ can be approximated within $\frac{1}{3deg(r)}$ in polynomial-time.*

Proof: Let G' be a planar embedding of G. Form a breadfirst search tree BT of G' rooted at r. Let w be a child of r which maximizes the total weight of the subtree BT_w of BT rooted at w. Note that the total weight of vertices of BT_w is at least $\frac{1}{deg(r)}$ of the total weight of vertices in G' different from r. Consider the subgraph G_w of G' induced by vertices of BT_w.

By Lemma 6, there are two vertices $v,\ u$ in BT_w such that the set C of vertices on the paths $P_v,\ P_u$ from v and u to the lowest common ancestor of v and u in BT_w splits G_w into two parts each having at least $1/3$ of the total weight of G_w.

For $j = 1, 2, ...$ place firefighters on the at most two vertices of P_v and P_u that are at distance of j from r.

It follows from Lemma 6 that the part of G_w between these two paths of total weight at least $1/3$ of that of G_w is saved in this way. □

5.3 Tradeoffs for firefighting on directed layered graphs

The dual, budget variant of the firefighter problem for the so called directed layered graphs have been studied in (Anshelevich et al. 2009).

A *directed layered graph* G with a source s is one whose vertices can be partitioned into l layers such that s is the only vertex in the 0 layer and for each directed edge (u, v) there is $0 < i \leq l$ where u belongs to the layer $i - 1$ and v belongs to the layer i.

Note that for a vertex on a layer i placements of firefighters in time steps greater than i cannot help in saving the vertex.

Definition 1 The (standard) firefighter problem for a directed graph G with a source vertex and the upper bound B on the number of vertices on which firefighters can be placed at a single time step is denoted by $FF_B(G)$.

By enumerating all feasible solutions to $FF_1(G)$, we obtain the following lemma.

Lemma 7 *Let G be a layered directed graph on n vertices, with l layers and a source vertex. An optimal solution to $FF_1(G)$ can be computed in time $O(n^2 + \Pi_{j=1}^{l} |V_j(G)|) \leq O(n^2 + \binom{n}{l})$, where $V_j(G)$ is the set of vertices of G on the layer j.*

By the following lemma and the known results on the approximability hardness of the classical firefighter problem for general undirected graphs, the hard instances have to have small radius. In the lemma, the *eccentricity* of a vertex means the maximum length of a shortest directed path from the vertex to another vertex in the graph.

Lemma 8 *Let G be a directed graph on n vertices with a source vertex and let d be the eccentricity of the source. $FF_1(G)$ can be approximated within n/d.*

Proof: Let P be a shortest directed path from the source vertex to a most distance vertex from it in G. Note that P has length d. In time step i, place a firefighter on the i-th vertex of P. Observe that all the d vertices on P different from the source vertex will be saved. □

By Lemmata 7, 8, we obtain the following tradeoff between the approximability and the exact time complexity in case of the classical firefighter problem on layered directed graphs.

Theorem 6 *Let G be a layered directed graph on n vertices with a source vertex. For each positive integer $k \geq 2$, $FF_1(G)$ can be approximated within n/k in polynomial time or an optimal solution to $FF_1(G)$ can be computed in time $O(n^k)$.*

Proof: Let d be the eccentricity of the source in G. If $k < d$ then by Lemma 8, we obtain an approximation within n/k in polynomial time. Otherwise, we obtain an optimal solution in time $O(n^k)$ by Lemma 7. □

The following combinatorial lemma valid for any directed graph with a source will be useful.

Lemma 9 *Let G be a directed graph on $n \geq 4$ vertices with a source vertex and let $\alpha > 1$. If the eccentricity of the source in G not less than $\lceil \alpha\sqrt{n} \rceil$ then there is $i \in \{1, ..., \lceil \alpha\sqrt{n} \rceil\}$ such that the number of vertices of G at distance of i from the root is at most $\lfloor 2i/\alpha^2 \rfloor$.*

Proof: If the theorem does not hold then the number of vertices of G different from the source vertex is not less than $\sum_{i=2}^{\lceil \alpha\sqrt{n} \rceil} 2i/\alpha^2$ which in turn is not less than $\frac{2}{\alpha^2} \times ((\lceil \alpha\sqrt{n} \rceil + 1)\lceil \alpha\sqrt{n} \rceil/2 - 1)$. Since the latter value is clearly greater than n, we obtain a contradiction. □

By using Lemma 9, we can also obtain another tradeoff for layered directed graphs between very close approximability in subexponential time and constant approximability in polynomial time.

Theorem 7 *Let G be a layered directed graph on n vertices with a source vertex. For any integer $k > 0$, an optimal solution to $FF_1(G)$ can be approximated within $(1 - \frac{1}{k})$ in time $n^{O(\sqrt{n})}$ or it can be approximated within $\frac{1}{k}$ in time $O(n^2)$.*

Proof: We may assume without loss of generality that the number of layers in G is not less than $\lceil \alpha\sqrt{n} \rceil$ since otherwise $FF_1(G)$ can be solved exactly in time $n^{O(\sqrt{n})}$ by Lemma 7. Apply Lemma 9 with $\alpha = \sqrt{2}$ to G. Let i be the number of the layer of G which has at most i vertices. Consider an algorithm which places during the first i time steps firefighters on the vertices on the i-th layer. If the number of vertices of G on the layers $j \geq i$ is at least $\frac{1}{k}$ of the optimum then the algorithm yields an $\frac{1}{k}$ approximation. Otherwise, more than $1 - \frac{1}{k}$ of the saved vertices are placed on the layers 1 through $i - 1$ in an optimal solution to $FF_1(G)$. Thus, there is at least one placement of at most $\binom{n}{i-1}$ feasible placements of firefighters on these levels which saves at least $1-\frac{1}{k}$ of the optimal number of vertices. Such a placement can be detected in time $n^{O(\sqrt{n})}$. □

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