250085 VU
TENSOR METHODS FOR DATA SCIENCE
AND SCIENTIFIC COMPUTING
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COURSE PROJECT 3

VLADIMIR KAZEEV

This is an **individual project**. All deliverables are to be submitted via Moodle not later than on the **31st March 2022**. Early submission is encouraged.

https://moodle.univie.ac.at/mod/assign/view.php?id=11741913

Multilevel MPS-TT Representation for a Poisson Problem in Two Dimensions

Project outline

The goal of the project is to apply the low-rank tensor approximation with BPX-type preconditioning for solving a Poisson boundary-value problem in two dimensions, which should result in a study analogous to the *1D study* shared via Moodle (see this Jupyter Notebook and its PDF version).

Specifically, we consider the problem of finding u such that

$$\begin{cases} -\triangle u = f \text{ on } D, \\ u\big|_{\varGamma} = 0\,, \\ \frac{\partial u}{\partial \mathsf{n}}\big|_{\partial D\setminus\varGamma} = 0\,, \end{cases}$$

posed on the unit square $D = (0,1)^2$, with

$$\varGamma = \{(x_1,x_2) \in \partial D \colon x_1 \cdot x_2 = 0\} = \big([0,1]\times\{0\}\big) \cup \big(\{0\}\times[0,1]\big)$$

and

$$\triangle = \partial_1^2 + \partial_2^2\,,$$

the Laplace operator.

Under the assumption $f \in H^{-1}(D)$, a weak formulation on the variational space

$$V = \{v \in \mathbb{H}^1(D) \colon v\big|_{\varGamma} = 0\}$$

in terms of the bilinear form $\mathsf{a} \colon V \times V \to \mathbb{R}$ and the linear form $\mathsf{f} \colon V \to \mathbb{R}$ given, for all $w, v \in V$, by

$$\mathsf{a}(w,v) = \int_D (\nabla w)^\mathsf{T} \nabla v = \langle \nabla w, \nabla v \rangle_{L^2(D)^2} \quad \text{and} \quad \mathsf{f}(v) = \langle f, v\rangle_{V^* \times V} = \int_D f v$$

is to be considered. The variational space is to be endowed with the norm $\|\cdot\|_V \colon V \to \mathbb{R}$ given by

$$\|v\|_V = \sqrt{\mathsf{a}(v,v)} \quad \text{for each} \quad v \in V\,.$$

As in the case of a single dimension, this norm coincides with the standard $H_0^1(D)$ seminorm.

The space V_L of continuous functions that are bilinear on each square element

$$(t_{L,i_1-1}, t_{L,i_1}) \times (t_{L,i_2-1}, t_{L,i_2}) \quad \text{with} \quad i_1, i_2 \in \{1,\ldots,2^L\}$$

of the L-level partition of D is to be employed for discretization with $L \in \mathbb{N}_0$ levels. For each $L \in \mathbb{N}_0$, the tensor products

$$\varphi_{L,j_1,j_2} = \varphi_{L,j_1} \otimes \varphi_{L,j_1} \quad \text{with} \quad j_1, j_2 \in \{1, \ldots, 2^L\}$$

of univariate “hat functions” (for a definition of the latter, see p.74 in *whiteboard.pdf* or the *1D study*) serve as a nodal basis for V_L. For $L, \ell \in \mathbb{N}_0$ such that $\ell \leq L$, they satisfy

$$\varphi_{\ell,j_1,j_2} = \sum_{j_1',j_2'=1}^{2^L} (\boldsymbol{P}_{L,\ell} \otimes \boldsymbol{P}_{L,\ell})_{\overline{i_1,i_2}\,\overline{j_1,j_2}}\, \varphi_{L,i_1,i_2} \quad \text{for all} \quad j_1, j_2 \in \{1, \ldots, 2^\ell\}\,,$$

where $\boldsymbol{P}_{L,\ell} \in \mathbb{R}^{2^L \times 2^\ell}$ is as given in Assignment 6. The discrete problem with $L \in \mathbb{N}_0$ discretization levels takes the form of a linear system with the matrix

$$\boldsymbol{A}_L \otimes \boldsymbol{\Phi}_L + \boldsymbol{\Phi}_L \otimes \boldsymbol{A}_L$$

in the nodal basis, see the *1D study* for a definition of $\boldsymbol{A}_L$ and $\boldsymbol{\Phi}_L$. It can be efficiently preconditioned by the BPX-type preconditioner

$$\boldsymbol{C}_L = \sum_{\ell=0}^{L} 2^{-\ell} (\boldsymbol{P}_{L,\ell} \otimes \boldsymbol{P}_{L,\ell})\,(\boldsymbol{P}_{L,\ell} \otimes \boldsymbol{P}_{L,\ell})^{\mathsf{T}} = \sum_{\ell=0}^{L} 2^{-\ell}\,(\boldsymbol{P}_{L,\ell}\,\boldsymbol{P}_{L,\ell}^{\mathsf{T}}) \otimes (\boldsymbol{P}_{L,\ell}\,\boldsymbol{P}_{L,\ell}^{\mathsf{T}})\,,$$

which is completely analogous in its properties and implementation to the preconditioner used in the *1D study*.

As a model solution, a suitable function u of the form

$$u(x_1, x_2) = u_1(x_1) \cdot u_2(x_2) \cdot w_2(x_2) \quad \text{for all} \quad (x_1, x_2) \in D\,,$$

where u_1 and u_2 are univariate algebraic polynomials of degree exactly two and w_2 is a trigonometric function (all different from the factors used in the *1D study*), should be used.

Consistently with the presentation of iterated low-rank refinement given in the lectures, *all* MPS-TT representations arising in this project should separate the refinement indices with respect to level and not physical dimension (see, for example, pp.49–52 in *whiteboard.pdf*). For example, approximations of functions obtained by L-level iterated refinement should be represented with L factors of mode size 4 (cf. 2 in the one-dimensional case) plus two trailing factors introduced in each decomposition for consistency (see how the trailing factors are handled in Assignment 6 and in the *1D study*). Similarly, the matrix $\boldsymbol{P}_{L,\ell} \otimes \boldsymbol{P}_{L,\ell}$ with $L, \ell \in \mathbb{N}_0$ should be represented with ℓ factors of mode size 4×4 and $L - \ell$ factors of mode size 4×1 (cf. 2×2 and 2×1 in the one-dimensional case), plus two trailing factors.

All the implementations and experiments included in the *1D notebook* should be reproduced in this project. The experiments involving full, entrywise representation need to be reproduced with smaller numbers of discretization levels. For each step of the study, the relevant functions from *1D notebook* should be copied (in Julia) or re-implemented (otherwise) and called in the newly implemented functions for the 2D case. No analytical derivation of the decompositions to be implemented is required. The implementation of preconditioning may rely on the explicitly hard-coded factors given in the *1D notebook* (see function`bpx_factors_explicit`). The presentation of any theoretical details related to the weak formulation of the problem (such as those included in the *1D notebook*) should serve to present the report in a self-contained way; giving a complete theoretical analysis of the problem is not a goal of the project.

The *1D study*, Assignment 6 and the whiteboard notes form a sufficient body of reference material. The papers [2, 1] may be used as additional references regarding the concepts involved and the technical details related to the project. (the *1D study* contains more specific references to these papers).

Deliverables

A complete submission should include the following.

1. A typeset report in the form of a single PDF file, at least eight pages long (excluding the cover-page information and the bibliography if there is any), presenting the problem considered, the work performed and the experimental results obtained.

2. The code of the complete implementation and of the experiments.

References

[1] Markus Bachmayr and Vladimir Kazeev. "Stability and preconditioning of elliptic PDEs with low-rank multilevel structure". English. In: *Foundations of Computational Mathematics* 20 (2020), pp. 1175–1236. ISSN: 1615-3383. DOI: `10.1007/s10208-020-09446-z`.

[2] Vladimir A. Kazeev and Boris N. Khoromskij. "Low-rank explicit QTT representation of the Laplace operator and its inverse". In: *SIAM Journal on Matrix Analysis and Applications* 33.3 (2012), pp. 742–758. DOI: `10.1137/100820479`.