Vector triangle inequality

The Cauchy-Schwarz inequality comes directly from what we know about the relationship between the dot product of a vector and its length, which we looked at in the lesson about dot products.

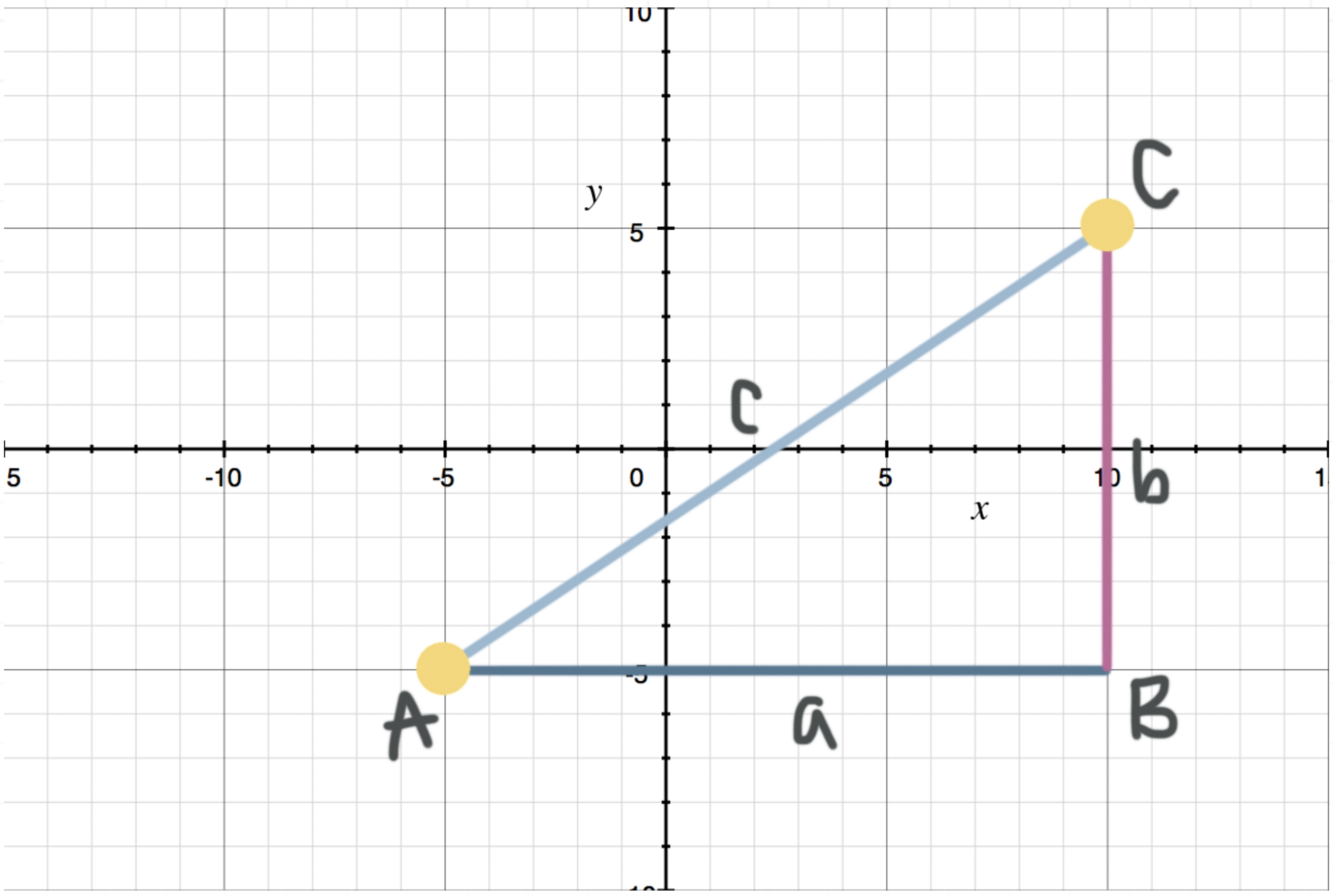
The reason the Cauchy-Schwarz inequality is useful is because, not only can we use it to test for linear independence, but we can build on it to come up with other theorems, including the vector triangle inequality.

The triangle inequality

The vector triangle inequality is identical to the triangle inequality you would have learned in an introductory geometry course, because it says that the sum of the lengths of two sides of a triangle will always be greater than or equal to the length of the third side. When you first learned it in geometry, that inequality looked like this:

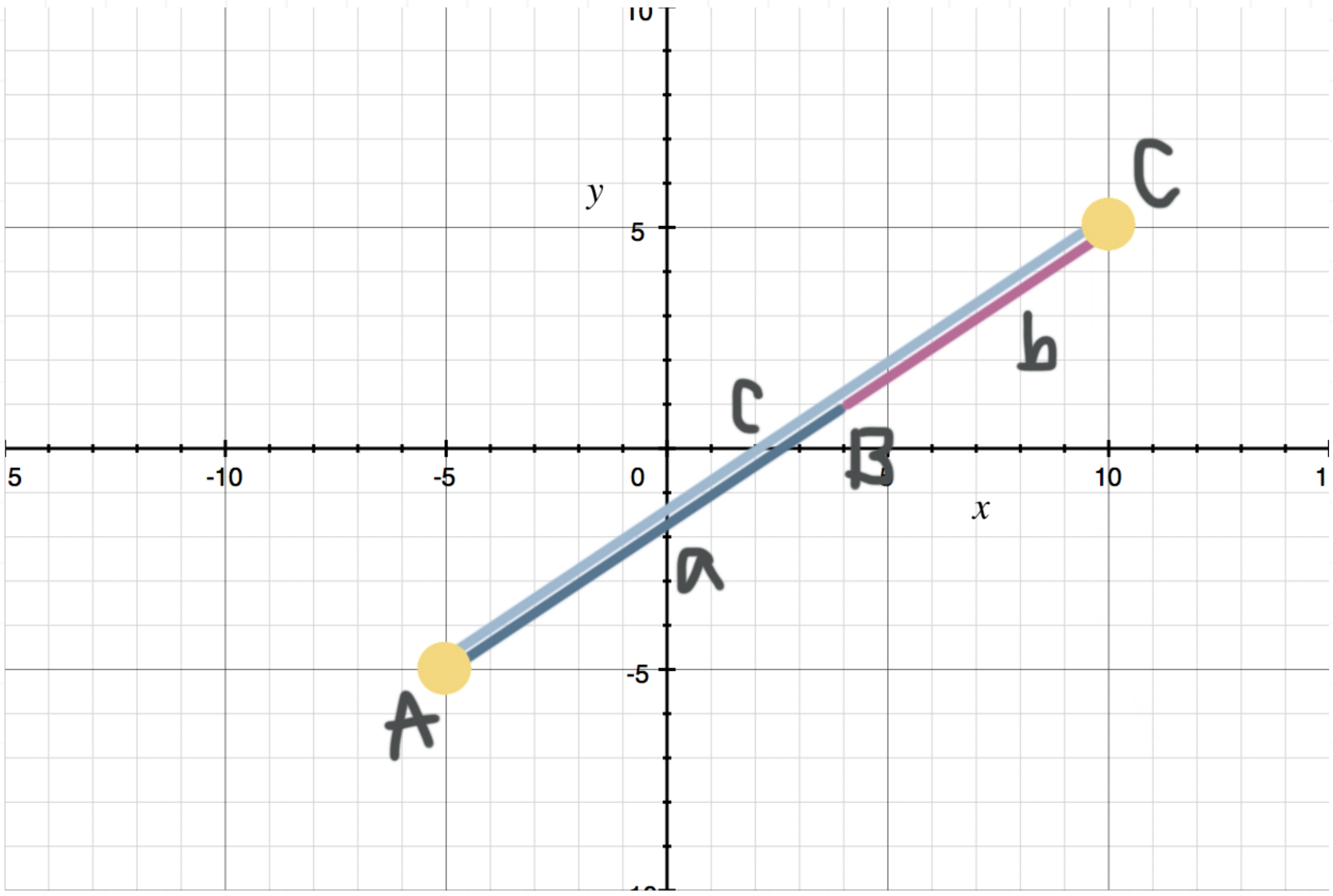
$$c \leq a + b$$

This inequality tells us that the length of c will always be less than or equal to the sum of the lengths of a and b. This makes intuitive sense if we sketch a triangle in the plane.



The length of c is the most direct, efficient route from point A to point C. If we start at A and travel to B first, and then to C, it will take us longer and we'll travel a further distance. So it makes sense that the length of c would naturally be less than the sum of the lengths of a and b.

Only when a and b are collinear is the sum of their lengths equivalent to the length of c.



The vector triangle inequality

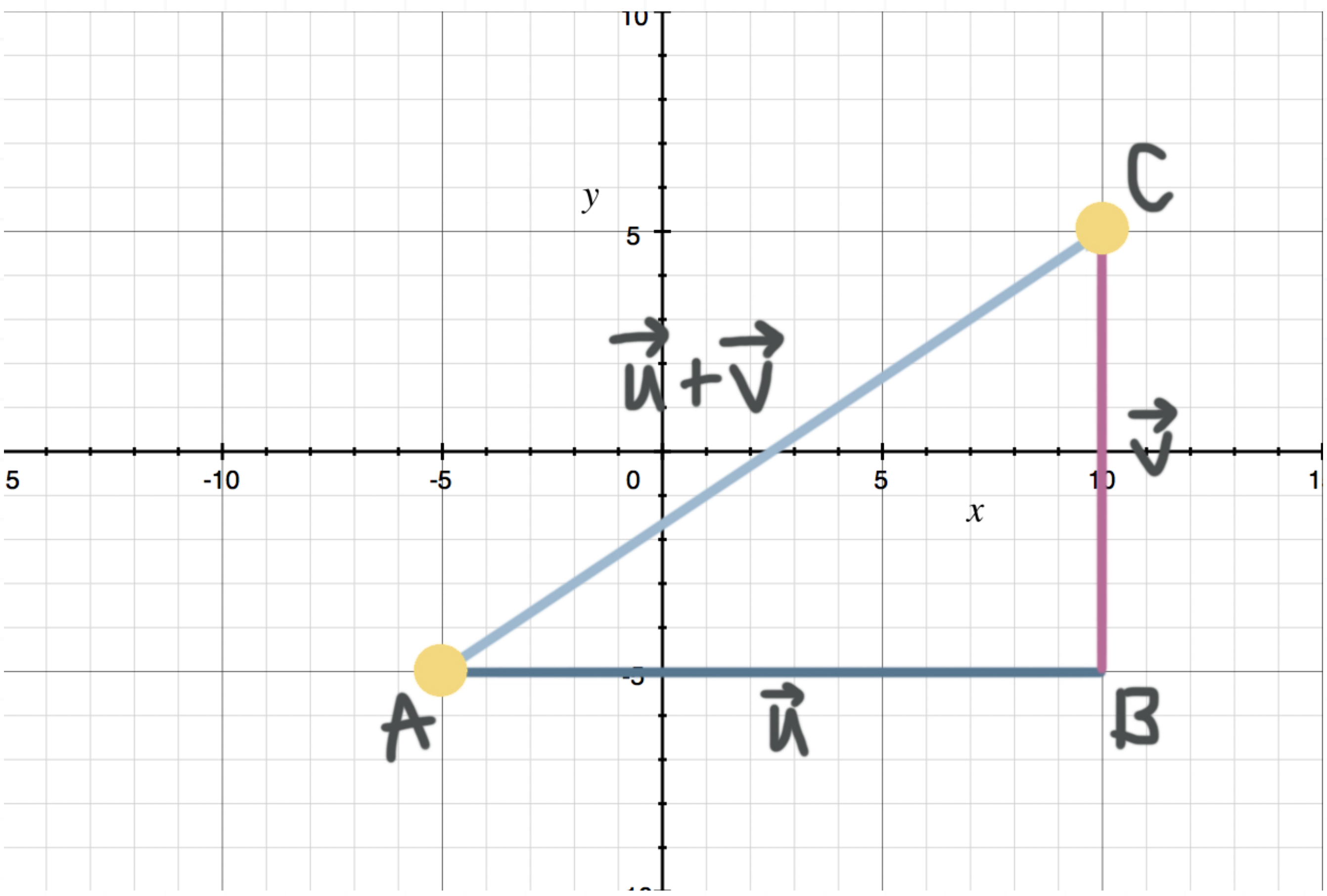
We can translate this triangle inequality so that it's written in terms of vectors, and when we do we call it the **vector triangle inequality**.

As we mentioned before, the reason we're talking about this now is because the vector triangle inequality can be derived directly from the Cauchy-Schwarz inequality.

Just like the basic triangle inequality from geometry, $c \leq a + b$, it tells us that the length of one side of the triangle is always less than or equal to the sum of the lengths of the other two sides.

$$||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$$

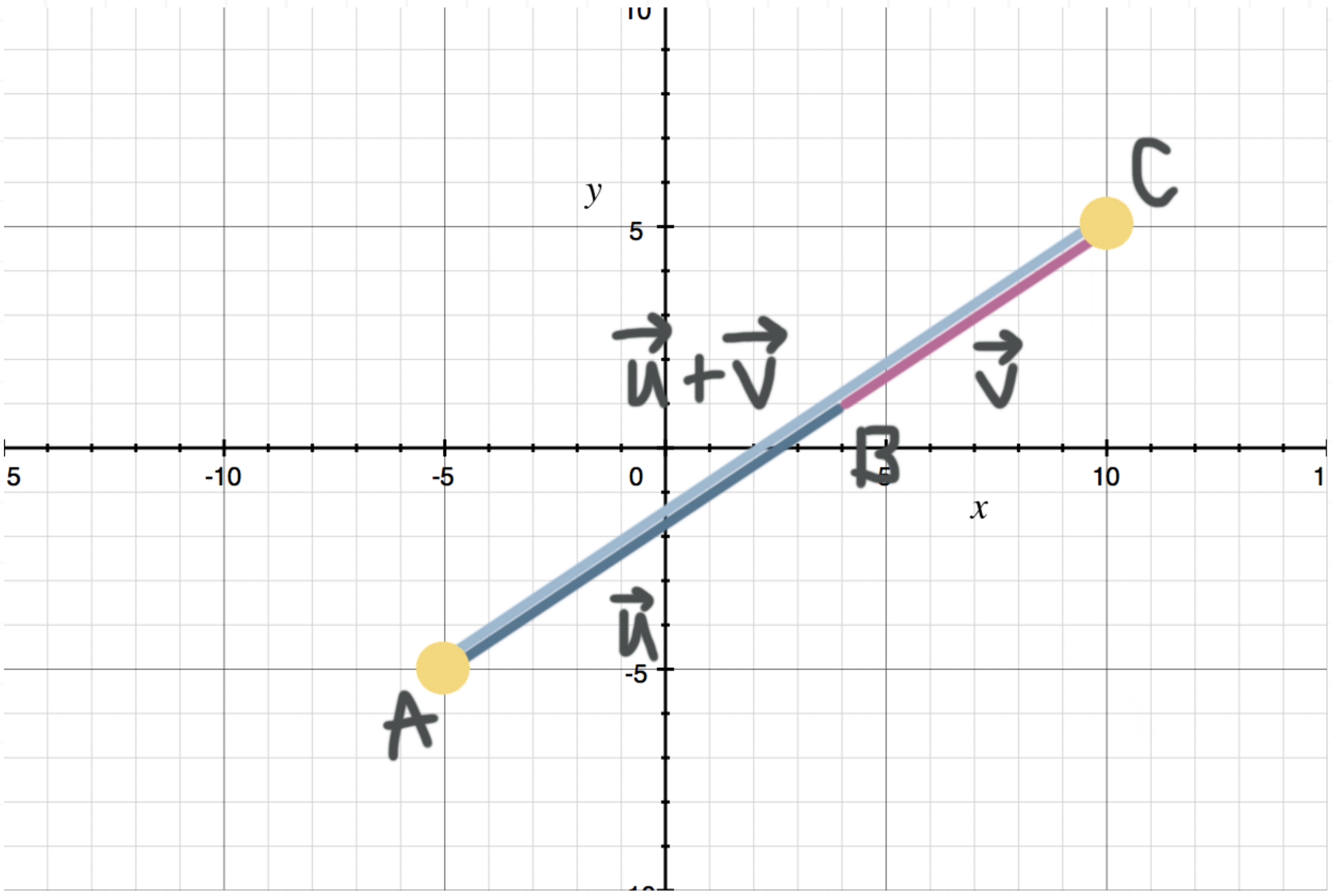
Instead of the triangle $\triangle ABC$ with side lengths a, b, and c, we sketch the triangle whose sides are the vectors $\vec{u}$, $\vec{v}$, and the sum of the vectors $\vec{u}+\vec{v}$.



Just like before, we can see that it's always a shorter, more efficient path to travel along $\vec{u}+\vec{v}$, than it is to travel along $\vec{u}$, and then along $\vec{v}$. Which is why the length of $\vec{u}+\vec{v}$, which we write as $||\vec{u}+\vec{v}||$, will be less than the sum of the lengths of $\vec{u}$ and $\vec{v}$, which we write as $||\vec{u}||+||\vec{v}||$.

And this is true for right triangles (like the ones we're sketching), but also for acute and obtuse triangles. It will always be a shorter path along the single longest side, than the path along the two shorter sides together.

Only when vector $\vec{u}$ and vector $\vec{v}$ are collinear is the sum of their lengths equivalent to the length of the vector $\vec{u}+\vec{v}$.



The triangle inequality for n dimensions

At this point, you might be wondering why we've bothered to redefine the triangle inequality in terms of vectors. After all, $c \le a + b$ looks much simpler than $||\vec{u} + \vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$, and we haven't added any new information to the inequality.

One of the reasons for doing this is because the triangle inequality is only useful in two dimensions. We can't use $c \le a + b$ in three dimensions or n dimensions.

But the vectors $\vec{u}$ and $\vec{v}$ can be defined in as many dimensions as we choose. Which means that translating the triangle inequality into vector form opens up its usefulness from only two dimensions, to n dimensions, and we can now use the inequality to make conclusions about relationships between vector length in space.

Testing for linear independence

But another reason to know the vector triangle inequality is that, just like the Cauchy-Schwarz inequality, it can be used to test for linear independence.

If two vectors $\vec{u}$ and $\vec{v}$ are collinear and point in the same direction, then the Cauchy-Schwarz inequality will give $||\vec{u}+\vec{v}||=||\vec{u}||+||\vec{v}||$. If the two vectors $\vec{u}$ and $\vec{v}$ are collinear, but they have the same length and point in exactly opposite directions, then $||\vec{u}+\vec{v}||$ will be 0.

Since we know that collinear vectors are linearly dependent, we can use the vector triangle inequality to test the linear (in)dependence of any pair of n-dimensional vectors.

Let's work through an example.

Example

Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ span $\mathbb{R}^2$.

$$\vec{u}=(2,-1),\ \vec{v}=(-1,4)$$

First, let's work on finding the value of the right side of the vector triangle inequality.

$$||\vec{u}||+||\vec{v}||$$

$$\sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{2^2+(-1)^2}+\sqrt{(-1)^2+4^2}$$

$$\sqrt{4+1}+\sqrt{1+16}$$

$$\sqrt{5}+\sqrt{17}$$

$$\approx 2.24+4.12$$

$$\approx 6.36$$

Second, let's work on finding the value of the left side of the vector triangle inequality.

$$||\vec{u}+\vec{v}||$$

$$\sqrt{(2-1)^2+(-1+4)^2}$$

$$\sqrt{1^2+3^2}$$

$$\sqrt{1+9}$$

$$\sqrt{10}$$

$$\approx 3.16$$

Since

$$3.16 < 6.36$$

$$||\vec{u}+\vec{v}|| < ||\vec{u}|| + ||\vec{v}||$$

we can conclude that $\vec{u}$ and $\vec{v}$ are not collinear, and that they therefore span $\mathbb{R}^2$.