Topic: Gram-Schmidt process for change of basis

Question: The subspace V is a plane in $\mathbb{R}^3$. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-2\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-3\\1\end{bmatrix}\right)$$

Answer choices:

A $\quad V_2 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\\frac{2}{3}\\\frac{1}{3}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{11}}\\-\frac{3}{\sqrt{11}}\\\frac{1}{\sqrt{11}}\end{bmatrix}\right)$

B $\quad V_2 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\\frac{2}{3}\\\frac{1}{3}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{14}}\\-\frac{3}{\sqrt{14}}\\\frac{1}{\sqrt{14}}\end{bmatrix}\right)$

C $\quad V_2 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\\frac{2}{3}\\\frac{1}{3}\end{bmatrix}, \begin{bmatrix}-\frac{5}{3\sqrt{10}}\\-\frac{7}{3\sqrt{10}}\\\frac{4}{3\sqrt{10}}\end{bmatrix}\right)$

D $\quad V_2 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\\frac{2}{3}\\\frac{1}{3}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\\\frac{1}{\sqrt{10}}\end{bmatrix}\right)$

Solution: C

Define $\vec{v}_1 = (-2,2,1)$ and $\vec{v}_2 = (-1,-3,1)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$ and $\vec{v}_2$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2)$$

Now all we need to do is replace $\vec{v}_2$ with a vector that's both orthogonal to $\vec{u}_1$, and normal. If we can do that, then the vector set that spans V will be orthonormal. We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right)\frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \frac{1}{9}\left(\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right)\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \frac{1}{9}((-1)(-2) + (-3)(2) + (1)(1))\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \frac{1}{9}(2 - 6 + 1)\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \frac{1}{9}(-3)\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 - \frac{2}{3} \\ -3 + \frac{2}{3} \\ 1 + \frac{1}{3} \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -\frac{5}{3} \\ -\frac{7}{3} \\ \frac{4}{3} \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it.

$$||\vec{w}_2|| = \sqrt{\left(-\frac{5}{3}\right)^2 + \left(-\frac{7}{3}\right)^2 + \left(\frac{4}{3}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{\frac{25}{9} + \frac{49}{9} + \frac{16}{9}}$$

$$||\vec{w}_2|| = \sqrt{\frac{90}{9}}$$

$$||\vec{w}_2|| = \sqrt{10}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -\frac{5}{3} \\ -\frac{7}{3} \\ \frac{4}{3} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$ and $\vec{u}_2$ form an orthonormal basis for V.

$$V_2 = \text{Span}\left(\frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{10}} \begin{bmatrix} -\frac{5}{3} \\ -\frac{7}{3} \\ -\frac{4}{3} \end{bmatrix}\right)$$

$$V_2 = \text{Span}\left(\begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{5}{3\sqrt{10}} \\ -\frac{7}{3\sqrt{10}} \\ \frac{4}{3\sqrt{10}} \end{bmatrix}\right)$$

Topic: Gram-Schmidt process for change of basis

Question: The subspace V is a space in $\mathbb{R}^3$. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}-1\\-2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right)$$

Answer choices:

A $\quad V_3 = \text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{5}}\\0\\-\frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}, \begin{bmatrix}\frac{2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{bmatrix}\right)$

B $\quad V_3 = \text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{5}}\\0\\-\frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}-\frac{1}{2}\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\-\frac{\sqrt{5}}{3}\\\sqrt{5}\end{bmatrix}\right)$

C $\quad V_3 = \text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{5}}\\0\\-\frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}-\frac{1}{2}\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\-\frac{3}{\sqrt{5}}\\\frac{9}{\sqrt{5}}\end{bmatrix}\right)$

D $\qquad V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

Solution: A

Define $\vec{v}_1 = (1,0,-2)$, $\vec{v}_2 = (-1,-2,2)$, and $\vec{v}_3 = (0,-1,3)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{1+0+4} = \sqrt{5}$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \frac{1}{5} \left(\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \frac{1}{5}((-1)(1) + (-2)(0) + (2)(-2)) \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \frac{1}{5}(-1 + 0 - 4) \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \frac{1}{5}(-5) \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{0^2 + (-2)^2 + 0^2}$$

$$||\vec{w}_2|| = \sqrt{0 + 4 + 0}$$

$$||\vec{w}_2|| = \sqrt{4}$$

$$||\vec{w}_2|| = 2$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u_1} - (\vec{v}_3 \cdot \vec{u}_2)\vec{u_2}$$

Plug in the values we already have.

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}\right)\frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{5}\left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{4}\left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{5}(0(1) - 1(0) + 3(-2))\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{4}(0(0) - 1(-2) + 3(0))\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{5}(0 + 0 - 6)\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{4}(0 + 2 + 0)\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + \frac{6}{5}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{6}{5} \\ 0 \\ -\frac{12}{5} \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 + \frac{6}{5} - 0 \\ -1 + 0 + 1 \\ 3 - \frac{12}{5} - 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} \frac{6}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$$

So $\vec{w}_3$ is orthogonal to $\vec{u}_2$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{6}{5}\right)^2 + 0^2 + \left(\frac{3}{5}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{36}{25} + 0 + \frac{9}{25}}$$

$$||\vec{w}_3|| = \sqrt{\frac{45}{25}}$$

$$||\vec{w}_3||=\sqrt{\frac{9}{5}}$$

$$||\vec{w}_3||=\frac{3}{\sqrt{5}}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3=\frac{1}{\frac{3}{\sqrt{5}}}\begin{bmatrix}\frac{6}{5}\\0\\\frac{3}{5}\end{bmatrix}$$

$$\vec{u}_3=\frac{\sqrt{5}}{3}\begin{bmatrix}\frac{6}{5}\\0\\\frac{3}{5}\end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3=\text{Span}\left(\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix},\frac{1}{2}\begin{bmatrix}0\\-2\\0\end{bmatrix},\frac{\sqrt{5}}{3}\begin{bmatrix}\frac{6}{5}\\0\\\frac{3}{5}\end{bmatrix}\right)$$

$$V_3=\text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{5}}\\0\\-\frac{2}{\sqrt{5}}\end{bmatrix},\begin{bmatrix}0\\-1\\0\end{bmatrix},\begin{bmatrix}\frac{2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{bmatrix}\right)$$

Topic: Gram-Schmidt process for change of basis

Question: The subspace V is a space in $\mathbb{R}^4$. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\1\\3\end{bmatrix}, \begin{bmatrix}2\\0\\-2\\0\end{bmatrix}\right)$$

Answer choices:

A $\quad V_3 = \text{Span}\left(\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}, \begin{bmatrix}0\\\frac{2}{\sqrt{10}}\\\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\\frac{1}{2}\end{bmatrix}\right)$

B $\quad V_3 = \text{Span}\left(\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}\right)$

C $\qquad V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \end{bmatrix}\right)$

D $\qquad V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{2}{5}} \\ -\sqrt{\frac{2}{5}} \\ \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\right)$

Solution: D

Define $\vec{v}_1 = (1,1,-1,1)$, $\vec{v}_2 = (0,2,1,3)$, and $\vec{v}_3 = (2,0,-2,0)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{1^2 + 1^2 + (-1)^2 + 1^2} = \sqrt{1+1+1+1} = \sqrt{4} = 2$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right)\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{4}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{4}(0(1) + 2(1) + 1(-1) + 3(1))\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{4}(0 + 2 - 1 + 3) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{(-1)^2 + 1^2 + 2^2 + 2^2}$$

$$||\vec{w}_2|| = \sqrt{1 + 1 + 4 + 4}$$

$$||\vec{w}_2|| = \sqrt{10}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have.

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right) \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{4} \left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{10} \left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right) \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{4}(2(1) + 0(1) - 2(-1) + 0(1)) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{10}(2(-1) + 0(1) - 2(2) + 0(2)) \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{4}(2 + 0 + 2 + 0) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{10}(-2 + 0 - 4 + 0) \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -\frac{3}{5} \\ \frac{3}{5} \\ \frac{6}{5} \\ \frac{6}{5} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}$$

The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{2}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{4}{25} + \frac{4}{25} + \frac{1}{25} + \frac{1}{25}}$$

$$||\vec{w}_3|| = \sqrt{\frac{10}{25}}$$

$$||\vec{w}_3|| = \sqrt{\frac{2}{5}}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{\sqrt{\frac{2}{5}}}\begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}$$

$$\vec{u}_3 = \sqrt{\frac{5}{2}}\begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{2}{5}} \\ -\sqrt{\frac{2}{5}} \\ \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\right)$$