The elimination matrix

We've seen how a collection of row operations can change a matrix into reduced row-echelon form, and how, if rref gives us the identity matrix, then we have one unique solution to the linear system represented by the matrix.

What we want to understand now is that each of those individual row operations can be represented by a matrix. Furthermore, the entire collection of all the row operations we perform can be brought together in one matrix, and we call that matrix the **elimination matrix**, E.

One row operation as a matrix

Let's use a matrix from the Gauss-Jordan elimination lesson, and call it A.

$$A = \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix}$$

To put this matrix into rref, our first step would be $-R_1 \to R_1$. But here's how we could accomplish this same row operation using a matrix that we multiply by A:

$$E_1A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix}$$

Here's what E_1 is telling us:

The second row of E_1, $[0 \quad 1 \quad 0]$, tells us "To get the new second row of A, give me 0 of the first row, plus 1 of the second row, plus 0 of the third row." And that makes sense, since we're not changing the second row of A.

The third row of E_1, $[0 \quad 0 \quad 1]$, tells us "To get the new third row of A, give me 0 of the first row, plus 0 of the second row, plus 1 of the third row." And that makes sense, since we're not changing the third row of A.

But the first row of E_1, $[-1 \quad 0 \quad 0]$, tells us "To get the new first row of A, give me the first row multiplied by a scalar of -1, plus 0 of the second row, plus 0 of the third row."

So E_1 accomplishes the row operation $-R_1 \to R_1$, without changing anything else about the rest of the matrix. And now the resulting matrix after this row operation is the matrix product,

$$E_1A = \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix} \right)$$

Once $-R_1 \to R_1$ is done, the next step toward reduced row-echelon form would be $5R_1 + R_2 \to R_2$. The only row we want to change is R_2, which means R_1 and R_3 will stay the same. To change the second row, we want a combination (the sum) of 5 of the first row, along with 1 of the second row, but 0 of the third row. So

$$E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix} \right)$$

The elimination matrix

Now we've applied the first two row operations to A. And we could simplify these two row operations into one elimination matrix, simply by multiplying E_2 by E_1.

$$E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2E_1 = \begin{bmatrix} 1(-1)+0(0)+0(0) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 5(-1)+1(0)+0(0) & 5(0)+1(1)+0(0) & 5(0)+1(0)+0(1) \\ 0(-1)+0(0)+1(0) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_2E_1 = \begin{bmatrix} -1+0+0 & 0+0+0 & 0+0+0 \\ -5+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$E_2E_1 = \begin{bmatrix} -1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The point here is that we could continue applying more and more row operations to matrix A, and all of them can be combined into one **elimination matrix** called E.

To get A into reduced row-echelon form, we'll actually need the following set of row operations:

1. $-R_1 \to R_1$
4. $R_3 - 2R_1 \to R_3$
7. $5R_2 + R_3 \to R_3$

2. $(1/5)R_2 \to R_2$ 5. $(1/4)R_2 \to R_2$ 8. $-R_3 \to R_3$

3. $R_1 + R_2 \to R_2$ 6. $-5R_2 + R_1 \to R_1$ 9. $R_1 + R_3 \to R_1$

There's a simple way to translate these kinds of row operations into the elimination matrix. If the operation is the multiplication of a row by a scalar, like these,

$-R_1 \to R_1$ $(1/4)R_2 \to R_2$

$(1/5)R_2 \to R_2$ $-R_3 \to R_3$

then we can put the scalar into the position given by the two subscripts that we see in the operation. For instance, for the row operation $-R_1 \to R_1$, we'd put the scalar -1 into $E_{1,1}$. Or for the row operation $(1/5)R_2 \to R_2$, we'd put the scalar $1/5$ into $E_{2,2}$.

For all other row operations, where we replace a row with the sum or difference of rows, like these,

$R_1 + R_2 \to R_2$ $-5R_2 + R_1 \to R_1$ $R_1 + R_3 \to R_1$

$R_3 - 2R_1 \to R_3$ $5R_2 + R_3 \to R_3$

then we put the coefficients from the left side into their corresponding columns, inside the row from the right side. For instance, for the row operation $R_1 + R_2 \to R_2$, we'd put the coefficients 1 and 1 into the first and second columns inside R_2. So we'd put a 1 in $E_{2,1}$ and a 1 in $E_{2,2}$. Or for the row operation $R_3 - 2R_1 \to R_3$, we'd put the coefficients 1 and -2 into the third and first columns inside R_3. So we'd put a 1 in $E_{3,3}$ and a -2 in $E_{3,1}$.

Let's restart the example we've been working with so that we can get more practice with this.

Example

Find the single elimination matrix E that puts A into reduced row-echelon form,

$$A = \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix}$$

where E accounts for the following set of row operations:

1. $-R_1 \to R_1$
2. $(1/5)R_2 \to R_2$
3. $R_1 + R_2 \to R_2$
4. $R_3 - 2R_1 \to R_3$
5. $(1/4)R_2 \to R_2$
6. $-5R_2 + R_1 \to R_1$
7. $5R_2 + R_3 \to R_3$
8. $-R_3 \to R_3$
9. $R_1 + R_3 \to R_1$

The row operation $-R_1 \to R_1$ means we're leaving the second and third rows alone, but multiplying the first row by a scalar of -1, so we'll put a -1 in $E_{1,1}$.

$$E_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $(1/5)R_2 \to R_2$ means we're leaving the first and third rows of the result alone, but multiplying the second row of the result by a scalar of $1/5$, so we'll put a $1/5$ in $E_{2,2}$.

$$E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $R_1 + R_2 \to R_2$ means we're leaving the first and third rows of the result alone, but replacing the second row of the result with 1 of the first row plus 1 of the second row, so we'll put a 1 in $E_{2,1}$ and a 1 in $E_{2,2}$.

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's consolidate what we have so far for $E_3E_2E_1$.

$$E_{1-3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(-1)+0(0)+0(0) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 0(-1)+\frac{1}{5}(0)+0(0) & 0(0)+\frac{1}{5}(1)+0(0) & 0(0)+\frac{1}{5}(0)+0(1) \\ 0(-1)+0(0)+1(0) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_{1-3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+\frac{1}{5}+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$E_{1-3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{1-3} = \begin{bmatrix} 1(-1)+0(0)+0(0) & 1(0)+0(1/5)+0(0) & 1(0)+0(0)+0(1) \\ 1(-1)+1(0)+0(0) & 1(0)+1(1/5)+0(0) & 1(0)+1(0)+0(1) \\ 0(-1)+0(0)+1(0) & 0(0)+0(1/5)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_{1-3} = \begin{bmatrix} -1+0+0 & 0+0+0 & 0+0+0 \\ -1+0+0 & 0+1/5+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$E_{1-3} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's do the next set of row operations. The row operation $R_3 - 2R_1 \to R_3$ means we're leaving the first and second rows of the result alone, but replacing the third row of the result with 1 of the third row and -2 of the first row, so we'll put a 1 in $E_{3,3}$ and a -2 in $E_{3,1}$.

$$E_4E_{1-3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $(1/4)R_2 \to R_2$ means we're leaving the first and third rows of the result alone, but replacing the second row of the result with $1/4$ of the second row, so we'll put a $1/4$ in $E_{2,2}$.

$$E_5E_4E_{1-3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $-5R_2 + R_1 \to R_1$ means we're leaving the second and third rows of the result alone, but replacing the first row of the result with

1 of the first row and -5 of the second row, so we'll put a -5 in $E_{1,2}$ and a 1 in $E_{1,1}$.

$$E_6E_5E_4E_{1-3} = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's consolidate what we have so far for $E_6E_5E_4E_{1-3}$.

$$E_{1-6} = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(-1)+0(-1)+0(0) & 1(0)+0(1/5)+0(0) & 1(0)+0(0)+0(1) \\ 0(-1)+1(-1)+0(0) & 0(0)+1(1/5)+0(0) & 0(0)+1(0)+0(1) \\ -2(-1)+0(-1)+1(0) & -2(0)+0(1/5)+1(0) & -2(0)+0(0)+1(1) \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1+0+0 & 0+0+0 & 0+0+0 \\ 0-1+0 & 0+1/5+0 & 0+0+0 \\ 2+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1/5 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(-1)+0(-1)+0(2) & 1(0)+0(1/5)+0(0) & 1(0)+0(0)+0(1) \\ 0(-1)+\frac{1}{4}(-1)+0(2) & 0(0)+\frac{1}{4}(1/5)+0(0) & 0(0)+\frac{1}{4}(0)+0(1) \\ 0(-1)+0(-1)+1(2) & 0(0)+0(1/5)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1+0+0 & 0+0+0 & 0+0+0 \\ 0-1/4+0 & 0+1/20+0 & 0+0+0 \\ 0+0+2 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -1/4 & 1/20 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} 1(-1)-5(-1/4)+0(2) & 1(0)-5(1/20)+0(0) & 1(0)-5(0)+0(1) \\ 0(-1)+1(-1/4)+0(2) & 0(0)+1(1/20)+0(0) & 0(0)+1(0)+0(1) \\ 0(-1)+0(-1/4)+1(2) & 0(0)+0(1/20)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} -1+5/4+0 & 0-1/4+0 & 0-0+0 \\ 0-1/4+0 & 0+1/20+0 & 0+0+0 \\ 0+0+2 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} 1/4 & -1/4 & 0 \\ -1/4 & 1/20 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Let's do the last set of row operations. The row operation $5R_2 + R_3 \to R_3$ means we're leaving the first and second rows of the result alone, but replacing the third row of the result with 5 of the second row and 1 of the third row, so we'll put a 5 in $E_{3,2}$ and a 1 in $E_{3,3}$.

$$E_7E_{1-6} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & -1/4 & 0 \\ -1/4 & 1/20 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

The row operation $-R_3 \to R_3$ means we're leaving the first and second rows of the result alone, but replacing the third row of the result with -1 of the third row, so we'll put a -1 in $E_{3,3}$.

$$E_8E_7E_{1-6} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & -1/4 & 0 \\ -1/4 & 1/20 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

The row operation $R_1 + R_3 \to R_1$ means we're leaving the second and third rows of the result alone, but replacing the first row of the result with 1 of the first row and 1 of the third row, so we'll put a 1 in $E_{1,1}$ and a 1 in $E_{1,3}$.

$$E_9E_8E_7E_{1-6} = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}\begin{bmatrix}1/4&-1/4&0\\-1/4&1/20&0\\2&0&1\end{bmatrix}$$

Let's consolidate the rest of the elimination matrices.

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1(1/4)+0(-1/4)+0(2)&1(-1/4)+0(1/20)+0(0)&1(0)+0(0)+0(1)\\0(1/4)+1(-1/4)+0(2)&0(-1/4)+1(1/20)+0(0)&0(0)+1(0)+0(1)\\0(1/4)+5(-1/4)+1(2)&0(-1/4)+5(1/20)+1(0)&0(0)+5(0)+1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1/4+0+0&-1/4+0+0&0+0+0\\0-1/4+0&0+1/20+0&0+0+0\\0-5/4+2&0+1/4+0&0+0+1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1/4&-1/4&0\\-1/4&1/20&0\\3/4&1/4&1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1(1/4)+0(-1/4)+0(3/4)&1(-1/4)+0(1/20)+0(1/4)&1(0)+0(0)+0(1)\\0(1/4)+1(-1/4)+0(3/4)&0(-1/4)+1(1/20)+0(1/4)&0(0)+1(0)+0(1)\\0(1/4)+0(-1/4)-1(3/4)&0(-1/4)+0(1/20)-1(1/4)&0(0)+0(0)-1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/4+0+0&-1/4+0+0&0+0+0\\0-1/4+0&0+1/20+0&0+0+0\\0+0-3/4&0+0-1/4&0+0-1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/4&-1/4&0\\-1/4&1/20&0\\-3/4&-1/4&-1\end{bmatrix}$$

$$E=\begin{bmatrix}1(1/4)+0(-1/4)+1(-3/4) & 1(-1/4)+0(1/20)+1(-1/4) & 1(0)+0(0)+1(-1)\\ 0(1/4)+1(-1/4)+0(-3/4) & 0(-1/4)+1(1/20)+0(-1/4) & 0(0)+1(0)+0(-1)\\ 0(1/4)+0(-1/4)+1(-3/4) & 0(-1/4)+0(1/20)+1(-1/4) & 0(0)+0(0)+1(-1)\end{bmatrix}$$

$$E=\begin{bmatrix}1/4+0-3/4 & -1/4+0-1/4 & 0+0-1\\ 0-1/4+0 & 0+1/20+0 & 0+0+0\\ 0+0-3/4 & 0+0-1/4 & 0+0-1\end{bmatrix}$$

$$E=\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & -1\\ -\frac{1}{4} & \frac{1}{20} & 0\\ -\frac{3}{4} & -\frac{1}{4} & -1\end{bmatrix}$$

We've found the elimination matrix, and we can check to make sure that it reduces A to the identity matrix.

$$EA=\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & -1\\ -\frac{1}{4} & \frac{1}{20} & 0\\ -\frac{3}{4} & -\frac{1}{4} & -1\end{bmatrix}\begin{bmatrix}-1 & -5 & 1\\ -5 & -5 & 5\\ 2 & 5 & -3\end{bmatrix}$$

$$EA=\begin{bmatrix}-\frac{1}{2}(-1)-\frac{1}{2}(-5)-1(2) & -\frac{1}{2}(-5)-\frac{1}{2}(-5)-1(5) & -\frac{1}{2}(1)-\frac{1}{2}(5)-1(-3)\\ -\frac{1}{4}(-1)+\frac{1}{20}(-5)+0(2) & -\frac{1}{4}(-5)+\frac{1}{20}(-5)+0(5) & -\frac{1}{4}(1)+\frac{1}{20}(5)+0(-3)\\ -\frac{3}{4}(-1)-\frac{1}{4}(-5)-1(2) & -\frac{3}{4}(-5)-\frac{1}{4}(-5)-1(5) & -\frac{3}{4}(1)-\frac{1}{4}(5)-1(-3)\end{bmatrix}$$

$$EA=\begin{bmatrix}\frac{1}{2}+\frac{5}{2}-2 & \frac{5}{2}+\frac{5}{2}-5 & -\frac{1}{2}-\frac{5}{2}+3\\ \frac{1}{4}-\frac{1}{4}+0 & \frac{5}{4}-\frac{1}{4}+0 & -\frac{1}{4}+\frac{1}{4}+0\\ \frac{3}{4}+\frac{5}{4}-2 & \frac{15}{4}+\frac{5}{4}-5 & -\frac{3}{4}-\frac{5}{4}+3\end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because multiplying the elimination matrix by A gives us the identity matrix, we know that we got the correct elimination matrix.

We won't talk about this until later in the course, but this elimination matrix we've found is called the inverse of A, which we write as A^{-1}. So

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -1 \\ -\frac{1}{4} & \frac{1}{20} & 0 \\ -\frac{3}{4} & -\frac{1}{4} & -1 \end{bmatrix}$$

As you may suspect, when you multiply an inverse of a matrix by the matrix itself, the result will always be the identity matrix.

$$A^{-1}A = I$$