Cross products

The **cross product** is a vector that's orthogonal to the two vectors you crossed. So given $\overrightarrow{a}=(a_1,a_2,a_3)$ and $\overrightarrow{b}=(b_1,b_2,b_3)$, their cross product $\overrightarrow{a}\times\overrightarrow{b}$ will be orthogonal to both $\overrightarrow{a}$ and $\overrightarrow{b}$. The formula for the cross product is

$$\overrightarrow{a}\times\overrightarrow{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$\overrightarrow{a}\times\overrightarrow{b}=\mathbf{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

$$\overrightarrow{a}\times\overrightarrow{b}=\mathbf{i}(a_2b_3-a_3b_2)-\mathbf{j}(a_1b_3-a_3b_1)+\mathbf{k}(a_1b_2-a_2b_1)$$

Notice how the value of the cross product looks like it's given as a matrix, but the matrix is bounded by straight lines instead of bracketed lines. That's because it's not actually a matrix, it's a determinant.

We'll devote an entire lesson section to determinants later on, but for now, we just want to know how to calculate a determinant.

Calculating a determinant

Notice first how the values in the top row of the 3×3 determinant become coefficients on the 2×2 determinants.

Second, notice how $\mathbf{i}$ is positive, $\mathbf{j}$ is negative, and $\mathbf{k}$ is positive. That's because the signs of the coefficients follow a checkerboard pattern. If you're pulling the value in the upper left-hand corner of the determinant, the sign will be positive, but then the signs alternate from there. So $\mathbf{i}$ is positive, $\mathbf{j}$ is negative, and $\mathbf{k}$ is positive because the checkerboard of signs is

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Third, the 2×2 determinants we attach to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are given by the values outside the row and column of the coefficient.

In other words, $\mathbf{i}$ is in the first row and first column, so eliminate the first row and column, and the 2×2 determinant with $\mathbf{i}$ is just everything left over (everything outside the first row and first column).

$$\begin{vmatrix} \mathbf{i} & \cdot & \cdot \\ \cdot & a_2 & a_3 \\ \cdot & b_2 & b_3 \end{vmatrix}$$

And $\mathbf{j}$ is in the first row and second column, so eliminate the first row and second column, and the 2×2 determinant with $\mathbf{j}$ is just everything left over (everything outside the first row and second column).

$$\begin{vmatrix} \cdot & \mathbf{j} & \cdot \\ a_1 & \cdot & a_3 \\ b_1 & \cdot & b_3 \end{vmatrix}$$

Also, $\mathbf{k}$ is in the first row and third column, so eliminate the first row and third column, and the 2×2 determinant with $\mathbf{k}$ is just everything left over (everything outside the first row and third column).

$$\begin{vmatrix} \cdot & \cdot & \mathbf{k} \\ a_1 & a_2 & \cdot \\ b_1 & b_2 & \cdot \end{vmatrix}$$

Let's do an example where we find the cross product of two three-dimensional vectors.

Example

Find the cross product of $\overrightarrow{a} = (1,2,3)$ and $\overrightarrow{b} = (-1,0,3)$.

The cross product would be

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 0 & 3 \end{vmatrix}$$

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i} \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}$$

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i}((2)(3) - (3)(0)) - \mathbf{j}((1)(3) - (3)(-1)) + \mathbf{k}((1)(0) - (2)(-1))$$

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i}(6 - 0) - \mathbf{j}(3 + 3) + \mathbf{k}(0 + 2)$$

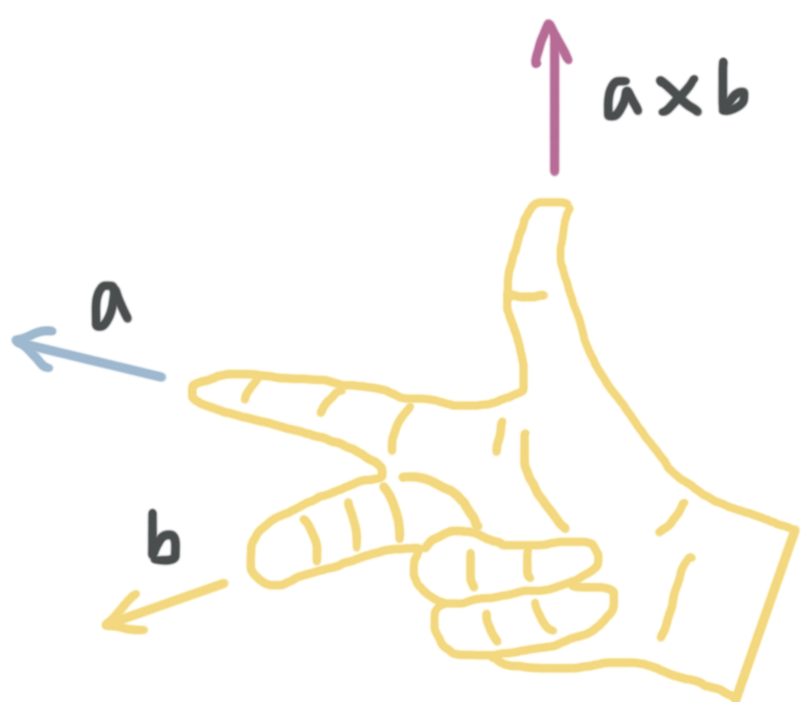
$$\overrightarrow{a} \times \overrightarrow{b} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$$

So the vector $\vec{a}\times\vec{b}=(6,-6,2)$ is orthogonal to both $\vec{a}=(1,2,3)$ and $\vec{b}=(-1,0,3)$.

The right-hand rule

Remember that an orthogonal vector can point in multiple directions. For instance, given a vector in two-dimensional space that points toward the positive direction of the x-axis, an orthogonal vector could point toward the positive direction of the y-axis, or toward the negative direction of the y-axis.

In general, the direction of the cross product vector is determined by the **right-hand rule**. If you find the cross product $\vec{a}\times\vec{b}$, it'll point in the direction that your right-hand thumb points, when your right-hand index finger points toward $\vec{a}$ and your right-hand middle finger points toward $\vec{b}$.



Since there are multiple vectors orthogonal to $\vec{a}$ and $\vec{b}$, the right-hand rule is important for determining the correct cross product.

Length of the cross product vector

Earlier we looked at a formula relating the dot product of two vectors to the product of their lengths:

$$\vec{u}\cdot\vec{v}=||\vec{u}||\ ||\vec{v}||\cos\theta$$

There's a similar formula for the cross product, which says that the length of the cross product vector is equivalent to the product of the two individual vectors and the sine of the angle between them.

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin\theta$$

The length of the cross product is also equivalent to the area of the parallelogram that's formed by the two vectors that were crossed to form the cross product.

Let's do an example of how to find the length of the cross product.

Example

Find the length of the cross product of $\vec{a}=(1,2,3)$ and $\vec{b}=(-1,0,3)$.

First, let's find the length of each vector individually.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{(-1)^2 + 0^2 + 3^2} = \sqrt{1 + 0 + 9} = \sqrt{10}$$

The angle between the vectors will be

$$\vec{a} \cdot \vec{b} = ||\vec{a}||\ ||\vec{b}|| \cos\theta$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \sqrt{14}\sqrt{10}\cos\theta$$

$$(1)(-1) + (2)(0) + (3)(3) = \sqrt{140}\cos\theta$$

$$-1 + 0 + 9 = 2\sqrt{35}\cos\theta$$

$$\frac{8}{2\sqrt{35}} = \cos\theta$$

$$\cos\theta = \frac{4}{\sqrt{35}}$$

$$\theta = \arccos\left(\frac{4}{\sqrt{35}}\right)$$

$$\theta \approx 47.46^\circ$$

Then the length of the cross product is given by

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||\ ||\vec{b}|| \sin\theta$$

$$||\vec{a} \times \vec{b}|| = \sqrt{14}\sqrt{10}\sin(47.46^\circ)$$

$$||\vec{a}\times\vec{b}||=\sqrt{140}\sin(47.46^\circ)$$

$$||\vec{a}\times\vec{b}||=2\sqrt{35}\sin(47.46^\circ)$$

Using a calculator, we see that the length of the cross product of $\vec{a}=(1,2,3)$ and $\vec{b}=(-1,0,3)$ is

$$||\vec{a}\times\vec{b}||\approx 8.72$$

We could have also tackled this last example by starting with the cross product vector itself, $\vec{a}\times\vec{b}=6\mathbf{i}-6\mathbf{j}+2\mathbf{k}$, which we found in the previous example. If we have the cross product, then all we have to do is plug its components into the formula for the length of a vector.

$$||\vec{a}\times\vec{b}||=\sqrt{6^2+(-6)^2+2^2}$$

$$||\vec{a}\times\vec{b}||=\sqrt{36+36+4}$$

$$||\vec{a}\times\vec{b}||=\sqrt{76}$$

$$||\vec{a}\times\vec{b}||\approx 8.72$$