Topic: Null and column spaces of the transpose

Question: Find the row space and left null space of A, and the dimensions of those spaces.

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 4 & 0 \end{bmatrix}$$

Answer choices:

A $\quad N(A^T) = \text{Span}\left(\begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(N(A^T)) = 1$

$\quad C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ in $\mathbb{R}^2$ $\quad \text{Dim}(C(A^T)) = 2$

B $\quad N(A^T) = \text{Span}\left(\begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix}\right)$ in $\mathbb{R}^2$ $\quad \text{Dim}(N(A^T)) = 2$

$\quad C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(C(A^T)) = 1$

C $\quad N(A^T) = \text{Span}\left(\begin{bmatrix} 4 \\ 12 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(N(A^T)) = 1$

$\quad C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}\right)$ in $\mathbb{R}^2$ $\quad \text{Dim}(C(A^T)) = 2$

D $\quad N(A^T) = \text{Span}\left(\begin{bmatrix} 4 \\ 12 \end{bmatrix}\right)$ in $\mathbb{R}^2$ $\quad \text{Dim}(N(A^T)) = 2$

$\quad C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(C(A^T)) = 1$

Solution: A

The transpose of A is

$$A^T = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & 0 \end{bmatrix}$$

To find the null space, we'll augment the matrix, and then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 4 & | & 0 \\ -3 & 1 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 & | & 0 \\ 0 & 1 & 12 & | & 0 \end{bmatrix}$$

Because we have pivot entries in the first two columns, we'll pull a system of equations from the matrix,

$$1x_1 + 0x_2 + 4x_3 = 0$$

$$0x_1 + 1x_2 + 12x_3 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = -4x_3$$

$$x_2 = -12x_3$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix}$$

Therefore, the left null space is

$$N(A^T) = \text{Span}\left(\begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix}\right)$$

The space of the null space of the transpose is always $\mathbb{R}^m$, where m is the number of rows in the original matrix, A. The original matrix has 3 rows, so the null space of the transpose $N(A^T)$ is a subspace of $\mathbb{R}^3$.

The column space of the transpose A^T, which is the same as the row space of A, is simply given by the columns in A^T that contain pivot entries when A^T is in reduced row-echelon form. So the column space of A^T is

$$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

The space of the column space of the transpose is always $\mathbb{R}^n$, where n is the number of columns in the original matrix, A. The original matrix has 2 columns, so the column space of the transpose $C(A^T)$ is a subspace of $\mathbb{R}^2$.

Because there's one vector that forms the basis of $N(A^T)$, the dimension of $N(A^T)$ is $\text{Dim}(N(A^T)) = 1$.

Because there are two vectors that form the basis of $C(A^T)$, the dimension of $C(A^T)$ is $\text{Dim}(C(A^T)) = 2$.

$$N(A^T) = \text{Span}\left(\begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix}\right) \text{ in } \mathbb{R}^3 \qquad \text{Dim}(N(A^T)) = 1$$

$$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \text{ in } \mathbb{R}^2 \qquad \text{Dim}(C(A^T)) = 2$$

Topic: Null and column spaces of the transpose

Question: Find the row space and left null space of B, and the dimensions of those spaces.

$$B = \begin{bmatrix} 2 & -2 & 1 & 0 \\ 1 & 3 & -3 & -2 \\ 0 & 0 & 4 & -4 \end{bmatrix}$$

Answer choices:

A $N(B^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\text{Dim}(N(B^T)) = 1$

$C(B^T) = \text{Span}\left(\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \\ -2 \end{bmatrix}\right)$ in $\mathbb{R}^4$ $\text{Dim}(C(B^T)) = 2$

B $N(B^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$ in $\mathbb{R}^4$ $\text{Dim}(N(B^T)) = 2$

$C(B^T) = \text{Span}\left(\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \\ -2 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\text{Dim}(C(B^T)) = 1$

C $\quad N(B^T) = \text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(N(B^T)) = 0$

$C(B^T) = \text{Span}\left(\begin{bmatrix}2\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-3\\-2\end{bmatrix}, \begin{bmatrix}0\\0\\4\\-4\end{bmatrix}\right)$ in $\mathbb{R}^4$ $\quad \text{Dim}(C(B^T)) = 3$

D $\quad N(B^T) = \text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ in $\mathbb{R}^4$ $\quad \text{Dim}(N(B^T)) = 3$

$C(B^T) = \text{Span}\left(\begin{bmatrix}2\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-3\\-2\end{bmatrix}, \begin{bmatrix}0\\0\\4\\-4\end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(C(B^T)) = 0$

Solution: C

The transpose of B is

$$B^T = \begin{bmatrix}2 & 1 & 0\\-2 & 3 & 0\\1 & -3 & 4\\0 & -2 & -4\end{bmatrix}$$

To find the null space, we'll augment the matrix, and then put it into reduced row-echelon form. Start by finding the pivot in the first column (by switching the first and third rows), and then zeroing out the rest of the first column.

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ -2 & 3 & 0 & | & 0 \\ 1 & -3 & 4 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 2 & 1 & 0 & | & 0 \\ -2 & 3 & 0 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ -2 & 3 & 0 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ 0 & -3 & 8 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix}$$

Find the pivot entry in the second column (by switching the second and fourth rows), and then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 0 & -2 & -4 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ 0 & -3 & 8 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ 0 & -3 & 8 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ 0 & -3 & 8 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & -22 & | & 0 \\ 0 & -3 & 8 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & -22 & | & 0 \\ 0 & 0 & 14 & | & 0 \end{bmatrix}$$

Find the pivot entry in the third column, then zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 14 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 14 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 14 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Because we have pivot entries in the first three columns, we'll pull a system of equations from the matrix,

$$1x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 1x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + 1x_3 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the left null space is

$$N(B^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

The space of the null space of the transpose is always $\mathbb{R}^m$, where m is the number of rows in the original matrix, B. The original matrix has 3 rows, so the null space of the transpose $N(B^T)$ is a subspace of $\mathbb{R}^3$.

The column space of the transpose B^T, which is the same as the row space of B, is simply given by the columns in B^T that contain pivot entries when B^T is in reduced row-echelon form. So the column space of B^T is

$$C(B^T) = \text{Span}\left(\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ -4 \end{bmatrix}\right)$$

The space of the column space of the transpose is always $\mathbb{R}^n$, where n is the number of columns in the original matrix, B. The original matrix has 4 columns, so the column space of the transpose $C(B^T)$ is a subspace of $\mathbb{R}^4$.

Because the zero vector is the only vector that forms the basis of $N(B^T)$, the dimension of $N(B^T)$ is $\text{Dim}(N(B^T)) = 0$.

Because there are three vectors that form the basis of $C(B^T)$, the dimension of $C(B^T)$ is $\text{Dim}(C(B^T)) = 3$.

$$N(B^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) \text{ in } \mathbb{R}^3 \qquad\qquad \text{Dim}(N(B^T)) = 0$$

$$C(B^T) = \text{Span}\left(\begin{bmatrix}2\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-3\\-2\end{bmatrix}, \begin{bmatrix}0\\0\\4\\-4\end{bmatrix}\right) \text{ in } \mathbb{R}^4 \qquad \text{Dim}(C(B^T)) = 3$$

Topic: Null and column spaces of the transpose

Question: Find the row space and left null space of C, and the dimensions of those spaces.

$$C = \begin{bmatrix} -1 & 5 & 0 \\ 1 & -2 & 3 \\ 0 & 0 & -4 \end{bmatrix}$$

Answer choices:

A $N(C^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\text{Dim}(N(C^T)) = 3$

$C(C^T) = \text{Span}\left(\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\text{Dim}(N(C^T)) = 0$

B $N(C^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\text{Dim}(N(C^T)) = 0$

$C(C^T) = \text{Span}\left(\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\text{Dim}(C(C^T)) = 3$

C $N(C^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\text{Dim}(N(C^T)) = 3$

$$C(C^T) = \text{Span}\left(\begin{bmatrix}-1\\5\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}\right) \quad \text{in } \mathbb{R}^3 \quad \text{Dim}(C(C^T)) = 1$$

D $$N(C^T) = \text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) \quad \text{in } \mathbb{R}^3 \quad \text{Dim}(N(C^T)) = 1$$

$$C(C^T) = \text{Span}\left(\begin{bmatrix}-1\\5\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}\right) \quad \text{in } \mathbb{R}^3 \quad \text{Dim}(C(C^T)) = 3$$

Solution: B

The transpose of C is

$$C^T = \begin{bmatrix}-1 & 1 & 0\\5 & -2 & 0\\0 & 3 & -4\end{bmatrix}$$

To find the null space, we'll augment the matrix, and then put it into reduced row-echelon form. Start by finding the pivot in the first column, and then zeroing out the rest of the first column.

$$\begin{bmatrix}-1 & 1 & 0 & | & 0\\5 & -2 & 0 & | & 0\\0 & 3 & -4 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -1 & 0 & | & 0\\5 & -2 & 0 & | & 0\\0 & 3 & -4 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 0 & | & 0\\0 & 3 & 0 & | & 0\\0 & 3 & -4 & | & 0\end{bmatrix}$$

Find the pivot entry in the second column, and then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & -4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & -4 & | & 0 \end{bmatrix}$$

Find the pivot entry in the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Because we have pivot entries in the first three columns, we'll pull a system of equations from the matrix,

$$1x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 1x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + 1x_3 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the left null space is

$$N(C^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

The space of the null space of the transpose is always $\mathbb{R}^m$, where m is the number of rows in the original matrix, C. The original matrix has 3 rows, so the null space of the transpose $N(C^T)$ is a subspace of $\mathbb{R}^3$.

The column space of the transpose C^T, which is the same as the row space of C, is simply given by the columns in C^T that contain pivot entries when C^T is in reduced row-echelon form. So the column space of C^T is

$$C(C^T) = \text{Span}\left(\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} \right)$$

The space of the column space of the transpose is always $\mathbb{R}^n$, where n is the number of columns in the original matrix, C. The original matrix has 3 columns, so the column space of the transpose $C(C^T)$ is a subspace of $\mathbb{R}^3$.

Because the zero vector is the only vector that forms the basis of $N(C^T)$, the dimension of $N(C^T)$ is $\text{Dim}(N(C^T)) = 0$.

Because there are three vectors that form the basis of $C(C^T)$, the dimension of $C(C^T)$ is $\text{Dim}(C(C^T)) = 3$.

$$N(C^T) = \text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) \text{ in } \mathbb{R}^3 \qquad \text{Dim}(N(C^T)) = 0$$

$$C(C^T) = \text{Span}\left(\begin{bmatrix}-1\\5\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\-4\end{bmatrix}\right) \text{ in } \mathbb{R}^3 \qquad \text{Dim}(C(C^T)) = 3$$