Orthogonal complements of the fundamental subspaces

Previously, we learned about four fundamental subspaces, which are defined for any matrix A and its transpose A^T:

- the column space $C(A)$, made of the column vectors of A.
- the null space $N(A)$, made of the vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{O}$.
- the row space $C(A^T)$, made of the column vectors of A^T, which are also the row vectors of A.
- the left null space $N(A^T)$, made of the vectors $\vec{x}$ that satisfy $\vec{x}A = \vec{O}$.

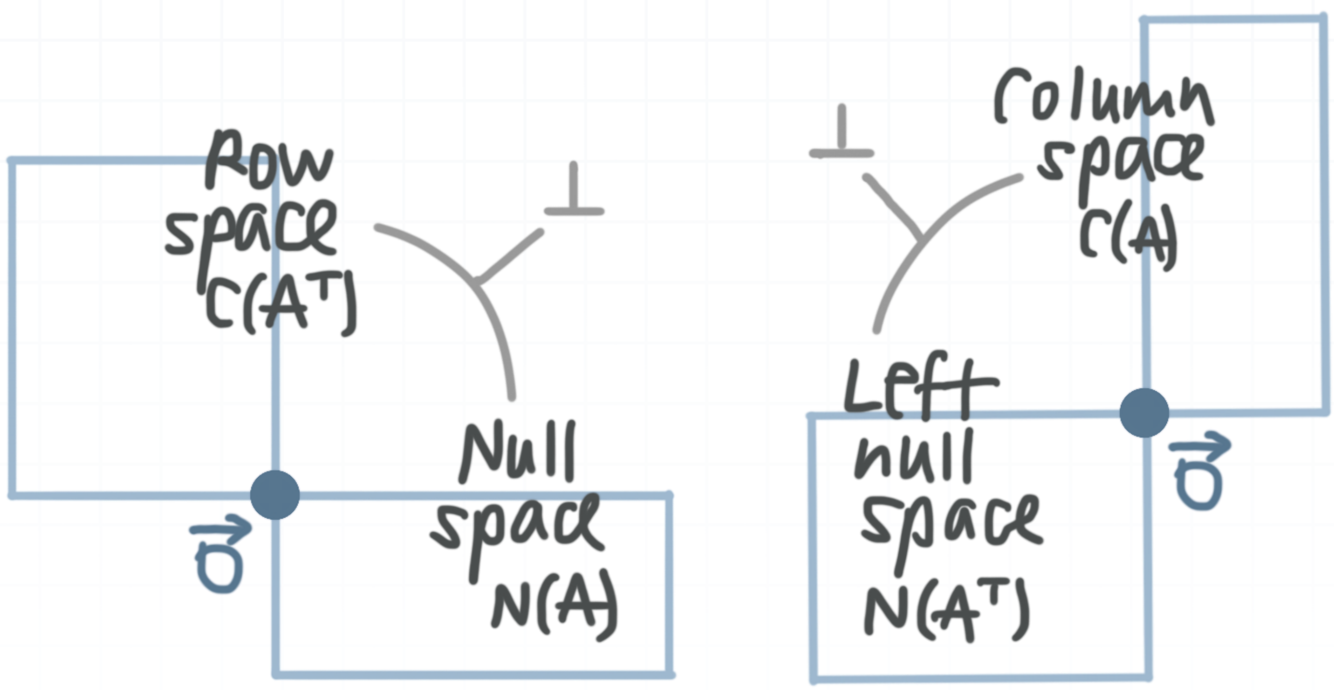
And we understand the relationship between a subspace V and its orthogonal complement $V^\perp$. Every vector $\vec{v}$ in V is orthogonal to every vector $\vec{x}$ in $V^\perp$.

Orthogonality of the fundamental subspaces

With all of this in mind, we can state two important facts:

1. The null space $N(A)$ and row space $C(A^T)$ are orthogonal complements, $N(A) = (C(A^T))^\perp$, or $(N(A))^\perp = C(A^T)$.
2. The left null space $N(A^T)$ and column space $C(A)$ are orthogonal complements, $N(A^T) = (C(A))^\perp$, or $(N(A^T))^\perp = C(A)$.

In other words, the row space is orthogonal to the null space, and vice versa, and the column space is orthogonal to the left null space, and vice versa. Because of this fact, we can say that the row space and null space only intersect at the zero vector, and similarly that the column space and left null space only intersect at the zero vector. In fact, there is never any overlap between V and $V^{\perp}$ other than the zero vector.



Dimension of the orthogonal complement

We also want to be able to define the dimension of any orthogonal complement. The good news is that there's actually an easy way to do this. Here's the fact we want to remember:

> The dimension of a subspace V and the dimension of its orthogonal complement $V^{\perp}$ will always sum to the dimension of the space $\mathbb{R}^n$ that contains them both.

$$\text{Dim}(V) + \text{Dim}(V^{\perp}) = n$$

In other words, if we remember that "dimension" really just means "the number of linearly independent vectors needed to form the basis," then we can say that the number of basis vectors of V, plus the number of basis vectors of $V^\perp$, will be equal to n, when both V and $V^\perp$ are in $\mathbb{R}^n$.

For instance, let's say V is a subspace of $\mathbb{R}^2$, and its dimension is 1. That means V is a line. Because we're in $\mathbb{R}^2$ (two dimensions), the dimension of $V^\perp$ must be $2-\text{Dim}(V)=2-1=1$. So $V^\perp$ is also a line.

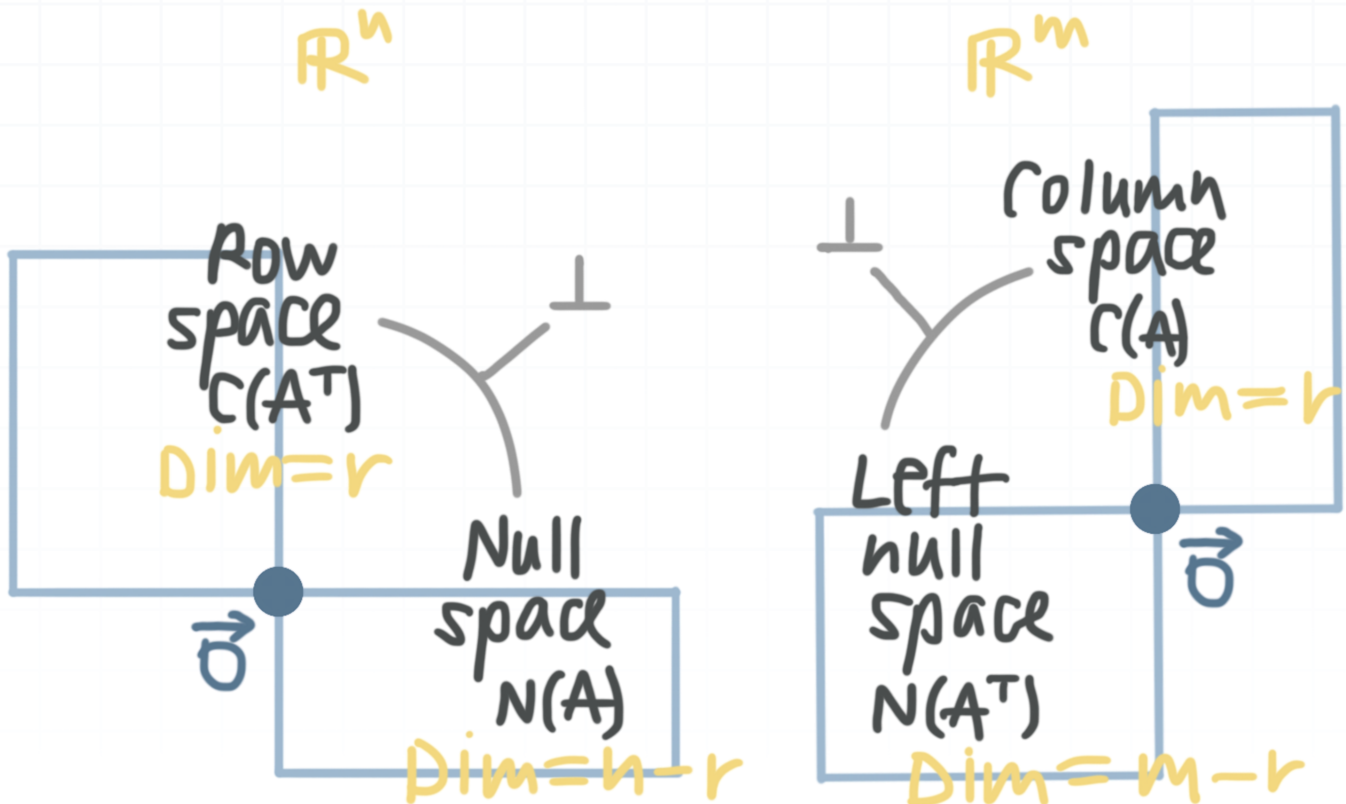
Or to take another example, let's say V is a subspace of $\mathbb{R}^3$, and its dimension is 2. That means V is a plane. Because we're in $\mathbb{R}^3$ (three dimensions), the dimension of $V^\perp$ must be $3-\text{Dim}(V)=3-2=1$. So $V^\perp$ is a line.

Of course, this same rule applies when we're talking about the four fundamental subspaces as orthogonal complements. If we define a matrix A as an $m\times n$ matrix (a matrix with m rows and n columns), then we know:

- The column space of A is its rank r (the number of linearly independent columns in A). The column space is defined in $\mathbb{R}^m$, because the number of linearly independent columns is given by the number of pivot entries, and the number of pivot entries can't possibly exceed the number of rows m in the matrix A. Because the column space and left null space are orthogonal complements, that means the dimension of the left null space must be $m-r$.

- The row space of A is its rank r (the number of linearly independent rows in A). The row space is defined in $\mathbb{R}^n$, because the number of linearly independent rows is given by the number of pivot entries, and the number of pivot entries can't possibly exceed the number of columns n in the matrix A. Because the row space and null space are orthogonal complements, that means the dimension of the null space must be $n - r$.

So we could add to the picture we sketched earlier, and this time include the dimensions of these fundamental subspaces:



Let's do an example to find the dimensions of orthogonal complements.

Example

For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 3 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

First, let's put A into reduced row-echelon form.

$$\begin{bmatrix}1 & -1 & 0\\ -2 & 0 & 0\\ 0 & 3 & -1\\ 1 & 1 & -2\end{bmatrix} \to \begin{bmatrix}1 & -1 & 0\\ 0 & -2 & 0\\ 0 & 3 & -1\\ 1 & 1 & -2\end{bmatrix} \to \begin{bmatrix}1 & -1 & 0\\ 0 & -2 & 0\\ 0 & 3 & -1\\ 0 & 2 & -2\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 3 & -1\\ 0 & 2 & -2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 3 & -1\\ 0 & 2 & -2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\\ 0 & 2 & -2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\\ 0 & 0 & -2\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & -2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

In reduced row-echelon form, we can see that there are three pivots, which means the rank of A is $r = 3$. And A is a 4×3 matrix, which means there are $m = 4$ rows and $n = 3$ columns. Therefore, the dimensions of the four fundamentals subspaces of A are:

column space, $C(A)$	$r = 3$
null space, $N(A)$	$n - r = 3 - 3 = 0$
row space, $C(A^T)$	$r = 3$
left null space, $N(A^T)$	$m - r = 4 - 3 = 1$

Notice how the dimension of the null space is 0. That means the null space contains only the zero vector.