Null space of a matrix

Now that we understand the general idea of a null space, we want to know how to calculate the null space of a particular matrix. In other words, given a matrix A, we want to be able to find the entire set of vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{O}$.

There may be only one vector in the null space, or there may be many. Either way, we want to be able to find the full set of vectors which are members of the null space.

To find the null space of a particular matrix, we'll plug the matrix into $A\vec{x} = \vec{O}$, rewrite the equation as an augmented matrix, put the augmented matrix into reduced row-echelon form, and then pull the solution from the matrix. The solution we find will be at least part of the null space of A.

Let's do an example.

Example

Find the null space of the matrix A.

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & -6 \\ 1 & -1 & -6 \end{bmatrix}$$

To find the null space of A, we need to find the vector set that satisfies $A\vec{x} = \vec{O}$, so we'll set up a matrix equation.

Because A has three columns, $\vec{x}$ needs to have three rows, so we'll use a 3-row column vector for $\vec{x}$. And multiplying the 3×3 matrix by the 3-row column vector will result in a 3×1 zero-vector, so the matrix equation must be

$$\begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & -6 \\ 1 & -1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this equation, we get a system of equations.

$$2x_1 + x_2 - 3x_3 = 0$$

$$4x_1 + 2x_2 - 6x_3 = 0$$

$$x_1 - x_2 - 6x_3 = 0$$

We can write the system as an augmented matrix,

$$\begin{bmatrix} 2 & 1 & -3 & | & 0 \\ 4 & 2 & -6 & | & 0 \\ 1 & -1 & -6 & | & 0 \end{bmatrix}$$

and then use Gaussian elimination to put it in reduced row-echelon form. Find the pivot entry in the first column by switching the first and third row.

$$\begin{bmatrix} 1 & -1 & -6 & | & 0 \\ 4 & 2 & -6 & | & 0 \\ 2 & 1 & -3 & | & 0 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -1 & -6 & | & 0 \\ 0 & 6 & 18 & | & 0 \\ 2 & 1 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -6 & | & 0 \\ 0 & 6 & 18 & | & 0 \\ 0 & 3 & 9 & | & 0 \end{bmatrix}$$

Find the pivot entry in the second column.

$$\begin{bmatrix} 1 & -1 & -6 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 3 & 9 & | & 0 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -3 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 3 & 9 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From the reduced row-echelon form of the matrix, we get

$$x_1 - 3x_3 = 0$$

$$x_2 + 3x_3 = 0$$

The pivot entries we found were for x_1 and x_2 (since they were in the first and second columns of our matrix), so we'll solve the system for x_1 and x_2.

$$x_1 = 3x_3$$

$$x_2 = -3x_3$$

This remaining system tells us that x_1 is equivalent to 3 times x_3, and that x_2 is equivalent to -3 times x_3. So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

Because $x_3 = 1x_3$, we were able to fill in the last row of the column matrix on the right with a 1. What this equation tells us is that we can choose any value for x_3, which is acting like a scalar. When we do, we'll get that the set of vectors that satisfy $A\vec{x} = \vec{O}$ is not only $\vec{x} = (3, -3, 1)$, but all linear combinations of $\vec{x} = (3, -3, 1)$. Which means the null space of A is all the linear combinations of $\vec{x} = (3, -3, 1)$.

But remember that when we say "all the linear combinations of a vector set," that's just the span of the vector set! So we can say that the null space of A is the span of $\vec{x} = (3, -3, 1)$.

$$N(A) = \text{Span}\left(\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}\right)$$

Keep in mind that, because the span of $\vec{x} = (3, -3, 1)$ is the null space of the reduced row-echelon form of A, we can also write the null space equation as

$$N(A) = N(\text{rref}(A))$$

This new equation for the null space tells us that, all we actually have to do to find the null space is put the original matrix A in reduced row-echelon form, and then find the linear combination equation from that rref matrix.

The set of vectors in the null space will be the span of the vector(s) in the linear combination equation.

Let's look at an example where we use this abbreviated process to find the null space of a particular matrix.

Example

Find the null space of K.

$$K=\begin{bmatrix}1&-2&1&3\\-3&6&-3&-9\\4&-8&4&12\end{bmatrix}$$

To find the null space, put the matrix K in reduced row-echelon form.

$$K=\begin{bmatrix}1&-2&1&3\\-3&6&-3&-9\\4&-8&4&12\end{bmatrix}$$

$$K=\begin{bmatrix}1&-2&1&3\\0&0&0&0\\4&-8&4&12\end{bmatrix}$$

$$K=\begin{bmatrix}1&-2&1&3\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Then set up the equation $(\text{rref}(K))\vec{x_n}=0$.

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the equation,

$$x_1 - 2x_2 + x_3 + 3x_4 = 0$$

which we can solve for the single pivot variable.

$$x_1 = 2x_2 - x_3 - 3x_4$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of K is the span of the vectors in this linear combination equation.

$$N(K) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$