**Topic**: Dimensionality, nullity, and rank

**Question**: Find the nullity of $M$.

$$M = \begin{bmatrix} 1 & -2 & 3 & -1 & 2 \\ -3 & 6 & -9 & 3 & -6 \\ -5 & 9 & -7 & 4 & 0 \end{bmatrix}$$

**Answer choices**:

A nullity$(M) = 2$

B nullity$(M) = 5$

C nullity$(M) = 3$

D nullity$(M) = 4$

**Solution**: C

To find the nullity of the matrix $M$, we need to first find the null space, so we'll set up the augmented matrix for $M\vec{x}=\vec{O}$, then put it in reduced row-echelon form.

$$\begin{bmatrix}1&-2&3&-1&2&|&0\\-3&6&-9&3&-6&|&0\\-5&9&-7&4&0&|&0\end{bmatrix}\to\begin{bmatrix}1&-2&3&-1&2&|&0\\0&0&0&0&0&|&0\\-5&9&-7&4&0&|&0\end{bmatrix}$$

$$\begin{bmatrix}1&-2&3&-1&2&|&0\\0&0&0&0&0&|&0\\0&-1&8&-1&10&|&0\end{bmatrix}\to\begin{bmatrix}1&-2&3&-1&2&|&0\\0&-1&8&-1&10&|&0\\0&0&0&0&0&|&0\end{bmatrix}$$

$$\begin{bmatrix}1&-2&3&-1&2&|&0\\0&1&-8&1&-10&|&0\\0&0&0&0&0&|&0\end{bmatrix}\to\begin{bmatrix}1&0&-13&1&-18&|&0\\0&1&-8&1&-10&|&0\\0&0&0&0&0&|&0\end{bmatrix}$$

Let's parse out a system of equations.

$$x_1-13x_3+x_4-18x_5=0$$

$$x_2-8x_3+x_4-10x_5=0$$

Solve for the pivot variables.

$$x_1=13x_3-x_4+18x_5$$

$$x_2=8x_3-x_4+10x_5$$

Rewrite the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 13 \\ 8 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 18 \\ 10 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of $M$ is the span of the vectors in this linear combination equation.

$$N(M) = \text{Span}\left( \begin{bmatrix} 13 \\ 8 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 18 \\ 10 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

We found $3$ spanning vectors that form a basis for the null space, which matches the dimension of the null space.

$$\text{Dim}(N(M)) = \text{nullity}(M) = 3$$

We can also get the nullity of $M$ from the number of free variables in rref$(M)$. Because there were $3$ free variables, $x_3$, $x_4$, and $x_5$, nullity$(M) = 3$.

**Topic**: Dimensionality, nullity, and rank

**Question**: Find the rank of $X$.

$$X = \begin{bmatrix} -2 & 10 & -4 \\ 1 & 3 & -6 \\ 1 & -5 & 9 \end{bmatrix}$$

**Answer choices**:

A rank$(X) = 1$

B rank$(X) = 0$

C rank$(X) = 3$

D rank$(X) = 2$

**Solution**: C

To find the rank of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} -2 & 10 & -4 \\ 1 & 3 & -6 \\ 1 & -5 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 2 \\ 1 & 3 & -6 \\ 1 & -5 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 2 \\ 0 & 8 & -8 \\ 1 & -5 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 2 \\ 0 & 8 & -8 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now that the matrix is in reduced row-echelon form, we can find the rank directly from the matrix. We can see that all three columns are pivot columns. So because there are three pivot variables, the rank is

$$\text{Dim}(C(X)) = \text{rank}(X) = 3$$

**Topic**: Dimensionality, nullity, and rank

**Question**: Find the nullity and the rank of $A$.

$$A = \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 2 & 5 & -4 & -7 & 3 \\ 1 & 4 & -3 & 5 & 4 \\ 1 & 2 & -2 & -6 & 3 \end{bmatrix}$$

**Answer choices**:

A nullity$(A) = 2$ and rank$(A) = 3$

B nullity$(A) = 3$ and rank$(A) = 3$

C nullity$(A) = 3$ and rank$(A) = 2$

D nullity$(A) = 1$ and rank$(A) = 4$

**Solution**: A

To find the nullity of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 2 & 5 & -4 & -7 & 3 \\ 1 & 4 & -3 & 5 & 4 \\ 1 & 2 & -2 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & -1 & 0 & -5 & 3 \\ 1 & 4 & -3 & 5 & 4 \\ 1 & 2 & -2 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & -1 & 0 & -5 & 3 \\ 0 & 1 & -1 & 6 & 4 \\ 1 & 2 & -2 & -6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & -1 & 0 & -5 & 3 \\ 0 & 1 & -1 & 6 & 4 \\ 0 & -1 & 0 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 1 & -1 & 6 & 4 \\ 0 & -1 & 0 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & -1 & 1 & 7 \\ 0 & -1 & 0 & -5 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & -1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -16 & 9 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & -1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -16 & 9 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -18 & -5 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since rref$(A)$ has three pivot columns and two free columns,

$$\text{rank}(A) = 3$$

$$\text{nullity}(A) = 2$$