Linear Algebra Workbook Solutions

Transposes

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TRANSPOSES AND THEIR DETERMINANTS

1. Find the transpose A^T.

$$A = \begin{bmatrix} 5 & 6 & 0 & 7 & 5 & -7 \end{bmatrix}$$

Solution:

To find the transpose of A, in order, turn each row of A into a column of A^T.

$$A^T = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 7 \\ 5 \\ -7 \end{bmatrix}$$

2. Find the transpose A^T.

$$A = \begin{bmatrix} 7 & 9 & -6 \\ 0 & -1 & 9 \end{bmatrix}$$

Solution:

To find the transpose of A, in order, turn each row of A into a column of A^T.

$$A^T = \begin{bmatrix} 7 & 0 \\ 9 & -1 \\ -6 & 9 \end{bmatrix}$$

■ 3. Find the transpose A^T.

$$A = \begin{bmatrix} -4 & -7 \\ 5 & 1 \\ 7 & -2 \\ 4 & -2 \end{bmatrix}$$

Solution:

To find the transpose of A, in order, turn each row of A into a column of A^T.

$$A^T = \begin{bmatrix} -4 & 5 & 7 & 4 \\ -7 & 1 & -2 & -2 \end{bmatrix}$$

■ 4. Find the determinant of the transpose of A.

$$A = \begin{bmatrix} 5 & 3 & 6 & -1 \\ 9 & 0 & 1 & -2 \\ 8 & -2 & -4 & 8 \\ 5 & 4 & 9 & 7 \end{bmatrix}$$

Solution:

The determinant of the transpose is always the same as the determinant of the original matrix, so we'll calculate the determinant of A, instead of bothering with the transpose. To find the determinant, work along the second row, since it includes a 0 that'll make the calculation simpler.

$$|A| = -9\begin{vmatrix} 3 & 6 & -1 \\ -2 & -4 & 8 \\ 4 & 9 & 7 \end{vmatrix} + 0\begin{vmatrix} 5 & 6 & -1 \\ 8 & -4 & 8 \\ 5 & 9 & 7 \end{vmatrix} - 1\begin{vmatrix} 5 & 3 & -1 \\ 8 & -2 & 8 \\ 5 & 4 & 7 \end{vmatrix} + (-2)\begin{vmatrix} 5 & 3 & 6 \\ 8 & -2 & -4 \\ 5 & 4 & 9 \end{vmatrix}$$

$$|A| = -9\begin{vmatrix} 3 & 6 & -1 \\ -2 & -4 & 8 \\ 4 & 9 & 7 \end{vmatrix} - \begin{vmatrix} 5 & 3 & -1 \\ 8 & -2 & 8 \\ 5 & 4 & 7 \end{vmatrix} - 2\begin{vmatrix} 5 & 3 & 6 \\ 8 & -2 & -4 \\ 5 & 4 & 9 \end{vmatrix}$$

Break the 3×3 determinants into 4×4 determinants.

$$|A| = -9\left[3\begin{vmatrix} -4 & 8 \\ 9 & 7 \end{vmatrix} - 6\begin{vmatrix} -2 & 8 \\ 4 & 7 \end{vmatrix} - 1\begin{vmatrix} -2 & -4 \\ 4 & 9 \end{vmatrix}\right]$$

$$-\left[5\begin{vmatrix} -2 & 8 \\ 4 & 7 \end{vmatrix} - 3\begin{vmatrix} 8 & 8 \\ 5 & 7 \end{vmatrix} - 1\begin{vmatrix} 8 & -2 \\ 5 & 4 \end{vmatrix}\right]$$

$$-2\left[5\begin{vmatrix} -2 & -4 \\ 4 & 9 \end{vmatrix} - 3\begin{vmatrix} 8 & -4 \\ 5 & 9 \end{vmatrix} + 6\begin{vmatrix} 8 & -2 \\ 5 & 4 \end{vmatrix}\right]$$

Calculate the 2×2 determinants.

$$|A| = -9\left[3((-4)(7) - (8)(9)) - 6((-2)(7) - (8)(4)) - 1((-2)(9) - (-4)(4))\right]$$

$$-\left[5((-2)(7) - (8)(4)) - 3((8)(7) - (8)(5)) - 1((8)(4) - (-2)(5))\right]$$

$$-2\left[5((-2)(9) - (-4)(4)) - 3((8)(9) - (-4)(5)) + 6((8)(4) - (-2)(5))\right]$$

$$|A| = -9\left[3(-28 - 72) - 6(-14 - 32) - 1(-18 + 16)\right]$$

$$-\left[5(-14-32)-3(56-40)-1(32+10)\right]$$

$$-2\left[5(-18+16)-3(72+20)+6(32+10)\right]$$

$$|A| = -9\left[3(-100)-6(-46)-1(-2)\right]-\left[5(-46)-3(16)-1(42)\right]$$

$$-2\left[5(-2)-3(92)+6(42)\right]$$

$$|A| = -9(-300+276+2)-(-230-48-42)-2(-10-276+252)$$

$$|A| = -9(-22)-(-320)-2(-34)$$

$$|A| = 198+320+68$$

$$|A| = 586$$

■ 5. Find the determinant of the transpose of A.

$$A = \begin{bmatrix} -9 & -3 & -1 \\ -4 & 7 & 3 \\ -4 & 8 & 7 \end{bmatrix}$$

Solution:

The determinant of the transpose is always the same as the determinant of the original matrix, so we'll calculate the determinant of A, instead of bothering with the transpose. To find the determinant, work along the first row.

$$|A| = -9\begin{vmatrix} 7 & 3 \\ 8 & 7 \end{vmatrix} - (-3)\begin{vmatrix} -4 & 3 \\ -4 & 7 \end{vmatrix} + (-1)\begin{vmatrix} -4 & 7 \\ -4 & 8 \end{vmatrix}$$

$$|A| = -9\begin{vmatrix} 7 & 3 \\ 8 & 7 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ -4 & 7 \end{vmatrix} - \begin{vmatrix} -4 & 7 \\ -4 & 8 \end{vmatrix}$$

Calculate the 2×2 determinants.

$$|A| = -9((7)(7) - (3)(8)) + 3((-4)(7) - (3)(-4)) - ((-4)(8) - (7)(-4))$$

$$|A| = -9(49 - 24) + 3(-28 + 12) - (-32 + 28)$$

$$|A| = -9(25) + 3(-16) - (-4)$$

$$|A| = -225 - 48 + 4$$

$$|A| = -269$$

6. Find the determinant of the transpose of A.

$$A = \begin{bmatrix} -8 & 6 & 8 \\ 3 & -9 & -1 \\ 4 & -9 & 9 \end{bmatrix}$$

Solution:

The determinant of the transpose is always the same as the determinant of the original matrix, so we'll calculate the determinant of A, instead of bothering with the transpose. To find the determinant, work along the first row.

$$|A| = -8\begin{vmatrix} -9 & -1 \\ -9 & 9 \end{vmatrix} - 6\begin{vmatrix} 3 & -1 \\ 4 & 9 \end{vmatrix} + 8\begin{vmatrix} 3 & -9 \\ 4 & -9 \end{vmatrix}$$

Calculate the 2×2 determinants.

$$|A| = -8((-9)(9) - (-1)(-9)) - 6((3)(9) - (-1)(4)) + 8((3)(-9) - (-9)(4))$$

$$|A| = -8(-81 - 9) - 6(27 + 4) + 8(-27 + 36)$$

$$|A| = -8(-90) - 6(31) + 8(9)$$

$$|A| = 720 - 186 + 72$$

$$|A| = 606$$

TRANSPOSES OF PRODUCTS, SUMS, AND INVERSES

1. Find $(AB)^T$.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix}$$

Solution:

Find the matrix AB,

$$AB = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1(-3) + 2(1) & -1(-2) + 2(2) \\ 2(-3) + 3(1) & 2(-2) + 3(2) \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 + 2 & 2 + 4 \\ -6 + 3 & -4 + 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 6 \\ -3 & 2 \end{bmatrix}$$

and then take its transpose by swapping the rows and columns.

$$(AB)^T = \begin{bmatrix} 5 & -3 \\ 6 & 2 \end{bmatrix}$$

2. Find $(AB)^T$.

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -4 & 1 \\ 0 & -3 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

Solution:

Start by taking the transposes individually by swapping rows and columns in A and B to get A^T and B^T.

$$A^T = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & -3 \\ -2 & 1 & 1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & 0 & -1 \\ -4 & -3 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

Find the product of these transposes.

$$B^TA^T = \begin{bmatrix} 2 & 0 & -1 \\ -4 & -3 & 1 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & -3 \\ -2 & 1 & 1 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 2(-1) + 0(2) - 1(-2) & 2(2) + 0(3) - 1(1) & 2(3) + 0(-3) - 1(1) \\ -4(-1) - 3(2) + 1(-2) & -4(2) - 3(3) + 1(1) & -4(3) - 3(-3) + 1(1) \\ 1(-1) - 2(2) + 2(-2) & 1(2) - 2(3) + 2(1) & 1(3) - 2(-3) + 2(1) \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} -2+0+2 & 4+0-1 & 6+0-1 \\ 4-6-2 & -8-9+1 & -12+9+1 \\ -1-4-4 & 2-6+2 & 3+6+2 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 0 & 3 & 5 \\ -4 & -16 & -2 \\ -9 & -2 & 11 \end{bmatrix}$$

We know the product $B^TA^T = (AB)^T$, so

$$(AB)^T = \begin{bmatrix} 0 & 3 & 5 \\ -4 & -16 & -2 \\ -9 & -2 & 11 \end{bmatrix}$$

■ 3. Find $(X+Y)^T$.

$$X = \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix}$$

Solution:

Find the sum $X+Y$,

$$X+Y = \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$X + Y = \begin{bmatrix} 4 + (-3) & 1 + 2 \\ -2 + 0 & 0 + (-1) \end{bmatrix}$$

$$X + Y = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}$$

and then take its transpose by swapping the rows and columns.

$$(X + Y)^T = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$$

■ 4. Find $(X + Y)^T$.

$$X = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \\ -2 & 0 & 3 \end{bmatrix}$$

$$Y = \begin{bmatrix} -1 & 2 & -3 \\ 0 & -1 & 2 \\ 4 & -1 & 0 \end{bmatrix}$$

Solution:

Find the sum $X + Y$,

$$X + Y = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \\ -2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 & -3 \\ 0 & -1 & 2 \\ 4 & -1 & 0 \end{bmatrix}$$

$$X+Y=\begin{bmatrix} 2+(-1) & 0+2 & 3+(-3) \\ 4+0 & 1+(-1) & -1+2 \\ -2+4 & 0+(-1) & 3+0 \end{bmatrix}$$

$$X+Y=\begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

and then take its transpose by swapping the rows and columns.

$$(X+Y)^T=\begin{bmatrix} 1 & 4 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

■ 5. Find $(X^T)^{-1}$.

$$X=\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$

Solution:

First transpose X.

$$X^T=\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$$

Augment X^T with I_2, and then put the left side of the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ -2 & 3 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 7 & | & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{3}{7} & -\frac{2}{7} \\ 0 & 1 & | & \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

Now that the left side of the augmented matrix is the identity matrix, the right side is the inverse $(X^T)^{-1}$.

$$(X^T)^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

■ 6. Find $(A^T)^{-1}$.

$$A = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix}$$

Solution:

First transpose A.

$$A^T = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & -1 \\ -3 & 1 & 4 \end{bmatrix}$$

Augment A^T with I_3, and then put the left side of the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 4 & 1 & 0 & | & 1 & 0 & 0 \\ 1 & 2 & -1 & | & 0 & 1 & 0 \\ -3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 4 & 1 & 0 & | & 1 & 0 & 0 \\ -3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & -7 & 4 & | & 1 & -4 & 0 \\ -3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & -7 & 4 & | & 1 & -4 & 0 \\ 0 & 7 & 1 & | & 0 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & -7 & 4 & | & 1 & -4 & 0 \\ 0 & 0 & 5 & | & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & -7 & 4 & | & 1 & -4 & 0 \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & | & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{7} & | & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & | & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{9}{35} & -\frac{4}{35} & -\frac{1}{35} \\ 0 & 1 & -\frac{4}{7} & | & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & \frac{9}{35} & -\frac{4}{35} & -\frac{1}{35} \\ 0 & 1 & 0 & | & -\frac{1}{35} & \frac{16}{35} & \frac{4}{35} \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

Now that the left side of the augmented matrix is the identity matrix, the right side is the inverse $(A^T)^{-1}$.

$$(A^T)^{-1} = \begin{bmatrix} \frac{9}{35} & -\frac{4}{35} & -\frac{1}{35} \\ -\frac{1}{35} & \frac{16}{35} & \frac{4}{35} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

NULL AND COLUMN SPACES OF THE TRANSPOSE

1. Find the null and column spaces of the transpose M^T, identify their spaces $\mathbb{R}^i$, and name the dimension of the subspaces.

$$M=\begin{bmatrix}-1 & 0\\ 2 & 4\\ -2 & -2\\ 0 & 4\end{bmatrix}$$

Solution:

The transpose of M is

$$M^T=\begin{bmatrix}-1 & 2 & -2 & 0\\ 0 & 4 & -2 & 4\end{bmatrix}$$

To find the null space of the transpose (the left null space), augment M^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}-1 & 2 & -2 & 0 & | & 0\\ 0 & 4 & -2 & 4 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & -2 & 2 & 0 & | & 0\\ 0 & 4 & -2 & 4 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 2 & 0 & | & 0\\ 0 & 1 & -\frac{1}{2} & 1 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 1 & 2 & | & 0\\ 0 & 1 & -\frac{1}{2} & 1 & | & 0\end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 + x_3 + 2x_4 = 0$$

$$x_2 - \frac{1}{2}x_3 + x_4 = 0$$

and then solve the system for the pivot variables.

$$x_1 = -x_3 - 2x_4$$

$$x_2 = \frac{1}{2}x_3 - x_4$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of the transpose (the left null space) is

$$N(M^T) = \text{Span}\left(\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The column space of the transpose is

$$C(M^T) = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right)$$

but only the first two columns of rref(M^T) are pivot columns, which means the column space of M^T can actually be spanned by just the first two column vectors.

$$C(M^T) = \text{Span}\left(\begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right)$$

The original matrix M has $m = 4$ rows and $n = 2$ columns, so the null space of the transpose $N(M^T)$ is a subspace of $\mathbb{R}^4$, and the column space of the transpose $C(M^T)$ is a subspace of $\mathbb{R}^2$. And the dimension of the null and column spaces of the transpose are

$$\text{Dim}(N(M^T)) = m - r = 4 - 2 = 2$$

$$\text{Dim}(C(M^T)) = r = 2$$

■ 2. Find the row space and left null space of A, and the dimensions of those spaces.

$$A = \begin{bmatrix}1 & 2\\0 & 1\\-4 & 0\end{bmatrix}$$

Solution:

The transpose of A is

$$A^T = \begin{bmatrix}1 & 0 & -4\\2 & 1 & 0\end{bmatrix}$$

To find the left null space (the null space of the transpose), augment A^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 2 & 1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & 8 & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 - 4x_3 = 0$$

$$x_2 + 8x_3 = 0$$

and then solve the system for the pivot variables.

$$x_1 = 4x_3$$

$$x_2 = -8x_3$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix}$$

Therefore, the left null space (the null space of the transpose) is

$$N(A^T) = \text{Span}\left(\begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix} \right)$$

The row space is

$$C(A^T) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-4\\0\end{bmatrix}\right)$$

but only the first two columns of rref(A^T) are pivot columns, which means the row space can actually be spanned by just the first two columns.

$$C(A^T) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$$

The original matrix A has $m = 3$ rows and $n = 2$ columns, so the left null $N(A^T)$ space is a subspace of $\mathbb{R}^3$, and the row space $C(A^T)$ is a subspace of $\mathbb{R}^2$. And the dimension of the left null and row spaces are

$$\text{Dim}(N(A^T)) = m - r = 3 - 2 = 1$$

$$\text{Dim}(C(A^T)) = r = 2$$

3. Find the row space and left null space of B, and the dimensions of those spaces.

$$B = \begin{bmatrix}2 & 3 & 1 & 0\\1 & -2 & -1 & 4\\0 & 0 & 2 & -2\end{bmatrix}$$

Solution:

The transpose of B is

$$B^T = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -2 & 0 \\ 1 & -1 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

To find the left null space (the null space of the transpose), augment B^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 3 & -2 & 0 & | & 0 \\ 1 & -1 & 2 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & | & 0 \\ 3 & -2 & 0 & | & 0 \\ 2 & 1 & 0 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 2 & 1 & 0 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 3 & -4 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 3 & -4 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 0 & 14 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 0 & 14 & | & 0 \\ 0 & 0 & 22 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 22 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 22 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 22 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the left null space (the null space of the transpose) is

$$N(B^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The row space is

$$C(B^T) = \text{Span}\left(\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix} \right)$$

The original matrix B has $m = 3$ rows and $n = 4$ columns, so the left null space $N(B^T)$ is a subspace of $\mathbb{R}^3$, and the row space $C(B^T)$ is a subspace of $\mathbb{R}^4$. And the dimension of the left null and row spaces are

$$\text{Dim}(N(B^T)) = m - r = 3 - 3 = 0$$

$$\text{Dim}(C(B^T)) = r = 3$$

4. Find the row space and left null space of C, and the dimensions of those spaces.

$$C = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

The transpose of C is

$$C^T = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 4 & 0 \\ 0 & 3 & 3 \end{bmatrix}$$

To find the left null space (the null space of the transpose), augment C^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} -1 & 1 & 0 & | & 0 \\ 2 & 4 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 2 & 4 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 6 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the left null space (the null space of the transpose) is

$$N(C^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The row space is

$$C(C^T) = \text{Span}\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right)$$

The original matrix C has $m = 3$ rows and $n = 3$ columns, so the left null space $N(C^T)$ is a subspace of $\mathbb{R}^3$, and the row space $C(C^T)$ is a subspace of $\mathbb{R}^3$. And the dimension of the left null and row spaces are

$$\text{Dim}(N(C^T)) = m - r = 3 - 3 = 0$$

$$\text{Dim}(C(C^T)) = r = 3$$

■ 5. Find the row space and left null space of A, and the dimensions of those spaces.

$$A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \\ 0 & -2 \end{bmatrix}$$

Solution:

The transpose of A is

$$A^T = \begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

To find the left null space (the null space of the transpose), augment A^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -3 & 0 & | & 0 \\ 3 & 1 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & | & 0 \\ 0 & 10 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & | & 0 \\ 0 & 1 & -\frac{1}{5} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{3}{5} & | & 0 \\ 0 & 1 & -\frac{1}{5} & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 - \frac{3}{5}x_3 = 0$$

$$x_2 - \frac{1}{5}x_3 = 0$$

and then solve the system for the pivot variables.

$$x_1 = \frac{3}{5}x_3$$

$$x_2 = \frac{1}{5}x_3$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

Therefore, the left null space (the null space of the transpose) is

$$N(A^T) = \text{Span}\left(\begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} \right)$$

The row space is

$$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right)$$

but only the first two columns of rref(A^T) are pivot columns, which means the row space can actually be spanned by just the first two columns.

$$C(A^T) = \text{Span}\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\right)$$

The original matrix A has $m = 3$ rows and $n = 2$ columns, so the left null $N(A^T)$ space is a subspace of $\mathbb{R}^3$, and the row space $C(A^T)$ is a subspace of $\mathbb{R}^2$. And the dimension of the left null and row spaces are

$$\text{Dim}(N(A^T)) = m - r = 3 - 2 = 1$$

$$\text{Dim}(C(A^T)) = r = 2$$

6. Find the null and column subspaces of the transpose M^T, identify their spaces $\mathbb{R}^i$, and name the dimension of the subspaces of M^T.

$$M = \begin{bmatrix}2 & 4\\1 & 0\\-1 & -1\\0 & 3\end{bmatrix}$$

Solution:

The transpose of M is

$$M^T = \begin{bmatrix}2 & 1 & -1 & 0\\4 & 0 & -1 & 3\end{bmatrix}$$

To find the null space of the transpose (the left null space), augment M^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & 1 & -1 & 0 & | & 0 \\ 4 & 0 & -1 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & | & 0 \\ 4 & 0 & -1 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & | & 0 \\ 0 & -2 & 1 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{3}{4} & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 - \frac{1}{4}x_3 + \frac{3}{4}x_4 = 0$$

$$x_2 - \frac{1}{2}x_3 - \frac{3}{2}x_4 = 0$$

and then solve the system for the pivot variables.

$$x_1 = \frac{1}{4}x_3 - \frac{3}{4}x_4$$

$$x_2 = \frac{1}{2}x_3 + \frac{3}{2}x_4$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{4} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of the transpose (the left null space) is

$$N(M^T) = \text{Span}\left(\begin{bmatrix}\frac{1}{4}\\ \frac{1}{2}\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}-\frac{3}{4}\\ \frac{3}{2}\\ 0\\ 1\end{bmatrix}\right)$$

The column space of the transpose is

$$C(M^T) = \text{Span}\left(\begin{bmatrix}2\\ 4\end{bmatrix}, \begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}-1\\ -1\end{bmatrix}, \begin{bmatrix}0\\ 3\end{bmatrix}\right)$$

but only the first two columns of rref(M^T) are pivot columns, which means the column space of M^T can actually be spanned by just the first two column vectors.

$$C(M^T) = \text{Span}\left(\begin{bmatrix}2\\ 4\end{bmatrix}, \begin{bmatrix}1\\ 0\end{bmatrix}\right)$$

The original matrix M has $m = 4$ rows and $n = 2$ columns, so the null space of the transpose $N(M^T)$ is a subspace of $\mathbb{R}^4$, and the column space of the transpose $C(M^T)$ is a subspace of $\mathbb{R}^2$. And the dimension of the null and column spaces of the transpose are

$$\text{Dim}(N(M^T)) = m - r = 4 - 2 = 2$$

$$\text{Dim}(C(M^T)) = r = 2$$

THE PRODUCT OF A MATRIX AND ITS TRANSPOSE

■ 1. Is A^TA invertible?

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$$

Solution:

The columns of A are linearly independent, so A^TA is invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & 0 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) + 0(0) + 3(3) & 1(-2) + 0(2) + 3(0) \\ -2(1) + 2(0) + 0(3) & -2(-2) + 2(2) + 0(0) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 0 + 9 & -2 + 0 + 0 \\ -2 + 0 + 0 & 4 + 4 + 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix}10 & -2\\ -2 & 8\end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix}10 & -2\\ -2 & 8\end{bmatrix} \to \begin{bmatrix}1 & -\frac{1}{5}\\ -2 & 8\end{bmatrix} \to \begin{bmatrix}1 & -\frac{1}{5}\\ 0 & \frac{38}{5}\end{bmatrix} \to \begin{bmatrix}1 & -\frac{1}{5}\\ 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Because we get the identity matrix, we can say that A^TA is invertible.

■ 2. Is A^TA invertible?

$$A = \begin{bmatrix}-12 & 6\\ 8 & -4\end{bmatrix}$$

Solution:

The columns of A aren't linearly independent, so A^TA is not invertible. We can confirm this by finding A^TA, and then verifying that A^TA doesn't simplify to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix}-12 & 8\\ 6 & -4\end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} -12 & 8 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} -12 & 6 \\ 8 & -4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -12(-12) + 8(8) & -12(6) + 8(-4) \\ 6(-12) - 4(8) & 6(6) - 4(-4) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 144 + 64 & -72 - 32 \\ -72 - 32 & 36 + 16 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 208 & -104 \\ -104 & 52 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 208 & -104 \\ -104 & 52 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ -104 & 52 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

■ 3. Is A^TA invertible?

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

Solution:

The columns of A are linearly independent, so A^TA is invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ -2 & 2 & -2 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1)+0(0)+1(1) & 1(1)+0(3)+1(0) & 1(-2)+0(2)+1(-2) \\ 1(1)+3(0)+0(1) & 1(1)+3(3)+0(0) & 1(-2)+3(2)+0(-2) \\ -2(1)+2(0)-2(1) & -2(1)+2(3)-2(0) & -2(-2)+2(2)-2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0+1 & 1+0+0 & -2+0-2 \\ 1+0+0 & 1+9+0 & -2+6+0 \\ -2+0-2 & -2+6+0 & 4+4+4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 10 & 4 \\ -4 & 4 & 12 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 10 & 4 \\ -4 & 4 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 4 \\ 2 & 1 & -4 \\ -4 & 4 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 4 \\ 0 & -19 & -12 \\ -4 & 4 & 12 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 10 & 4 \\ 0 & -19 & -12 \\ 0 & 44 & 28 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 4 \\ 0 & 1 & \frac{12}{19} \\ 0 & 44 & 28 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{44}{19} \\ 0 & 1 & \frac{12}{19} \\ 0 & 44 & 28 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{44}{19} \\ 0 & 1 & \frac{12}{19} \\ 0 & 0 & \frac{4}{19} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{44}{19} \\ 0 & 1 & \frac{12}{19} \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{12}{19} \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because we got to the identity matrix, we can say that A^TA is invertible.

■ 4. Is A^TA invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

Solution:

The columns of A are not linearly independent, which means A^TA won't be invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{bmatrix}$$

Then the product A^TA is

$$A^TA=\begin{bmatrix}1&0\\0&1\\-2&3\end{bmatrix}\begin{bmatrix}1&0&-2\\0&1&3\end{bmatrix}$$

$$A^TA=\begin{bmatrix}1(1)+0(0)&1(0)+0(1)&1(-2)+0(3)\\0(1)+1(0)&0(0)+1(1)&0(-2)+1(3)\\-2(1)+3(0)&-2(0)+3(1)&-2(-2)+3(3)\end{bmatrix}$$

$$A^TA=\begin{bmatrix}1+0&0+0&-2+0\\0+0&0+1&0+3\\-2+0&0+3&4+9\end{bmatrix}$$

$$A^TA=\begin{bmatrix}1&0&-2\\0&1&3\\-2&3&13\end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA=\begin{bmatrix}1&0&-2\\0&1&3\\-2&3&13\end{bmatrix}\to\begin{bmatrix}1&0&-2\\0&1&3\\0&3&9\end{bmatrix}\to\begin{bmatrix}1&0&-2\\0&1&3\\0&0&0\end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

■ 5. Is A^TA invertible?

$$A=\begin{bmatrix}4&-2\\-6&3\end{bmatrix}$$

Solution:

The columns of A aren't linearly independent, so A^TA is not invertible. We can confirm this by finding A^TA, and then verifying that A^TA doesn't simplify to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4(4) - 6(-6) & 4(-2) - 6(3) \\ -2(4) + 3(-6) & -2(-2) + 3(3) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 16 + 36 & -8 - 18 \\ -8 - 18 & 4 + 9 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 52 & -26 \\ -26 & 13 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 52 & -26 \\ -26 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ -26 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

6. Is A^TA invertible?

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 3 & 3 \end{bmatrix}$$

Solution:

The columns of A are not linearly independent, which means A^TA won't be invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} -1 & 0 \\ 0 & 3 \\ 2 & 3 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} -1 & 0 \\ 0 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 3 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -1(-1)+0(0) & -1(0)+0(3) & -1(2)+0(3) \\ 0(-1)+3(0) & 0(0)+3(3) & 0(2)+3(3) \\ 2(-1)+3(0) & 2(0)+3(3) & 2(2)+3(3) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0 & 0+0 & -2+0 \\ 0+0 & 0+9 & 0+9 \\ -2+0 & 0+9 & 4+9 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 9 & 9 \\ -2 & 9 & 13 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 9 & 9 \\ -2 & 9 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 9 & 9 \\ 0 & 9 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 9 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

W W W . K R I S T A K I N G M A T H . C O M