Topic: The column space and Ax=b

Question: Find the null space, then find the column space of A.

$$A = \begin{bmatrix} 1 & -5 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 1 & -1 & 2 & 4 \end{bmatrix}$$

Answer choices:

A $\quad N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \end{bmatrix}\right), C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right)$

B $\quad N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}\right), C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right)$

C $\quad N(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right), C(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$

D $\quad N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right), C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right)$

Solution: D

Find the null space of A by first putting the matrix into reduced row-echelon form.

$$\begin{bmatrix}1&-5&2&4\\3&0&1&2\\1&-1&2&4\end{bmatrix}\to\begin{bmatrix}1&-5&2&4\\0&15&-5&-10\\1&-1&2&4\end{bmatrix}\to\begin{bmatrix}1&-5&2&4\\0&15&-5&-10\\0&4&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&-5&2&4\\0&1&-\frac{1}{3}&-\frac{2}{3}\\0&4&0&0\end{bmatrix}\to\begin{bmatrix}1&-5&2&4\\0&1&-\frac{1}{3}&-\frac{2}{3}\\0&0&\frac{4}{3}&\frac{8}{3}\end{bmatrix}\to\begin{bmatrix}1&0&\frac{1}{3}&\frac{2}{3}\\0&1&-\frac{1}{3}&-\frac{2}{3}\\0&0&\frac{4}{3}&\frac{8}{3}\end{bmatrix}$$

$$\begin{bmatrix}1&0&\frac{1}{3}&\frac{2}{3}\\0&1&-\frac{1}{3}&-\frac{2}{3}\\0&0&1&2\end{bmatrix}\to\begin{bmatrix}1&0&\frac{1}{3}&\frac{2}{3}\\0&1&0&0\\0&0&1&2\end{bmatrix}\to\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\end{bmatrix}$$

Set up the matrix equation.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

From this matrix equation, we get

$$x_1=0$$

$$x_2=0$$

$x_3 + 2x_4 = 0$, or $x_3 = -2x_4$

Then the null space of A is all the linear combinations given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

which means the null space is the span of the single column vector.

$$N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right)$$

The column space is all the linear combinations of the column vectors.

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} \right)$$

Topic: The column space and Ax=b

Question: Find the column space of M in terms of its basis.

$$M=\begin{bmatrix}-1 & 2 & 6 & 5\\ 0 & 3 & -7 & 9\\ 3 & -6 & -18 & -15\end{bmatrix}$$

Answer choices:

A $C(M)=\text{Span}\left(\begin{bmatrix}-1\\0\\3\end{bmatrix},\begin{bmatrix}2\\3\\-6\end{bmatrix},\begin{bmatrix}6\\-7\\-18\end{bmatrix},\begin{bmatrix}5\\9\\-15\end{bmatrix}\right)$

B $C(M)=\text{Span}\left(\begin{bmatrix}-1\\0\\3\end{bmatrix},\begin{bmatrix}6\\-7\\-18\end{bmatrix}\right)$

C $C(M)=\text{Span}\left(\begin{bmatrix}-1\\0\\3\end{bmatrix},\begin{bmatrix}2\\3\\-6\end{bmatrix}\right)$

D $C(M)=\text{Span}\left(\begin{bmatrix}-1\\0\\3\end{bmatrix},\begin{bmatrix}2\\3\\-6\end{bmatrix},\begin{bmatrix}6\\-7\\-18\end{bmatrix}\right)$

Solution: C

Put M into row-echelon form.

$$\begin{bmatrix} -1 & 2 & 6 & 5 \\ 0 & 3 & -7 & 9 \\ 3 & -6 & -18 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -5 \\ 0 & 3 & -7 & 9 \\ 3 & -6 & -18 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -5 \\ 0 & 3 & -7 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -5 \\ 0 & 1 & -\frac{7}{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first and second columns of the matrix are the pivot columns, then the first and second columns of the original matrix form a basis for the column space of M.

$$C(M) = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}\right)$$

Topic: The column space and Ax=b

Question: Find the column space of A in terms of its basis.

$$A = \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 3 & 5 & 7 \\ -3 & 6 & 3 & 9 \\ 2 & -4 & -2 & -6 \end{bmatrix}$$

Answer choices:

A $\quad C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}\right)$

B $\quad C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 9 \\ -6 \end{bmatrix}\right)$

C $\quad C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 9 \\ -6 \end{bmatrix}\right)$

D $\quad C(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 3 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 9 \\ -6 \end{bmatrix}\right)$

Solution: A

Put A into row-echelon form.

$$\begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 3 & 5 & 7 \\ -3 & 6 & 3 & 9 \\ 2 & -4 & -2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 15 & -6 \\ 2 & -4 & -2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 15 & -6 \\ 0 & 0 & -10 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & 0 & 15 & -6 \\ 0 & 0 & -10 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{2}{5} \\ 0 & 0 & -10 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first, second, and third columns of the matrix are the pivot columns, then the first, second, and third columns of the original matrix form a basis for the column space of A.

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix} \right)$$