Topic: Orthonormal bases

Question: Which of the vector sets is orthonormal?

$$\vec{v}_1 = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_2 = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$$\vec{v}_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Answer choices:

A $V = \{\vec{v}_1, \vec{v}_2\}$

B $V = \{\vec{v}_1, \vec{v}_3\}$

C $V = \{\vec{v}_2, \vec{v}_3\}$

D None of these

Solution: C

For the set to be orthonormal, each vector needs to have length 1.

$$||\vec{v}_1||^2=\vec{v}_1\cdot\vec{v}_1=\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)$$

$$=\frac{1}{4}+\frac{1}{4}+\frac{1}{2}=1$$

$$||\vec{v}_2||^2=\vec{v}_2\cdot\vec{v}_2=\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)+\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)+\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$$

$$=\frac{1}{9}+\frac{4}{9}+\frac{4}{9}=1$$

$$||\vec{v}_3||^2=\vec{v}_3\cdot\vec{v}_3=(0)(0)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$=0+\frac{1}{2}+\frac{1}{2}=1$$

Each vector has length 1, so now we need to check which of the vectors are orthogonal.

$$\vec{v}_1\cdot\vec{v}_2=\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)+\left(\frac{1}{2}\right)\left(-\frac{2}{3}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{2}{3}\right)$$

$$=\frac{1}{6}-\frac{2}{6}-\frac{2}{3\sqrt{2}}=-\frac{1}{6}-\frac{2}{3\sqrt{2}}$$

$$\vec{v}_1 \cdot \vec{v}_3 = \left(-\frac{1}{2}\right)(0) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= 0 + \frac{1}{2\sqrt{2}} - \frac{1}{2} = \frac{1}{2\sqrt{2}} - \frac{1}{2}$$

$$\vec{v}_2 \cdot \vec{v}_3 = \left(-\frac{1}{3}\right)(0) + \left(-\frac{2}{3}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= 0 - \frac{2}{3\sqrt{2}} + \frac{2}{3\sqrt{2}} = 0$$

So $V = \{\vec{v}_1, \vec{v}_2\}$ and $V = \{\vec{v}_1, \vec{v}_3\}$ are not orthonormal sets since their dot products are nonzero, but $V = \{\vec{v}_2, \vec{v}_3\}$ is an orthonormal set because its dot product is zero.

Topic: Orthonormal bases

Question: Convert $\vec{x} = (-12,6)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{5}{6} \\ -\frac{\sqrt{11}}{6} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{\sqrt{11}}{6} \\ \frac{5}{6} \end{bmatrix}$$

Answer choices:

A $\quad [\vec{x}]_B = \begin{bmatrix} -14 + \sqrt{11} \\ 5 + 2\sqrt{11} \end{bmatrix}$

B $\quad [\vec{x}]_B = \begin{bmatrix} -10 - \sqrt{11} \\ 5 - 2\sqrt{11} \end{bmatrix}$

C $\quad [\vec{x}]_B = \begin{bmatrix} -10 - 2\sqrt{11} \\ 5 - \sqrt{11} \end{bmatrix}$

D $\quad [\vec{x}]_B = \begin{bmatrix} -14 + \sqrt{11} \\ 5 - 2\sqrt{11} \end{bmatrix}$

Solution: B

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2 = \left(\frac{5}{6}\right)^2 + \left(-\frac{\sqrt{11}}{6}\right)^2 = \frac{25}{36} + \frac{11}{36} = \frac{36}{36} = 1$$

$$||\vec{v}_2||^2 = \left(\frac{\sqrt{11}}{6}\right)^2 + \left(\frac{5}{6}\right)^2 = \frac{11}{36} + \frac{25}{36} = \frac{36}{36} = 1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = \left(\frac{5}{6}\right)\left(\frac{\sqrt{11}}{6}\right) + \left(-\frac{\sqrt{11}}{6}\right)\left(\frac{5}{6}\right)$$

$$= \frac{5\sqrt{11}}{36} - \frac{5\sqrt{11}}{36} = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x} = (-12,6)$ can be converted to the alternate basis B with dot products. In other words, instead of solving

$$\begin{bmatrix} \frac{5}{6} & \frac{\sqrt{11}}{6} \\ -\frac{\sqrt{11}}{6} & \frac{5}{6} \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} -12 \\ 6 \end{bmatrix}$$

which would require us to put the augmented matrix into reduced row-echelon form, we can simply take dot products to get the value of $[\vec{x}]_B$.

$$[\vec{x}]_B = \begin{bmatrix} \frac{5}{6}(-12) - \frac{\sqrt{11}}{6}(6) \\ \frac{\sqrt{11}}{6}(-12) + \frac{5}{6}(6) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -10 - \sqrt{11} \\ -2\sqrt{11} + 5 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -10 - \sqrt{11} \\ 5 - 2\sqrt{11} \end{bmatrix}$$

Topic: Orthonormal bases

Question: Convert $\vec{x}=(\sqrt{66},\sqrt{6},\sqrt{11})$ from the standard basis to the alternate basis $B=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$.

$$\vec{v}_1=\begin{bmatrix}\frac{4}{\sqrt{66}}\\ -\frac{7}{\sqrt{66}}\\ \frac{1}{\sqrt{66}}\end{bmatrix},\ \vec{v}_2=\begin{bmatrix}-\frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix},\ \vec{v}_3=\begin{bmatrix}-\frac{1}{\sqrt{11}}\\ -\frac{1}{\sqrt{11}}\\ -\frac{3}{\sqrt{11}}\end{bmatrix}$$

Answer choices:

A $\quad [\vec{x}]_B=\begin{bmatrix}\frac{4\sqrt{66}-7\sqrt{6}+\sqrt{11}}{\sqrt{66}}\\ \frac{-2\sqrt{66}-\sqrt{6}+\sqrt{11}}{\sqrt{6}}\\ \frac{-\sqrt{66}-\sqrt{6}-3\sqrt{11}}{\sqrt{11}}\end{bmatrix}$

B $\quad [\vec{x}]_B=\begin{bmatrix}\frac{-4\sqrt{66}+7\sqrt{6}-\sqrt{11}}{\sqrt{66}}\\ \frac{2\sqrt{66}+\sqrt{6}-\sqrt{11}}{\sqrt{6}}\\ \frac{\sqrt{66}+\sqrt{6}+3\sqrt{11}}{\sqrt{11}}\end{bmatrix}$

C $\quad [\vec{x}]_B=\begin{bmatrix}\frac{4\sqrt{66}+7\sqrt{6}+\sqrt{11}}{\sqrt{66}}\\ \frac{2\sqrt{66}+\sqrt{6}+\sqrt{11}}{\sqrt{6}}\\ \frac{\sqrt{66}+\sqrt{6}+3\sqrt{11}}{\sqrt{11}}\end{bmatrix}$

D $\quad [\vec{x}]_B=\begin{bmatrix}\frac{-4\sqrt{66}-7\sqrt{6}-\sqrt{11}}{\sqrt{66}}\\ \frac{-2\sqrt{66}-\sqrt{6}-\sqrt{11}}{\sqrt{6}}\\ \frac{-\sqrt{66}-\sqrt{6}-3\sqrt{11}}{\sqrt{11}}\end{bmatrix}$

Solution: A

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2=\left(\frac{4}{\sqrt{66}}\right)^2+\left(-\frac{7}{\sqrt{66}}\right)^2+\left(\frac{1}{\sqrt{66}}\right)^2=\frac{16}{66}+\frac{49}{66}+\frac{1}{66}=\frac{66}{66}=1$$

$$||\vec{v}_2||^2=\left(-\frac{2}{\sqrt{6}}\right)^2+\left(-\frac{1}{\sqrt{6}}\right)^2+\left(\frac{1}{\sqrt{6}}\right)^2=\frac{4}{6}+\frac{1}{6}+\frac{1}{6}=\frac{6}{6}=1$$

$$||\vec{v}_3||^2=\left(-\frac{1}{\sqrt{11}}\right)^2+\left(-\frac{1}{\sqrt{11}}\right)^2+\left(-\frac{3}{\sqrt{11}}\right)^2=\frac{1}{11}+\frac{1}{11}+\frac{9}{11}=\frac{11}{11}=1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1\cdot\vec{v}_2=\frac{4}{\sqrt{66}}\left(-\frac{2}{\sqrt{6}}\right)-\frac{7}{\sqrt{66}}\left(-\frac{1}{\sqrt{6}}\right)+\frac{1}{\sqrt{66}}\left(\frac{1}{\sqrt{6}}\right)$$

$$=-\frac{8}{\sqrt{396}}+\frac{7}{\sqrt{396}}+\frac{1}{\sqrt{396}}=\frac{0}{\sqrt{396}}=0$$

$$\vec{v}_1\cdot\vec{v}_3=\frac{4}{\sqrt{66}}\left(-\frac{1}{\sqrt{11}}\right)-\frac{7}{\sqrt{66}}\left(-\frac{1}{\sqrt{11}}\right)+\frac{1}{\sqrt{66}}\left(-\frac{3}{\sqrt{11}}\right)$$

$$=-\frac{4}{\sqrt{726}}+\frac{7}{\sqrt{726}}-\frac{3}{\sqrt{726}}=\frac{0}{\sqrt{726}}=0$$

$$\vec{v}_2 \cdot \vec{v}_3 = -\frac{2}{\sqrt{6}}\left(-\frac{1}{\sqrt{11}}\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{11}}\right) + \frac{1}{\sqrt{6}}\left(-\frac{3}{\sqrt{11}}\right)$$

$$= \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} - \frac{3}{\sqrt{66}} = \frac{0}{\sqrt{66}} = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x} = (\sqrt{66}, \sqrt{6}, \sqrt{11})$ can be converted to the alternate basis B with dot products. In other words, instead of solving

$$\begin{bmatrix} \frac{4}{\sqrt{66}} & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{11}} \\ -\frac{7}{\sqrt{66}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{66}} & \frac{1}{\sqrt{6}} & -\frac{3}{\sqrt{11}} \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} \sqrt{66} \\ \sqrt{6} \\ \sqrt{11} \end{bmatrix}$$

which would require us to put the augmented matrix into reduced row-echelon form, we can simply take dot products to get the value of $[\vec{x}]_B$.

$$[\vec{x}]_B = \begin{bmatrix} \frac{4}{\sqrt{66}}(\sqrt{66}) - \frac{7}{\sqrt{66}}(\sqrt{6}) + \frac{1}{\sqrt{66}}(\sqrt{11}) \\ -\frac{2}{\sqrt{6}}(\sqrt{66}) - \frac{1}{\sqrt{6}}(\sqrt{6}) + \frac{1}{\sqrt{6}}(\sqrt{11}) \\ -\frac{1}{\sqrt{11}}(\sqrt{66}) - \frac{1}{\sqrt{11}}(\sqrt{6}) - \frac{3}{\sqrt{11}}(\sqrt{11}) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{4\sqrt{66}}{\sqrt{66}} - \frac{7\sqrt{6}}{\sqrt{66}} + \frac{\sqrt{11}}{\sqrt{66}} \\ -\frac{2\sqrt{66}}{\sqrt{6}} - \frac{\sqrt{6}}{\sqrt{6}} + \frac{\sqrt{11}}{\sqrt{6}} \\ -\frac{\sqrt{66}}{\sqrt{11}} - \frac{\sqrt{6}}{\sqrt{11}} - \frac{3\sqrt{11}}{\sqrt{11}} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{4\sqrt{66} - 7\sqrt{6} + \sqrt{11}}{\sqrt{66}} \\ \frac{-2\sqrt{66} - \sqrt{6} + \sqrt{11}}{\sqrt{6}} \\ \frac{-\sqrt{66} - \sqrt{6} - 3\sqrt{11}}{\sqrt{11}} \end{bmatrix}$$