Topic: Eigenvalues, eigenvectors, eigenspaces

Question: Find the eigenvalues of the transformation matrix A.

$$A = \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix}$$

Answer choices:

A $\lambda = 3, \lambda = 4$

B $\lambda = -3, \lambda = 4$

C $\lambda = 3, \lambda = -4$

D $\lambda = -3, \lambda = -4$

Solution: B

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 3 & 0 - 0 \\ 0 - 1 & \lambda - 4 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 3 & 0 \\ -1 & \lambda - 4 \end{bmatrix} \right|$$

The determinant is

$$(\lambda + 3)(\lambda - 4) - (0)(-1)$$

$$(\lambda + 3)(\lambda - 4)$$

$\lambda = -3$ or $\lambda = 4$

Topic: Eigenvalues, eigenvectors, eigenspaces

Question: For the transformation matrix A, find the eigenvectors associated with each eigenvalue, $\lambda = -3$ and $\lambda = 4$.

$$A = \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix}$$

$$|\lambda I_n - A| = \begin{bmatrix} \lambda + 3 & 0 \\ -1 & \lambda - 4 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 7 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

B $\begin{bmatrix} -7 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

C $\begin{bmatrix} 7 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

D $\begin{bmatrix} -7 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution: D

With $\lambda = -3$ and $\lambda = 4$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda + 3 & 0 \\ -1 & \lambda - 4 \end{bmatrix}\right)$$

we get

$$E_{-3} = N\left(\begin{bmatrix} -3 + 3 & 0 \\ -1 & -3 - 4 \end{bmatrix}\right)$$

$$E_{-3} = N\left(\begin{bmatrix} 0 & 0 \\ -1 & -7 \end{bmatrix}\right)$$

and

$$E_4 = N\left(\begin{bmatrix} 4 + 3 & 0 \\ -1 & 4 - 4 \end{bmatrix}\right)$$

$$E_4 = N\left(\begin{bmatrix} 7 & 0 \\ -1 & 0 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-3} will satisfy

$$\begin{bmatrix} 0 & 0 \\ -1 & -7 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & | & 0 \\ -1 & -7 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -7 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + 7v_2 = 0$$

So with $v_1 = -7v_2$, we'll substitute $v_2 = t$, and say that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -7 \\ 1 \end{bmatrix}$$

Which means that E_{-3} is defined by

$$E_{-3} = \text{Span}\left(\begin{bmatrix} -7 \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_4 will satisfy

$$\begin{bmatrix} 7 & 0 \\ -1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & | & 0 \\ -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & | & 0 \\ 7 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 \\ 7 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + 0v_2 = 0$$

$$v_1 = 0v_2$$

And with $v_1 = 0v_2$, we'll substitute $v_2 = t$, and say that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Which means that E_4 is defined by

$$E_4 = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} -7 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Topic: Eigenvalues, eigenvectors, eigenspaces

Question: Find the eigenvectors of the transformation matrix.

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

B $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

C $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

D $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Solution: D

Find the determinant $|\lambda I_n - A|$.

$$\left|\lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\begin{bmatrix}2 & -3\\0 & 5\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}-\begin{bmatrix}2 & -3\\0 & 5\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-2 & 0-(-3)\\0-0 & \lambda-5\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-2 & 3\\0 & \lambda-5\end{bmatrix}\right|$$

The determinant is

$$(\lambda-2)(\lambda-5)-(3)(0)$$

$$(\lambda-2)(\lambda-5)$$

$\lambda = 2$ or $\lambda = 5$

With $\lambda = 2$ and $\lambda = 5$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix}\lambda-2 & 3\\0 & \lambda-5\end{bmatrix}\right)$$

we get

$$E_2 = N\left(\begin{bmatrix} 2-2 & 3 \\ 0 & 2-5 \end{bmatrix}\right)$$

$$E_2 = N\left(\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\right)$$

and

$$E_5 = N\left(\begin{bmatrix} 5-2 & 3 \\ 0 & 5-5 \end{bmatrix}\right)$$

$$E_5 = N\left(\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_2 will satisfy

$$\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & | & 0 \\ 0 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_2 = 0$$

So the eigenvector for E_2 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And the eigenvectors in the eigenspace E_5 will satisfy

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = 0$$

$$v_1 = -v_2$$

So the eigenvector for E_5 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$