Linear Algebra Workbook Solutions

Eigenvalues and Eigenvectors

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MATH

EIGENVALUES, EIGENVECTORS, EIGENSPACES

■ 1. Find the eigenvalues of the transformation matrix A.

$$A = \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 2 & -2 \\ 0 & \lambda + 5 \end{bmatrix} \right|$$

The determinant is

$$(\lambda + 2)(\lambda + 5) - (-2)(0)$$

$$(\lambda + 2)(\lambda + 5)$$

$\lambda = -2$ or $\lambda = -5$

2. For the transformation matrix A, find the eigenvectors associated with each eigenvalue, $\lambda = -2$ and $\lambda = -5$.

$$A = \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix}$$

$$\lambda I_n - A = \begin{bmatrix} \lambda + 2 & -2 \\ 0 & \lambda + 5 \end{bmatrix}$$

Solution:

With $\lambda = -2$ and $\lambda = -5$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda + 2 & -2 \\ 0 & \lambda + 5 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2 + 2 & -2 \\ 0 & -2 + 5 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix}\right)$$

and

$$E_{-5} = N\left(\begin{bmatrix} -5 + 2 & -2 \\ 0 & -5 + 5 \end{bmatrix}\right)$$

$$E_{-5} = N\left(\begin{bmatrix} -3 & -2 \\ 0 & 0 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & | & 0 \\ 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_2 = 0$$

So, substituting $v_1 = t$, the eigenvector for E_{-2} will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

And the eigenvectors in the eigenspace E_{-5} will satisfy

$$\begin{bmatrix} -3 & -2 \\ 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + \frac{2}{3}v_2 = 0$$

$$v_1 = -\frac{2}{3}v_2$$

So, substituting $v_2 = t$, the eigenvector for E_{-5} will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which means that E_{-5} is defined by

$$E_{-5} = \text{Span}\left(\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

■ 3. Find the eigenvalues of the transformation matrix A.

$$A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 3 & 1 \\ 5 & \lambda + 1 \end{bmatrix} \right|$$

The determinant is

$$(\lambda - 3)(\lambda + 1) - (1)(5)$$

$$\lambda^2 - 3\lambda + \lambda - 3 - 5$$

$$\lambda^2 - 2\lambda - 8$$

$$(\lambda + 2)(\lambda - 4)$$

$\lambda = -2$ or $\lambda = 4$

4. For the transformation matrix A, find the eigenvectors associated with each eigenvalue, $\lambda = -2$ and $\lambda = 4$.

$$A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$$

$$\lambda I_n - A = \begin{bmatrix} \lambda - 3 & 1 \\ 5 & \lambda + 1 \end{bmatrix}$$

Solution:

With $\lambda = -2$ and $\lambda = 4$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 3 & 1 \\ 5 & \lambda + 1 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2 - 3 & 1 \\ 5 & -2 + 1 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix}\right)$$

and

$$E_4 = N\left(\begin{bmatrix} 4 - 3 & 1 \\ 5 & 4 + 1 \end{bmatrix}\right)$$

$$E_4 = N\left(\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 & | & 0 \\ 5 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} & | & 0 \\ 5 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 - \frac{1}{5}v_2 = 0$$

$$v_1 = \frac{1}{5}v_2$$

So, substituting $v_2 = t$, the eigenvector for E_{-2} will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}\right)$$

And the eigenvectors in the eigenspace E_4 will satisfy

$$\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & | & 0 \\ 5 & 5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = 0$$

$$v_1 = -v_2$$

So, substituting $v_2 = t$, the eigenvector for E_4 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Which means that E_4 is defined by

$$E_4 = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

■ 5. Find the eigenvectors of the transformation matrix.

$$A = \begin{bmatrix} 5 & 0 \\ -4 & 3 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ -4 & 3 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ -4 & 3 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 5 & 0 \\ 4 & \lambda - 3 \end{bmatrix} \right|$$

The determinant is

$$(\lambda - 5)(\lambda - 3) - (0)(4)$$

$$(\lambda - 5)(\lambda - 3)$$

$\lambda = 5$ or $\lambda = 3$

With $\lambda = 5$ and $\lambda = 3$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 5 & 0 \\ 4 & \lambda - 3 \end{bmatrix} \right)$$

we get

$$E_5 = N\left(\begin{bmatrix} 5-5 & 0 \\ 4 & 5-3 \end{bmatrix}\right)$$

$$E_5 = N\left(\begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3-5 & 0 \\ 4 & 3-3 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} -2 & 0 \\ 4 & 0 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_5 will satisfy

$$\begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & | & 0 \\ 4 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + \frac{1}{2}v_2 = 0$$

$$v_1 = -\frac{1}{2}v_2$$

So, substituting $v_2 = t$, the eigenvector for E_5 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which means that E_5 is defined by

$$E_5 = \text{Span}\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_3 will satisfy

$$\begin{bmatrix} -2 & 0 \\ 4 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & | & 0 \\ 4 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 \\ 4 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = 0$$

So, substituting $v_2 = t$, the eigenvector for E_5 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Which means that E_5 is defined by

$$E_5 = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

■ 6. Find the eigenvectors of the transformation matrix.

$$A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 6 & 2 \\ -2 & \lambda - 1 \end{bmatrix} \right|$$

The determinant is

$$(\lambda - 6)(\lambda - 1) - (2)(-2)$$

$$\lambda^2 - 6\lambda - \lambda + 6 + 4$$

$$\lambda^2 - 7\lambda + 10$$

$$(\lambda - 2)(\lambda - 5)$$

$\lambda = 2$ or $\lambda = 5$

With $\lambda = 2$ and $\lambda = 5$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 6 & 2 \\ -2 & \lambda - 1 \end{bmatrix}\right)$$

we get

$$E_2 = N\left(\begin{bmatrix} 2 - 6 & 2 \\ -2 & 2 - 1 \end{bmatrix}\right)$$

$$E_2 = N\left(\begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix}\right)$$

and

$$E_5 = N\left(\begin{bmatrix} 5 - 6 & 2 \\ -2 & 5 - 1 \end{bmatrix}\right)$$

$$E_5 = N\left(\begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_2 will satisfy

$$\begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & | & 0 \\ -2 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & | & 0 \\ -2 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 - \frac{1}{2}v_2 = 0$$

$$v_1 = \frac{1}{2}v_2$$

So, substituting $v_2 = t$, the eigenvector for E_2 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which means that E_2 is defined by

$$E_2 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_5 will satisfy

$$\begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & | & 0 \\ -2 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & | & 0 \\ -2 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 - 2v_2 = 0$$

$$v_1 = 2v_2$$

So, substituting $v_2 = t$, the eigenvector for E_5 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Which means that E_5 is defined by

$$E_5 = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

EIGEN IN THREE DIMENSIONS

■ 1. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 2 & -4 & -3 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 5 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda + 2) \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 5 \end{vmatrix} - 0 \begin{vmatrix} -4 & -3 \\ 0 & \lambda + 5 \end{vmatrix} + 0 \begin{vmatrix} -4 & -3 \\ \lambda - 1 & 0 \end{vmatrix}$$

$$(\lambda+2)[(\lambda-1)(\lambda+5)-(0)(0)]-0+0$$

$$(\lambda+2)(\lambda-1)(\lambda+5)$$

$\lambda=-5$, $\lambda=-2$, or $\lambda=1$

With $\lambda=-5$, $\lambda=-2$, and $\lambda=1$, we'll have two eigenspaces, given by $E_\lambda=N(\lambda I_n-A)$. With

$$E_\lambda=N\left(\begin{bmatrix}\lambda+2 & -4 & -3\\ 0 & \lambda-1 & 0\\ 0 & 0 & \lambda+5\end{bmatrix}\right)$$

we get

$$E_{-5}=N\left(\begin{bmatrix}-5+2 & -4 & -3\\ 0 & -5-1 & 0\\ 0 & 0 & -5+5\end{bmatrix}\right)$$

$$E_{-5}=N\left(\begin{bmatrix}-3 & -4 & -3\\ 0 & -6 & 0\\ 0 & 0 & 0\end{bmatrix}\right)$$

and

$$E_{-2}=N\left(\begin{bmatrix}-2+2 & -4 & -3\\ 0 & -2-1 & 0\\ 0 & 0 & -2+5\end{bmatrix}\right)$$

$$E_{-2}=N\left(\begin{bmatrix}0 & -4 & -3\\ 0 & -3 & 0\\ 0 & 0 & 3\end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1+2 & -4 & -3 \\ 0 & 1-1 & 0 \\ 0 & 0 & 1+5 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 3 & -4 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-5} will satisfy

$$\begin{bmatrix} -3 & -4 & -3 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -4 & -3 & | & 0 \\ 0 & -6 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & 1 & | & 0 \\ 0 & -6 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + v_3 = 0$$

$$v_2 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -v_3$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-5} is defined by

$$E_{-5} = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} 0 & -4 & -3 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 & -3 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & \frac{3}{4} & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & \frac{3}{4} & | & 0 \\ 0 & 0 & \frac{9}{4} & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & \frac{3}{4} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_2 = 0$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 3 & -4 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -4 & -3 & | & 0 \\ 0 & 0 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} & -1 & | & 0 \\ 0 & 0 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & -1 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 - \frac{4}{3}v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = \frac{4}{3}v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix}\right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$$

■ 2. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left|\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right|$$

$$\left|\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right|$$

$$\left| \begin{bmatrix} \lambda-2 & 2 & -1 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda-2)\begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda+1 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 \\ 0 & \lambda+1 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ \lambda-2 & 0 \end{vmatrix}$$

$$(\lambda-2)[(\lambda-2)(\lambda+1)-(0)(0)]-0+0$$

$$(\lambda-2)(\lambda-2)(\lambda+1)$$

$\lambda = -1$ or $\lambda = 2$

With $\lambda = -1$ or $\lambda = 2$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda-2 & 2 & -1 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix} \right)$$

we get

$$E_{-1} = N\left(\begin{bmatrix} -1-2 & 2 & -1 \\ 0 & -1-2 & 0 \\ 0 & 0 & -1+1 \end{bmatrix} \right)$$

$$E_{-1} = N\left(\begin{bmatrix} -3 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

and

$$E_2 = N\left(\begin{bmatrix} 2-2 & 2 & -1 \\ 0 & 2-2 & 0 \\ 0 & 0 & 2+1 \end{bmatrix}\right)$$

$$E_2 = N\left(\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-1} will satisfy

$$\begin{bmatrix} -3 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & -1 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + \frac{1}{3}v_3 = 0$$

$$v_2 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -\frac{1}{3}v_3$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-1} is defined by

$$E_{-1} = \text{Span}\left(\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_2 will satisfy

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & -1 & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_2 = 0$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_2 is defined by

$$E_2 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

■ 3. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -3 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 3 & 0 & 0 \\ 4 & \lambda - 1 & 0 \\ 0 & -5 & \lambda - 2 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda + 3) \begin{vmatrix} \lambda - 1 & 0 \\ -5 & \lambda - 2 \end{vmatrix} - 0 \begin{vmatrix} 4 & 0 \\ 0 & \lambda - 2 \end{vmatrix} + 0 \begin{vmatrix} 4 & \lambda - 1 \\ 0 & -5 \end{vmatrix}$$

$$(\lambda + 3)[(\lambda - 1)(\lambda - 2) - (0)(-5)] - 0 + 0$$

$$(\lambda + 3)(\lambda - 1)(\lambda - 2)$$

$\lambda = -3$ or $\lambda = 1$ or $\lambda = 2$

With $\lambda = -3$, $\lambda = 1$ and $\lambda = 2$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda + 3 & 0 & 0 \\ 4 & \lambda - 1 & 0 \\ 0 & -5 & \lambda - 2 \end{bmatrix}\right)$$

we get

$$E_{-3} = N\left(\begin{bmatrix} -3 + 3 & 0 & 0 \\ 4 & -3 - 1 & 0 \\ 0 & -5 & -3 - 2 \end{bmatrix}\right)$$

$$E_{-3} = N\left(\begin{bmatrix} 0 & 0 & 0 \\ 4 & -4 & 0 \\ 0 & -5 & -5 \end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1 + 3 & 0 & 0 \\ 4 & 1 - 1 & 0 \\ 0 & -5 & 1 - 2 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 4 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & -5 & -1 \end{bmatrix}\right)$$

and

$$E_2 = N\left(\begin{bmatrix} 2+3 & 0 & 0 \\ 4 & 2-1 & 0 \\ 0 & -5 & 2-2 \end{bmatrix}\right)$$

$$E_2 = N\left(\begin{bmatrix} 5 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -5 & 0 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-3} will satisfy

$$\begin{bmatrix} 0 & 0 & 0 \\ 4 & -4 & 0 \\ 0 & -5 & -5 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 4 & -4 & 0 & | & 0 \\ 0 & -5 & -5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -4 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & -5 & -5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -4 & 0 & | & 0 \\ 0 & -5 & -5 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & -5 & -5 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + v_3 = 0$$

$$v_2 + v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -v_3$$

$$v_2 = -v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Which means that E_{-3} is defined by

$$E_{-3} = \text{Span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 4 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & -5 & -1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 & | & 0 \\ 4 & 0 & 0 & | & 0 \\ 0 & -5 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 4 & 0 & 0 & | & 0 \\ 0 & -5 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & -5 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -5 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & \frac{1}{5} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 + \frac{1}{5}v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = 0$$

$$v_2 = -\frac{1}{5}v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_2 will satisfy

$$\begin{bmatrix} 5 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -5 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 & | & 0 \\ 4 & 1 & 0 & | & 0 \\ 0 & -5 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 4 & 1 & 0 & | & 0 \\ 0 & -5 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & -5 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_2 is defined by

$$E_2 = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

■ 4. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ -2 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ -2 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 2 & \lambda + 3 & 0 \\ -3 & -1 & \lambda + 5 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda-4)\begin{vmatrix}\lambda+3 & 0\\ -1 & \lambda+5\end{vmatrix}-0\begin{vmatrix}2 & 0\\ -3 & \lambda+5\end{vmatrix}+0\begin{vmatrix}2 & \lambda+3\\ -3 & -1\end{vmatrix}$$

$$(\lambda-4)[(\lambda+3)(\lambda+5)-(0)(-1)]-0+0$$

$$(\lambda-4)(\lambda+3)(\lambda+5)$$

$\lambda=-5$ or $\lambda=-3$ or $\lambda=4$

With $\lambda=-5$, $\lambda=-3$ and $\lambda=4$, we'll have three eigenspaces, given by $E_\lambda=N(\lambda I_n-A)$. With

$$E_\lambda=N\left(\begin{bmatrix}\lambda-4 & 0 & 0\\ 2 & \lambda+3 & 0\\ -3 & -1 & \lambda+5\end{bmatrix}\right)$$

we get

$$E_{-5}=N\left(\begin{bmatrix}-5-4 & 0 & 0\\ 2 & -5+3 & 0\\ -3 & -1 & -5+5\end{bmatrix}\right)$$

$$E_{-5}=N\left(\begin{bmatrix}-9 & 0 & 0\\ 2 & -2 & 0\\ -3 & -1 & 0\end{bmatrix}\right)$$

and

$$E_{-3} = N\left(\begin{bmatrix} -3-4 & 0 & 0 \\ 2 & -3+3 & 0 \\ -3 & -1 & -3+5 \end{bmatrix}\right)$$

$$E_{-3} = N\left(\begin{bmatrix} -7 & 0 & 0 \\ 2 & 0 & 0 \\ -3 & -1 & 2 \end{bmatrix}\right)$$

and

$$E_{4} = N\left(\begin{bmatrix} 4-4 & 0 & 0 \\ 2 & 4+3 & 0 \\ -3 & -1 & 4+5 \end{bmatrix}\right)$$

$$E_{4} = N\left(\begin{bmatrix} 0 & 0 & 0 \\ 2 & 7 & 0 \\ -3 & -1 & 9 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-5} will satisfy

$$\begin{bmatrix} -9 & 0 & 0 \\ 2 & -2 & 0 \\ -3 & -1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -9 & 0 & 0 & | & 0 \\ 2 & -2 & 0 & | & 0 \\ -3 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 2 & -2 & 0 & | & 0 \\ -3 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -2 & 0 & | & 0 \\ -3 & -1 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -2 & 0 & | & 0 \\ 0 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-5} is defined by

$$E_{-5} = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_{-3} will satisfy

$$\begin{bmatrix} -7 & 0 & 0 \\ 2 & 0 & 0 \\ -3 & -1 & 2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 0 & 0 & | & 0 \\ 2 & 0 & 0 & | & 0 \\ -3 & -1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 2 & 0 & 0 & | & 0 \\ -3 & -1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ -3 & -1 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ -3 & -1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 - 2v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = 0$$

$$v_2 = 2v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Which means that E_{-3} is defined by

$$E_{-3} = \text{Span}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_4 will satisfy

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 7 & 0 \\ -3 & -1 & 9 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 2 & 7 & 0 & | & 0 \\ -3 & -1 & 9 & | & 0 \end{bmatrix} \to \begin{bmatrix} -3 & -1 & 9 & | & 0 \\ 2 & 7 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} & -3 & | & 0 \\ 2 & 7 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & -3 & | & 0 \\ 0 & \frac{19}{3} & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} & -3 & | & 0 \\ 0 & 1 & \frac{18}{19} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{63}{19} & | & 0 \\ 0 & 1 & \frac{18}{19} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{63}{19} \\ 0 & 1 & \frac{18}{19} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 - \frac{63}{19}v_3 = 0$$

$$v_2 + \frac{18}{19}v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = \frac{63}{19}v_3$$

$$v_2 = -\frac{18}{19}v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} \frac{63}{19} \\ -\frac{18}{19} \\ 1 \end{bmatrix}$$

Which means that E_4 is defined by

$$E_4 = \text{Span}\left(\begin{bmatrix} \frac{63}{19} \\ -\frac{18}{19} \\ 1 \end{bmatrix} \right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} \frac{63}{19} \\ -\frac{18}{19} \\ 1 \end{bmatrix}$$

5. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 3 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda - 1)\begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 3 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 0 \\ 0 & \lambda + 3 \end{vmatrix} + 0\begin{vmatrix} -2 & \lambda - 1 \\ 0 & 0 \end{vmatrix}$$

$$(\lambda - 1)[(\lambda - 1)(\lambda + 3) - (0)(0)] + 2[(-2)(\lambda + 3) - (0)(0)] + 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda + 3) - 4(\lambda + 3)$$

$$(\lambda + 3)((\lambda - 1)^2 - 4)$$

$$(\lambda + 3)(\lambda^2 - \lambda - \lambda + 1 - 4)$$

$$(\lambda + 3)(\lambda^2 - 2\lambda - 3)$$

$$(\lambda + 3)(\lambda + 1)(\lambda - 3)$$

$\lambda = -3$ or $\lambda = -1$ or $\lambda = 3$

With $\lambda = -3$, $\lambda = -1$ and $\lambda = 3$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 3 \end{bmatrix}\right)$$

we get

$$E_{-3} = N\left(\begin{bmatrix} -3 - 1 & -2 & 0 \\ -2 & -3 - 1 & 0 \\ 0 & 0 & -3 + 3 \end{bmatrix}\right)$$

$$E_{-3} = N\left(\begin{bmatrix} -4 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

and

$$E_{-1} = N\left(\begin{bmatrix} -1 - 1 & -2 & 0 \\ -2 & -1 - 1 & 0 \\ 0 & 0 & -1 + 3 \end{bmatrix}\right)$$

$$E_{-1} = N\left(\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3 - 1 & -2 & 0 \\ -2 & 3 - 1 & 0 \\ 0 & 0 & 3 + 3 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-3} will satisfy

$$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 & 0 & | & 0 \\ -2 & -4 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 0 & | & 0 \\ -2 & -4 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-3} is defined by

$$E_{-3} = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_{-1} will satisfy

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 0 & | & 0 \\ -2 & -2 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ -2 & -2 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_{-1} is defined by

$$E_{-1} = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_3 will satisfy

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 0 & | & 0 \\ -2 & 2 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ -2 & 2 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

We get the system of equations

$$v_1 - v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_3 is defined by

$$E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

■ 6. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 4 & -3 & 0 \\ -3 & \lambda + 4 & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda + 4) \begin{vmatrix} \lambda + 4 & 0 \\ 0 & \lambda + 1 \end{vmatrix} - (-3) \begin{vmatrix} -3 & 0 \\ 0 & \lambda + 1 \end{vmatrix} - 0 \begin{vmatrix} -3 & \lambda + 4 \\ 0 & 0 \end{vmatrix}$$

$$(\lambda + 4)[(\lambda + 4)(\lambda + 1) - (0)(0)] + 3[(-3)(\lambda + 1) - (0)(0)] - 0$$

$$(\lambda + 4)(\lambda + 4)(\lambda + 1) - 9(\lambda + 1)$$

$$(\lambda + 1)((\lambda + 4)^2 - 9)$$

$$(\lambda + 1)(\lambda^2 + 8\lambda + 16 - 9)$$

$$(\lambda + 1)(\lambda^2 + 8\lambda + 7)$$

$$(\lambda+1)(\lambda+1)(\lambda+7)$$

$$\lambda=-1 \text{ or } \lambda=-7$$

With $\lambda=-1$ and $\lambda=-7$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda+4 & -3 & 0 \\ -3 & \lambda+4 & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix}\right)$$

we get

$$E_{-7} = N\left(\begin{bmatrix} -7+4 & -3 & 0 \\ -3 & -7+4 & 0 \\ 0 & 0 & -7+1 \end{bmatrix}\right)$$

$$E_{-7} = N\left(\begin{bmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix}\right)$$

and

$$E_{-1} = N\left(\begin{bmatrix} -1+4 & -3 & 0 \\ -3 & -1+4 & 0 \\ 0 & 0 & -1+1 \end{bmatrix}\right)$$

$$E_{-1} = N\left(\begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-7} will satisfy

$$\begin{bmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 & 0 & | & 0 \\ -3 & -3 & 0 & | & 0 \\ 0 & 0 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ -3 & -3 & 0 & | & 0 \\ 0 & 0 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & -6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_{-7} is defined by

$$E_{-7} = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-1} will satisfy

$$\begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 0 & | & 0 \\ -3 & 3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ -3 & 3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the equation

$$v_1 - v_2 = 0$$

and then we solve it for the pivot variables.

$$v_1 = v_2$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_{-1} is defined by

$$E_{-1} = \text{Span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix}-1\\1\\0\end{bmatrix} \text{ and } \begin{bmatrix}1\\1\\0\end{bmatrix}$$

W W W . K R I S T A K I N G M A T H . C O M