Linear Algebra Workbook

Matrices as vectors

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VECTORS

■ 1. For the matrix A, find the row vectors, the space, $\mathbb{R}^n$, that contains the row vectors, and the dimension of the space they form.

$$A = \begin{bmatrix} -4 & 8 & 6 & 12 & -1 \\ 3 & -2 & 18 & 0 & -3 \\ 12 & -17 & -4 & 1 & 1 \end{bmatrix}$$

■ 2. For the matrix B, find the column vectors, the space, $\mathbb{R}^n$, that contains the column vectors, and the dimension of the space they form.

$$B = \begin{bmatrix} 12 & 0 & 9 \\ 3 & -21 & -1 \\ -7 & 4 & 13 \end{bmatrix}$$

■ 3. Sketch the vectors in standard position.

$$\overrightarrow{w} = (1,2),\ \overrightarrow{x} = (-5,0),\ \overrightarrow{y} = (-3,-4),\ \overrightarrow{z} = (0,-1)$$

■ 4. Sketch the vectors in order from tip to tail (where the terminal point of one is the initial point of the next), starting at the origin, and determine the shape they form.

$\overrightarrow{a}_1 = (1,2)$ $\qquad$ $\overrightarrow{a}_3 = (1,-2)$ $\qquad$ $\overrightarrow{a}_5 = (-2,0)$

$\vec{a}_2 = (2,0)$ $\quad$ $\vec{a}_4 = (-1, -2)$ $\quad$ $\vec{a}_6 = (-1,2)$

■ 5. Find $\vec{b}_1 + \vec{b}_2$, $\vec{b}_1 - \vec{b}_2$, and $2\vec{b}_2$.

$$\vec{b}_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}$$

■ 6. Is the product of $\vec{b}_1$ and $\vec{b}_2$ defined? Why or why not?

$$\vec{b}_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}$$

VECTOR OPERATIONS

■ 1. Find $\vec{u}+\vec{w}$, $\vec{x}-\vec{y}$, and $\vec{v}-(\vec{w}+\vec{u})$.

$\vec{u}=(-3,5)$ $\vec{w}=(5,-13)$ $\vec{y}=(1,4,2)$

$\vec{v}=(2,1)$ $\vec{x}=(4,5,-7)$

■ 2. Sketch $\vec{u}+\vec{w}$, $\vec{x}-\vec{y}$, and $\vec{v}-(\vec{w}+\vec{u})$.

$\vec{u}=(-3,5)$ $\vec{w}=(5,-13)$ $\vec{y}=(1,4,2)$

$\vec{v}=(2,1)$ $\vec{x}=(4,5,-7)$

■ 3. Find $b\vec{x}$, $c\vec{u}+b\vec{u}$, and $(c+b)\vec{u}$. What can you say about the relationship between $c\vec{u}+b\vec{u}$ and $(c+b)\vec{u}$.

$\vec{u}=(-3,5)$ $b=-1$

$\vec{x}=(4,5,-7)$ $c=3$

■ 4. Find $\vec{x}+b\vec{y}-c\vec{x}-\vec{y}$.

$\vec{x}=(4,5,-7)$ $b=-1$

$\vec{y}=(1,4,2)$ $c=3$

■ 5. Sketch the individual vectors from tip to tail.

$$\vec{x} = (4,5,-7)$$

$$\vec{y} = (1,4,2)$$

■ 6. Find $\vec{x} \cdot \vec{y}$, $\vec{w} \cdot \vec{w}$, and $b(\vec{u} \cdot \vec{v})$.

$\vec{u} = (-3,5)$ $\quad$ $\vec{w} = (5,-13)$ $\quad$ $\vec{y} = (1,4,2)$

$\vec{v} = (2,1)$ $\quad$ $\vec{x} = (4,5,-7)$ $\quad$ $b = -1$

UNIT VECTORS AND BASIS VECTORS

■ 1. Change each vector to a unit vector.

$$\vec{a}=(3,-4)$$

$$\vec{b}=(12,2)$$

$$\vec{c}=(0,7,1)$$

■ 2. Confirm that the vectors each have length 1.

$$\hat{u}_a=\frac{1}{5}\begin{bmatrix}3\\-4\end{bmatrix}$$

$$\hat{u}_b=\frac{1}{\sqrt{37}}\begin{bmatrix}6\\1\end{bmatrix}$$

$$\hat{u}_c=\frac{1}{5\sqrt{2}}\begin{bmatrix}0\\7\\1\end{bmatrix}$$

■ 3. What are the basis vectors for $\mathbb{R}^4$?

■ 4. Express the vectors as linear combinations of the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$.

$$\hat{u}_a = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\hat{u}_b = \frac{1}{\sqrt{37}}\begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$\hat{u}_c = \frac{1}{5\sqrt{2}}\begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

■ 5. Express $\overrightarrow{v} = (x, 2x, -1)$ in terms of the standard basis vectors.

■ 6. Sketch the basis vectors $\hat{i}$ and $\hat{j}$ in $\mathbb{R}^2$, and the vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ in $\mathbb{R}^3$.

LINEAR COMBINATIONS AND SPAN

■ 1. Say whether each of the following is a linear combination. If it isn't, say why.

$$-\pi\vec{x} - e\vec{y}$$

$$\vec{x} \cdot \vec{y}$$

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{u} = \frac{1}{\sqrt{2}}((3,0) - (1,1))$$

$$||\vec{b}||$$

■ 2. Do the vectors span $\mathbb{R}^4$?

$$\left\{ \begin{bmatrix} 3 \\ \frac{1}{2} \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -\pi \\ \pi \\ \pi \\ -\pi \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ 8 \\ 22 \\ 9 \end{bmatrix} \right\}$$

■ 3. Do the vectors span $\mathbb{R}^4$?

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

■ 4. Do the vectors span $\mathbb{R}^2$?

$$\left\{\begin{bmatrix}44\\-8\end{bmatrix}, \begin{bmatrix}11\\-2\end{bmatrix}\right\}$$

■ 5. What is the zero vector $\vec{O}$ in $\mathbb{R}^5$? What is its span?

■ 6. Prove that any vector $\vec{v}=(v_1, v_2, v_3)$ in $\mathbb{R}^3$ can be reached by a linear combination of $\hat{i}$, $\hat{j}$, and $\hat{k}$.

LINEAR INDEPENDENCE IN TWO DIMENSIONS

■ 1. Are the column vectors of the following matrix linearly independent?

$$A = \begin{bmatrix} 2 & 6 & 7 \\ -1 & 11 & 3 \end{bmatrix}$$

■ 2. Show how one of the vectors could be written as a linear combination of the other two.

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{z} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

■ 3. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

■ 4. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{a} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

5. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$$

6. Use a matrix to say whether the vectors are linearly dependent or linearly independent.

$$\vec{x} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \vec{y} = \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$$

LINEAR INDEPENDENCE IN THREE DIMENSIONS

1. Use a matrix to say whether the vector set is linearly independent.

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$$

2. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$$

3. Use a matrix to say whether the vector set is linearly independent.

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

4. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

5. Is the vector set linearly independent? Why or why not?

$$\vec{u}=\begin{bmatrix}1\\4\\5\end{bmatrix}, \vec{v}=\begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{w}=\begin{bmatrix}3\\6\\8\end{bmatrix}, \vec{x}=\begin{bmatrix}-2\\7\\1\end{bmatrix}$$

6. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{u}=\begin{bmatrix}1\\4\\5\end{bmatrix}, \vec{v}=\begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{w}=\begin{bmatrix}3\\6\\8\end{bmatrix}, \vec{x}=\begin{bmatrix}-2\\7\\1\end{bmatrix}$$

LINEAR SUBSPACES

1. What are the criteria that define a subspace? Which criteria is logically part of another criteria?

2. Sketch the graph of each space.

$$V_a = \left\{(x, y) \in \mathbb{R}^2 \mid x, y \leq -1\right\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid y < x^2\right\}$$

$$V_c = \left\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, y \leq x\right\}$$

3. What space is being described by each of the sets?

$$V_a = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0\right\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0,\ x = y\right\}$$

$$V_c = \left\{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\right\}$$

4. Are these spaces subspaces?

$$V_a = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0\right\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0,\ x = y\right\}$$

$$V_c = \left\{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\right\}$$

■ 5. Show that each space is not a subspace.

$$V_a = \left\{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - 7z = 3\right\}$$

$$V_b = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \le 0\right\}$$

$$V_c = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\right\}$$

■ 6. Prove that the zero vector $\overrightarrow{O} = (0,0,0)$ is a subspace of $\mathbb{R}^3$.

SPANS AS SUBSPACES

1. Sketch the spans together on the same set of axes.

$$V = \text{Span}\left(\begin{bmatrix}1\\-3\end{bmatrix}\right)$$

$$V = \text{Span}\left(\begin{bmatrix}0\\2\end{bmatrix}\right)$$

2. Show that that spans are subspaces.

$$V = \text{Span}\left(\begin{bmatrix}1\\-3\end{bmatrix}\right)$$

$$V = \text{Span}\left(\begin{bmatrix}0\\2\end{bmatrix}\right)$$

3. Prove that the span forms a subspace of $\mathbb{R}^3$.

$$\text{Span}\left(\begin{bmatrix}-6\\5\\1\end{bmatrix}\right)$$

4. Write the line $y = 3x + 2$ in set notation, and then write it as a single vector, only using x.

5. Write the line $2y + 4x = 0$ in set notation, and then write it as a single vector, only using y.

6. Write the line $y = 3x + 2$ as the linear combination of two vectors. Then plug in $x = -1$, $x = 0$, and $x = 1$, and sketch all three in the same plane.

BASIS

1. What requirements must be met in order for a vector set to form the basis for a space?

2. What's the standard basis for $\mathbb{R}^4$?

3. Say whether or not the vector set V forms a basis for $\mathbb{R}^2$.

$$V = \text{Span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$$

4. Which scalars c_1 and c_2 would you need to form the vector $\vec{v} = (7, -3)$ as a linear combination of the vectors in the span?

$$V = \text{Span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$$

5. Say whether the span forms a basis for $\mathbb{R}^3$.

$$V = \text{Span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\4\end{bmatrix}\right)$$

6. What scalars would you need to get the vector $\vec{v}=(2,0,-5)$ from a linear combination of the set V?

$$V=\text{Span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\1\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\4\end{bmatrix}\right)$$