Linear Algebra Workbook Solutions

Matrices as vectors

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VECTORS

1. For the matrix A, find the row vectors, the space, $\mathbb{R}^n$, that contains the row vectors, and the dimension of the space they form.

$$A = \begin{bmatrix} -4 & 8 & 6 & 12 & -1 \\ 3 & -2 & 18 & 0 & -3 \\ 12 & -17 & -4 & 1 & 1 \end{bmatrix}$$

Solution:

The row vectors of A are

$$a_1 = [-4 \quad 8 \quad 6 \quad 12 \quad -1]$$

$$a_2 = [3 \quad -2 \quad 18 \quad 0 \quad -3]$$

$$a_3 = [12 \quad -17 \quad -4 \quad 1 \quad 1]$$

Since the row vectors each have five components, they're contained in $\mathbb{R}^5$ space, and because there are three vectors, they form a three-dimensional space, like $\mathbb{R}^3$, within $\mathbb{R}^5$.

2. For the matrix B, find the column vectors, the space, $\mathbb{R}^n$, that contains the column vectors, and the dimension of the space they form.

$$B = \begin{bmatrix} 12 & 0 & 9 \\ 3 & -21 & -1 \\ -7 & 4 & 13 \end{bmatrix}$$

Solution:

The column vectors of B are

$$b_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}, b_3 = \begin{bmatrix} 9 \\ -1 \\ 13 \end{bmatrix}$$

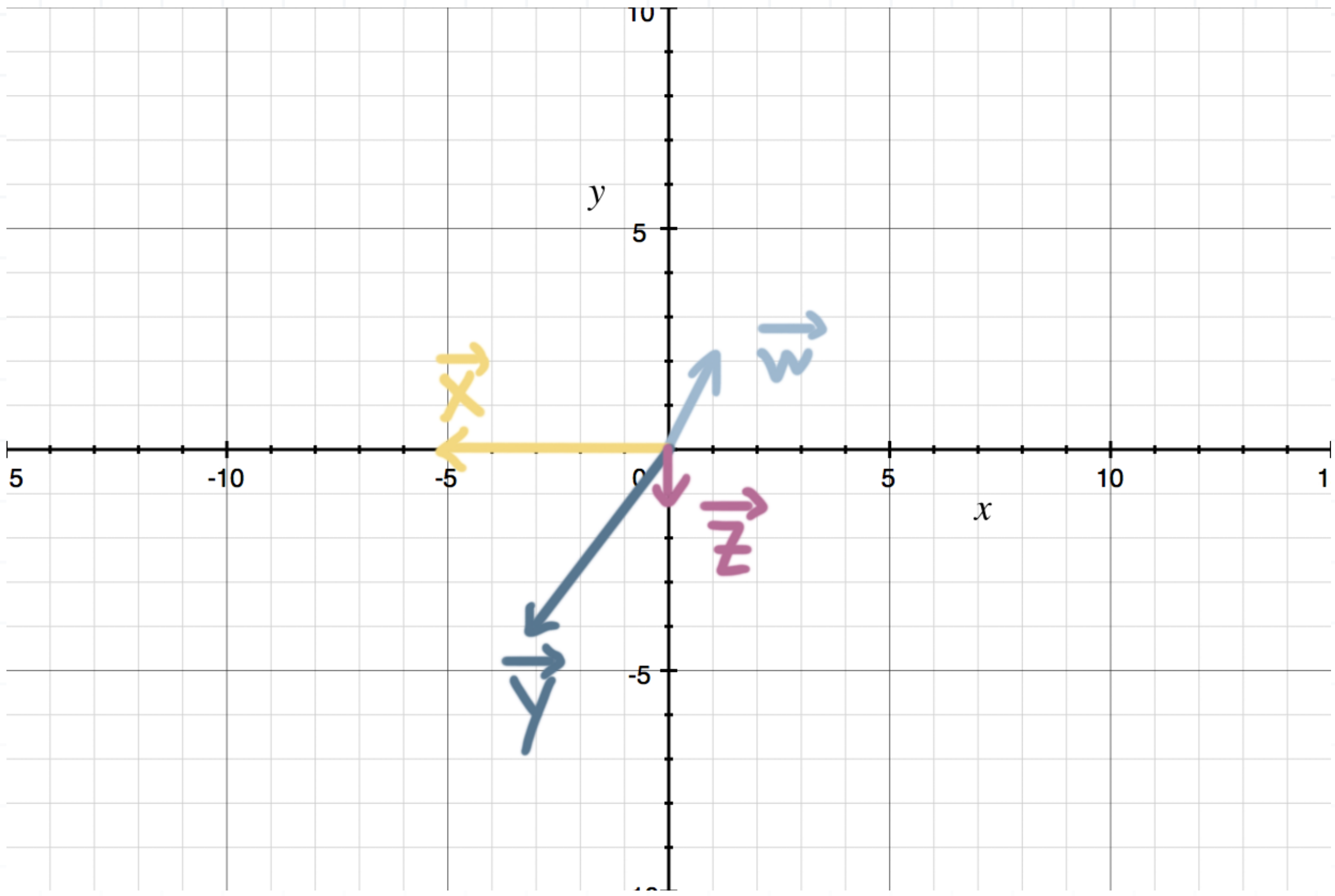
Since the column vectors each have three components, they're contained in $\mathbb{R}^3$ space, and because there are three vectors, they form a three-dimensional space in $\mathbb{R}^3$.

3. Sketch the vectors in standard position.

$$\overrightarrow{w} = (1,2), \ \overrightarrow{x} = (-5,0), \ \overrightarrow{y} = (-3,-4), \ \vec{z} = (0,-1)$$

Solution:

A sketch of the vectors in standard position is



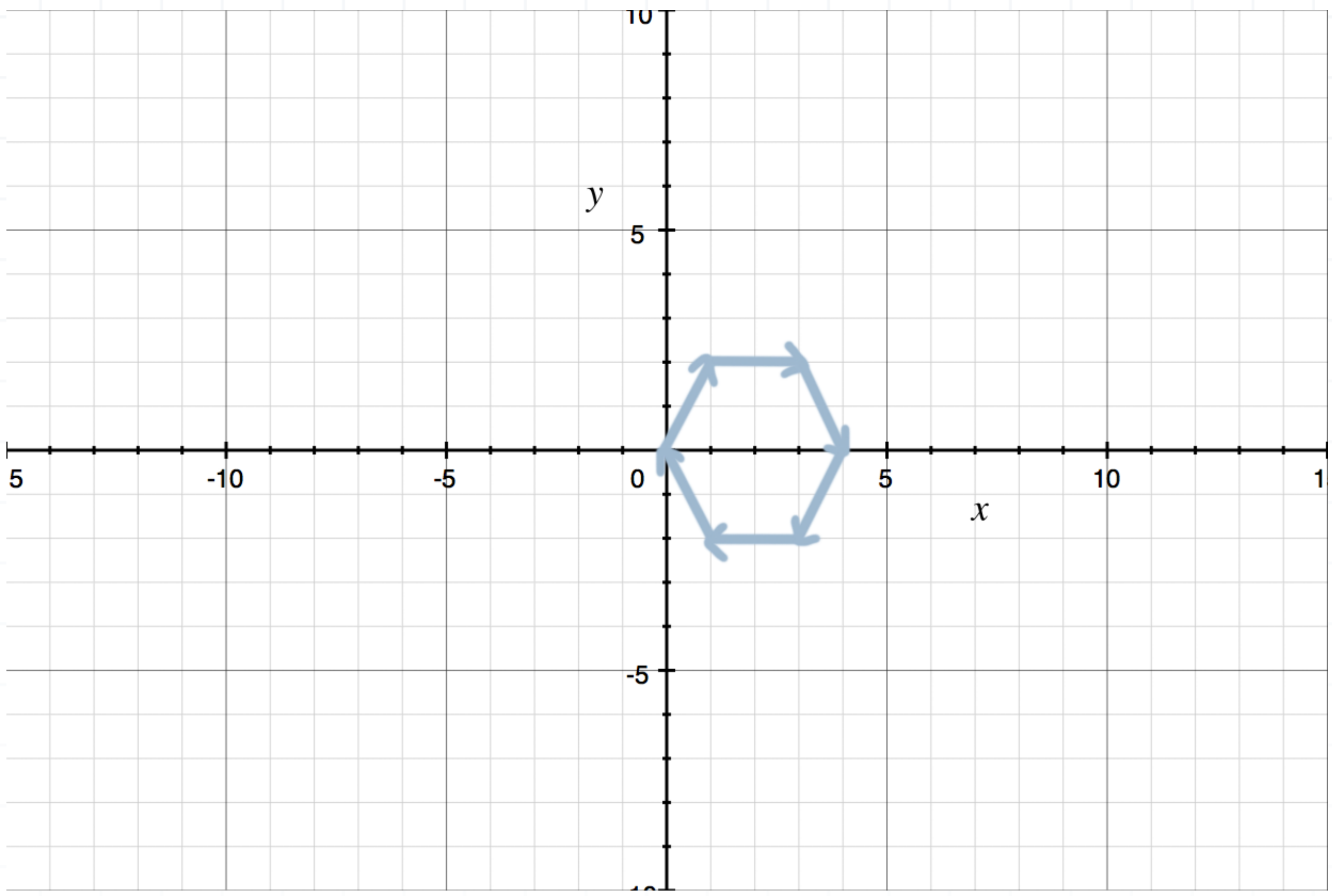
4. Sketch the vectors in order from tip to tail (where the terminal point of one is the initial point of the next), starting at the origin, and determine the shape they form.

$\vec{a}_1 = (1,2)$ $\quad$ $\vec{a}_3 = (1,-2)$ $\quad$ $\vec{a}_5 = (-2,0)$

$\vec{a}_2 = (2,0)$ $\quad$ $\vec{a}_4 = (-1,-2)$ $\quad$ $\vec{a}_6 = (-1,2)$

Solution:

The vectors form a hexagon, and a sketch of all of them together is



■ 5. Find $\vec{b}_1 + \vec{b}_2$, $\vec{b}_1 - \vec{b}_2$, and $2\vec{b}_2$.

$$\vec{b}_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}$$

Solution:

The value $\vec{b}_1 + \vec{b}_2$ is

$$\vec{b}_1 + \vec{b}_2 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix} + \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix} = \begin{bmatrix} 12+0 \\ 3-21 \\ -7+4 \end{bmatrix} = \begin{bmatrix} 12 \\ -18 \\ -3 \end{bmatrix}$$

The value $\vec{b}_1 - \vec{b}_2$ is

$$\vec{b}_1 - \vec{b}_2 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix} - \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 - 0 \\ 3 - (-21) \\ -7 - 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \\ -11 \end{bmatrix}$$

The value of $2\vec{b}_2$ is

$$2\vec{b}_2 = 2\begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(0) \\ 2(-21) \\ 2(4) \end{bmatrix} = \begin{bmatrix} 0 \\ -42 \\ 8 \end{bmatrix}$$

■ 6. Is the product of $\vec{b}_1$ and $\vec{b}_2$ defined? Why or why not?

$$\vec{b}_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}$$

Solution:

The product of $\vec{b}_1$ and $\vec{b}_2$ isn't defined, because they're both column matrices, which means they both have dimensions 3×1. When you multiply two matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix. But $\vec{b}_1$ has one column, and $\vec{b}_2$ has three rows, so as they're written, they can't be multiplied.

VECTOR OPERATIONS

■ 1. Find $\vec{u}+\vec{w}$, $\vec{x}-\vec{y}$, and $\vec{v}-(\vec{w}+\vec{u})$.

$\vec{u}=(-3,5)$ $\vec{w}=(5,-13)$ $\vec{y}=(1,4,2)$

$\vec{v}=(2,1)$ $\vec{x}=(4,5,-7)$

Solution:

The value of $\vec{u}+\vec{w}$ is

$$\vec{u}+\vec{w}=(-3,5)+(5,-13)$$

$$\vec{u}+\vec{w}=(-3+5,5-13)$$

$$\vec{u}+\vec{w}=(2,-8)$$

The value of $\vec{x}-\vec{y}$ is

$$\vec{x}-\vec{y}=(4,5,-7)-(1,4,2)$$

$$\vec{x}-\vec{y}=(4-1,5-4,-7-2)$$

$$\vec{x}-\vec{y}=(3,1,-9)$$

The value of $\vec{v}-(\vec{w}+\vec{u})$ is

$$\vec{v}-(\vec{w}+\vec{u})=(2,1)-((5,-13)+(-3,5))$$

$$\vec{v}-(\vec{w}+\vec{u})=(2,1)-(5-3,-13+5)$$

$$\vec{v} - (\vec{w} + \vec{u}) = (2,1) - (2, -8)$$

$$\vec{v} - (\vec{w} + \vec{u}) = (2 - 2, 1 - (-8))$$

$$\vec{v} - (\vec{w} + \vec{u}) = (0,9)$$

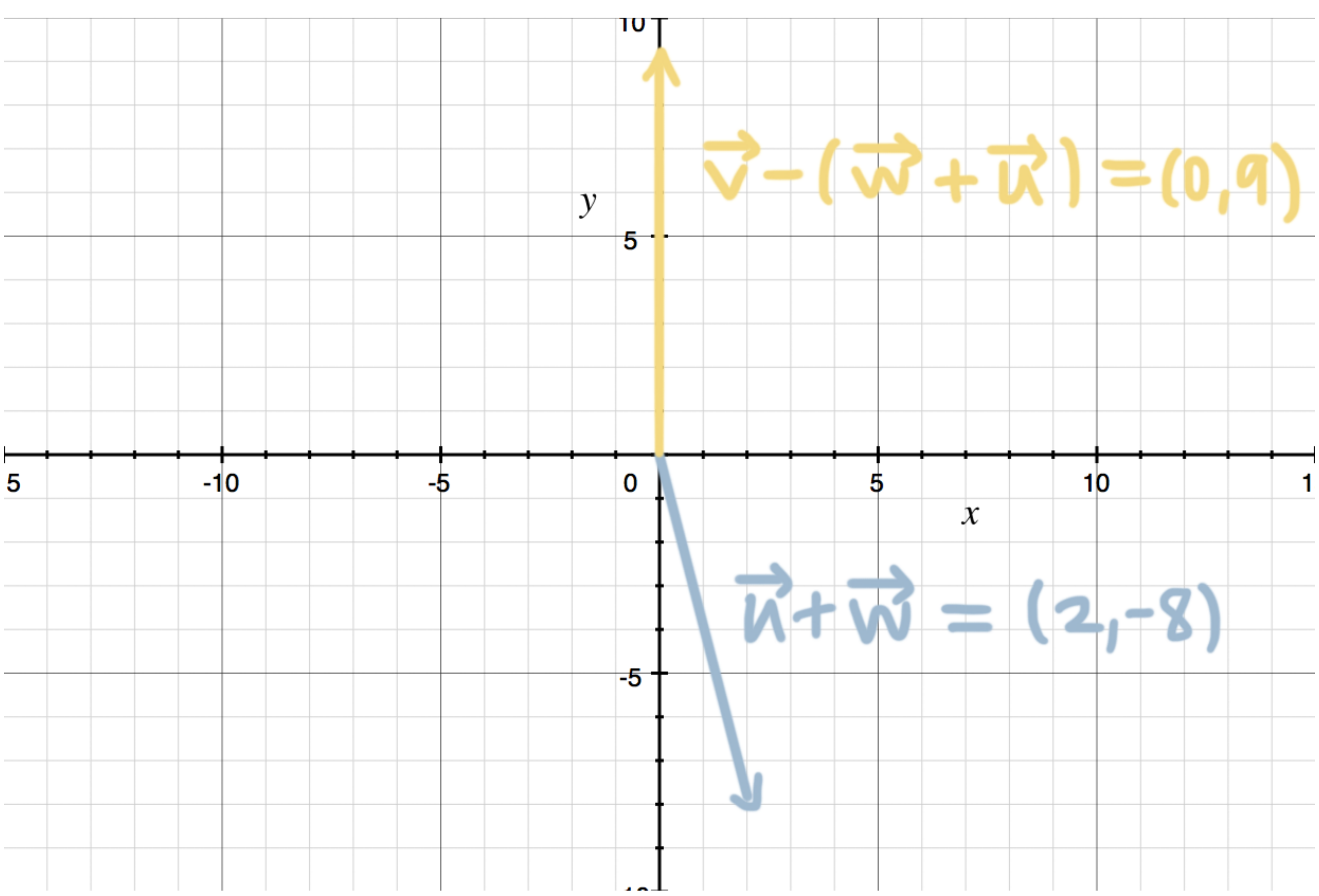
■ 2. Sketch $\vec{u} + \vec{w}$, $\vec{x} - \vec{y}$, and $\vec{v} - (\vec{w} + \vec{u})$.

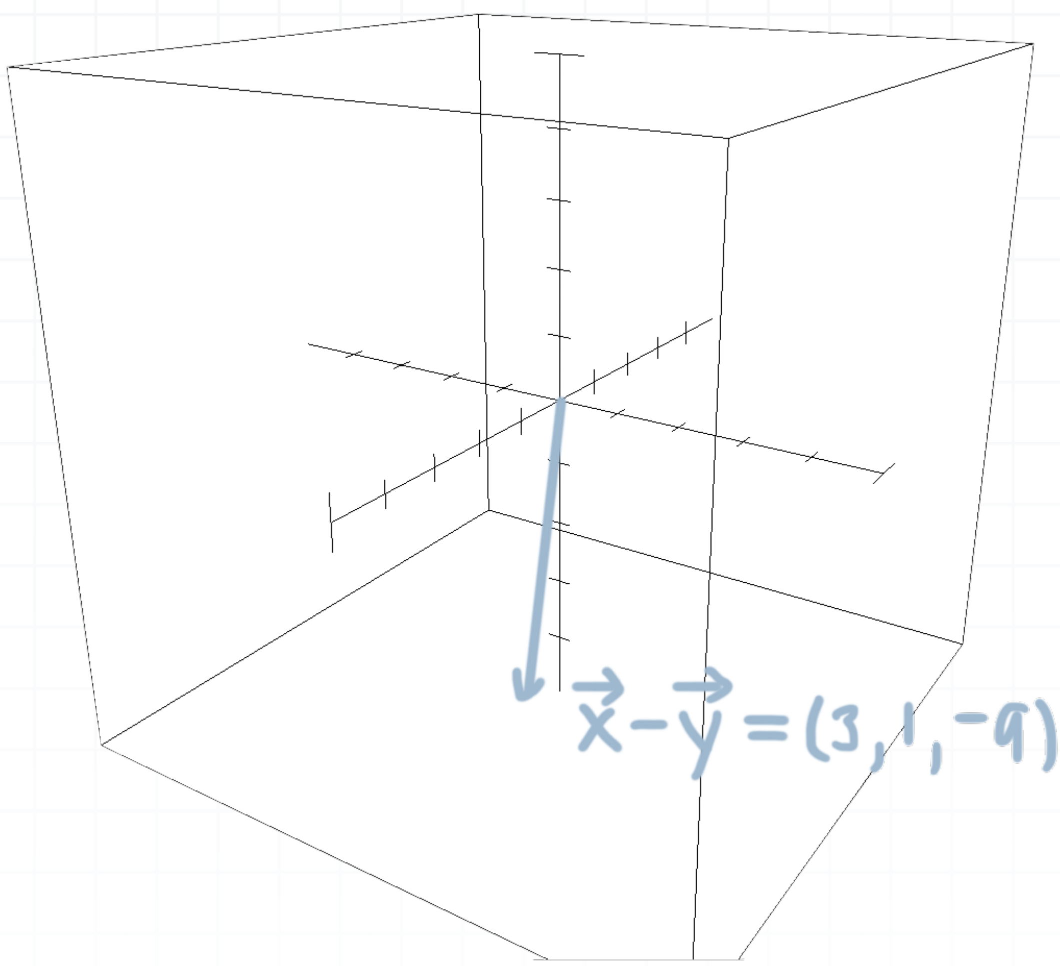
$\vec{u} = (-3,5)$ $\qquad$ $\vec{w} = (5, -13)$ $\qquad$ $\vec{y} = (1,4,2)$

$\vec{v} = (2,1)$ $\qquad$ $\vec{x} = (4,5, -7)$

Solution:

A sketch of $\vec{u} + \vec{w}$, $\vec{x} - \vec{y}$, and $\vec{v} - (\vec{w} + \vec{u})$ in the same plane is





■ 3. Find $b\vec{x}$, $c\vec{u}+b\vec{u}$, and $(c+b)\vec{u}$. What can you say about the relationship between $c\vec{u}+b\vec{u}$ and $(c+b)\vec{u}$.

$$\vec{u}=(-3,5) \qquad b=-1$$

$$\vec{x}=(4,5,-7) \qquad c=3$$

Solution:

The value of $b\vec{x}$ is

$$b\vec{x}=-1(4,5,-7)$$

$$b\vec{x}=(-4,-5,7)$$

The value of $c\vec{u}+b\vec{u}$ is

$$c\vec{u}+b\vec{u}=3(-3,5)-1(-3,5)$$

$$c\vec{u}+b\vec{u}=(-9,15)-(-3,5)$$

$$c\vec{u}+b\vec{u}=(-9-(-3),15-5)$$

$$c\vec{u}+b\vec{u}=(-6,10)$$

The value of $(c+b)\vec{u}$ is

$$(c+b)\vec{u}=(3+(-1))(-3,5)$$

$$(c+b)\vec{u}=2(-3,5)$$

$$(c+b)\vec{u}=(-6,10)$$

The values of $c\vec{u}+b\vec{u}$ and $(c+b)\vec{u}$ are equal because the distributive property applies to vector addition, which means that, from $c\vec{u}+b\vec{u}$, the vector $\vec{u}$ can be factored out, and the expression can be rewritten as $(c+b)\vec{u}$.

■ 4. Find $\vec{x}+b\vec{y}-c\vec{x}-\vec{y}$.

$$\vec{x}=(4,5,-7) \qquad b=-1$$

$$\vec{y}=(1,4,2) \qquad c=3$$

Solution:

First find $b\vec{y}$ and $c\vec{x}$.

$$b\vec{y} = (-1)(1,4,2)$$

$$b\vec{y} = (-1, -4, -2)$$

and

$$c\vec{x} = 3(4,5, -7)$$

$$c\vec{x} = (12,15, -21)$$

Now move from left to right, starting with $\vec{x} + b\vec{y}$, adding more terms as we go.

$$\vec{x} + b\vec{y} = (4,5, -7) + (-1, -4, -2)$$

$$\vec{x} + b\vec{y} = (4 - 1,5 - 4, -7 - 2)$$

$$\vec{x} + b\vec{y} = (3,1, -9)$$

Then

$$\vec{x} + b\vec{y} - c\vec{x} = (3,1, -9) - (12,15, -21)$$

$$\vec{x} + b\vec{y} - c\vec{x} = (3 - 12,1 - 15, -9 - (-21))$$

$$\vec{x} + b\vec{y} - c\vec{x} = (-9, -14,12)$$

Then

$$\vec{x} + b\vec{y} - c\vec{x} - \vec{y} = (-9, -14,12) - (1,4,2)$$

$$\vec{x} + b\vec{y} - c\vec{x} - \vec{y} = (-9 - 1, -14 - 4,12 - 2)$$

$$\overrightarrow{x} + b\overrightarrow{y} - c\overrightarrow{x} - \overrightarrow{y} = (-10, -18, 10)$$

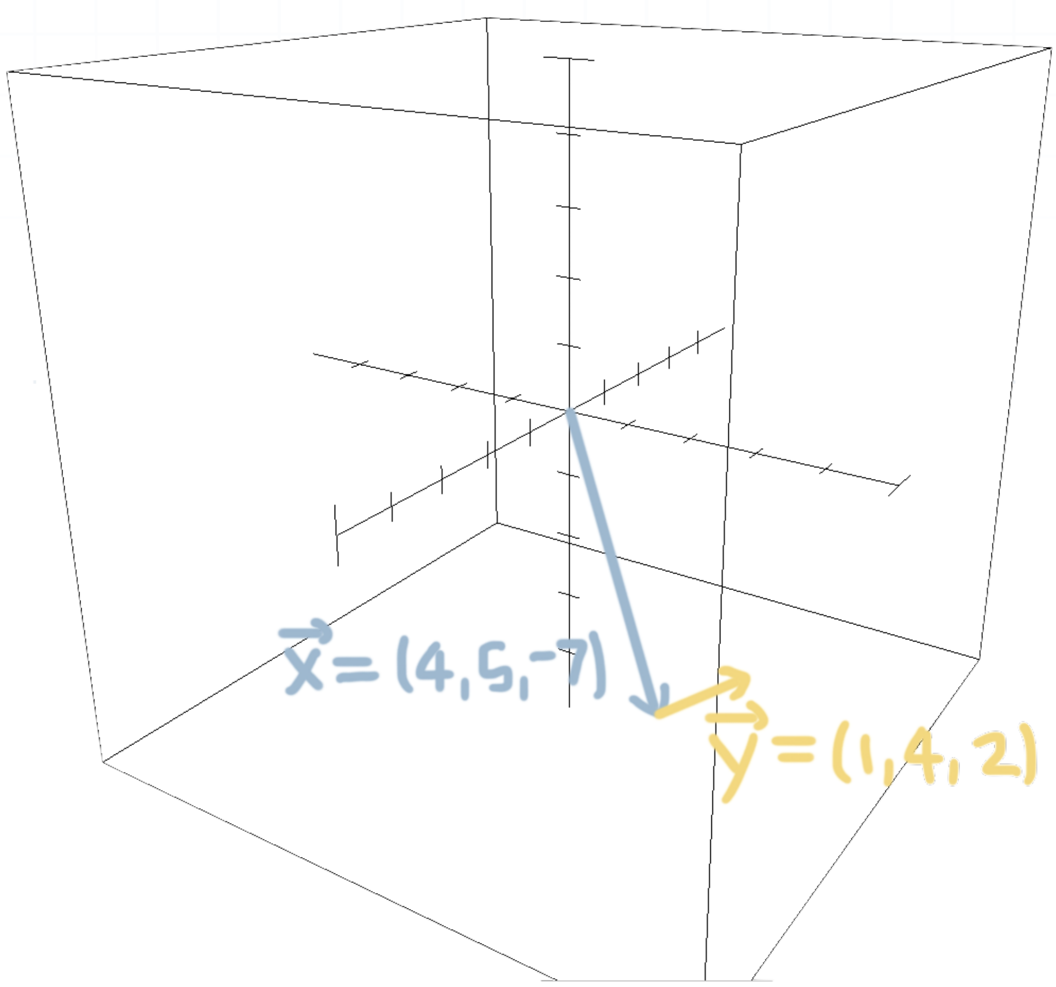
■ 5. Sketch the individual vectors from tip to tail.

$$\overrightarrow{x} = (4, 5, -7)$$

$$\overrightarrow{y} = (1, 4, 2)$$

Solution:

A sketch of the vectors is



■ 6. Find $\vec{x} \cdot \vec{y}$, $\vec{w} \cdot \vec{w}$, and $b(\vec{u} \cdot \vec{v})$.

$\vec{u} = (-3,5)$ $\vec{w} = (5,-13)$ $\vec{y} = (1,4,2)$

$\vec{v} = (2,1)$ $\vec{x} = (4,5,-7)$ $b = -1$

Solution:

The value of $\vec{x} \cdot \vec{y}$ is

$$\vec{x} \cdot \vec{y} = (4,5,-7) \cdot (1,4,2)$$

$$\vec{x} \cdot \vec{y} = (4)(1) + (5)(4) + (-7)(2)$$

$$\vec{x} \cdot \vec{y} = 4 + 20 - 14$$

$$\vec{x} \cdot \vec{y} = 10$$

The value of $\vec{w} \cdot \vec{w}$ is

$$\vec{w} \cdot \vec{w} = (5,-13) \cdot (5,-13)$$

$$\vec{w} \cdot \vec{w} = (5)(5) + (-13)(-13)$$

$$\vec{w} \cdot \vec{w} = 25 + 169$$

$$\vec{w} \cdot \vec{w} = 194$$

The value of $b(\vec{u} \cdot \vec{v})$ is

$$b(\vec{u} \cdot \vec{v}) = (-1)((-3,5) \cdot (2,1))$$

$$b(\vec{u} \cdot \vec{v}) = (-1)((-3)(2) + (5)(1))$$

$$b(\overrightarrow{u} \cdot \overrightarrow{v}) = (-1)(-6 + 5)$$

$$b(\overrightarrow{u} \cdot \overrightarrow{v}) = (-1)(-1)$$

$$b(\overrightarrow{u} \cdot \overrightarrow{v}) = 1$$

UNIT VECTORS AND BASIS VECTORS

1. Change each vector to a unit vector.

$$\vec{a} = (3, -4)$$

$$\vec{b} = (12,2)$$

$$\vec{c} = (0,7,1)$$

Solution:

Find the magnitude of $\vec{a}$.

$$||\vec{a}|| = ||(3, -4)||$$

$$||\vec{a}|| = \sqrt{3^2 + (-4)^2}$$

$$||\vec{a}|| = \sqrt{9 + 16}$$

$$||\vec{a}|| = \sqrt{25}$$

$$||\vec{a}|| = 5$$

Find the magnitude of $\vec{b}$.

$$||\vec{b}|| = ||(12,2)||$$

$$||\vec{b}|| = \sqrt{12^2 + 2^2}$$

$$||\vec{b}|| = \sqrt{144 + 4}$$

$$||\vec{b}|| = \sqrt{148}$$

$$||\vec{b}|| = 2\sqrt{37}$$

Find the magnitude of $\vec{c}$.

$$||\vec{c}|| = ||(0,7,1)||$$

$$||\vec{c}|| = \sqrt{0^2 + 7^2 + 1^2}$$

$$||\vec{c}|| = \sqrt{49 + 1}$$

$$||\vec{c}|| = \sqrt{50}$$

$$||\vec{c}|| = 5\sqrt{2}$$

Then the unit vectors are

$$\hat{u}_a = \frac{1}{||\vec{a}||}\vec{a} = \frac{1}{5}\begin{bmatrix}3\\-4\end{bmatrix}$$

$$\hat{u}_b = \frac{1}{||\vec{b}||}\vec{b} = \frac{1}{2\sqrt{37}}\begin{bmatrix}12\\2\end{bmatrix} = \frac{1}{\sqrt{37}}\begin{bmatrix}6\\1\end{bmatrix}$$

$$\hat{u}_c = \frac{1}{||\vec{c}||}\vec{c} = \frac{1}{5\sqrt{2}}\begin{bmatrix}0\\7\\1\end{bmatrix}$$

■ 2. Confirm that the vectors each have length 1.

$$\hat{u}_a = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\hat{u}_b = \frac{1}{\sqrt{37}}\begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$\hat{u}_c = \frac{1}{5\sqrt{2}}\begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

Solution:

Find the length of $\hat{u}_a$.

$$||\hat{u}_a|| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2}$$

$$||\hat{u}_a|| = \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$||\hat{u}_a|| = \sqrt{\frac{25}{25}}$$

$$||\hat{u}_a|| = 1$$

Find the length of $\hat{u}_b$.

$$||\hat{u}_b|| = \sqrt{\left(\frac{6}{\sqrt{37}}\right)^2 + \left(\frac{1}{\sqrt{37}}\right)^2}$$

$$||\hat{u}_b|| = \sqrt{\frac{36}{37} + \frac{1}{37}}$$

$$||\hat{u}_b|| = \sqrt{\frac{37}{37}}$$

$$||\hat{u}_b|| = 1$$

Find the length of $\hat{u}_c$.

$$||\hat{u}_c|| = \sqrt{0^2 + \left(\frac{7}{5\sqrt{2}}\right)^2 + \left(\frac{1}{5\sqrt{2}}\right)^2}$$

$$||\hat{u}_c|| = \sqrt{\frac{49}{25(2)} + \frac{1}{25(2)}}$$

$$||\hat{u}_c|| = \sqrt{\frac{49}{50} + \frac{1}{50}}$$

$$||\hat{u}_c|| = \sqrt{\frac{50}{50}}$$

$$||\hat{u}_c|| = 1$$

■ 3. What are the basis vectors for $\mathbb{R}^4$?

Solution:

$$\hat{r}_1 = (1,0,0,0)$$

$$\hat{r}_2 = (0,1,0,0)$$

$$\hat{r}_3 = (0,0,1,0)$$

$$\hat{r}_4 = (0,0,0,1)$$

■ 4. Express the vectors as linear combinations of the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$.

$$\hat{u}_a = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\hat{u}_b = \frac{1}{\sqrt{37}}\begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$\hat{u}_c = \frac{1}{5\sqrt{2}}\begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

Solution:

The vector $\hat{u}_a$ can be rewritten in terms of the standard basis vectors as

$$\begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{4}{5} \end{bmatrix} = \frac{3}{5}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

The vector $\hat{u}_b$ can be rewritten in terms of the standard basis vectors as

$$\begin{bmatrix}\frac{6}{\sqrt{37}}\\ \frac{1}{\sqrt{37}}\end{bmatrix}=\begin{bmatrix}\frac{6}{\sqrt{37}}\\ 0\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{\sqrt{37}}\end{bmatrix}=\frac{6}{\sqrt{37}}\begin{bmatrix}1\\ 0\end{bmatrix}+\frac{1}{\sqrt{37}}\begin{bmatrix}0\\ 1\end{bmatrix}=\frac{6}{\sqrt{37}}\hat{i}+\frac{1}{\sqrt{37}}\hat{j}$$

The vector $\hat{u}_c$ can be rewritten in terms of the standard basis vectors as

$$\begin{bmatrix}0\\ \frac{7}{5\sqrt{2}}\\ \frac{1}{5\sqrt{2}}\end{bmatrix}=0+\begin{bmatrix}0\\ \frac{7}{5\sqrt{2}}\\ 0\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{1}{5\sqrt{2}}\end{bmatrix}=\frac{7}{5\sqrt{2}}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+\frac{1}{5\sqrt{2}}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

$$=\frac{7}{5\sqrt{2}}\hat{j}+\frac{1}{5\sqrt{2}}\hat{k}=\frac{7}{5\sqrt{2}}\hat{j}+\frac{1}{5\sqrt{2}}\hat{k}$$

■ 5. Express $\overrightarrow{v}=(x,2x,-1)$ in terms of the standard basis vectors.

Solution:

Rewrite $\overrightarrow{v}$ in terms of the standard basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$.

$$\overrightarrow{v}=(x,2x,-1)$$

$$\overrightarrow{v}=\begin{bmatrix}x\\ 0\\ 0\end{bmatrix}+\begin{bmatrix}0\\ 2x\\ 0\end{bmatrix}+\begin{bmatrix}0\\ 0\\ -1\end{bmatrix}$$

$$\vec{v} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} 2x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

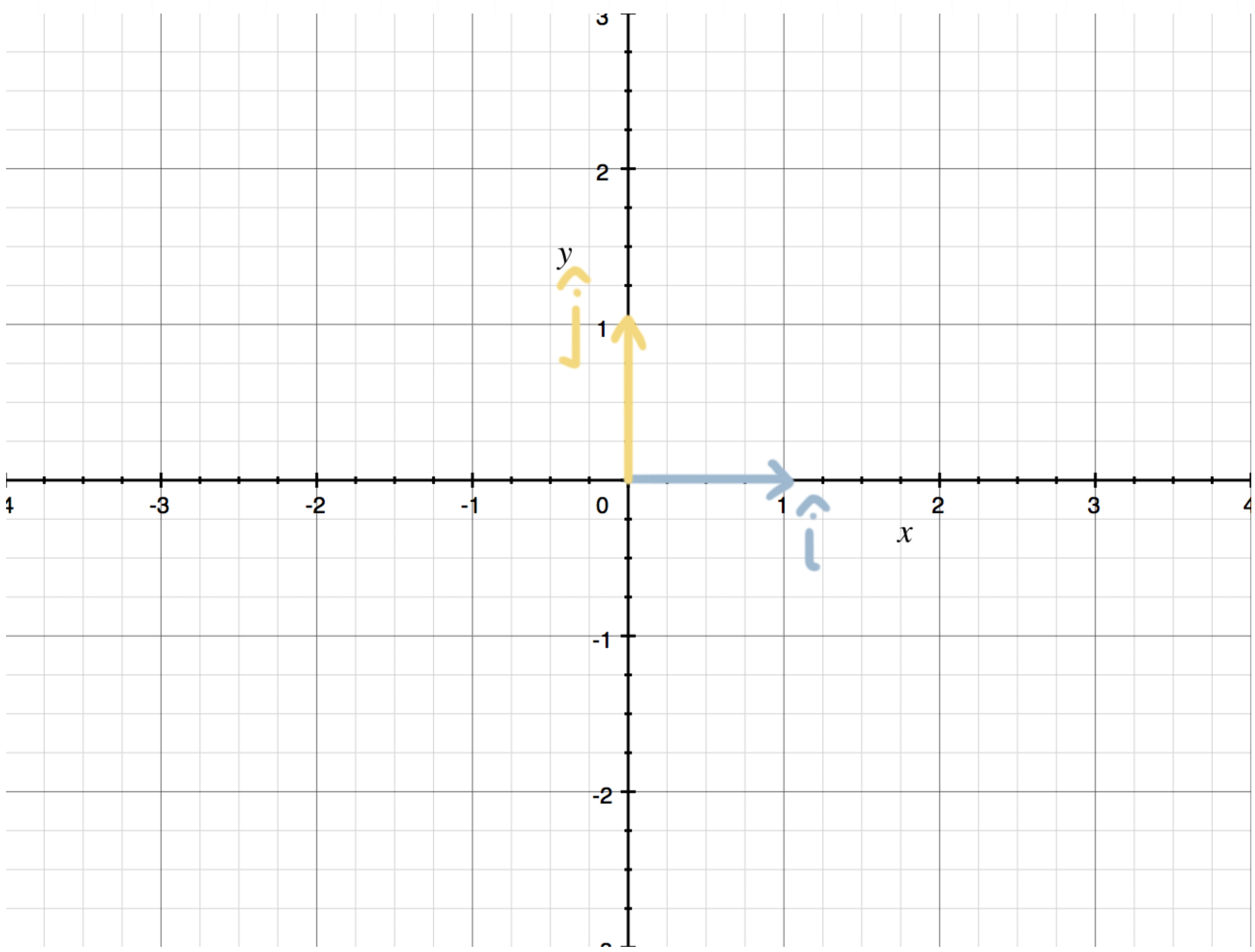
$$\vec{v} = x\hat{i} + 2x\hat{j} - 1\hat{k}$$

$$\vec{v} = x\hat{i} + 2x\hat{j} - \hat{k}$$

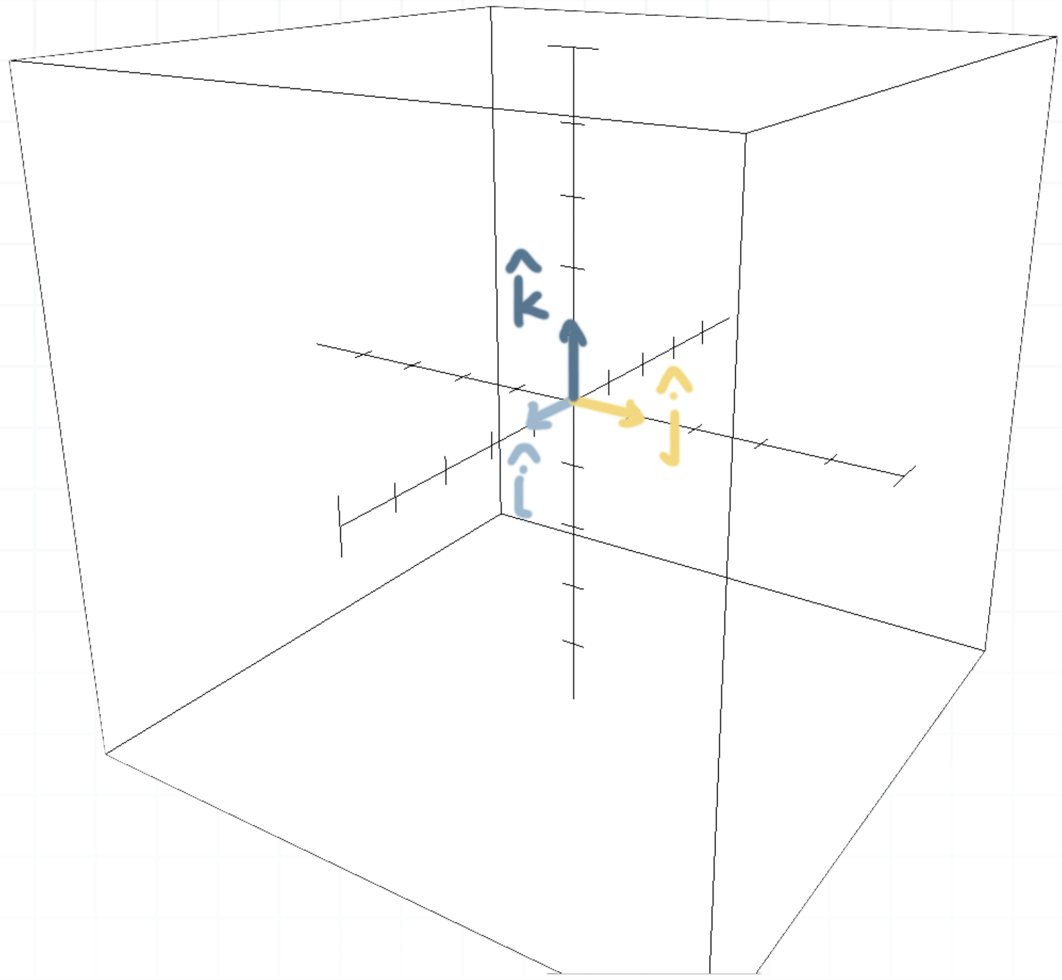
■ 6. Sketch the basis vectors $\hat{i}$ and $\hat{j}$ in $\mathbb{R}^2$, and the vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ in $\mathbb{R}^3$.

Solution:

A sketch of the standard basis vectors in $\mathbb{R}^2$ is



And a sketch of the standard basis vectors in $\mathbb{R}^3$ is



LINEAR COMBINATIONS AND SPAN

■ 1. Say whether each of the following is a linear combination. If it isn't, say why.

$$-\pi\vec{x}-e\vec{y}$$

$$\vec{x}\cdot\vec{y}$$

$$\vec{a}=\hat{i}+\hat{j}+\hat{k}$$

$$\vec{u}=\frac{1}{\sqrt{2}}((3,0)-(1,1))$$

$$||\vec{b}||$$

Solution:

The expressions

$$-\pi\vec{x}-e\vec{y}$$

$$\vec{a}=\hat{i}+\hat{j}+\hat{k}$$

$$\vec{u}=\frac{1}{\sqrt{2}}((3,0)-(1,1))$$

all represent linear combinations.

But $\vec{x} \cdot \vec{y}$ is not a linear combination, because the dot product returns a scalar, so $\vec{x} \cdot \vec{y}$ can't be a linear combination, since it isn't a vector.

And $||\vec{b}||$ isn't a linear combination, because, like the dot product, the magnitude returns a single scalar, so $||\vec{b}||$ can't be a linear combination.

■ 2. Do the vectors span $\mathbb{R}^4$?

$$\left\{\begin{bmatrix}3\\ \frac{1}{2}\\ 0\\ -1\end{bmatrix}, \begin{bmatrix}-\pi\\ \pi\\ \pi\\ -\pi\end{bmatrix}, \begin{bmatrix}-\frac{2}{3}\\ 8\\ 22\\ 9\end{bmatrix}\right\}$$

Solution:

The vector set doesn't span $\mathbb{R}^4$, because at least 4 vectors are needed to span $\mathbb{R}^4$.

■ 3. Do the vectors span $\mathbb{R}^4$?

$$\left\{\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}\right\}$$

Solution:

The vector set spans $\mathbb{R}^4$, because these are the standard basis vectors for $\mathbb{R}^4$.

■ 4. Do the vectors span $\mathbb{R}^2$?

$$\left\{\begin{bmatrix}44\\-8\end{bmatrix},\begin{bmatrix}11\\-2\end{bmatrix}\right\}$$

Solution:

The vector set doesn't span $\mathbb{R}^2$. While two vectors may be enough to span $\mathbb{R}^2$, the two vectors can't be collinear if they're going to be a spanning set.

The first vector, $\vec{v}_1 = (44, -8)$, can be written as $\vec{v}_1 = 4(11, -2)$, which means that the two vectors are collinear. Because they're collinear, they can't span $\mathbb{R}^2$.

■ 5. What is the zero vector $\vec{O}$ in $\mathbb{R}^5$? What is its span?

Solution:

The zero vector in $\mathbb{R}^5$ is $\vec{O} = (0,0,0,0,0)$, and its span is simply the origin in $\mathbb{R}^5$.

$$\text{Span}\{\vec{O}\} = (0,0,0,0,0)$$

■ 6. Prove that any vector $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ can be reached by a linear combination of $\hat{i}$, $\hat{j}$, and $\hat{k}$.

Solution:

Let c_1, c_2, and c_3 be any real numbers ($c_1, c_2, c_3 \in \mathbb{R}$). We want to show that any vector, $\vec{v}$, can be written as $\vec{v} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If we rewrite this equation with column vectors, we get

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

This means that $v_1 = c_1$, $v_2 = c_2$, and $v_3 = c_3$, such that, no matter what vector $\vec{v}$ is given to us, we can just pick $c_1 = v_1$, $c_2 = v_2$, and $c_3 = v_3$ for our weights, and we'll be able to span $\mathbb{R}^3$ using those weights and the basis vectors.

LINEAR INDEPENDENCE IN TWO DIMENSIONS

1. Are the column vectors of the following matrix linearly independent?

$$A = \begin{bmatrix} 2 & 6 & 7 \\ -1 & 11 & 3 \end{bmatrix}$$

Solution:

First, let's get the column vectors.

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 6 \\ 11 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

Since each column vector has two components, we know that they're in $\mathbb{R}^2$. However, since there are three column vectors, we know that at least one of them can be made from a linear combination of the other two, and so they must be linearly dependent.

2. Show how one of the vectors could be written as a linear combination of the other two.

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{z} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Solution:

Since $\vec{x}$ and $\vec{y}$ are the standard basis vectors for $\mathbb{R}^2$, they can easily be used to make $\vec{z}$.

$$\vec{z} = 5\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}5(1)\\5(0)\end{bmatrix} + \begin{bmatrix}3(0)\\3(1)\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}5+0\\0+3\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}5\\3\end{bmatrix}$$

■ 3. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{x} = \begin{bmatrix}2\\0\end{bmatrix}, \vec{y} = \begin{bmatrix}1\\1\end{bmatrix}$$

Solution:

To test for linear independence, set up the vector equation.

$$c_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Break this up into a system of linear equations.

$$2c_1 + c_2 = 0$$

$$c_2 = 0$$

We can already see that c_2 is zero, and if we plug $c_2 = 0$ into the first equation, we can see that

$$2c_1 + 0 = 0$$

$$2c_1 = 0$$

$$c_1 = 0$$

Because the only values of c_1 and c_2 that make the vector equation true are $c_1 = 0$ and $c_2 = 0$, we know the vectors must be linearly independent.

■ 4. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{a} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Solution:

To test for linear independence, set up the vector equation.

$$c_1 \begin{bmatrix} 6 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Break this up into a system of linear equations.

$$6c_1 - 2c_2 = 0$$

$$3c_1 + 5c_2 = 0$$

Divide the first equation by -2 to get

$$-3c_1 + c_2 = 0$$

$$3c_1 + 5c_2 = 0$$

Then add the equations to cancel c_1 and solve for c_2.

$$-3c_1 + c_2 + (3c_1 + 5c_2) = 0 + (0)$$

$$-3c_1 + c_2 + 3c_1 + 5c_2 = 0$$

$$c_2 + 5c_2 = 0$$

$$6c_2 = 0$$

$$c_2 = 0$$

Now we can plug $c_2 = 0$ into the $3c_1 + 5c_2 = 0$ to solve for c_1.

$$3c_1 + 5(0) = 0$$

$$3c_1 = 0$$

$$c_1 = 0$$

Because the only values of c_1 and c_2 that make the vector equation true are $c_1 = 0$ and $c_2 = 0$, we know the vectors must be linearly independent.

■ 5. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$$

Solution:

To test for linear independence, set up the vector equation.

$$c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Break this up into a system of linear equations.

$$3c_1 - 6c_2 = 0$$

$$2c_1 - 4c_2 = 0$$

Solve the system with an augmented matrix.

$$\begin{bmatrix} 3 & -6 & | & 0 \\ 2 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & | & 0 \\ 2 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

The reduced row-echelon form of the augmented matrix gives the relationship

$$c_1 - 2c_2 = 0$$

$$c_1 = 2c_2$$

Which means we can pick any value that we choose for c_2, and we'll get an associated value of c_1 that satisfies the system. For instance, $(c_1, c_2) = (2,1)$ is a solution. Because $(c_1, c_2) = (0,0)$ is not the only solution, the vectors must be linearly dependent.

6. Use a matrix to say whether the vectors are linearly dependent or linearly independent.

$$\vec{x} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \vec{y} = \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$$

Solution:

Set up an augmented matrix of the vectors as column vectors.

$$\begin{bmatrix} 2 & -\frac{1}{2} & | & 0 \\ 8 & -2 & | & 0 \end{bmatrix}$$

Use Gaussian elimination to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & -\frac{1}{2} & | & 0 \\ 8 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{4} & | & 0 \\ 8 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{4} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

The reduced row-echelon form of the matrix gives the relationship

$$c_1 - \frac{1}{4}c_2 = 0$$

$$c_1 = \frac{1}{4}c_2$$

$$4c_1 = c_2$$

Which means we can pick any value that we choose for c_1, and we'll get an associated value of c_2 that satisfies the system. For instance, $(c_1, c_2) = (1,4)$ is a solution. Because $(c_1, c_2) = (0,0)$ is not the only solution, the vectors must be linearly dependent.

LINEAR INDEPENDENCE IN THREE DIMENSIONS

■ 1. Use a matrix to say whether the vector set is linearly independent.

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$$

Solution:

Form a matrix of the vectors as column vectors.

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

Use Gaussian elimination to work toward reduced row-echelon form.

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 5 \\ 0 & -6 & -18 \\ 1 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -6 & -18 \\ 1 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -6 & -18 \\ 0 & -\frac{7}{2} & -\frac{21}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 3 \\ 0 & -\frac{7}{2} & -\frac{21}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & -\frac{7}{2} & -\frac{21}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

We could continue on, but because we get an entire row of zeros in the matrix, we know that the vectors must be linearly dependent.

■ 2. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$$

Solution:

For a set of three vectors to span $\mathbb{R}^3$, they must be linearly independent of one another. Since we determined in the last problem that these vectors are a linearly dependent set, that means they're coplanar, and they can't span $\mathbb{R}^3$.

■ 3. Use a matrix to say whether the vector set is linearly independent.

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

Solution:

Form a matrix of the vectors as column vectors.

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 1 & 6 \\ 5 & 0 & 8 \end{bmatrix}$$

Use Gaussian elimination to work toward reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 1 & 6 \\ 5 & 0 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -3 & -6 \\ 5 & 0 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -3 & -6 \\ 0 & -5 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because we've ended up with the identity matrix, we know the vectors must form a linearly independent set.

4. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

Solution:

For a set of three vectors to span $\mathbb{R}^3$, they must be linearly independent of one another. Since we determined in the last problem that these vectors are a linearly independent set, that means they do span $\mathbb{R}^3$.

5. Is the vector set linearly independent? Why or why not?

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}, \vec{x} = \begin{bmatrix} -2 \\ 7 \\ 1 \end{bmatrix}$$

Solution:

The vector set can't be linearly independent. Although each vector has three components, there are four vectors in total. That means we know that one of them can be made from a linear combination of the other three, so they must be a linearly dependent set.

6. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}, \vec{x} = \begin{bmatrix} -2 \\ 7 \\ 1 \end{bmatrix}$$

Solution:

For a set of three vectors to span $\mathbb{R}^3$, they must be vectors in $\mathbb{R}^3$, and they must be linearly independent of one another. Since we determined earlier that the first three vectors are a linearly independent set, that means the first three vectors do span $\mathbb{R}^3$. Adding the vector $\vec{x}$ to the set would change the set from linearly independent to linearly dependent, but it doesn't change the fact that the set still spans $\mathbb{R}^3$.

LINEAR SUBSPACES

1. What are the criteria that define a subspace? Which criteria is logically part of another criteria?

Solution:

In order for a space to be considered a subspace, it must

1. include the zero vector,
2. be closed under scalar multiplication, and
3. be closed under addition.

The first requirement, that the space include the zero vector, is logically included in the second requirement about scalar multiplication, because, if a space is closed under scalar multiplication, then by definition it includes the zero vector.

2. Sketch the graph of each space.

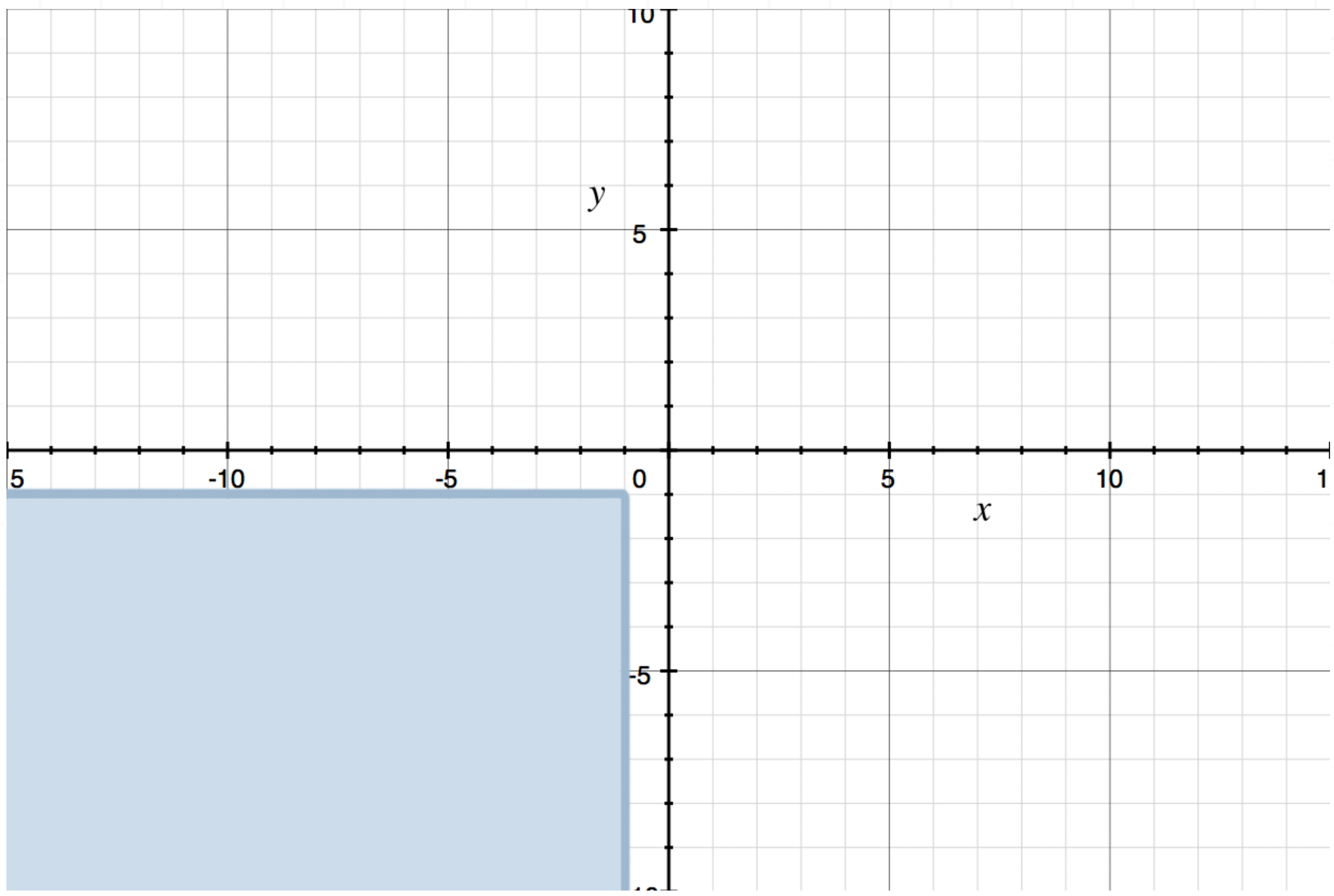
$$V_a = \left\{(x, y) \in \mathbb{R}^2 \mid x, y \leq -1\right\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid y < x^2\right\}$$

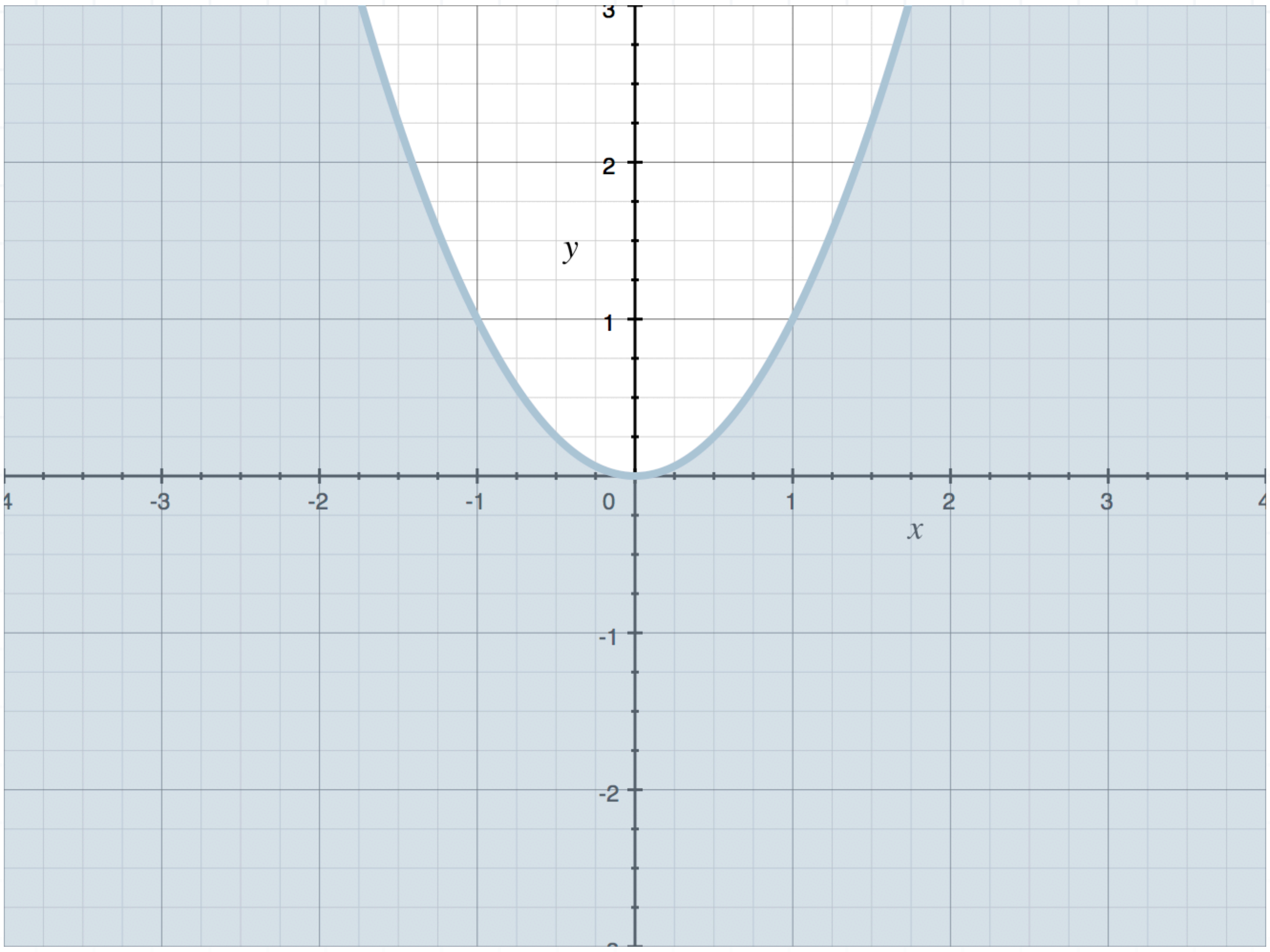
$$V_c = \left\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, y \leq x\right\}$$

Solution:

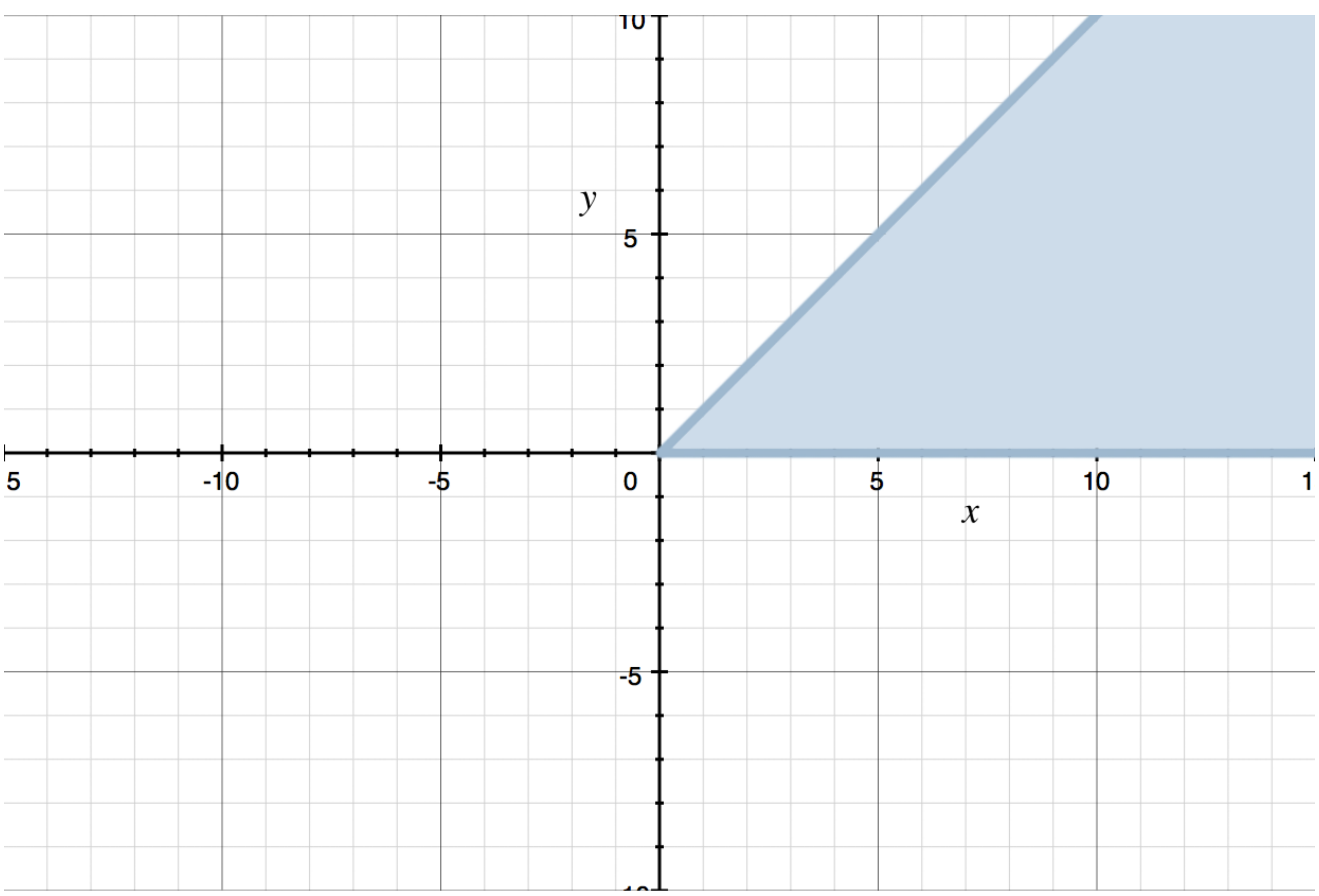
The space V_a is defined when both x and y are less than or equal to -1, so a sketch of that space is



The space V_b is defined when y is less than x^2, so a sketch of that space is



The space V_c is defined when y is less than or equal to x, but when x and y are both still greater than or equal to 0, so a sketch of that space is



3. What space is being described by each of the sets?

$$V_a = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$$

$$V_b = \{(x, y) \in \mathbb{R}^2 \mid xy = 0, \ x = y\}$$

$$V_c = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\}$$

Solution:

The space V_a is the major axes in $\mathbb{R}^2$, since $xy = 0$ is only true if either $x = 0$ and/or $y = 0$.

Similarly, the space V_b is only the zero vector in $\mathbb{R}^2$, $\vec{O} = (0,0)$. That's because, in order for $xy = 0$ to be true, the space must be limited to the major axes, but if $x = y$ is also true, that means the only vector included in the space will exist where the major axes intersect each other, which is only at the origin.

The space V_c is all of $\mathbb{R}^3$ space, since the only requirement is that x, y, and z are all in $\mathbb{R}^3$.

4. Are these spaces subspaces?

$$V_a = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0,\ x = y\right\}$$

$$V_c = \left\{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\right\}$$

Solution:

The space V_a is not a subspace, because V_a isn't closed under addition.

The space V_b is a subspace, because the zero vector is always a subspace. By definition, it always includes the zero vector, and is closed under addition and scalar multiplication.

The space V_c is a subspace, because an entire space $\mathbb{R}^n$ is always a subspace of itself. So all of $\mathbb{R}^3$ is a subspace because it includes the zero vector, and it's closed under addition and scalar multiplication.

■ 5. Show that each space is not a subspace.

$$V_a = \left\{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - 7z = 3\right\}$$

$$V_b = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \leq 0\right\}$$

$$V_c = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\right\}$$

Solution:

For V_a, we start by checking for the zero vector, $\vec{O} = (0,0,0)$.

$$2x + y - 7z = 3$$

$$2(0) + 0 - 7(0) = 3$$

$$0 = 3$$

This tells us that V_a doesn't include the zero vector, so V_a cannot be a subspace.

For V_b, we can say that the set includes the zero vector, so let's check scalar multiplication. If we pick $\vec{v}_b = (0, -1,0)$, which is in the set, and multiply it by $c = -1$, we get $-\vec{v}_b = (0,1,0)$, which is not in the set, which means V_b isn't closed under scalar multiplication, and is therefore not a subspace.

For V_c, the set includes the zero vector and is closed under scalar multiplication. To check to see whether the set is closed under addition. If we pick two vectors in the set, $\vec{v}_1 = (1,0)$ and $\vec{v}_2 = (0,1)$ and add them, we get

$$\vec{v}_1 + \vec{v}_2 = (1,0) + (0,1)$$

$$\vec{v}_1 + \vec{v}_2 = (1 + 0,0 + 1)$$

$$\vec{v}_1 + \vec{v}_2 = (1,1)$$

Since neither the x-value nor the y-value is a zero in this vector, we can say that the set is not closed under addition, and therefore that V_c is not a subspace.

■ 6. Prove that the zero vector $\vec{O}=(0,0,0)$ is a subspace of $\mathbb{R}^3$.

Solution:

We need to look at all three parts of the definition of a subspace. Since we're looking at the zero vector, the zero vector is obviously included, so the first part of the definition of a subspace is satisfied.

To check to see whether the set is closed under scalar multiplication, we'll multiply $\vec{O}=(0,0,0)$ by a scalar.

$$c\vec{O}=c(0,0,0)$$

$$c\vec{O}=(0c,0c,0c)$$

$$c\vec{O}=(0,0,0)$$

We can tell that, no matter which scalar we pick, we'll always get the zero vector again, so the set is closed under scalar multiplication.

To check to see whether the set is closed under addition, we add two vectors from the set together, but the only vector in the set is $\vec{O}=(0,0,0)$, so we get

$$\vec{O}+\vec{O}=(0,0,0)+(0,0,0)$$

$$\vec{O}+\vec{O}=(0+0,0+0,0+0)$$

$$\vec{O}+\vec{O}=(0,0,0)$$

And since we only get the zero vector (of course), we can say that $\overrightarrow{O} = (0,0,0)$ is closed under addition.

Therefore, because we've shown that the set includes the zero vector, and that the set is closed under scalar multiplication and closed under addition, we can say that the zero vector $\overrightarrow{O} = (0,0,0)$ is a subspace of $\mathbb{R}^3$.

SPANS AS SUBSPACES

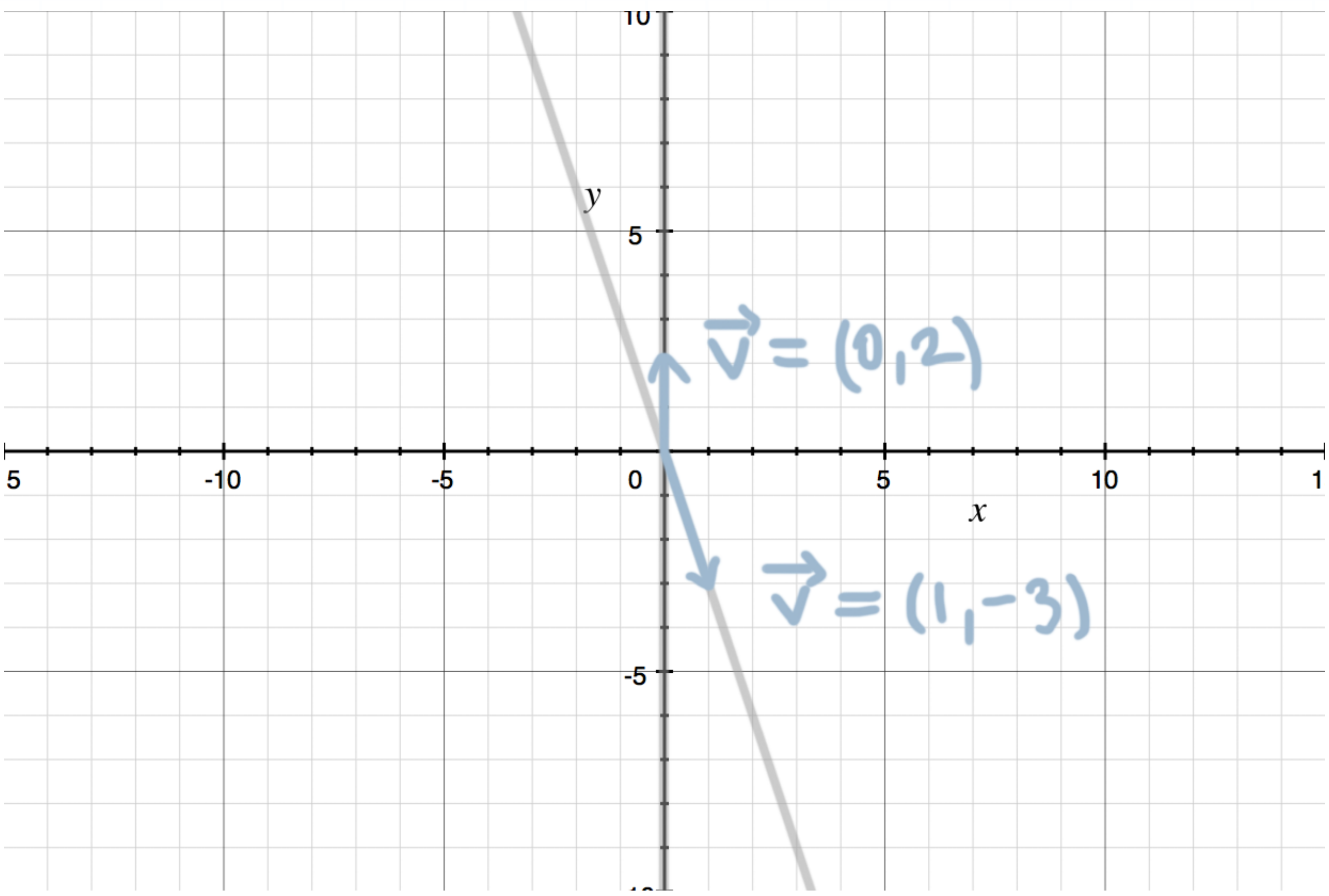
1. Sketch the spans together on the same set of axes.

$$V = \text{Span}\left(\begin{bmatrix}1\\-3\end{bmatrix}\right)$$

$$V = \text{Span}\left(\begin{bmatrix}0\\2\end{bmatrix}\right)$$

Solution:

The span of $\vec{v} = (1,-3)$ will be the line that runs through $\vec{v} = (1,-3)$, and the span of $\vec{v} = (0,2)$ will be the line that runs through $\vec{v} = (0,2)$.



■ 2. Show that that spans are subspaces.

$$V = \text{Span}\left(\begin{bmatrix}1\\-3\end{bmatrix}\right)$$

$$V = \text{Span}\left(\begin{bmatrix}0\\2\end{bmatrix}\right)$$

Solution:

We know that any line in $\mathbb{R}^2$ through the origin is a subspace of $\mathbb{R}^2$, so by that fact alone, we know that each span is a subspace. But let's take each part of the definition individually.

First, because both lines pass through the origin, we know the zero vector is included in both spans. Second, if we multiply either of the vectors by a scalar, we'll still get a vector that's along the line, so both spans are closed under scalar multiplication. And third, if we add two vectors that fall along the same line, we'll still get a vector that's along the line, so both spans are closed under addition.

■ 3. Prove that the span forms a subspace of $\mathbb{R}^3$.

$$\text{Span}\left(\begin{bmatrix}-6\\5\\1\end{bmatrix}\right)$$

Solution:

The span is just a linear combination of the vectors included in the span, so

$$\text{Span}\left(\begin{bmatrix}-6\\5\\1\end{bmatrix}\right)=c\begin{bmatrix}-6\\5\\1\end{bmatrix}$$

If we choose $c=0$, we get the zero vector, so the span includes the zero vector. Since we can multiply c by any other scalar to just get another scalar, we know that the set is closed under scalar multiplication. And since we get

$$c_1\begin{bmatrix}-6\\5\\1\end{bmatrix}+c_2\begin{bmatrix}-6\\5\\1\end{bmatrix}$$

$$(c_1+c_2)\begin{bmatrix}-6\\5\\1\end{bmatrix}$$

$$c_3\begin{bmatrix}-6\\5\\1\end{bmatrix}$$

we can see that the set is also closed under addition. So the span is a subspace of $\mathbb{R}^3$.

4. Write the line $y = 3x + 2$ in set notation, and then write it as a single vector, only using x.

Solution:

A line is simply a series of points in $\mathbb{R}^2$, so writing it in set notation is simply

$$\left\{(x, y) \in \mathbb{R}^2 \mid y = 3x + 2\right\}$$

To write the line as a vector using only x, we see that y is already defined in terms of x, and so we can write

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x + 2 \end{bmatrix}$$

5. Write the line $2y + 4x = 0$ in set notation, and then write it as a single vector, only using y.

Solution:

A line is simply a series of points in $\mathbb{R}^2$, so writing it in set notation is simply

$$\left\{(x, y) \in \mathbb{R}^2 \mid 2y + 4x = 0\right\}$$

To write the line as a vector using only y, solve the equation of the line for x.

$$2y + 4x = 0$$

$$4x = -2y$$

$$x = -\frac{1}{2}y$$

Which means we can write the set in vector form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}y \\ y \end{bmatrix}$$

■ 6. Write the line $y = 3x + 2$ as the linear combination of two vectors. Then plug in $x = -1$, $x = 0$, and $x = 1$, and sketch all three in the same plane.

Solution:

We've already written $y = 3x + 2$ as the vector

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x + 2 \end{bmatrix}$$

Now we can break this up.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 0 \\ 3x + 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

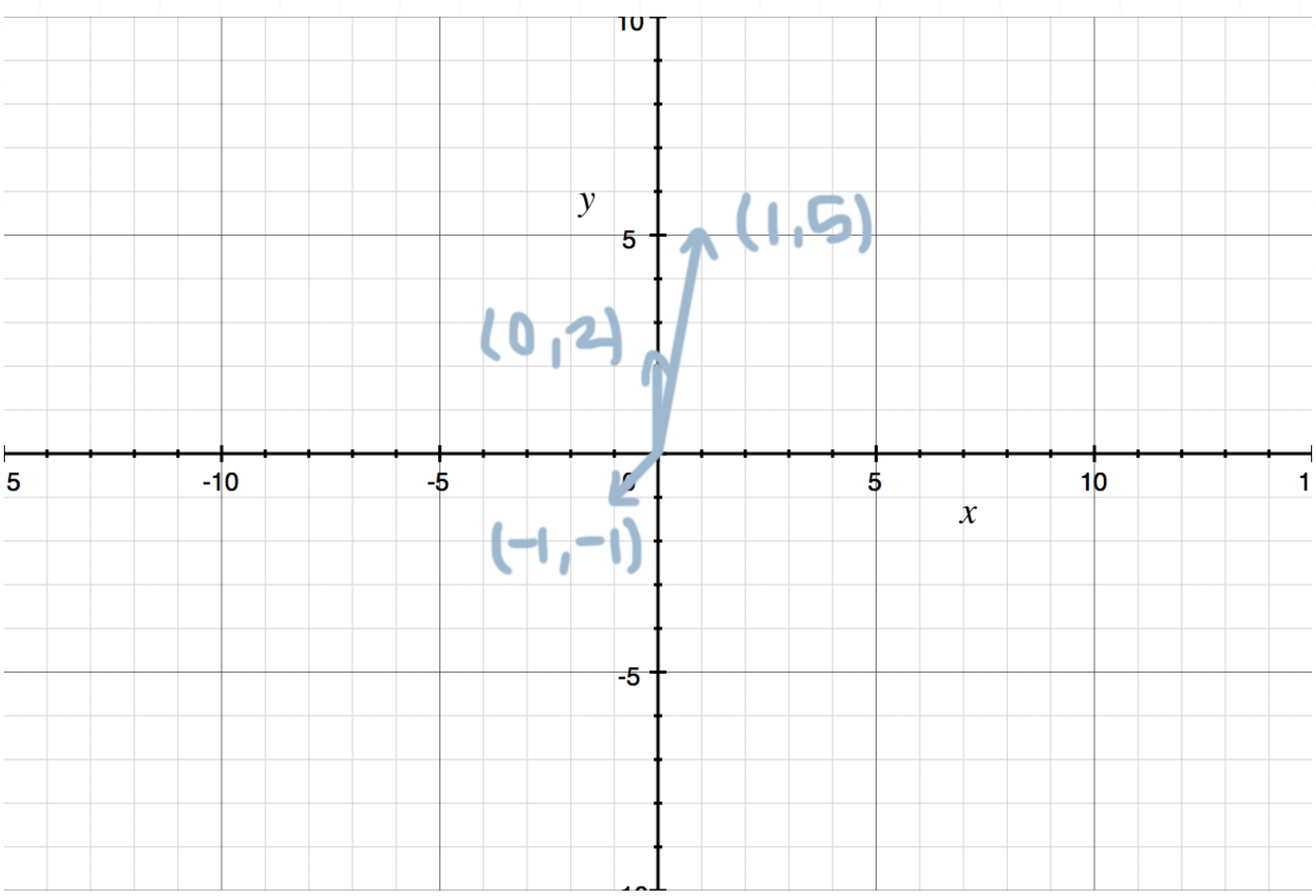
Then for each of the x-values, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+0 \\ -3+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

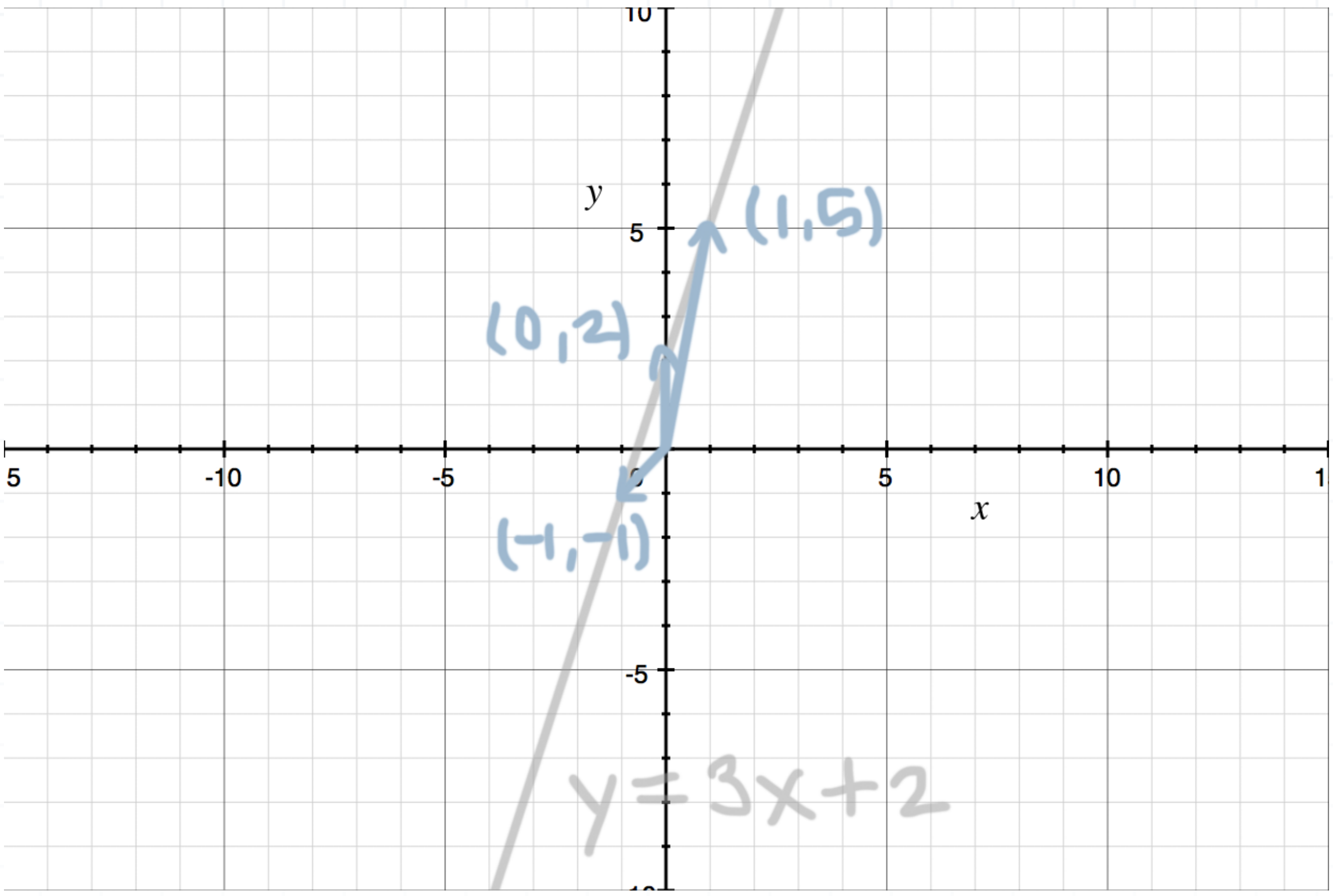
$$\begin{bmatrix} x \\ y \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 3+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

A sketch of the three vectors in the same plane is



Together, they form the line $y = 3x + 2$.


10
y
5
(1,5)
(0,2)
-10
-5
5
10
x
(-1,-1)
-5
y=3x+2

BASIS

■ 1. What requirements must be met in order for a vector set to form the basis for a space?

Solution:

A vector set can form the basis for a space if it 1) spans the space, and 2) is a linearly independent set.

■ 2. What's the standard basis for $\mathbb{R}^4$?

Solution:

The standard basis for $\mathbb{R}^4$ is given by four vectors, each with four components:

$$\vec{v}_1 = (1,0,0,0)$$

$$\vec{v}_2 = (0,1,0,0)$$

$$\vec{v}_3 = (0,0,1,0)$$

$$\vec{v}_4 = (0,0,0,1)$$

3. Say whether or not the vector set V forms a basis for $\mathbb{R}^2$.

$$V = \text{Span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$$

Solution:

First, we check to see if the vectors span $\mathbb{R}^2$, by confirming that we can get to any vector in $\mathbb{R}^2$ using a linear combination of the vectors in the set.

$$c_1\begin{bmatrix}-2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

Change the linear combination equation into an augmented matrix, then put it into reduced row-echelon form.

$$\begin{bmatrix}-2 & 1 & | & x\\1 & 0 & | & y\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & | & y\\-2 & 1 & | & x\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & | & y\\0 & 1 & | & x+2y\end{bmatrix}$$

From the matrix, we get

$$c_1 = y$$

$$c_2 = x + 2y$$

This system tells us that we can pick any point (x, y) that we want to reach in $\mathbb{R}^2$, and we'll be able to plug that point into the system to find the corresponding values of c_1 and c_2 that we need to use. Therefore, the vector set V spans $\mathbb{R}^2$.

Now we need to show that the vectors are linearly independent, which we can do by setting $(x, y) = (0,0)$.

$$c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put the linear combination equation into an augmented matrix, then put it into reduced row-echelon form.

$$\begin{bmatrix} -2 & 1 & | & 0 \\ 1 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ -2 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

From the matrix, we get

$$c_1 = 0$$

$$c_2 = 0$$

Because the only combination of c_1 and c_2 that gives $(x, y) = (0,0)$ is $(c_1, c_2) = (0,0)$, we know the vectors are linearly independent.

Therefore, because the vectors span all of $\mathbb{R}^2$ and are linearly independent, we can say that V forms a basis for $\mathbb{R}^2$.

■ 4. Which scalars c_1 and c_2 would you need to form the vector $\vec{v} = (7, -3)$ as a linear combination of the vectors in the span?

$$V = \text{Span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$$

Solution:

Using this set, we already found

$$c_1 = y$$

$$c_2 = x + 2y$$

So to form $\vec{v} = (7, -3)$, we'll need

$$c_1 = -3$$

$$c_2 = 7 + 2(-3) = 7 - 6 = 1$$

We can double check that $(c_1, c_2) = (-3,1)$ gives $\vec{v} = (7, -3)$ by plugging into the linear combination equation.

$$\begin{bmatrix}x\\y\end{bmatrix} = c_1\begin{bmatrix}-2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

■ 5. Say whether the span forms a basis for $\mathbb{R}^3$.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\right)$$

Solution:

First, we check to see if the vectors span $\mathbb{R}^3$ by verifying that we can get to any vector in $\mathbb{R}^3$ using a linear combination of the vectors in the set. We set up the linear combination equation

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Put the system into an augmented matrix,

$$\begin{bmatrix} 1 & 2 & -2 & | & x \\ 0 & 1 & 1 & | & y \\ -1 & -1 & 4 & | & z \end{bmatrix}$$

and then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -2 & | & x \\ 0 & 1 & 1 & | & y \\ -1 & -1 & 4 & | & z \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & | & x \\ 0 & 1 & 1 & | & y \\ 0 & 1 & 2 & | & x+z \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & x-2y \\ 0 & 1 & 1 & | & y \\ 0 & 1 & 2 & | & x+z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & | & x-2y \\ 0 & 1 & 1 & | & y \\ 0 & 0 & 1 & | & x-y+z \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 5x-6y+4z \\ 0 & 1 & 1 & | & y \\ 0 & 0 & 1 & | & x-y+z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 5x-6y+4z \\ 0 & 1 & 0 & | & -x+2y-z \\ 0 & 0 & 1 & | & x-y+z \end{bmatrix}$$

From the matrix, we get

$$c_1 = 5x - 6y + 4z$$

$$c_2 = -x + 2y - z$$

$$c_3 = x - y + z$$

So we can see that it won't matter which vector $\vec{v} = (x, y, z)$ we pick; we know that it will span $\mathbb{R}^3$, because plugging the vector into the system will simply give us the values of c_1, c_2, and c_3 that we need to use to get $\vec{v} = (x, y, z)$.

Now we need to show that the vectors are linearly independent. We do this by setting $(x, y, z) = (0,0,0)$. We'll augment the matrix with this zero vector, and then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ -1 & -1 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The matrix gives $(c_1, c_2, c_3) = (0,0,0)$ as the only set of scalars that make the linear combination equation give the zero vector, which tells us that the vector set is linearly independent.

Therefore, because the vectors span all of $\mathbb{R}^3$ and are linearly independent, we can say that V forms a basis for $\mathbb{R}^3$.

■ 6. What scalars would you need to get the vector $\vec{v} = (2,0,-5)$ from a linear combination of the set V?

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\right)$$

Solution:

We already found

$$c_1 = 5x - 6y + 4z$$

$$c_2 = -x + 2y - z$$

$$c_3 = x - y + z$$

To get $\overrightarrow{v} = (2,0,-5)$, we plug the values from $\overrightarrow{v}$ into the system for c_1, c_2, and c_3.

$$c_1 = 5(2) - 6(0) + 4(-5) = 10 - 20 = -10$$

$$c_2 = -2 + 2(0) - (-5) = -2 + 5 = 3$$

$$c_3 = 2 - 0 + (-5) = 2 - 5 = -3$$

We can double check that $(c_1, c_2, c_3) = (-10,3,-3)$ gives $\overrightarrow{v} = (2,0,-5)$ by plugging into the linear combination equation.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -10 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} - \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 + 6 + 6 \\ 0 + 3 - 3 \\ 10 - 3 - 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$

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