Linear independence in three dimensions

We can also define linear dependence and independence in three dimensions. If three vectors in three-dimensional space are linearly independent, it means that two of the vectors would be linearly independent of one another in two-dimensional space, and then the third vector would lie outside the plane of the first two vectors.

Only 3 three-dimensional linearly independent vectors are needed to span $\mathbb{R}^3$. Which means, given a set of 4 three-dimensional vectors, the set will always be linearly dependent, since at least one of the vectors could be made from some linear combination of the other three.

Testing for linear independence

To test for linear independence in three dimensions, we can use the same method we used in two dimensions, which was setting the sum of the linear combination of the vectors equal to the zero vector, and then solving the system.

If the only solution is one in which all three constants are 0, then the vectors are linearly independent. But if there's a solution to the system in which one or more of the constants is non-zero, then the vectors are linearly dependent.

Example

Say whether the vectors are linearly dependent or linearly independent.

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \ v_3 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

Set up an equation in which the sum of the linear combination is set equal to the zero vector.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's use a matrix to solve the system. As we saw in the previous lesson, we really only need the column vectors on the left to go into the matrix.

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -3 \\ 3 & 4 & 0 \end{bmatrix}$$

If we can use Gaussian elimination to put this matrix into reduced row-echelon form, then we'll know the vector set $\{v_1, v_2, v_3\}$ is linearly independent.

Start by zeroing out the first column below the pivot entry.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 3 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 4 & -6 \end{bmatrix}$$

Finish zeroing out the rest of the second column around the pivot.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 0 & 22 \end{bmatrix}$$

Find the pivot in the third column.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because we were able to get to the identity matrix with Gaussian elimination, we know $(c_1, c_2, c_3) = (0,0,0)$, and therefore that the vector set $\{v_1, v_2, v_3\}$ is linearly independent. Which means we could also say that $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$.

Matrices are great for solving systems of linear equations, but of course we could have also solved the system using the old elimination and substitution method:

[1] $c_1 + 2c_3 = 0$

[2] $2c_1 + c_2 - 3c_3 = 0$

[3] $3c_1 + 4c_2 = 0$

Solve [1] for c_3 in terms of c_1,

$$c_1 + 2c_3 = 0$$

$$2c_3 = -c_1$$

$$c_3 = -\frac{1}{2}c_1$$

and then solve [3] for c_2 in terms of c_1.

$$3c_1 + 4c_2 = 0$$

$$4c_2 = -3c_1$$

$$c_2 = -\frac{3}{4}c_1$$

Now we can substitute $c_2 = -(3/4)c_1$ and $c_3 = -(1/2)c_1$ into [2] to get an equation only in terms of c_1.

$$2c_1 + c_2 - 3c_3 = 0$$

$$2c_1 - \frac{3}{4}c_1 - 3\left(-\frac{1}{2}c_1\right) = 0$$

$$2c_1 - \frac{3}{4}c_1 + \frac{3}{2}c_1 = 0$$

$$\left(2 - \frac{3}{4} + \frac{3}{2}\right)c_1 = 0$$

$$\frac{11}{4}c_1 = 0$$

$$c_1 = 0$$

Then

$$c_2 = -\frac{3}{4}c_1$$

$$c_2 = -\frac{3}{4}(0)$$

$$c_2 = 0$$

and

$$c_3 = -\frac{1}{2}c_1$$

$$c_3 = -\frac{1}{2}(0)$$

$$c_3 = 0$$

This way too, we see that $(c_1, c_2, c_3) = (0,0,0)$, which means that the vector set $\{v_1, v_2, v_3\}$ is linearly independent.

Let's do another example in which the vector set is linearly dependent.