Linear transformations as matrix-vector products

For now, the kinds of transformations we're interested in are linear transformations, and all the transformations we've looked at up to this point are linear transformations.

We've already seen hints of this in the last couple of lessons, but now we'll say explicitly that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if, for any two vectors $\vec{u}$ and $\vec{v}$ that are both in $\mathbb{R}^n$, and for a c that's also in $\mathbb{R}$ (c is any real number), then

- the transformation of their sum is equivalent to the sum of their individual transformations, $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, and
- the transformation of a scalar multiple the vector is equivalent to the product of the scalar and the transformation of the original vector, $T(c\vec{u}) = cT(\vec{u})$ and $T(c\vec{v}) = cT(\vec{v})$.

Matrix-vector products

You can always represent a linear transformation as a matrix-vector product. For instance, let's say we're transforming vectors from $\mathbb{R}^2$ to vectors in $\mathbb{R}^3$, so $T : \mathbb{R}^2 \to \mathbb{R}^3$, and that the transformation is described as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_2 - x_1 \\ -x_1 + 2x_2 \\ 3x_1 + x_2 \end{bmatrix}$$

To turn this transformation into a matrix-vector product, we need to start by thinking about the space we're transforming *from*. In this case, T is transforming vectors from $\mathbb{R}^2$. The identity matrix for $\mathbb{R}^2$ is I_2,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so the next step is to apply T to I_2, which we'll do one column at a time.

$$\begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix}$$

For the transformation of the first column of the identity matrix, we plug into T to get

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3(0) - 1 \\ -1 + 2(0) \\ 3(1) + 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

For the transformation of the second column of the identity matrix, we plug into T to get

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3(1) - 0 \\ -0 + 2(1) \\ 3(0) + 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Now we get the matrix

$$\begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -1 & 3 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now, whenever we want to find the transformation T of a vector in $\mathbb{R}^2$, we can simply perform a matrix multiplication problem.

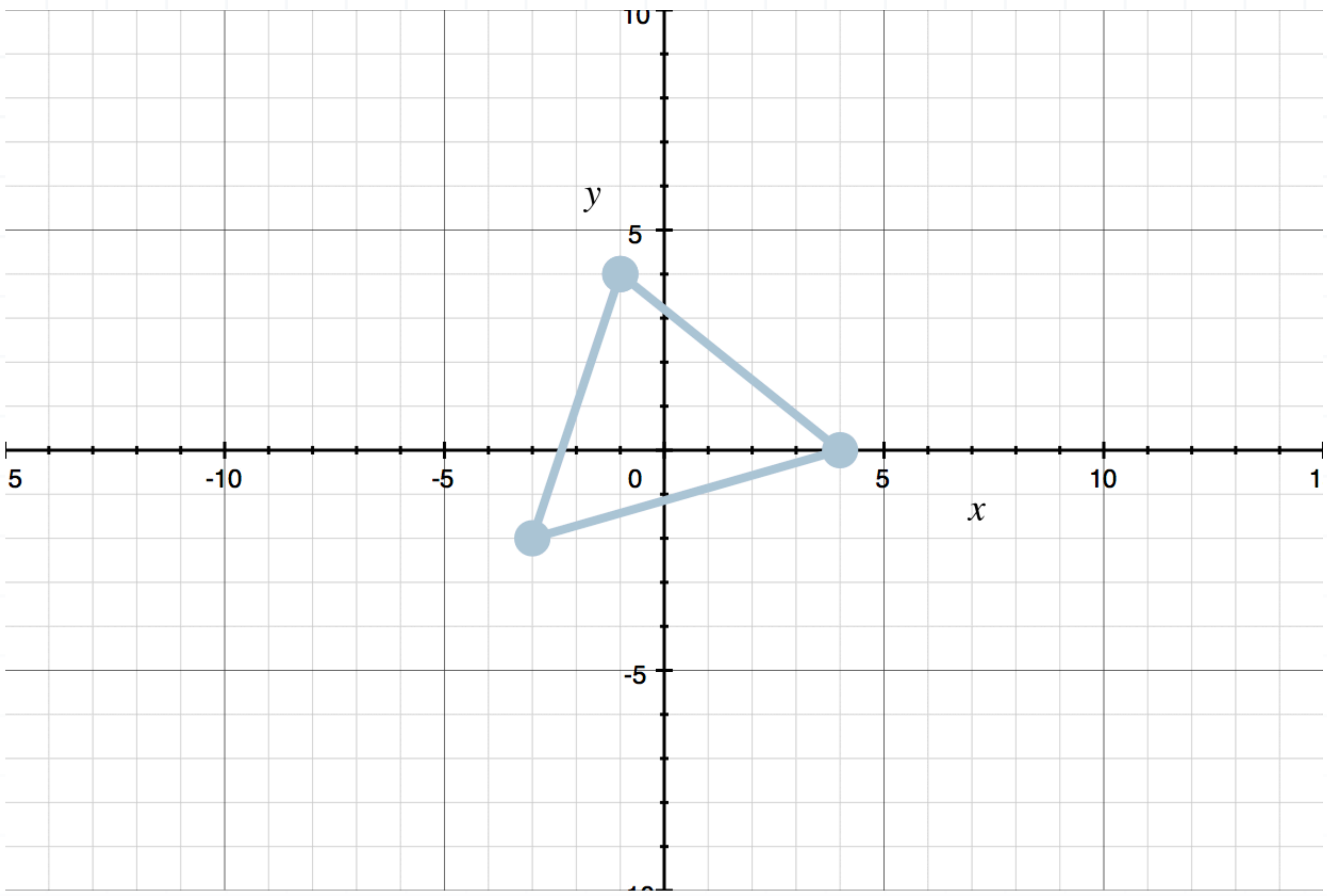
Notice first that we've expressed the transformation as a matrix-vector product, $T(\vec{x}) = A\vec{x}$. So now we know that, if we want to transform any vector $\vec{x}$ by T, we can simply multiply the transformation matrix A by the vector $\vec{x}$ that we're trying to transform.

Notice also that if we multiply this 3×2 transformation matrix A by the 2×1 vector $\vec{x}$, we'll get a 3×1 vector as the result. It will always be true that the product of a matrix and a vector will be a vector, and the dimensions of the resulting vector are predictable. We said that this particular transformation T is transforming from $\mathbb{R}^2$ to $\mathbb{R}^3$, so we already know the resulting vector will be in $\mathbb{R}^3$, and we see just by looking at the dimensions of the matrix-vector product that the dimensions of the resulting vector are 3×1, so the resulting vector is, in fact, in $\mathbb{R}^3$.

Example

Use a matrix-vector product to transform the triangle with vertices $(-1,4)$, $(-3,-2)$, and $(4,0)$. The transformation T should include a reflection over the x-axis and a horizontal stretch by a factor of 3.

We don't have to sketch the triangle in order to do the problem, but it'll help us visualize what we're working with.



If each point in the triangle is given by (x, y), a reflection over the x-axis means we'll take the y-coordinate of each point in the triangle and multiply it by -1. So after the reflection, each transformed point will be $(x, -y)$.

To stretch horizontally by a factor of 3, we'll need to multiply every x-value by 3. So after both the reflection and the stretch, each transformed point will be $(3x, -y)$.

Therefore, if a position vector

$$\overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original triangle, then a position vector

$$\vec{v}=\begin{bmatrix}3v_1\\-v_2\end{bmatrix}$$

represents the corresponding point in the transformed triangle. So a transformation T that expresses the reflection and the stretch for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}3v_1\\-v_2\end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}3(1)\\-0\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}3(0)\\-1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the triangle, $(-1,4)$, $(-3,-2)$, and $(4,0)$.

$$T\left(\begin{bmatrix}-1\\4\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&-1\end{bmatrix}\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}3(-1)+0(4)\\0(-1)-1(4)\end{bmatrix}=\begin{bmatrix}-3\\-4\end{bmatrix}$$

$$T\left(\begin{bmatrix}-3\\-2\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&-1\end{bmatrix}\begin{bmatrix}-3\\-2\end{bmatrix}=\begin{bmatrix}3(-3)+0(-2)\\0(-3)-1(-2)\end{bmatrix}=\begin{bmatrix}-9\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\0\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}=\begin{bmatrix}3(4)+0(0)\\0(4)-1(0)\end{bmatrix}=\begin{bmatrix}12\\0\end{bmatrix}$$

Therefore, we can sketch the transformed triangle, and see at a glance that it does look roughly like it's been reflected over the x-axis and stretched horizontally to three times its original width.