**Topic**: Compositions of linear transformations

**Question**: If $S : X \to Y$ and $T : Y \to Z$, then what is $T(S(\overrightarrow{x}))$?

$$S(\overrightarrow{x}) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

$$T(\overrightarrow{x}) = \begin{bmatrix} 2x_1 - x_2 \\ -2x_2 \end{bmatrix}$$

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

**Answer choices**:

A $\quad \vec{z} = (-7,6)$

B $\quad \vec{z} = (7, -6)$

C $\quad \vec{z} = (-7, -6)$

D $\quad \vec{z} = (7,6)$

**Solution**: C

Apply the transformation $S$ to each column of the $I_2$ identity matrix.

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1+0\\3(1)\end{bmatrix}=\begin{bmatrix}-1\\3\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-0+1\\3(0)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

So the transformation $S$ can be written as

$$S(\vec{x})=\begin{bmatrix}-1&1\\3&0\end{bmatrix}\vec{x}$$

Apply the transformation $T$ to each column of the $I_2$ identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2(1)-0\\-2(0)\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}2(0)-1\\-2(1)\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

So the transformation $T$ can be written as

$$T(\vec{y})=\begin{bmatrix}2&-1\\0&-2\end{bmatrix}\vec{y}$$

Then the composition $T\circ S$ can be written as

$$T(S(\vec{x}))=\begin{bmatrix}2&-1\\0&-2\end{bmatrix}\begin{bmatrix}-1&1\\3&0\end{bmatrix}\vec{x}$$

$$T(S(\overrightarrow{x})) = \begin{bmatrix} 2(-1)-1(3) & 2(1)-1(0) \\ 0(-1)-2(3) & 0(1)-2(0) \end{bmatrix} \overrightarrow{x}$$

$$T(S(\overrightarrow{x})) = \begin{bmatrix} -2-3 & 2-0 \\ 0-6 & 0-0 \end{bmatrix} \overrightarrow{x}$$

$$T(S(\overrightarrow{x})) = \begin{bmatrix} -5 & 2 \\ -6 & 0 \end{bmatrix} \overrightarrow{x}$$

Transform $\overrightarrow{x} = (1, -1)$.

$$T\left(S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -5 & 2 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -5(1)+2(-1) \\ -6(1)+0(-1) \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -5-2 \\ -6+0 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -7 \\ -6 \end{bmatrix}$$

Therefore, we can say that the vector $\overrightarrow{x} = (1, -1)$ in the subset $X$ is transformed into the vector $\vec{z} = (-7, -6)$ in the subset $Z$.

**Topic**: Compositions of linear transformations

**Question**: If $S : X \to Y$ and $T : Y \to Z$, then what is $T(S(\overrightarrow{x}))$?

$$S(\overrightarrow{x}) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ -4x_3 \\ x_2 - x_1 \end{bmatrix}$$

$$T(\overrightarrow{x}) = \begin{bmatrix} -3x_1 \\ -2x_2 + x_3 \\ 4x_3 \end{bmatrix}$$

$$\overrightarrow{x} = \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}$$

**Answer choices**:

A $\quad \vec{z} = (-24, -6, -24)$

B $\quad \vec{z} = (-24, -6, 24)$

C $\quad \vec{z} = (-24, 6, -24)$

D $\quad \vec{z} = (24, 6, 24)$

**Solution**: A

Apply the transformation $S$ to each column of the $I_3$ identity matrix.

$$S\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2(1)-0+0\\-4(0)\\0-1\end{bmatrix}=\begin{bmatrix}2\\0\\-1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2(0)-1+0\\-4(0)\\1-0\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}2(0)-0+1\\-4(1)\\0-0\end{bmatrix}=\begin{bmatrix}1\\-4\\0\end{bmatrix}$$

So the transformation $S$ can be written as

$$S(\vec{x})=\begin{bmatrix}2&-1&1\\0&0&-4\\-1&1&0\end{bmatrix}\vec{x}$$

Apply the transformation $T$ to each column of the $I_3$ identity matrix.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}-3(1)\\-2(0)+0\\4(0)\end{bmatrix}=\begin{bmatrix}-3\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-3(0)\\-2(1)+0\\4(0)\end{bmatrix}=\begin{bmatrix}0\\-2\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-3(0)\\-2(0)+1\\4(1)\end{bmatrix}=\begin{bmatrix}0\\1\\4\end{bmatrix}$$

So the transformation $T$ can be written as

$$T(\vec{y})=\begin{bmatrix}-3&0&0\\0&-2&1\\0&0&4\end{bmatrix}\vec{y}$$

Then the composition $T\circ S$ can be written as

$$T(S(\vec{x}))=\begin{bmatrix}-3&0&0\\0&-2&1\\0&0&4\end{bmatrix}\begin{bmatrix}2&-1&1\\0&0&-4\\-1&1&0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}-3(2)+0(0)+0(-1)&-3(-1)+0(0)+0(1)&-3(1)+0(-4)+0(0)\\0(2)-2(0)+1(-1)&0(-1)-2(0)+1(1)&0(1)-2(-4)+1(0)\\0(2)+0(0)+4(-1)&0(-1)+0(0)+4(1)&0(1)+0(-4)+4(0)\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}-6+0+0&3+0+0&-3+0+0\\0-0-1&0-0+1&0+8+0\\0+0-4&0+0+4&0+0+0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}-6&3&-3\\-1&1&8\\-4&4&0\end{bmatrix}\vec{x}$$

Transform $\vec{x}=(2,-4,0)$.

$$T\left(S\left(\begin{bmatrix}2\\-4\\0\end{bmatrix}\right)\right)=\begin{bmatrix}-6&3&-3\\-1&1&8\\-4&4&0\end{bmatrix}\begin{bmatrix}2\\-4\\0\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} -6(2) + 3(-4) - 3(0) \\ -1(2) + 1(-4) + 8(0) \\ -4(2) + 4(-4) + 0(0) \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} -12 - 12 - 0 \\ -2 - 4 + 0 \\ -8 - 16 + 0 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} -24 \\ -6 \\ -24 \end{bmatrix}$$

Therefore, we can say that the vector $\vec{x} = (2, -4, 0)$ in the subset $X$ is transformed into the vector $\vec{z} = (-24, -6, -24)$ in the subset $Z$.

**Topic**: Compositions of linear transformations

**Question**: If $S: X \to Y$ and $T: Y \to Z$, then what is $T(S(\overrightarrow{x}))$?

$$S(\overrightarrow{x}) = \begin{bmatrix} -5x_3 \\ 2x_3 \\ x_1 + x_2 \end{bmatrix}$$

$$T(\overrightarrow{x}) = \begin{bmatrix} 3x_2 \\ -2x_1 \\ 4x_3 \end{bmatrix}$$

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

**Answer choices**:

A $\vec{z} = (12,20,20)$

B $\vec{z} = (-12,20,-20)$

C $\vec{z} = (12,-20,-20)$

D $\vec{z} = (-12,-20,20)$

**Solution**: D

Apply the transformation $S$ to each column of the $I_3$ identity matrix.

$$S\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}-5(0)\\2(0)\\1+0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-5(0)\\2(0)\\0+1\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-5(1)\\2(1)\\0+0\end{bmatrix}=\begin{bmatrix}-5\\2\\0\end{bmatrix}$$

So the transformation $S$ can be written as

$$S(\vec{x})=\begin{bmatrix}0&0&-5\\0&0&2\\1&1&0\end{bmatrix}\vec{x}$$

Apply the transformation $T$ to each column of the $I_3$ identity matrix.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}3(0)\\-2(1)\\4(0)\end{bmatrix}=\begin{bmatrix}0\\-2\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}3(1)\\-2(0)\\4(0)\end{bmatrix}=\begin{bmatrix}3\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3(0)\\-2(0)\\4(1)\end{bmatrix}=\begin{bmatrix}0\\0\\4\end{bmatrix}$$

So the transformation $T$ can be written as

$$T(\vec{y})=\begin{bmatrix}0&3&0\\-2&0&0\\0&0&4\end{bmatrix}\vec{y}$$

Then the composition $T\circ S$ can be written as

$$T(S(\vec{x}))=\begin{bmatrix}0&3&0\\-2&0&0\\0&0&4\end{bmatrix}\begin{bmatrix}0&0&-5\\0&0&2\\1&1&0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0(0)+3(0)+0(1)&0(0)+3(0)+0(1)&0(-5)+3(2)+0(0)\\-2(0)+0(0)+0(1)&-2(0)+0(0)+0(1)&-2(-5)+0(2)+0(0)\\0(0)+0(0)+4(1)&0(0)+0(0)+4(1)&0(-5)+0(2)+4(0)\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0+0+0&0+0+0&0+6+0\\0+0+0&0+0+0&10+0+0\\0+0+4&0+0+4&0+0+0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0&0&6\\0&0&10\\4&4&0\end{bmatrix}\vec{x}$$

Transform $\vec{x}=(1,4,-2)$.

$$T\left(S\left(\begin{bmatrix}1\\4\\-2\end{bmatrix}\right)\right)=\begin{bmatrix}0&0&6\\0&0&10\\4&4&0\end{bmatrix}\begin{bmatrix}1\\4\\-2\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}1\\4\\-2\end{bmatrix}\right)\right)=\begin{bmatrix}0(1)+0(4)+6(-2)\\0(1)+0(4)+10(-2)\\4(1)+4(4)+0(-2)\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}1\\4\\-2\end{bmatrix}\right)\right)=\begin{bmatrix}0+0-12\\0+0-20\\4+16+0\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}1\\4\\-2\end{bmatrix}\right)\right)=\begin{bmatrix}-12\\-20\\20\end{bmatrix}$$

Therefore, we can say that the vector $\overrightarrow{x}=(1,4,-2)$ in the subset $X$ is transformed into the vector $\vec{z}=(-12,-20,20)$ in the subset $Z$.