



Linear Algebra Final Exam Solutions

Linear Algebra Final Exam Answer Key

1. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input checked="" type="checkbox"/>	<input type="checkbox"/> D	<input type="checkbox"/> E
2. (5 pts)	<input checked="" type="checkbox"/>	<input type="checkbox"/> B	<input type="checkbox"/> C	<input type="checkbox"/> D	<input type="checkbox"/> E
3. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input type="checkbox"/> C	<input type="checkbox"/> D	<input checked="" type="checkbox"/>
4. (5 pts)	<input type="checkbox"/> A	<input checked="" type="checkbox"/>	<input type="checkbox"/> C	<input type="checkbox"/> D	<input type="checkbox"/> E
5. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input type="checkbox"/> C	<input type="checkbox"/> D	<input checked="" type="checkbox"/>
6. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input checked="" type="checkbox"/>	<input type="checkbox"/> D	<input type="checkbox"/> E
7. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input type="checkbox"/> C	<input type="checkbox"/> D	<input checked="" type="checkbox"/>
8. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input type="checkbox"/> C	<input checked="" type="checkbox"/>	<input type="checkbox"/> E



9. (15 pts)

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \end{bmatrix}$$

10. (15 pts)

$$T(S(\vec{x})) = \begin{bmatrix} 8 \\ -8 \\ 14 \end{bmatrix}$$

$$S(T(\vec{x})) = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$$

11. (15 pts)

$$\det(B) = -21$$

12. (15 pts)

$$V_3 = \text{Span}\left(\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \frac{2}{\sqrt{14}} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ -1 \end{bmatrix}\right)$$



Linear Algebra Final Exam Solutions

1. C. The matrix for the system is

$$\begin{bmatrix} 1 & -4 & 1 & 20 \\ -1 & 0 & 1 & 10 \\ 4 & 1 & -2 & -25 \end{bmatrix}$$

Start by working on the first column.

$$\begin{bmatrix} 1 & -4 & 1 & 20 \\ 0 & -4 & 2 & 30 \\ 4 & 1 & -2 & -25 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 1 & 20 \\ 0 & -4 & 2 & 30 \\ 0 & 17 & -6 & -105 \end{bmatrix}$$

Find the pivot entry in the second column, then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -4 & 1 & 20 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 17 & -6 & -105 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -10 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 17 & -6 & -105 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & -1 & -10 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 0 & \frac{5}{2} & \frac{45}{2} \end{bmatrix}$$

Find the pivot entry in the third column, then zero out the rest of the third column.



$$\begin{bmatrix} 1 & 0 & -1 & -10 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 0 & 1 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 0 & 1 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (-1, -3, 9)$.

2. A. To find $A - C$ by subtracting matrices, we subtract corresponding entries from each matrix.

$$A - C = \begin{bmatrix} 5 & -4 \\ -3 & 9 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 7 \\ -3 & 3 \end{bmatrix}$$

$$A - C = \begin{bmatrix} 5 - 3 & -4 - 2 \\ -3 - (-1) & 9 - 7 \\ 0 - (-3) & 4 - 3 \end{bmatrix}$$

$$A - C = \begin{bmatrix} 2 & -6 \\ -2 & 2 \\ 3 & 1 \end{bmatrix}$$

To find $(A - C)B$, multiply each row of $A - C$ by the first column of B .

$$(A - C)B = \begin{bmatrix} 2 & -6 \\ -2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & 7 \\ 3 & 7 & 8 \end{bmatrix}$$

$$(A - C)B = \begin{bmatrix} 2(4) - 6(3) & \dots & \dots \\ -2(4) + 2(3) & \dots & \dots \\ 3(4) + 1(3) & \dots & \dots \end{bmatrix}$$



$$(A - C)B = \begin{bmatrix} 8 - 18 & \dots & \dots \\ -8 + 6 & \dots & \dots \\ 12 + 3 & \dots & \dots \end{bmatrix}$$

$$(A - C)B = \begin{bmatrix} -10 & \dots & \dots \\ -2 & \dots & \dots \\ 15 & \dots & \dots \end{bmatrix}$$

Multiply each row of $A - C$ by the second column of B .

$$(A - C)B = \begin{bmatrix} -10 & 2(-1) - 6(7) & \dots \\ -2 & -2(-1) + 2(7) & \dots \\ 15 & 3(-1) + 1(7) & \dots \end{bmatrix}$$

$$(A - C)B = \begin{bmatrix} -10 & -2 - 42 & \dots \\ -2 & 2 + 14 & \dots \\ 15 & -3 + 7 & \dots \end{bmatrix}$$

$$(A - C)B = \begin{bmatrix} -10 & -44 & \dots \\ -2 & 16 & \dots \\ 15 & 4 & \dots \end{bmatrix}$$

Multiply each row of $A - C$ by the third column of B .

$$(A - C)B = \begin{bmatrix} -10 & -44 & 2(7) - 6(8) \\ -2 & 16 & -2(7) + 2(8) \\ 15 & 4 & 3(7) + 1(8) \end{bmatrix}$$

$$(A - C)B = \begin{bmatrix} -10 & -44 & 14 - 48 \\ -2 & 16 & -14 + 16 \\ 15 & 4 & 21 + 8 \end{bmatrix}$$



$$(A - C)B = \begin{bmatrix} -10 & -44 & -34 \\ -2 & 16 & 2 \\ 15 & 4 & 29 \end{bmatrix}$$

3. E. The vector sum is

$$\vec{u} = \vec{a} + 2\vec{b} - 3\vec{c} - \vec{d}$$

$$\vec{u} = (-3, 5, -1) + 2(4, 2, 7) - 3(0, 2, 1) - (-1, 3, 5)$$

Apply the scalars to each vector.

$$\vec{u} = (-3, 5, -1) + (8, 4, 14) + (0, -6, -3) + (1, -3, -5)$$

Add each of the vector components.

$$\vec{u} = (-3 + 8 + 0 + 1, 5 + 4 - 6 - 3, -1 + 14 - 3 - 5)$$

$$\vec{u} = (6, 0, 5)$$

Then, find the length of \vec{u} .

$$||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

$$||\vec{u}|| = \sqrt{6^2 + 0^2 + 5^2}$$

$$||\vec{u}|| = \sqrt{36 + 0 + 25}$$

$$||\vec{u}|| = \sqrt{61}$$

Then the unit vector in the direction of $\vec{u} = (6, 0, 5)$ is



$$\vec{v} = \frac{1}{||\vec{u}||} \vec{u}$$

$$\vec{v} = \frac{1}{\sqrt{61}} \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} \frac{6}{\sqrt{61}} \\ 0 \\ \frac{5}{\sqrt{61}} \end{bmatrix}$$

4. B. The angle θ between two vectors is given by a relationship between the dot product of the vectors and the lengths of the vectors.

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos \theta$$

$$\vec{u} \cdot \vec{v} = (2, -3, 0, 5) \cdot (12, 3, 8, -3)$$

$$\vec{u} \cdot \vec{v} = (2)(12) + (-3)(3) + (0)(8) + (5)(-3)$$

$$\vec{u} \cdot \vec{v} = 24 - 9 + 0 - 15$$

$$\vec{u} \cdot \vec{v} = 0$$

Because the dot product is 0, \vec{u} and \vec{v} are orthogonal to one another and the angle between them is $\theta = 90^\circ$.



5. E. The cross product would be

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & -1 \\ 0 & -2 & 2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & -1 \\ 0 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 3 \\ 0 & -2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i}((3)(2) - (-1)(-2)) - \mathbf{j}((-2)(2) - (-1)(0)) + \mathbf{k}((-2)(-2) - (3)(0))$$

$$\vec{a} \times \vec{b} = \mathbf{i}(6 - 2) - \mathbf{j}(-4 + 0) + \mathbf{k}(4 - 0)$$

$$\vec{a} \times \vec{b} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\vec{a} \times \vec{b} = (4, 4, 4)$$

6. C. We can rewrite W as

$$W = \left\{ x \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \mid x, y, z \in \mathbb{R}^4 \right\}$$

The subspace W is a space in \mathbb{R}^4 , spanned by the three vectors $\vec{w}_1 = (0, -2, 1, 1)$, $\vec{w}_2 = (1, 1, 0, 0)$ and $\vec{w}_3 = (1, 0, -1, 0)$. Therefore, its orthogonal complement W^\perp is the set of vectors which are orthogonal to $\vec{w}_1 = (0, -2, 1, 1)$, $\vec{w}_2 = (1, 1, 0, 0)$ and $\vec{w}_3 = (1, 0, -1, 0)$.



$$W^\perp = \{ \vec{x} \in \mathbb{R}^4 \mid \vec{x} \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = 0 \text{ and } \vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \text{ and } \vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = 0 \}$$

If we let $\vec{x} = (x_1, x_2, x_3, x_4)$, we get three equations from these dot products.

$$-2x_2 + x_3 + x_4 = 0$$

$$x_1 + x_2 = 0$$

$$x_1 - x_3 = 0$$

Put these equations into an augmented matrix,

$$\left[\begin{array}{cccc|c} 0 & -2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{array} \right]$$

then put it into reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array} \right] \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array} \right] \rightarrow$$



$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & \frac{1}{3} & | & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & \frac{1}{3} & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & | & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & | & 0 \\ 0 & 0 & 1 & \frac{1}{3} & | & 0 \end{bmatrix}$$

This rref form gives the system of equations

$$x_1 + \frac{1}{3}x_4 = 0$$

$$x_2 - \frac{1}{3}x_4 = 0$$

$$x_3 + \frac{1}{3}x_4 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = -\frac{1}{3}x_4$$

$$x_2 = \frac{1}{3}x_4$$

$$x_3 = -\frac{1}{3}x_4$$

We can express this system as



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$W^\perp = \text{Span}\left(\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right)$$

7. E. The transpose of A is

$$A^T = \begin{bmatrix} -1 & 2 & 6 \\ -3 & 4 & 0 \end{bmatrix}$$

To find the null space, we'll augment the matrix, and then put it into reduced row-echelon form.

$$\left[\begin{array}{ccc|c} -1 & 2 & 6 & 0 \\ -3 & 4 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -6 & 0 \\ -3 & 4 & 0 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 0 \\ 0 & -2 & -18 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -6 & 0 \\ 0 & 1 & 9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 12 & 0 \\ 0 & 1 & 9 & 0 \end{array}\right]$$



Because we have pivot entries in the first two columns, we'll pull a system of equations from the matrix,

$$x_1 + 12x_3 = 0$$

$$x_2 + 9x_3 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = -12x_3$$

$$x_2 = -9x_3$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -12 \\ -9 \\ 1 \end{bmatrix}$$

Therefore, the left null space is

$$N(A^T) = \text{Span}\left(\begin{bmatrix} -12 \\ -9 \\ 1 \end{bmatrix}\right)$$

The space of the null space of the transpose is always \mathbb{R}^m , where m is the number of rows in the original matrix, A . The original matrix has 3 rows, so the null space of the transpose $N(A^T)$ is a subspace of \mathbb{R}^3 .

The column space of the transpose A^T , which is the same as the row space of A , is simply given by the columns in A^T that contain pivot



entries when A^T is in reduced row-echelon form. So the column space of A^T is

$$C(A^T) = \text{Span}\left(\begin{bmatrix} -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

The space of the column space of the transpose is always \mathbb{R}^n , where n is the number of columns in the original matrix, A . The original matrix has 2 columns, so the column space of the transpose $C(A^T)$ is a subspace of \mathbb{R}^2 .

Because there's one vector that forms the basis of $N(A^T)$, the dimension of $N(A^T)$ is $\text{Dim}(N(A^T)) = 1$.

Because there are two vectors that form the basis of $C(A^T)$, the dimension of $C(A^T)$ is $\text{Dim}(C(A^T)) = 2$.

$$N(A^T) = \text{Span}\left(\begin{bmatrix} -12 \\ -9 \\ 1 \end{bmatrix}\right) \text{ in } \mathbb{R}^3$$

$$\text{Dim}(N(A^T)) = 1$$

$$C(A^T) = \text{Span}\left(\begin{bmatrix} -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) \text{ in } \mathbb{R}^2$$

$$\text{Dim}(C(A^T)) = 2$$

8. D. Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 0 \\ 3 & 1 \end{bmatrix} \right|$$



$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -5 & 0 \\ 3 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - (-5) & 0 - 0 \\ 0 - 3 & \lambda - 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 5 & 0 \\ -3 & \lambda - 1 \end{bmatrix} \right|$$

The determinant is

$$(\lambda + 5)(\lambda - 1) - (-3)(0)$$

$$(\lambda + 5)(\lambda - 1)$$

$$\lambda = -5 \text{ or } \lambda = 1$$

With $\lambda = -5$ and $\lambda = 1$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda + 5 & 0 \\ -3 & \lambda - 1 \end{bmatrix}\right)$$

we get

$$E_{-5} = N\left(\begin{bmatrix} -5 + 5 & 0 \\ -3 & -5 - 1 \end{bmatrix}\right)$$

$$E_{-5} = N\left(\begin{bmatrix} 0 & 0 \\ -3 & -6 \end{bmatrix}\right)$$

and



$$E_1 = N \left(\begin{bmatrix} 1+5 & 0 \\ -3 & 1-1 \end{bmatrix} \right)$$

$$E_1 = N \left(\begin{bmatrix} 6 & 0 \\ -3 & 0 \end{bmatrix} \right)$$

Therefore, the eigenvectors in the eigenspace E_{-5} will satisfy

$$\begin{bmatrix} 0 & 0 \\ -3 & -6 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -3 & -6 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 0 \end{array} \right]$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + 2v_2 = 0$$

$$v_1 = -2v_2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So the eigenvector for E_{-5} will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

And the eigenvectors in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 6 & 0 \\ -3 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$\left[\begin{array}{cc|c} 6 & 0 & 0 \\ -3 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ -3 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = 0$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So the eigenvector for E_1 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

9. Put the matrix A into reduced row-echelon form.

$$\left[\begin{array}{cccc} 1 & -2 & -5 & -3 \\ 3 & -1 & -5 & -4 \\ 0 & -5 & -10 & -5 \end{array} \right] \rightarrow \left[\begin{array}{cccc} 1 & -2 & -5 & -3 \\ 0 & 5 & 10 & 5 \\ 0 & -5 & -10 & -5 \end{array} \right] \rightarrow$$

$$\left[\begin{array}{cccc} 1 & -2 & -5 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -10 & -5 \end{array} \right] \rightarrow \left[\begin{array}{cccc} 1 & -2 & -5 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow$$



$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To find the complementary solution, augment $\text{rref}(A)$ with the zero vector to get a system of equations.

$$x_1 - x_3 - x_4 = 0$$

$$x_2 + 2x_3 + x_4 = 0$$

Solve for the pivot variables in terms of the free variables.

$$x_1 = x_3 + x_4$$

$$x_2 = -2x_3 - x_4$$

The vectors that satisfy the null space are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

We could therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

To find the particular solution, augment A with $\vec{b} = (b_1, b_2, b_3)$, then put it in reduced row-echelon form.



$$\begin{bmatrix} 1 & -2 & -5 & -3 & | & b_1 \\ 3 & -1 & -5 & -4 & | & b_2 \\ 0 & -5 & -10 & -5 & | & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -5 & -3 & | & b_1 \\ 0 & 5 & 10 & 5 & | & b_2 - 3b_1 \\ 0 & -5 & -10 & -5 & | & b_3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -2 & -5 & -3 & | & b_1 \\ 0 & 1 & 2 & 1 & | & \frac{1}{5}(b_2 - 3b_1) \\ 0 & -5 & -10 & -5 & | & b_3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & | & b_1 + \frac{2}{5}(b_2 - 3b_1) \\ 0 & 1 & 2 & 1 & | & \frac{1}{5}(b_2 - 3b_1) \\ 0 & -5 & -10 & -5 & | & b_3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & | & b_1 + \frac{2}{5}(b_2 - 3b_1) \\ 0 & 1 & 2 & 1 & | & \frac{1}{5}(b_2 - 3b_1) \\ 0 & 0 & 0 & 0 & | & b_3 + b_2 - 3b_1 \end{bmatrix}$$

From the third row, the system is constrained.

$$-3b_1 + b_2 + b_3 = 0$$

$$b_2 = 3b_1 - b_3$$

Choose values of b_1 , b_2 , and b_3 that make this equation true, like $b_2 = 1$, $b_3 = 2$, and $b_1 = 1$. Then the matrix is



$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 1 + \frac{2}{5}(1 - 3(1)) \\ 0 & 1 & 2 & 1 & \frac{1}{5}(1 - 3(1)) \\ 0 & 0 & 0 & 0 & 2 + 1 - 3(1) \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & \frac{1}{5} \\ 0 & 1 & 2 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

This gives a system of equations

$$x_1 - x_3 - x_4 = \frac{1}{5}$$

$$x_2 + 2x_3 + x_4 = -\frac{2}{5}$$

Because x_3 and x_4 are free variables, set $x_3 = 0$ and $x_4 = 0$.

$$x_1 - 0 - 0 = \frac{1}{5}$$

$$x_2 + 2(0) + 0 = -\frac{2}{5}$$

The system becomes

$$x_1 = \frac{1}{5}$$

$$x_2 = -\frac{2}{5}$$

So the particular solution is



$$\vec{x}_p = \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \end{bmatrix}$$

The general solution is the sum of the complementary and particular solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \end{bmatrix}$$

10. Apply the transformation S to each column of the I_3 identity matrix.

$$S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -0 - 3(1) \\ 0 - 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 - 3(0) \\ 1 - 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -0 - 3(0) \\ 0 - 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$



So the transformation S can be written as

$$S(\vec{x}) = \begin{bmatrix} -3 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}$$

Apply the transformation T to each column of the I_3 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 - 2(0) + 0 \\ 0 - 1 \\ 2(1) + 0 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2(1) + 0 \\ 0 - 0 \\ 2(0) + 1 - 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2(0) + 1 \\ 1 - 0 \\ 2(0) + 0 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{x}) = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \vec{x}$$

Then the composition $T \circ S$ can be written as

$$T(S(\vec{x})) = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -3 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}$$



$$T(S(\vec{x})) = \begin{bmatrix} -3 + 0 + 0 & -1 - 2 + 1 & 0 + 2 + 0 \\ 3 + 0 + 0 & 1 + 0 + 1 & 0 + 0 + 0 \\ -6 + 0 + 0 & -2 + 1 - 1 & 0 - 1 + 0 \end{bmatrix} \vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix} -3 & -2 & 2 \\ 3 & 2 & 0 \\ -6 & -2 & -1 \end{bmatrix} \vec{x}$$

Transform $\vec{x} = (-2, -1, 0)$.

$$T\left(S\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} -3 & -2 & 2 \\ 3 & 2 & 0 \\ -6 & -2 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} -3(-2) - 2(-1) + 2(0) \\ 3(-2) + 2(-1) + 0(0) \\ -6(-2) + (-2)(-1) + (-1)(0) \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} 6 + 2 + 0 \\ -6 - 2 + 0 \\ 12 + 2 + 0 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} 8 \\ -8 \\ 14 \end{bmatrix}$$

Then the composition $S \circ T$ can be written as

$$S(T(\vec{x})) = \begin{bmatrix} -3 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \vec{x}$$



$$S(T(\vec{x})) = \begin{bmatrix} -3 + 1 + 0 & 6 - 0 + 0 & -3 - 1 + 0 \\ 0 - 1 - 2 & 0 + 0 - 1 & 0 + 1 + 1 \\ 0 - 1 + 0 & 0 + 0 + 0 & 0 + 1 + 0 \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} -2 & 6 & -4 \\ -3 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \vec{x}$$

Transform $\vec{x} = (-2, -1, 0)$.

$$S\left(T\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} -2 & 6 & -4 \\ -3 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} (-2)(-2) + 6(-1) - 4(0) \\ (-3)(-2) + (-1)(-1) + 2(0) \\ (-1)(-2) + 0(-1) + 1(0) \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} 4 - 6 - 0 \\ 6 + 1 + 0 \\ 2 + 0 + 0 \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$$

11. The matrices A and C are identical, other than two changes. Matrix A has rows 2 and 3 that are swapped, compared to matrix C . When matrices are identical other than a swapped row, the determinant of one is equal to the negative determinant of the other.



The second change is that the second row of C has been multiplied by 3, compared to matrix A . If we have a row multiplied by a constant k , then the determinant of the new matrix is multiplied by k .

Putting these two changes together, we get

$$\det(C) = -3\det(A)$$

$$\det(C) = -3(7) = -21$$

We also see that $B = C^T$. The determinant of a transpose of a square matrix will always be equal to the determinant of the original matrix, which means $\det(B) = \det(C) = -21$.

12. Define $\vec{v}_1 = (-1, 1, 1, -1)$, $\vec{v}_2 = (-2, -1, 0, 1)$, and $\vec{v}_3 = (1, 0, -2, -1)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of \vec{v}_1 is

$$\|\vec{v}_1\| = \sqrt{(-1)^2 + (1)^2 + (1)^2 + (-1)^2} = \sqrt{1 + 1 + 1 + 1} = \sqrt{4} = 2$$

Then if \vec{u}_1 is the normalized version of \vec{v}_1 , we can say

$$\vec{u}_1 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

So we can say that V is spanned by \vec{u}_1 , \vec{v}_2 , and \vec{v}_3 .



$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name \vec{w}_2 as the vector that connects $\text{Proj}_{V_1} \vec{v}_2$ to \vec{v}_2 .

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1} \vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4} \left(\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4}((-2)(-1) + (-1)(1) + (0)(1) + (1)(-1)) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4}(0) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$



$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So \vec{w}_2 is orthogonal to \vec{u}_1 , but it hasn't been normalized, so let's normalize it. The length of \vec{w}_2 is

$$||\vec{w}_2|| = \sqrt{(-2)^2 + (-1)^2 + (-0)^2 + (1)^2}$$

$$||\vec{w}_2|| = \sqrt{4 + 1 + 0 + 1}$$

$$||\vec{w}_2|| = \sqrt{6}$$

Then the normalized version of \vec{w}_2 is \vec{u}_2 .

$$\vec{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So we can say that V is spanned by \vec{u}_1 , \vec{u}_2 , and \vec{v}_3 . Then the vector \vec{w}_3 is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1} \vec{v}_3 - \text{Proj}_{V_2} \vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have.



$$\vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} - \frac{1}{4} \left(\begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{6} \left(\begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} - \frac{1}{4}((1)(-1) + (0)(1) + (-2)(1) + (-1)(-1)) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$- \frac{1}{6}((1)(-2) + (0)(-1) + (-2)(0) + (-1)(1)) \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} - \frac{1}{4}(-2) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{6}(-3) \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$



$$\vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ -1 \end{bmatrix}$$

The length of \vec{w}_3 is

$$||\vec{w}_3|| = \sqrt{\left(-\frac{1}{2}\right)^2 + 0^2 + \left(-\frac{3}{2}\right)^2 + (-1)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{1}{4} + 0 + \frac{9}{4} + 1}$$

$$||\vec{w}_3|| = \frac{\sqrt{14}}{2}$$

Then the normalized version of \vec{w}_3 is \vec{u}_3 :

$$\vec{u}_3 = \frac{2}{\sqrt{14}} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ -1 \end{bmatrix}$$



Therefore, we can say that \vec{u}_1 , \vec{u}_2 , and \vec{u}_3 form an orthonormal basis for V .

$$V_3 = \text{Span}\left(\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \frac{2}{\sqrt{14}} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ -1 \end{bmatrix}\right)$$



