Transformation matrix for a basis

Earlier we learned that a linear transformation could always be represented as a matrix-vector product. So we'd say that the linear transformation $T(\vec{x})$ could be written as

$$T(\vec{x}) = A\vec{x}$$

But up to now, we've always been working in the standard basis. Which means that the linear transformation T took vectors $\vec{x}$ that were given in the standard basis, and transformed them using the matrix A into another vector $T(\vec{x})$ in the standard basis.

Transforming from an alternate basis

Now we want to learn how to use the same transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ to transform vectors $[\vec{x}]_B$ given in an alternate basis, into vectors $[T(\vec{x})]_B$ in the alternate basis.

Whether we're transforming vectors in the standard basis or some alternate basis B for the subspace V, the transformation is still linear. Which means that, in the same way we represent a transformation in the standard basis as $T(\vec{x}) = A\vec{x}$, we can represent a transformation in the alternate basis B as

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

If we also define another invertible matrix C as the change of basis matrix that converts vectors between the standard basis and the alternate basis,

$$C[\vec{x}]_B = \vec{x}$$

then we can define a relationship between the matrices A, M, and C. Specifically, we know that $M = C^{-1}AC$.

Let's walk through an example, so that we can see how to use this $M = C^{-1}AC$ relationship.

Example

Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (2,1)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \end{bmatrix}\right)$$

In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}=(1,1)$ and $\vec{w}=(-3,-2)$, so

$$C=\begin{bmatrix}1 & -3\\ 1 & -2\end{bmatrix}$$

Now we'll find C^{-1}.

$$[C\ |\ I]=\begin{bmatrix}1 & -3 & | & 1 & 0\\ 1 & -2 & | & 0 & 1\end{bmatrix}$$

$$[C\ |\ I]=\begin{bmatrix}1 & -3 & | & 1 & 0\\ 0 & 1 & | & -1 & 1\end{bmatrix}$$

$$[C\ |\ I]=\begin{bmatrix}1 & 0 & | & -2 & 3\\ 0 & 1 & | & -1 & 1\end{bmatrix}$$

Once the left side of the augmented matrix is I, the right side is the inverse C^{-1}, so

$$C^{-1}=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}$$

With A, C, and C^{-1}, we can find $M=C^{-1}AC$.

$$M=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ -3 & 0\end{bmatrix}\begin{bmatrix}1 & -3\\ 1 & -2\end{bmatrix}$$

$$M=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}2(1)-1(1) & 2(-3)-1(-2)\\ -3(1)+0(1) & -3(-3)+0(-2)\end{bmatrix}$$

$$M=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}2-1 & -6+2\\ -3+0 & 9+0\end{bmatrix}$$

$$M=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -4\\ -3 & 9\end{bmatrix}$$

$$M=\begin{bmatrix}-2(1)+3(-3) & -2(-4)+3(9)\\ -1(1)+1(-3) & -1(-4)+1(9)\end{bmatrix}$$

$$M=\begin{bmatrix}-2-9 & 8+27\\ -1-3 & 4+9\end{bmatrix}$$

$$M=\begin{bmatrix}-11 & 35\\ -4 & 13\end{bmatrix}$$

The result here is the matrix M from $[T(\vec{x})]_B=M[\vec{x}]_B$ that will transform vectors $[\vec{x}]_B$ in the alternate basis in the domain into vectors $[T(\vec{x})]_B$ in the alternate basis in the codomain.

Now that we have M, given any vector $[\vec{x}]_B$ defined in the alternate basis in the domain, we can simply multiply M by the vector to get another vector, also defined in the alternate basis, in the codomain.

We've been asked to transform $[\vec{x}]_B=(2,1)$, so we'll multiply M by this vector.

$$[T(\vec{x})]_B=M[\vec{x}]_B$$

$$[T(\vec{x})]_B=\begin{bmatrix}-11 & 35\\ -4 & 13\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}-11(2)+35(1)\\ -4(2)+13(1)\end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -22+35 \\ -8+13 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 13 \\ 5 \end{bmatrix}$$

In other words, $[\vec{x}]_B = (2,1)$ is defined in the alternate basis in the domain, and the transformation T maps that vector to $[T(\vec{x})]_B = (13,5)$ in the alternate basis in the codomain.

What if we want to transform a vector in the standard basis in the domain into a vector in the alternate basis in the codomain. We can either,

1. transform the vector in the standard basis in the domain into a vector in the standard basis in the codomain, and then transform the result from the standard basis in the codomain to the alternate basis in the codomain, $\vec{x} \to T(\vec{x}) \to [T(\vec{x})]_B$, where $T(\vec{x}) = A\vec{x}$ and $[T(\vec{x})]_B = C^{-1}T(\vec{x})$, or you can

2. transform the vector in the standard basis in the domain into a vector in the alternate basis in the domain, and then transform the result from the alternate basis in the domain to the alternate basis in the codomain, $\vec{x} \to [\vec{x}]_B \to [T(\vec{x})]_B$, where $[\vec{x}]_B = C^{-1}\vec{x}$ and $[T(\vec{x})]_B = M[\vec{x}]_B$.

We can visualize the pathways between the domain and codomain, and the standard and alternate bases, as

$$
\begin{tikzcd}
 & \text{Domain} & \text{Codomain} \\
\text{Standard Basis} & \vec{x} \arrow[r, "A"] \arrow[d, "C^{-1}"'] & T(\vec{x}) \arrow[d, "C^{-1}"] \\
\text{Alternate Basis} & [\vec{x}]_B \arrow[r, "M"'] & [T(\vec{x})]_B
\end{tikzcd}
$$