The column space and Ax=b

We've already talked about the null space of the matrix A, and $A\vec{x}=\vec{O}$, where $A\vec{x}$ is a matrix-vector product. Now we want to talk about another matrix-vector product, $A\vec{x}=\vec{b}$, and its relationship to the column space of the matrix A.

It turns out that $A\vec{x}=\vec{b}$ has a solution when $\vec{b}$ is a member of the column space of A.

The column space

The **column space** of a matrix A is defined as all possible linear combinations (the span) of the column vectors in A. So if matrix A is an $m \times n$ matrix with column vectors v_1, v_2, ... v_n,

$$A = [v_1 \quad v_2 \quad v_3 \quad \dots \quad v_n]$$

then the column space of A is given as

$$C(A) = \text{Span}(v_1, v_2, v_3, \dots v_n)$$

So in $A\vec{x}=\vec{b}$, if $\vec{b}$ is a linear combination of the columns of A (if $\vec{b}$ is in the column space), then we'll be able to find a solution, which means that we'll be able to find some vector $\vec{x}$ that satisfies the equation $A\vec{x}=\vec{b}$.

Notice also that we've just defined the column space as a span, and remember that any span is a valid subspace. So, just like the null space, the column space of a matrix is always a valid subspace.

Linear independence and the null space

Remember before that when we talked about the null space, we said that,

- if the null space $N(A)$ includes only the zero vector, then the columns of A are linearly independent, but
- if the null space $N(A)$ includes any other vector in addition to the zero vector, then the columns of A are linearly dependent.

This is important to the column space because we're essentially saying that we can use the vectors in the null space to make a claim about the linear (in)dependence of the column vectors in the matrix.

If we're able to say that the column vectors of A are linearly independent, then we can also say that the column vectors of A form a basis for the column space. Otherwise, if what we know about the null space tells us that the column vectors of A are linearly dependent, then we know that the column vectors of A cannot form a basis.

In other words, if the column vectors $v_1, v_2, v_3, \dots v_n$ in A is a linearly independent set of vectors, then $(v_1, v_2, v_3, \dots v_n)$ would form a basis for the column space of A.

And so to say whether the column vectors of A are linearly independent, we can simply start by finding the null space of A.

Dimensions of the null and column space

For any $m \times n$ matrix A, the null space of A is a subspace in $\mathbb{R}^n$, and the column space of A is a subspace in $\mathbb{R}^m$. More specifically,

- $C(A)$ will be an r-dimensional subspace in $\mathbb{R}^m$, where r is the number of pivot columns in A and m is the total number of rows in A.

- $N(A)$ will be an $(n-r)$-dimensional subspace in $\mathbb{R}^n$, where $(n-r)$ is the number of non-pivot columns (free columns) in A and n is the total number of columns in A.

Let's do an example of how to find the column space of a matrix by first finding the null space.

Example

Find the null space and its dimension, then find the column space of A.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

Let's find the null space of A by first putting the matrix into reduced row-echelon form. Find the pivot entry in the first column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & -4 & 2 & 2 \\ 5 & 6 & -2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & -4 & 2 & 2 \\ 0 & \frac{7}{2} & -\frac{19}{2} & -\frac{11}{2} \end{bmatrix}$$

Find the pivot entry in the second column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{7}{2} & -\frac{19}{2} & -\frac{11}{2} \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{3}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{7}{2} & -\frac{19}{2} & -\frac{11}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{3}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{31}{4} & -\frac{15}{4} \end{bmatrix}$$

Find the pivot entry in the third column.

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{3}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{15}{31} \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{31} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{15}{31} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{31} \\ 0 & 1 & 0 & -\frac{8}{31} \\ 0 & 0 & 1 & \frac{15}{31} \end{bmatrix}$$

Then to find the null space of A, set up the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{31} \\ 0 & 1 & 0 & -\frac{8}{31} \\ 0 & 0 & 1 & \frac{15}{31} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can see from the matrix that x_1, x_2, and x_3 are pivot columns, since those are the columns containing pivot entries, and that x_4 is a free variable. This gives the system of equations

$$x_1 - \frac{3}{31}x_4 = 0$$

$$x_2 - \frac{8}{31}x_4 = 0$$

$$x_3 + \frac{15}{31}x_4 = 0$$

which can be solved for the pivot variables as

$$x_1 = \frac{3}{31}x_4$$

$$x_2 = \frac{8}{31}x_4$$

$$x_3 = -\frac{15}{31}x_4$$

Then the null space of A is all the linear combinations given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{3}{31} \\ \frac{8}{31} \\ -\frac{15}{31} \\ 1 \end{bmatrix}$$

which means the null space is the span of the single column vector.

$$N(A) = N(\text{rref}(A)) = \text{Span}\left(\begin{bmatrix} \frac{3}{31} \\ \frac{8}{31} \\ -\frac{15}{31} \\ 1 \end{bmatrix}\right)$$

Since the null space is the span of the single column vector, the dimensions of the null space is $\text{Dim}(N(A)) = 1$.

In the case of A, x_4 is "free," which means we can set it equal to any value. So it just becomes a scalar, and we can pick infinitely many values of x_4 to get infinitely many vectors that satisfy

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{3}{31} \\ \frac{8}{31} \\ -\frac{15}{31} \\ 1 \end{bmatrix}$$

So the null space of A doesn't just contain the zero vector, which means there's more than one solution to $A\vec{x} = \vec{O}$, which means that the column vectors of A are a linearly dependent set.

Because the vector set

$$\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

is not linearly independent, they don't form a basis for the column space of A. Of course, they span the column space of A, but they can't form a basis for A's column space because they're linearly dependent.

So while the column vectors of A aren't linearly independent, the column space is still just all the linear combinations of the column vectors, which is

$$C(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}\right)$$

Finding the basis for the column space

So if the given vector set isn't a basis for the column space of A, what is? We just need to find the vectors in the set that are redundant, and then remove those, and we'll have the basis. We can do that by setting up an equation with the column vectors from A. Using the same matrix from the last example, we'd get

$$x_1 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = 0$$

We said that x_4 was "free," which means we can set it equal to any value that we choose. We'll set $x_4 = 0$, which will cancel that vector from the equation.

$$x_1 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = 0$$

$$x_1 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix} = 0$$

We want to determine if there's any solution to this equation other than $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$.

$$2x_1 + x_2 + 3x_3 = 0$$

$$4x_1 - 2x_2 + 8x_3 = 0$$

$$5x_1 + 6x_2 - 2x_3 = 0$$

Let's try to solve the system with an augmented matrix and Gaussian elimination.

$$\begin{bmatrix} 2 & 1 & 3 & | & 0 \\ 4 & -2 & 8 & | & 0 \\ 5 & 6 & -2 & | & 0 \end{bmatrix}$$

Find the pivot entry in the first column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & | & 0 \\ 4 & -2 & 8 & | & 0 \\ 5 & 6 & -2 & | & 0 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & | & 0 \\ 0 & -4 & 2 & | & 0 \\ 5 & 6 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & | & 0 \\ 0 & -4 & 2 & | & 0 \\ 0 & \frac{7}{2} & -\frac{19}{2} & | & 0 \end{bmatrix}$$

Find the pivot entry in the second column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & \frac{7}{2} & -\frac{19}{2} & | & 0 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & \frac{7}{2} & -\frac{19}{2} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{7}{4} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & -\frac{31}{4} & | & 0 \end{bmatrix}$$

Find the pivot entry in the third column.

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

This reduced row-echelon matrix tells us that the only solution to the system is $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. Which means that the first three column vectors of A are linearly independent of one another. None of them are redundant, because we can't make any of the vectors using a linear combination of the others. Therefore, instead of the vector set

$$\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

we can eliminate the last column vector (the one associated with x_4 that's redundant), and then we'll have three linearly independent vectors remaining.

$$\begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-2\\6\end{bmatrix}, \begin{bmatrix}3\\8\\-2\end{bmatrix}$$

Because this set is linearly independent, they can form the basis of the column space of A.

$$\text{Basis of } C(A) \text{ is } \left(\begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-2\\6\end{bmatrix}, \begin{bmatrix}3\\8\\-2\end{bmatrix}\right)$$

Which means that, instead of writing the column space as

$$C(A) = \text{Span}\left(\begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-2\\6\end{bmatrix}, \begin{bmatrix}3\\8\\-2\end{bmatrix}, \begin{bmatrix}1\\4\\-3\end{bmatrix}\right)$$

we could write it as

$$C(A) = \text{Span}\left(\begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-2\\6\end{bmatrix}, \begin{bmatrix}3\\8\\-2\end{bmatrix}\right)$$

We've only eliminated a dependent vector, which means both vectors sets will span the same space. But the second set forms a basis for the column space, while the first set doesn't, since the first set isn't linearly independent. With three vectors forming the basis for the column space of A, the dimension of the column space is $\text{Dim}(C(A)) = 3$.