Linear Algebra Final Exam Solutions

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Linear Algebra Final Exam Answer Key

1. (5 pts)	■	B	C	D	E
2. (5 pts)	A	B	C	D	■
3. (5 pts)	■	B	C	D	E
4. (5 pts)	A	■	C	D	E
5. (5 pts)	A	B	C	■	E
6. (5 pts)	A	B	■	D	E
7. (5 pts)	A	B	C	D	■
8. (5 pts)	A	■	C	D	E

9. (15 pts) $C(M) = \text{Span}\left(\begin{bmatrix} 3 \\ -3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ -1 \end{bmatrix}\right)$ $\mathbb{R}^4$ Dim $= 2$

$N(M) = \text{Span}\left(\begin{bmatrix} -2 \\ -6 \\ 1 \end{bmatrix}\right)$ $\mathbb{R}^3$ Dim $= 1$

$C(M^T) = \text{Span}\left(\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}\right)$ $\mathbb{R}^3$ Dim $= 2$

$N(M^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right)$ $\mathbb{R}^4$ Dim $= 2$

10. (15 pts) $\vec{x}^* = \left(-\frac{14}{3}, -\frac{1}{3}\right)$

11. (15 pts) $V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}\right)$

12. (15 pts) $E_{-1} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$ and $E_2 = \text{Span}\left(\begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix}\right)$

- since $\lambda = 2$ in the eigenspace E_2, any vector $\vec{v}$ in E_2, under the transformation T, will be scaled by 2, meaning that $T(\vec{v}) = \lambda\vec{v} = 2\vec{v}$, and

- since $\lambda = -1$ in the eigenspace E_{-1}, any vector $\vec{v}$ in E_{-1}, under the transformation T, will be scaled by -1, meaning that $T(\vec{v}) = \lambda\vec{v} = -\vec{v}$.

Linear Algebra Final Exam Solutions

1. A. Rewrite the system as an augmented matrix.

$$\begin{bmatrix} -1 & 1 & 2 & 7 & | & 4 \\ 3 & -1 & -4 & 5 & | & 8 \\ 2 & 4 & 3 & -1 & | & 4 \\ 0 & -1 & -1 & -13 & | & 5 \end{bmatrix}$$

To put A into reduced row-echelon form, start by working on the first column.

$$\begin{bmatrix} 1 & -1 & -2 & -7 & | & -4 \\ 3 & -1 & -4 & 5 & | & 8 \\ 2 & 4 & 3 & -1 & | & 4 \\ 0 & -1 & -1 & -13 & | & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 & -7 & | & -4 \\ 0 & 2 & 2 & 26 & | & 20 \\ 2 & 4 & 3 & -1 & | & 4 \\ 0 & -1 & -1 & -13 & | & 5 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -1 & -2 & -7 & | & -4 \\ 0 & 2 & 2 & 26 & | & 20 \\ 0 & 6 & 7 & 13 & | & 12 \\ 0 & -1 & -1 & -13 & | & 5 \end{bmatrix}$$

Find the pivot entry in the second column, then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -1 & -2 & -7 & | & -4 \\ 0 & 1 & 1 & 13 & | & 10 \\ 0 & 6 & 7 & 13 & | & 12 \\ 0 & -1 & -1 & -13 & | & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 6 & | & 6 \\ 0 & 1 & 1 & 13 & | & 10 \\ 0 & 6 & 7 & 13 & | & 12 \\ 0 & -1 & -1 & -13 & | & 5 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 & 6 & | & 6 \\ 0 & 1 & 1 & 13 & | & 10 \\ 0 & 0 & 1 & -65 & | & -48 \\ 0 & -1 & -1 & -13 & | & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 6 & | & 6 \\ 0 & 1 & 1 & 13 & | & 10 \\ 0 & 0 & 1 & -65 & | & -48 \\ 0 & 0 & 0 & 0 & | & 15 \end{bmatrix}$$

Find the pivot entry in the third column, then zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & -59 & | & -42 \\ 0 & 1 & 1 & 13 & | & 10 \\ 0 & 0 & 1 & -65 & | & -48 \\ 0 & 0 & 0 & 0 & | & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -59 & | & -42 \\ 0 & 1 & 0 & 78 & | & 58 \\ 0 & 0 & 1 & -65 & | & -48 \\ 0 & 0 & 0 & 0 & | & 15 \end{bmatrix}$$

The fourth row shows us that $0 = 15$, which can't be true. Therefore, the system has no solutions.

2. E. First multiply the matrix A by the matrix C, multiplying each row of A by each column of C.

$$AC = \begin{bmatrix} 7 & 3 & -5 \\ -5 & -8 & -4 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ -7 & -2 \\ 2 & 3 \end{bmatrix}$$

$$AC = \begin{bmatrix} 7(6) + 3(-7) - 5(2) & 7(1) + 3(-2) - 5(3) \\ -5(6) - 8(-7) - 4(2) & -5(1) - 8(-2) - 4(3) \end{bmatrix}$$

$$AC = \begin{bmatrix} 42 - 21 - 10 & 7 - 6 - 15 \\ -30 + 56 - 8 & -5 + 16 - 12 \end{bmatrix}$$

$$AC = \begin{bmatrix} 11 & -14 \\ 18 & -1 \end{bmatrix}$$

Now multiply the matrix B by the matrix AC, multiplying each row of B by each column of AC.

$$B(AC) = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 11 & -14 \\ 18 & -1 \end{bmatrix}$$

$$B(AC) = \begin{bmatrix} -2(11) + 3(18) & -2(-14) + 3(-1) \\ 4(11) - 1(18) & 4(-14) - 1(-1) \end{bmatrix}$$

$$B(AC) = \begin{bmatrix} -22 + 54 & 28 - 3 \\ 44 - 18 & -56 + 1 \end{bmatrix}$$

$$B(AC) = \begin{bmatrix} 32 & 25 \\ 26 & -55 \end{bmatrix}$$

3. A. First, we need to apply the scalars to the vectors to find $-2\overrightarrow{u}$ and $4\overrightarrow{v}$.

$$-2\overrightarrow{u} = -2(-3,2,1,0) = (6, -4, -2,0)$$

$$4\overrightarrow{v} = 4(1, -5, -4,1) = (4, -20, -16,4)$$

Then the sum $-2\overrightarrow{u} + 4\overrightarrow{v}$ is

$$-2\overrightarrow{u} + 4\overrightarrow{v} = \begin{bmatrix} 6 \\ -4 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ -20 \\ -16 \\ 4 \end{bmatrix}$$

$$-2\overrightarrow{u}+4\overrightarrow{v}=\begin{bmatrix}6+4\\-4-20\\-2-16\\0+4\end{bmatrix}$$

$$-2\overrightarrow{u}+4\overrightarrow{v}=\begin{bmatrix}10\\-24\\-18\\4\end{bmatrix}$$

Calculate the dot product of $\overrightarrow{w}$ and $-2\overrightarrow{u}+4\overrightarrow{v}$.

$$\overrightarrow{w}\cdot(-2\overrightarrow{u}+4\overrightarrow{v})=[0\quad -1\quad 1\quad 2]\begin{bmatrix}10\\-24\\-18\\4\end{bmatrix}$$

$$\overrightarrow{w}\cdot(-2\overrightarrow{u}+4\overrightarrow{v})=0(10)-1(-24)+1(-18)+2(4)$$

$$\overrightarrow{w}\cdot(-2\overrightarrow{u}+4\overrightarrow{v})=0+24-18+8$$

$$\overrightarrow{w}\cdot(-2\overrightarrow{u}+4\overrightarrow{v})=14$$

4. B. First find the normal vector to the plane.

$$\overrightarrow{AB}=(0-2,-3-(-1),2-4)$$

$$\overrightarrow{AB}=(-2,-2,-2)$$

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-2(x - 2) - 2(y - (-1)) - 2(z - 4) = 0$$

Now we'll simplify to get the equation of the plane into standard form.

$$-2(x - 2) - 2(y + 1) - 2(z - 4) = 0$$

$$-2x + 4 - 2y - 2 - 2z + 8 = 0$$

$$-2x - 2y - 2z + 10 = 0$$

$$x + y + z = 5$$

5. D. Put the matrix A, augmented with the zero vector, into reduced row-echelon form.

$$\begin{bmatrix} 2 & -4 & 6 & | & 0 \\ 3 & -5 & -2 & | & 0 \\ -5 & 7 & 18 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & | & 0 \\ 3 & -5 & -2 & | & 0 \\ -5 & 7 & 18 & | & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -11 & | & 0 \\ -5 & 7 & 18 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -11 & | & 0 \\ 0 & -3 & 33 & | & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -19 & | & 0 \\ 0 & 1 & -11 & | & 0 \\ 0 & -3 & 33 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -19 & | & 0 \\ 0 & 1 & -11 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

To find the complementary solution, pull out a system of equations from rref(A).

$$x_1 - 19x_3 = 0$$

$$x_2 - 11x_3 = 0$$

Solve for the pivot variables in terms of the free variable.

$$x_1 = 19x_3$$

$$x_2 = 11x_3$$

The vector that satisfies the null space is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 19 \\ 11 \\ 1 \end{bmatrix}$$

We could therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} 19 \\ 11 \\ 1 \end{bmatrix}$$

To find the particular solution, augment A with $\vec{b} = (b_1, b_2, b_3)$, then put it in reduced row-echelon form.

$$\begin{bmatrix} 2 & -4 & 6 & | & b_1 \\ 3 & -5 & -2 & | & b_2 \\ -5 & 7 & 18 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & | & \frac{1}{2}b_1 \\ 3 & -5 & -2 & | & b_2 \\ -5 & 7 & 18 & | & b_3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & 3 & | & \frac{1}{2}b_1 \\ 0 & 1 & -11 & | & -\frac{3}{2}b_1+b_2 \\ -5 & 7 & 18 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & | & \frac{1}{2}b_1 \\ 0 & 1 & -11 & | & -\frac{3}{2}b_1+b_2 \\ 0 & -3 & 33 & | & \frac{5}{2}b_1+b_3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -19 & | & -\frac{5}{2}b_1+2b_2 \\ 0 & 1 & -11 & | & -\frac{3}{2}b_1+b_2 \\ 0 & -3 & 33 & | & \frac{5}{2}b_1+b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -19 & | & -\frac{5}{2}b_1+2b_2 \\ 0 & 1 & -11 & | & -\frac{3}{2}b_1+b_2 \\ 0 & 0 & 0 & | & -2b_1+3b_2+b_3 \end{bmatrix}$$

Substitute the values from $\overrightarrow{b} = (1,1,-1)$.

$$\begin{bmatrix} 1 & 0 & -19 & | & -\frac{5}{2}(1)+2(1) \\ 0 & 1 & -11 & | & -\frac{3}{2}(1)+1 \\ 0 & 0 & 0 & | & -2(1)+3(1)-1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -19 & | & -\frac{1}{2} \\ 0 & 1 & -11 & | & -\frac{1}{2} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Rewrite the matrix as a system of equations.

$$x_1 - 19x_3 = -\frac{1}{2}$$

$$x_2 - 11x_3 = -\frac{1}{2}$$

Now, because x_3 is a free variable, set $x_3 = 0$.

$$x_1 - 19(0) = -\frac{1}{2}$$

$$x_2 - 11(0) = -\frac{1}{2}$$

The system becomes

$$x_1 = -\frac{1}{2}$$

$$x_2 = -\frac{1}{2}$$

So the particular solution is

$$\vec{x_p} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

The general solution is the sum of the complementary and particular solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 19 \\ 11 \\ 1 \end{bmatrix}$$

6. C. Given $\vec{x} = (-2,5)$ and

$$S(\vec{x}) = \begin{bmatrix} x_1 + x_2 \\ 2x_2 - x_1 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} x_1 + 3x_2 \\ 2x_2 \end{bmatrix}$$

start by using S to transform the standard basis vectors.

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 + 0 \\ 2(0) - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 + 1 \\ 2(1) - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So the transformation S can be written as the matrix-vector product

$$S(\vec{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \vec{x}$$

Use T to transform the standard basis vectors.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 + 3(0) \\ 2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 + 3(1) \\ 2(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

So the transformation T can be written as the matrix-vector product

$$T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \vec{x}$$

If we call the matrix from S

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

and we call the matrix from T

$$B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

then the composition of the transformations is

$$S(T(\vec{x})) = AB\vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} 1(1)+1(0) & 1(3)+1(2) \\ -1(1)+2(0) & -1(3)+2(2) \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \vec{x}$$

To transform $\vec{x} = (-2,5)$, multiply this transformation matrix by $\vec{x} = (-2,5)$.

$$S\left(T\left(\begin{bmatrix} -2 \\ 5 \end{bmatrix}\right)\right) = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} -2 \\ 5 \end{bmatrix}\right)\right) = \begin{bmatrix} 1(-2)+5(5) \\ -1(-2)+1(5) \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} -2 \\ 5 \end{bmatrix}\right)\right) = \begin{bmatrix} 23 \\ 7 \end{bmatrix}$$

7. E. The subspace V is a plane in $\mathbb{R}^4$, spanned by the two vectors $\vec{v}_1=(-1,0,-2,3)$ and $\vec{v}_2=(-1,-2,0,-5)$. Its orthogonal complement $V^\perp$ is the set of vectors which are orthogonal to both $\vec{v}_1=(-1,0,-2,3)$ and $\vec{v}_2=(-1,-2,0,-5)$.

$$V^\perp=\{\vec{x}\in\mathbb{R}^4 \mid \vec{x}\cdot\begin{bmatrix}-1\\0\\-2\\3\end{bmatrix}=0\ ,\ \vec{x}\cdot\begin{bmatrix}-1\\-2\\0\\-5\end{bmatrix}=0\}$$

Let $\vec{x}=(x_1,x_2,x_3,x_4)$ to get two equations from these dot products.

$$-x_1-2x_3+3x_4=0$$

$$-x_1-2x_2-5x_4=0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix}-1 & 0 & -2 & 3 & | & 0\\ -1 & -2 & 0 & -5 & | & 0\end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix}1 & 0 & 2 & -3 & | & 0\\ -1 & -2 & 0 & -5 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 2 & -3 & | & 0\\ 0 & -2 & 2 & -8 & | & 0\end{bmatrix}\to$$

$$\begin{bmatrix}1 & 0 & 2 & -3 & | & 0\\ 0 & 1 & -1 & 4 & | & 0\end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + 2x_3 - 3x_4 = 0$$

$$x_2 - x_3 + 4x_4 = 0$$

Solve the system for the pivot variables, x_1 and x_2.

$$x_1 = -2x_3 + 3x_4$$

$$x_2 = x_3 - 4x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

The orthogonal complement is

$$V^{\perp} = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right)$$

8. B. In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (-2,3)$ and $\vec{v}_2 = (-4,0)$, so

$$C = \begin{bmatrix} -2 & -4 \\ 3 & 0 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} -2 & -4 & | & 1 & 0 \\ 3 & 0 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 2 & | & -\frac{1}{2} & 0 \\ 3 & 0 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 2 & | & -\frac{1}{2} & 0 \\ 0 & -6 & | & \frac{3}{2} & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 2 & | & -\frac{1}{2} & 0 \\ 0 & 1 & | & -\frac{1}{4} & -\frac{1}{6} \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & | & 0 & \frac{1}{3} \\ 0 & 1 & | & -\frac{1}{4} & -\frac{1}{6} \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{4} & -\frac{1}{6} \end{bmatrix}$$

With A, C, and C^{-1}, we can find $M = C^{-1}AC$.

$$M = \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{4} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} -3 & 10 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ 3 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{4} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} -3(-2) + 10(3) & -3(-4) + 10(0) \\ 0(-2) + 4(3) & 0(-4) + 4(0) \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{4} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 6 + 30 & 12 + 0 \\ 0 + 12 & 0 + 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{4} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 36 & 12 \\ 12 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0(36) + \frac{1}{3}(12) & 0(12) + \frac{1}{3}(0) \\ -\frac{1}{4}(36) - \frac{1}{6}(12) & -\frac{1}{4}(12) - \frac{1}{6}(0) \end{bmatrix}$$

$$M = \begin{bmatrix} 0 + 4 & 0 + 0 \\ -9 - 2 & -3 - 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 4 & 0 \\ -11 & -3 \end{bmatrix}$$

We've been asked to transform $[\vec{x}]_B=(-1,4)$, so we'll multiply M by this vector.

$$[T(\vec{x})]_B=M[\vec{x}]_B$$

$$[T(\vec{x})]_B=\begin{bmatrix}4 & 0\\ -11 & -3\end{bmatrix}\begin{bmatrix}-1\\ 4\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}4(-1)+0(4)\\ -11(-1)-3(4)\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}-4+0\\ 11-12\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}-4\\ -1\end{bmatrix}$$

9. First put M into reduced row-echelon form.

$$\begin{bmatrix}3 & 0 & 6\\ -3 & 1 & 0\\ 6 & -2 & 0\\ 0 & -1 & -6\end{bmatrix}\to\begin{bmatrix}1 & 0 & 2\\ -3 & 1 & 0\\ 6 & -2 & 0\\ 0 & -1 & -6\end{bmatrix}\to\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 6\\ 6 & -2 & 0\\ 0 & -1 & -6\end{bmatrix}\to$$

$$\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 6\\ 0 & -2 & -12\\ 0 & -1 & -6\end{bmatrix}\to\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 6\\ 0 & 0 & 0\\ 0 & -1 & -6\end{bmatrix}\to\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 6\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

In rref(M), the pivot columns are the first and second columns, which means $C(M)$ is given by the span of the first and second columns of M.

$$C(M) = \text{Span}\left(\begin{bmatrix}3\\-3\\6\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-2\\-1\end{bmatrix}\right)$$

If we augment rref(M) with the zero vector, we get the system of equations

$$x_1 + 2x_3 = 0$$

$$x_2 + 6x_3 = 0$$

Solve the system for the pivot variables in terms of the free variable.

$$x_1 = -2x_3$$

$$x_2 = -6x_3$$

Then the solution to the system is

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\begin{bmatrix}-2\\-6\\1\end{bmatrix}$$

which means the null space is given as

$$N(M) = \text{Span}\left(\begin{bmatrix}-2\\-6\\1\end{bmatrix}\right)$$

Find M^T,

$$M^T = \begin{bmatrix} 3 & -3 & 6 & 0 \\ 0 & 1 & -2 & -1 \\ 6 & 0 & 0 & -6 \end{bmatrix}$$

then put M^T into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -2 & -1 \\ 6 & 0 & 0 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 6 & -12 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 6 & -12 & -6 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In rref(M^T), the pivot columns are the first and second columns, which means $C(M^T)$ is given by the span of the first and second columns of M^T.

$$C(M^T) = \text{Span}\left(\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right)$$

If we augment rref(M^T) with the zero vector, we get the system of equations

$$x_1 - x_4 = 0$$

$$x_2 - 2x_3 - x_4 = 0$$

Solve the system for the pivot variables in terms of the free variables.

$$x_1 = x_4$$

$$x_2 = 2x_3 + x_4$$

Then the solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

which means the left null space is given as

$$N(M^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Because M is an $m \times n = 4 \times 3$ matrix, the row space and null space are defined in $\mathbb{R}^n = \mathbb{R}^3$, and the column space and left null space are defined in $\mathbb{R}^m = \mathbb{R}^4$.

The dimension of the column space and row space is the rank of M, $r = 2$. The dimension of the null space is $n - r = 3 - 2 = 1$, and the dimension of the left null space is $m - r = 4 - 2 = 2$.

In summary, the four fundamental subspaces are

$$C(M) = \text{Span}\left(\begin{bmatrix} 3 \\ -3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ -1 \end{bmatrix} \right) \qquad \mathbb{R}^4 \qquad \text{Dim} = 2$$

$$N(M) = \text{Span}\left(\begin{bmatrix}-2\\-6\\1\end{bmatrix}\right)$$ $\mathbb{R}^3$ Dim = 1

$$C(M^T) = \text{Span}\left(\begin{bmatrix}3\\0\\6\end{bmatrix}, \begin{bmatrix}-3\\1\\0\end{bmatrix}\right)$$ $\mathbb{R}^3$ Dim = 2

$$N(M^T) = \text{Span}\left(\begin{bmatrix}0\\2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\right)$$ $\mathbb{R}^4$ Dim = 2

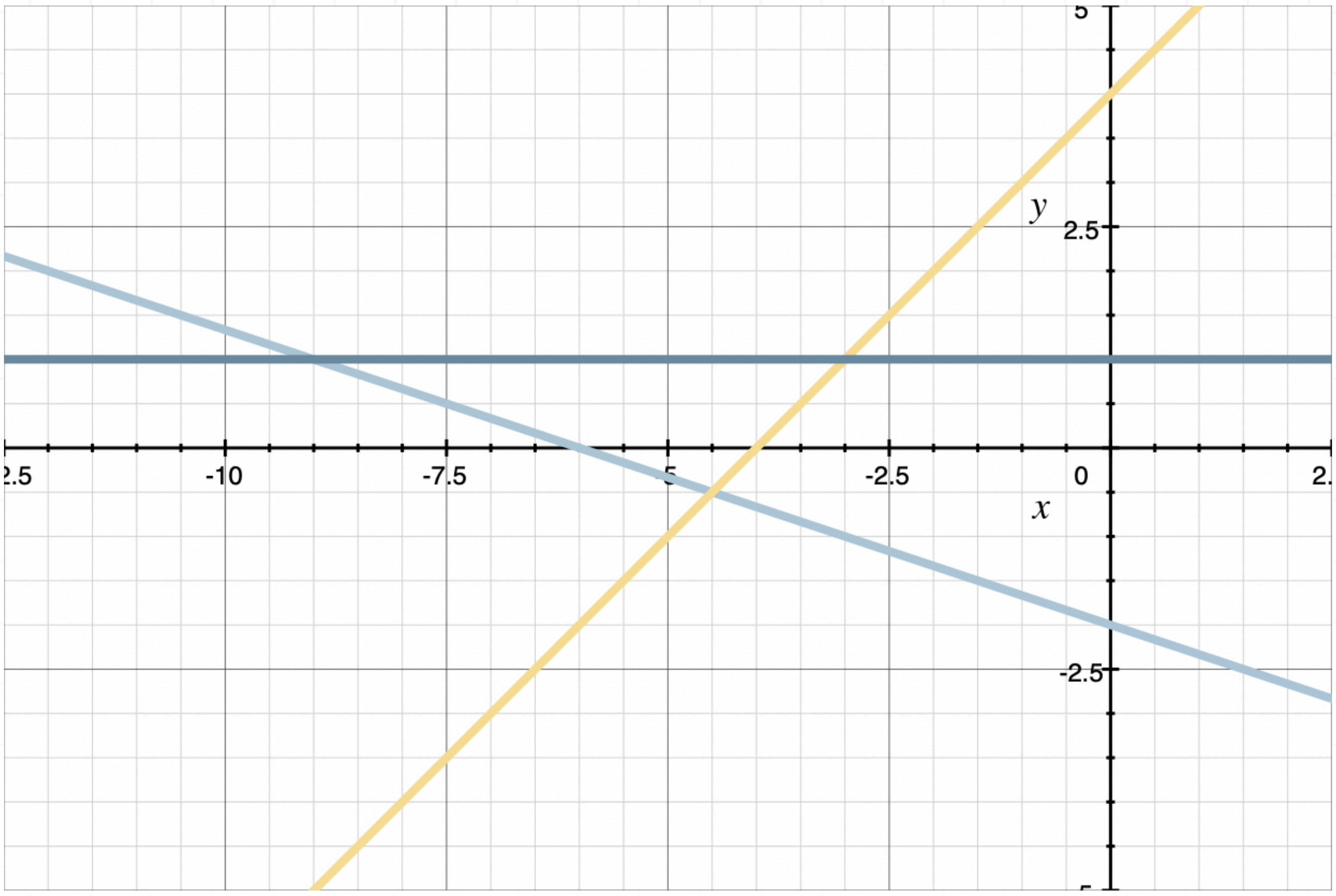
10. Put each line into slope-intercept form,

$$y = -\frac{1}{3}x - 2$$

$$y = x + 4$$

$$y = 1$$

then graph all three in the same plane.



While there are three points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x}=\vec{b}$, which means there's no vector $\vec{x}=(x,y)$ that satisfies the equation $A\vec{x}=\vec{b}$,

$$\begin{bmatrix}1 & 3\\-1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-6\\4\\1\end{bmatrix}$$

In other words, $\vec{b}=(-6,4,1)$ is not in the column space of A. The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A=\begin{bmatrix}1 & 3\\-1 & 1\\0 & 1\end{bmatrix}$$

Now we'll find A^T.

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

Then A^TA is

$$A^TA = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) - 1(-1) + 0(0) & 1(3) - 1(1) + 0(1) \\ 3(1) + 1(-1) + 1(0) & 3(3) + 1(1) + 1(1) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 1 + 0 & 3 - 1 + 0 \\ 3 - 1 + 0 & 9 + 1 + 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 2 \\ 2 & 11 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b} = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 4 \\ 1 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1(-6) - 1(4) + 0(1) \\ 3(-6) + 1(4) + 1(1) \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -6 - 4 + 0 \\ -18 + 4 + 1 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -10 \\ -13 \end{bmatrix}$$

Then we get

$$A^T A \vec{x}^* = A^T \vec{b}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 11 \end{bmatrix} \vec{x}^* = \begin{bmatrix} -10 \\ -13 \end{bmatrix}$$

Then to find $\vec{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & 2 & | & -10 \\ 2 & 11 & | & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & -5 \\ 2 & 11 & | & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & -5 \\ 0 & 9 & | & -3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & | & -5 \\ 0 & 1 & | & -\frac{1}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & -\frac{14}{3} \\ 0 & 1 & | & -\frac{1}{3} \end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(-\frac{14}{3}, -\frac{1}{3}\right)$$

11. Define $\vec{v}_1 = (2,0,-2)$, $\vec{v}_2 = (0,-2,4)$, and $\vec{v}_3 = (-2,-2,3)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{2^2 + 0^2 + (-2)^2} = \sqrt{4 + 0 + 4} = \sqrt{8} = 2\sqrt{2}$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix}0\\-2\\4\end{bmatrix} - \left(\begin{bmatrix}0\\-2\\4\end{bmatrix} \cdot \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\0\\-2\end{bmatrix}\right)\frac{1}{2\sqrt{2}}\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}0\\-2\\4\end{bmatrix} - \frac{1}{8}\left(\begin{bmatrix}0\\-2\\4\end{bmatrix} \cdot \begin{bmatrix}2\\0\\-2\end{bmatrix}\right)\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}0\\-2\\4\end{bmatrix} - \frac{1}{8}((0)(2) + (-2)(0) + (4)(-2))\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}0\\-2\\4\end{bmatrix} - \frac{1}{8}(0 + 0 - 8)\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}0\\-2\\4\end{bmatrix} - \frac{1}{8}(-8)\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{2^2 + (-2)^2 + 2^2}$$

$$||\vec{w}_2|| = \sqrt{4 + 4 + 4}$$

$$||\vec{w}_2|| = \sqrt{12}$$

$$||\vec{w}_2|| = 2\sqrt{3}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$.

$$\vec{u}_2 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have.

$$\vec{w}_3=\begin{bmatrix}-2\\-2\\3\end{bmatrix}-\left(\begin{bmatrix}-2\\-2\\3\end{bmatrix}\cdot\frac{1}{2\sqrt{2}}\begin{bmatrix}2\\0\\-2\end{bmatrix}\right)\frac{1}{2\sqrt{2}}\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$-\left(\begin{bmatrix}-2\\-2\\3\end{bmatrix}\cdot\frac{1}{2\sqrt{3}}\begin{bmatrix}2\\-2\\2\end{bmatrix}\right)\frac{1}{2\sqrt{3}}\begin{bmatrix}2\\-2\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-2\\-2\\3\end{bmatrix}-\frac{1}{8}\left(\begin{bmatrix}-2\\-2\\3\end{bmatrix}\cdot\begin{bmatrix}2\\0\\-2\end{bmatrix}\right)\begin{bmatrix}2\\0\\-2\end{bmatrix}-\frac{1}{12}\left(\begin{bmatrix}-2\\-2\\3\end{bmatrix}\cdot\begin{bmatrix}2\\-2\\2\end{bmatrix}\right)\begin{bmatrix}2\\-2\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-2\\-2\\3\end{bmatrix}-\frac{1}{8}(-2(2)-2(0)+3(-2))\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$-\frac{1}{12}(-2(2)-2(-2)+3(2))\begin{bmatrix}2\\-2\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-2\\-2\\3\end{bmatrix}-\frac{1}{8}(-4+0-6)\begin{bmatrix}2\\0\\-2\end{bmatrix}-\frac{1}{12}(-4+4+6)\begin{bmatrix}2\\-2\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-2\\-2\\3\end{bmatrix}+\frac{5}{4}\begin{bmatrix}2\\0\\-2\end{bmatrix}-\frac{1}{2}\begin{bmatrix}2\\-2\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-2\\-2\\3\end{bmatrix}+\begin{bmatrix}\frac{5}{2}\\0\\-\frac{5}{2}\end{bmatrix}-\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} -2 + \frac{5}{2} - 1 \\ -2 + 0 + 1 \\ 3 - \frac{5}{2} - 1 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix}$$

So $\vec{w}_3$ is orthogonal to $\vec{u}_2$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(-\frac{1}{2}\right)^2 + (-1)^2 + \left(-\frac{1}{2}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}}$$

$$||\vec{w}_3|| = \sqrt{\frac{3}{2}}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$.

$$\vec{u}_3 = \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{2\sqrt{2}}\begin{bmatrix}2\\0\\-2\end{bmatrix}, \frac{1}{2\sqrt{3}}\begin{bmatrix}2\\-2\\2\end{bmatrix}, \sqrt{\frac{2}{3}}\begin{bmatrix}-\frac{1}{2}\\-1\\-\frac{1}{2}\end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{6}}\\-\frac{\sqrt{2}}{\sqrt{3}}\\-\frac{1}{\sqrt{6}}\end{bmatrix}\right)$$

12. Starting with

$$A = \begin{bmatrix}-1 & 4 & -2\\0 & 2 & 0\\0 & 0 & -1\end{bmatrix}$$

we'll first find the determinant $|\lambda I_n - A|$.

$$\left|\lambda\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}-1 & 4 & -2\\0 & 2 & 0\\0 & 0 & -1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda & 0 & 0\\0 & \lambda & 0\\0 & 0 & \lambda\end{bmatrix} - \begin{bmatrix}-1 & 4 & -2\\0 & 2 & 0\\0 & 0 & -1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda + 1 & -4 & 2\\0 & \lambda - 2 & 0\\0 & 0 & \lambda + 1\end{bmatrix}\right|$$

Then let's work along the first column, since it includes two 0 values, to find the determinant of this resulting matrix.

$$(\lambda+1)\begin{vmatrix}\lambda-2 & 0\\ 0 & \lambda+1\end{vmatrix}-0\begin{vmatrix}-4 & 2\\ 0 & \lambda+1\end{vmatrix}+0\begin{vmatrix}-4 & 2\\ \lambda-2 & 0\end{vmatrix}$$

The last two determinants cancel, leaving us with just

$$(\lambda+1)\begin{vmatrix}\lambda-2 & 0\\ 0 & \lambda+1\end{vmatrix}$$

$$(\lambda+1)[(\lambda-2)(\lambda+1)-(0)(0)]$$

$$(\lambda+1)^2(\lambda-2)$$

Remember that we're trying to satisfy $|\lambda I_n - A| = 0$, so we can set this characteristic polynomial equal to 0 to get the characteristic equation, and then we'll solve for λ.

$$(\lambda+1)^2(\lambda-2)=0$$

$\lambda = -1$ or $\lambda = 2$

With these three eigenvalues, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. Given

$$E_\lambda = N\left(\begin{bmatrix}\lambda+1 & -4 & 2\\ 0 & \lambda-2 & 0\\ 0 & 0 & \lambda+1\end{bmatrix}\right)$$

we get

$$E_{-1} = N\left(\begin{bmatrix} -1+1 & -4 & 2 \\ 0 & -1-2 & 0 \\ 0 & 0 & -1+1 \end{bmatrix}\right)$$

$$E_{-1} = N\left(\begin{bmatrix} 0 & -4 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

and

$$E_{2} = N\left(\begin{bmatrix} 2+1 & -4 & 2 \\ 0 & 2-2 & 0 \\ 0 & 0 & 2+1 \end{bmatrix}\right)$$

$$E_{2} = N\left(\begin{bmatrix} 3 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-1} will satisfy

$$\begin{bmatrix} 0 & -4 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 & 2 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & -\frac{3}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_2 = 0$$

$$v_3 = 0$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_{-1} is defined by

$$E_{-1} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_2 will satisfy

$$\begin{bmatrix} 3 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 - \frac{4}{3}v_2 = 0$$

$$v_3 = 0$$

or

$$v_1 = \frac{4}{3}v_2$$

$$v_3 = 0$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_2 is defined by

$$E_2 = \text{Span}\left(\begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix} \right)$$

Let's put these results together. For the eigenvalues $\lambda = -1$ and $\lambda = 2$, respectively, we got

$$E_{-1} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \text{ and } E_2 = \text{Span}\left(\begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix} \right)$$

Each of these spans represents a line in $\mathbb{R}^3$. So for any vector $\vec{v}$ along any of these lines, when we apply the transformation T to the vector $\vec{v}$, $T(\vec{v})$ will be a vector along the same line, it might just be scaled up or scaled down. Specifically,

- since $\lambda = 2$ in the eigenspace E_2, any vector $\vec{v}$ in E_2, under the transformation T, will be scaled by 2, meaning that $T(\vec{v}) = \lambda\vec{v} = 2\vec{v}$, and

- since $\lambda = -1$ in the eigenspace E_{-1}, any vector $\vec{v}$ in E_{-1}, under the transformation T, will be scaled by -1, meaning that $T(\vec{v}) = \lambda\vec{v} = -\vec{v}$.

W W W . K R I S T A K I N G M A T H . C O M