The null space and Ax=O

In the last lesson, we talked about matrix-vector products. In this lesson we want to focus on one really important matrix-vector product, $A\vec{x}=\vec{O}$, where A is a matrix, $\vec{x}$ is a vector, and $\vec{O}$ is the zero vector.

The null space

We talked before about the definition of a subspace, and we said that a set of vectors S was only a subspace if the following three things were all true:

1. S contains the zero vector.
2. S is closed under addition.
3. S is closed under scalar multiplication.

In this lesson, we want to define the **null space**, which is the specific subspace in which the entire universe of vectors $\vec{x}$ in $\mathbb{R}^n$ satisfy the homogeneous equation $A\vec{x}=\vec{O}$ (homogeneous because one side of the equation is 0), where A is any matrix with dimensions $m \times n$.

We indicate the null space with the capital N, and we write

$$N=\left\{\vec{x}\in\mathbb{R}^n \ \mid \ A\vec{x}=\vec{O}\right\}$$

This set notation says that the null space N is all the vectors $\vec{x}$ in $\mathbb{R}^n$ that satisfy $A\vec{x}=\vec{O}$. If we use the matrix A, then we call the null space "the null space of A," or $N(A)$; if we use the matrix B, then we call the null space "the null space of B," or $N(B)$, etc.

The null space as a subspace

The null space N satisfies each of the three conditions of a subspace, so the null space is always a subspace.

First, N contains the zero vector. For instance, an example of the matrix A multiplied by the zero vector could look like this:

$$A\vec{x} = \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & \dots & a_{(1,n)} \\ a_{(2,1)} & \cdot & & \\ \vdots & & \cdot & \\ & & & \cdot \\ a_{(m,1)} & & & a_{(m,n)} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} a_{(1,1)}(0) + a_{(1,2)}(0) + \dots + a_{(1,n)}(0) \\ a_{(2,1)}(0) + a_{(2,2)}(0) + \dots + a_{(2,n)}(0) \\ \vdots \\ a_{(m,1)}(0) + a_{(m,2)}(0) + \dots + a_{(m,n)}(0) \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 0 + 0 + \dots + 0 \\ 0 + 0 + \dots + 0 \\ \vdots \\ 0 + 0 + \dots + 0 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A\vec{x} = \vec{O}$$

Because we get the zero vector when we multiply any matrix A by the zero vector $\vec{x}$, the null space N contains the zero vector.

Second, the null space is closed under addition. Remember that in order to be closed under addition, the sum of any two vectors in the subspace must also be in the subspace. So for instance, if we pick any two vectors $\vec{x}_1$ and $\vec{x}_2$ that are in the null space, then the sum $\vec{x}_1 + \vec{x}_2$ must also be in the null space.

We can show that $\vec{x}_1 + \vec{x}_2$ is in the null space if $\vec{x}_1$ and $\vec{x}_2$ are in the null space. Since we chose $\vec{x}_1$ and $\vec{x}_2$ as vectors in the null space, that means they satisfy $A\vec{x} = \vec{O}$, which means both of these equations are true:

$$A\vec{x}_1 = \vec{O}$$

$$A\vec{x}_2 = \vec{O}$$

If the vector $\vec{x}_1 + \vec{x}_2$ is also in the null space, then $A(\vec{x}_1 + \vec{x}_2) = \vec{O}$. We know that matrix multiplication (and by extension matrix-vector multiplication, since vectors can be written as matrices) is distributive, which means we can rewrite the equation.

$$A(\vec{x}_1 + \vec{x}_2) = \vec{O}$$

$$A\vec{x}_1 + A\vec{x}_2 = \vec{O}$$

We already know that $A\vec{x}_1 = \vec{O}$ and $A\vec{x}_2 = \vec{O}$, so this equation becomes

$$\vec{O} + \vec{O} = \vec{O}$$

$$\vec{O} = \vec{O}$$

So we've shown that the null space N contains $\vec{x}_1 + \vec{x}_2$ if it contains $\vec{x}_1$ and $\vec{x}_2$, which means the null space is closed under addition.

Third, the null space is closed under scalar multiplication. If we again take a vector in N, called $\vec{x}_1$, then $c\vec{x}_1$ must also be in N. Which means $c\vec{x}_1$ has to satisfy $A\vec{x} = \vec{O}$, such that

$$Ac\vec{x}_1 = \vec{O}$$

This equation can always be rewritten as

$$cA\vec{x}_1 = \vec{O}$$

And we already established that $A\vec{x}_1 = \vec{O}$, so we get

$$c\vec{O} = \vec{O}$$

$$\vec{O} = \vec{O}$$

And now we've shown that the null space contains $c\vec{x}_1$ if it contains $\vec{x}_1$, which means the null space is closed under scalar multiplication.

Because the null space always contains the zero vector, is always closed under addition, and is always closed under scalar multiplication, the null space N will always be a valid subspace of $\mathbb{R}^n$.

We'll talk more about this in the next lesson, but let's walk through a simple example of how to show that a vector is in the null space.

Example

Show that $\vec{x}=(2,4)$ is in the null space of A.

$$A=\begin{bmatrix}4&-2\\2&-1\end{bmatrix}$$

If $\vec{x}=(2,4)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x}=\vec{O}$$

$$\begin{bmatrix}4&-2\\2&-1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation here, we should get the zero vector.

$$\begin{bmatrix}4(2)-2(4)\\2(2)-1(4)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}8-8\\4-4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Because we get a true equation, we know that $\vec{x}=(2,4)$ is in the null space of A.

Linear independence and the null space

The null space of a matrix A tells us about the linear independence of the column vectors of A.

- If the null space $N(A)$ includes only the zero vector, then the columns of A are linearly independent.

- If the null space $N(A)$ includes any other vector in addition to the zero vector, then the columns of A are linearly dependent.

And the reasoning for that makes sense. If there's a nonzero vector in the null space, it tells us that there's a linear combination of the columns of A that gives the zero vector, which means that the columns of A can only be linearly dependent.

The opposite is also true. If all of the column vectors in the matrix are linearly independent of one another, then the zero vector is the only member of the null space. But if the column vectors of the matrix are linearly dependent, then the null space will include at least one nonzero vector.