Using determinants to find area

Now we want to look at an application of matrix determinants. In this lesson, we'll talk about a geometric property of the determinant, which is that the column vectors of a matrix form a parallelogram whose area is the absolute value of the determinant.

Furthermore, we can use this geometric property to find the area of other figures as well, not just parallelograms.

Forming the parallelogram

Given a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if you separate the matrix into its column vectors, and call them v_1 and v_2,

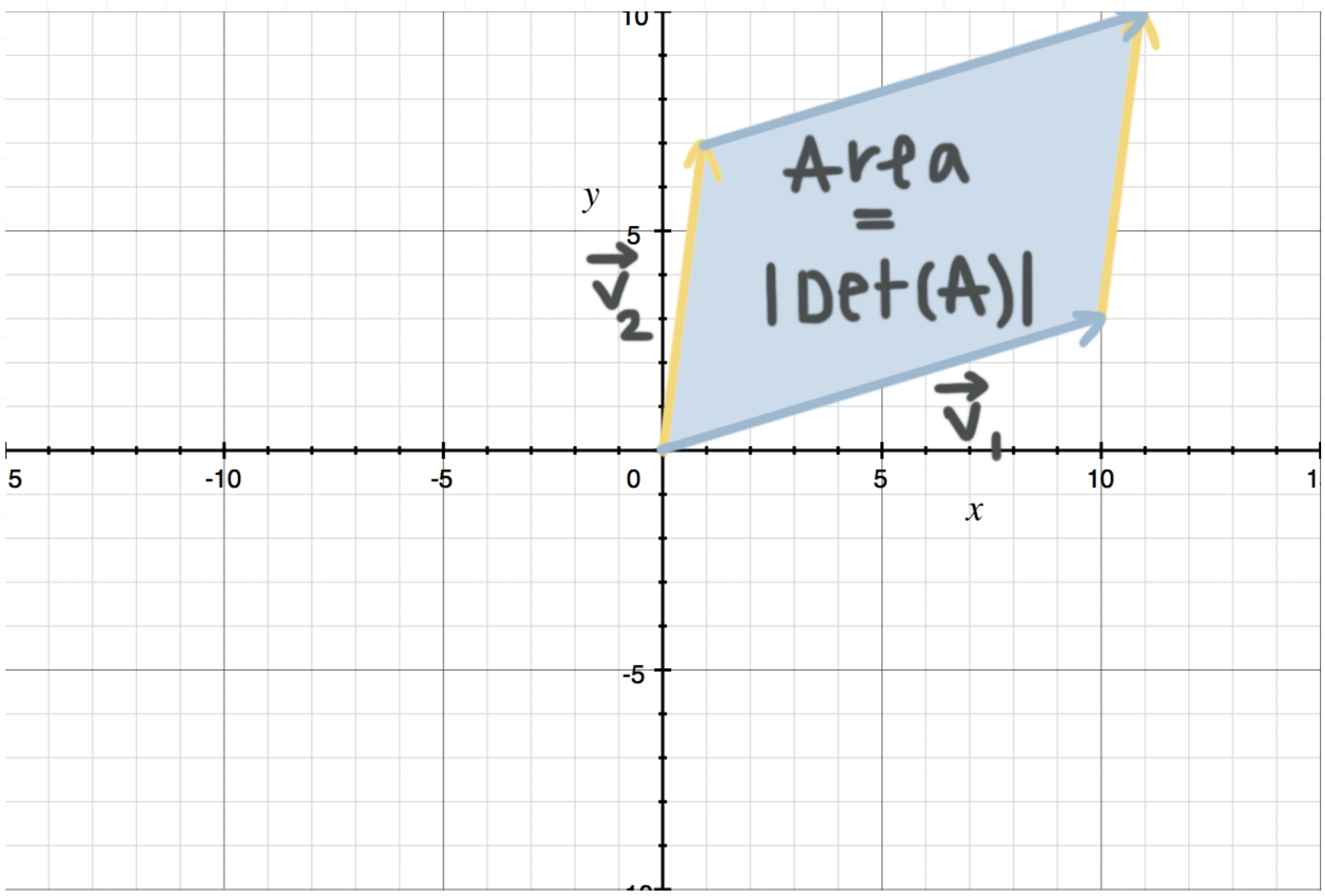
$$v_1 = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} b \\ d \end{bmatrix}$$

then you can plot v_1 and v_2 in the xy-plane, and create a parallelogram using them as two of its adjacent sides.

The area of that parallelogram is simply the absolute value of the determinant of the matrix A.

$$\text{Area} = |\text{Det}(A)|$$

We're using $\text{Det}(A)$ instead of $|A|$ to avoid confusion between the bars around A that signify the determinant, and the bars in Area $= |\text{Det}(A)|$ that are there to indicate the absolute value.



Area under a transformation

Furthermore, let's say we're given a figure in the xy-plane, whether it's a parallelogram or any other figure, and let's call that figure f. We could always decide to apply some transformation matrix to f. Let's call the transformation matrix T. Applying T to f will create a new figure, and let's call the new figure g.

We know of course that the area of each figure will be the determinant of a matrix that describes the figure. So if matrix F defines f, then the area of f

is $\text{Area}_f = |\text{Det}(F)|$. And if matrix G defines g, then the area of g would be $\text{Area}_g = |\text{Det}(G)|$.

But we can also describe the area of g in terms of f and the transformation matrix T. As it turns out, the area of g can also be given by

$$\text{Area}_g = |\text{Area}_f \cdot \text{Det}(T)|$$

And this is really helpful, because once we get the figure g after the transformation, we don't have to find its area or calculate its determinant. We can just use the area of f (the figure we started with), multiply that by the determinant of the transformation matrix, and then take the absolute value of that product. In other words, we can find the area of g before we even have the matrix that defines g!

Example

The rectangle R with vertices $(1,1)$, $(1,6)$, $(-3,6)$, and $(-3,1)$ is transformed by T. Find the area of the transformed figure P.

$$T(\vec{x}) = \begin{bmatrix} -3 & 1 \\ 4 & 0 \end{bmatrix} \vec{x}$$

All we've been given are the vertices of R and the transformation matrix T, and that's all we need. We don't have to sketch R, we don't have to represent R as a matrix, we don't have to actually apply T to R, nor do we need to know anything else about the transformed figure. All we need to find is the area of R, and the determinant of T.

Horizontally, the rectangle R is defined between -3 and 1, so its width is 4. Vertically, the rectangle R is defined between 1 and 6, so its height is 5. Therefore, the area of R is $\text{Area}_R = 4(5) = 20$.

The determinant of T is

$$\text{Det}(T) = \begin{vmatrix} -3 & 1 \\ 4 & 0 \end{vmatrix} = (-3)(0) - (1)(4) = 0 - 4 = -4$$

Therefore, the area of the transformed figure is

$$\text{Area}_P = |\text{Area}_R \cdot \text{Det}(T)|$$

$$\text{Area}_P = |20(-4)|$$

$$\text{Area}_P = |-80|$$

$$\text{Area}_P = 80$$

The area of the original rectangle R is 20, but after undergoing the transformation T, the area of the new transformed figure P is 80.

So the area of the parallelogram can be given by the determinant of the 2×2 matrix, but it's also true that the determinant of the 3×3 matrix will give the volume of the parallelepiped (three-dimensional parallelogram).