Topic: Linear transformations as rotations

Question: Find the rotation of $\vec{x} = (-1,4)$ by an angle of $\theta = 270^\circ$.

Answer choices:

A $\quad \text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = \begin{bmatrix}4\\1\end{bmatrix}$

B $\quad \text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = \begin{bmatrix}-4\\-1\end{bmatrix}$

C $\quad \text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = \begin{bmatrix}4\\-1\end{bmatrix}$

D $\quad \text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = \begin{bmatrix}-4\\1\end{bmatrix}$

Solution: A

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^2$ by 270° is

$$\text{Rot}_{270^\circ}(\vec{x}) = \begin{bmatrix}\cos(270^\circ) & -\sin(270^\circ)\\ \sin(270^\circ) & \cos(270^\circ)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 270^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix}\cos(270^\circ) & -\sin(270^\circ)\\ \sin(270^\circ) & \cos(270^\circ)\end{bmatrix} = \begin{bmatrix}0 & -(-1)\\ -1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

So the transformation for a 270° rotation is

$$\text{Rot}_{270^\circ}(\vec{x}) = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x} = (-1,4)$.

$$\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\ 4\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}-1\\ 4\end{bmatrix}$$

$$\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\ 4\end{bmatrix}\right) = \begin{bmatrix}0(-1) + 1(4)\\ -1(-1) + 0(4)\end{bmatrix}$$

$$\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\ 4\end{bmatrix}\right) = \begin{bmatrix}0 + 4\\ 1 + 0\end{bmatrix}$$

$$\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\ 4\end{bmatrix}\right) = \begin{bmatrix}4\\ 1\end{bmatrix}$$

Topic: Linear transformations as rotations

Question: Find the rotation of $\vec{x}=(2,0,-3)$ by an angle of $\theta=60^\circ$ about the x-axis.

Answer choices:

A $\quad \text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}2\\-\frac{3\sqrt{3}}{2}\\\frac{3}{2}\end{bmatrix}$

B $\quad \text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}2\\\frac{3\sqrt{3}}{2}\\-\frac{3}{2}\end{bmatrix}$

C $\quad \text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}-2\\\frac{3\sqrt{3}}{2}\\\frac{3}{2}\end{bmatrix}$

D $\quad \text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}-2\\-\frac{3\sqrt{3}}{2}\\\frac{3}{2}\end{bmatrix}$

Solution: B

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^3$ by 60° around the x-axis is

$$\text{Rot}_{60^\circ \text{ around } x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(60^\circ) & -\sin(60^\circ) \\ 0 & \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 60^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(60^\circ) & -\sin(60^\circ) \\ 0 & \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

So the transformation for a 60° rotation around the x-axis is

$$\text{Rot}_{60^\circ \text{ around } x}(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x} = (2,0,-3)$.

$$\text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

$$\text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}1(2)+0(0)+0(-3)\\0(2)+\frac{1}{2}(0)-\frac{\sqrt{3}}{2}(-3)\\0(2)+\frac{\sqrt{3}}{2}(0)+\frac{1}{2}(-3)\end{bmatrix}$$

$$\text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}2+0+0\\0+0+\frac{3\sqrt{3}}{2}\\0+0-\frac{3}{2}\end{bmatrix}$$

$$\text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}2\\\frac{3\sqrt{3}}{2}\\-\frac{3}{2}\end{bmatrix}$$

Topic: Linear transformations as rotations

Question: Find the rotation of $\vec{x} = (-2,3,-1)$ by an angle of $\theta = 225^\circ$ about the z-axis.

Answer choices:

A $\quad \text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}-\frac{5\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\\1\end{bmatrix}$

B $\quad \text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}\frac{5\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\-1\end{bmatrix}$

C $\quad \text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}-\frac{5\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\1\end{bmatrix}$

D $\quad \text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}\frac{5\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\\-1\end{bmatrix}$

Solution: D

The transformation to rotate any vector $\overrightarrow{x}$ in $\mathbb{R}^3$ by 225° around the z-axis is

$$\text{Rot}_{225^\circ \text{ around } z} = \begin{bmatrix} \cos(225^\circ) & -\sin(225^\circ) & 0 \\ \sin(225^\circ) & \cos(225^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 225^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix} \cos(225^\circ) & -\sin(225^\circ) & 0 \\ \sin(225^\circ) & \cos(225^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\left(-\frac{\sqrt{2}}{2}\right) & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the transformation for a 225° rotation around the z-axis is

$$\text{Rot}_{225^\circ \text{ around } z}(\overrightarrow{x}) = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Apply this specific rotation matrix to $\overrightarrow{x} = (-2,3,-1)$.

$$\text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$\text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right)=\begin{bmatrix}-\frac{\sqrt{2}}{2}(-2)+\frac{\sqrt{2}}{2}(3)+0(-1)\\-\frac{\sqrt{2}}{2}(-2)-\frac{\sqrt{2}}{2}(3)+0(-1)\\0(-2)+0(3)+1(-1)\end{bmatrix}$$

$$\text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right)=\begin{bmatrix}\sqrt{2}+\frac{3\sqrt{2}}{2}+0\\\sqrt{2}-\frac{3\sqrt{2}}{2}+0\\0+0-1\end{bmatrix}$$

$$\text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right)=\begin{bmatrix}\frac{5\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\\-1\end{bmatrix}$$