



# Linear Algebra Final Exam Solutions

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## Linear Algebra Final Exam Answer Key

1. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input type="checkbox"/> C	<input checked="" type="checkbox"/>	<input type="checkbox"/> E
2. (5 pts)	<input checked="" type="checkbox"/>	<input type="checkbox"/> B	<input type="checkbox"/> C	<input type="checkbox"/> D	<input type="checkbox"/> E
3. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input checked="" type="checkbox"/>	<input type="checkbox"/> D	<input type="checkbox"/> E
4. (5 pts)	<input type="checkbox"/> A	<input checked="" type="checkbox"/>	<input type="checkbox"/> C	<input type="checkbox"/> D	<input type="checkbox"/> E
5. (5 pts)	<input checked="" type="checkbox"/>	<input type="checkbox"/> B	<input type="checkbox"/> C	<input type="checkbox"/> D	<input type="checkbox"/> E
6. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input type="checkbox"/> C	<input type="checkbox"/> D	<input checked="" type="checkbox"/>
7. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input checked="" type="checkbox"/>	<input type="checkbox"/> D	<input type="checkbox"/> E
8. (5 pts)	<input type="checkbox"/> A	<input type="checkbox"/> B	<input checked="" type="checkbox"/>	<input type="checkbox"/> D	<input type="checkbox"/> E



9. (15 pts)  $\vec{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

10. (15 pts)  $C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}\right)$   $\mathbb{R}^3$  Dim = 3

$N(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$   $\mathbb{R}^4$  Dim = 1

$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}\right)$   $\mathbb{R}^4$  Dim = 3

$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$   $\mathbb{R}^3$  Dim = 0

11. (15 pts)  $V^\perp = \text{Span}\left(\begin{bmatrix} 2 \\ \frac{10}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}\right)$



12. (15 pts)  $E_0 = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right)$ ,  $E_2 = \text{Span}\left(\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}\right)$ ,  $E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

- Since  $\lambda = 0$  in the eigenspace  $E_0$ , any vector  $\vec{v}$  in  $E_0$ , under the transformation  $T$ , will be scaled down to the zero vector, meaning that  $T(\vec{v}) = \lambda \vec{v} = 0 \vec{v} = \vec{0}$ .
- Since  $\lambda = 2$  in the eigenspace  $E_2$ , any vector  $\vec{v}$  in  $E_2$ , under the transformation  $T$ , will be scaled by 2, meaning that  $T(\vec{v}) = \lambda \vec{v} = 2 \vec{v}$ .
- Since  $\lambda = 3$  in the eigenspace  $E_3$ , any vector  $\vec{v}$  in  $E_3$ , under the transformation  $T$ , will be scaled by 3, meaning that  $T(\vec{v}) = \lambda \vec{v} = 3 \vec{v}$ .



# Linear Algebra Final Exam Solutions

1. D. To put  $A$  into reduced row-echelon form, start by working on the first column.

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ -2 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 3 & 1 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & -6 & 5 & 1 \\ 0 & 2 & -1 & 0 \\ 3 & 1 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & -6 & 5 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 10 & -4 & -2 \end{bmatrix}$$

Find the pivot entry in the second column, then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 2 & -1 & 0 \\ 0 & 10 & -4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 2 & -1 & 0 \\ 0 & 10 & -4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 10 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{13}{3} & -\frac{1}{3} \end{bmatrix}$$

Find the pivot entry in the third column, then zero out the rest of the third column.



$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{13}{3} & -\frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{13}{3} & -\frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{13}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. A. Multiply each row of  $A$  by the first column of  $B$ .

$$AB = \begin{bmatrix} -2 & 1 & 2 & 5 \\ 0 & -1 & 4 & 2 \\ 1 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -5 & 0 & 2 \\ 2 & -3 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -2(1) + 1(-5) + 2(2) + 5(2) & \dots & \dots \\ 0(1) - 1(-5) + 4(2) + 2(2) & \dots & \dots \\ 1(1) - 3(-5) + 1(2) - 1(2) & \dots & \dots \end{bmatrix}$$



$$AB = \begin{bmatrix} -2 - 5 + 4 + 10 & \dots & \dots \\ 0 + 5 + 8 + 4 & \dots & \dots \\ 1 + 15 + 2 - 2 & \dots & \dots \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & \dots & \dots \\ 17 & \dots & \dots \\ 16 & \dots & \dots \end{bmatrix}$$

Multiply each row of  $A$  by the second column of  $B$ .

$$AB = \begin{bmatrix} 7 & -2(1) + 1(0) + 2(-3) + 5(0) & \dots \\ 17 & 0(1) - 1(0) + 4(-3) + 2(0) & \dots \\ 16 & 1(1) - 3(0) + 1(-3) - 1(0) & \dots \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -2 + 0 - 6 + 0 & \dots \\ 17 & 0 - 0 - 12 + 0 & \dots \\ 16 & 1 - 0 - 3 - 0 & \dots \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -8 & \dots \\ 17 & -12 & \dots \\ 16 & -2 & \dots \end{bmatrix}$$

Multiply each row of  $A$  by the third column of  $B$ .

$$AB = \begin{bmatrix} 7 & -8 & -2(2) + 1(2) + 2(-1) + 5(-1) \\ 17 & -12 & 0(2) - 1(2) + 4(-1) + 2(-1) \\ 16 & -2 & 1(2) - 3(2) + 1(-1) - 1(-1) \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -8 & -4 + 2 - 2 - 5 \\ 17 & -12 & 0 - 2 - 4 - 2 \\ 16 & -2 & 2 - 6 - 1 + 1 \end{bmatrix}$$



$$AB = \begin{bmatrix} 7 & -8 & -9 \\ 17 & -12 & -8 \\ 16 & -2 & -4 \end{bmatrix}$$

3. C. The vector sum is

$$2\vec{a} - 3\vec{b} + 5\vec{c} - \vec{d}$$

$$2(2, 6, -1) - 3(-3, 1, 1) + 5(0, 5, -2) - (1, 4, -4)$$

Apply the scalars to each vector.

$$(4, 12, -2) + (9, -3, -3) + (0, 25, -10) + (-1, -4, 4)$$

Add each of the vector components.

$$(4 + 9 + 0 - 1, 12 - 3 + 25 - 4, -2 - 3 - 10 + 4)$$

$$(12, 30, -11)$$

4. B. The length of  $\vec{x} = (4, -2, 1, 0)$  is given by

$$||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

$$||\vec{x}|| = \sqrt{4^2 + (-2)^2 + 1^2 + 0^2}$$

$$||\vec{x}|| = \sqrt{16 + 4 + 1 + 0}$$

$$||\vec{x}|| = \sqrt{21}$$





5. A. The equation of a plane with normal vector  $\vec{n} = (a, b, c) = (2, -6, 1)$ , which passes through  $(x_0, y_0, z_0) = (5, 2, -3)$ , is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$2(x - 5) - 6(y - 2) + 1(z - (-3)) = 0$$

Simplify to get the equation of the plane into standard form.

$$2x - 10 - 6y + 12 + z + 3 = 0$$

$$2x - 6y + z + 5 = 0$$

$$2x - 6y + z = -5$$

6. E. Given  $\vec{x} = (4, -1)$  and

$$S(\vec{x}) = \begin{bmatrix} -2x_1 + x_2 \\ 3x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} x_1 - 4x_2 \\ -4x_2 \end{bmatrix}$$

start by using  $S$  to transform the standard basis vectors.

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2(1) + 0 \\ 3(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2(0) + 1 \\ 3(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



So the transformation  $S$  can be written as the matrix-vector product

$$S(\vec{x}) = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} \vec{x}$$

Use  $T$  to transform the standard basis vectors.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 - 4(0) \\ -4(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 4(1) \\ -4(1) \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

So the transformation  $T$  can be written as the matrix-vector product

$$T(\vec{y}) = \begin{bmatrix} 1 & -4 \\ 0 & -4 \end{bmatrix} \vec{y}$$

If we call the matrix from  $S$

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$$

and we call the matrix from  $T$

$$B = \begin{bmatrix} 1 & -4 \\ 0 & -4 \end{bmatrix}$$

then the composition of the transformations is

$$T(S(\vec{x})) = BA\vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix} 1 & -4 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} \vec{x}$$



$$T(S(\vec{x})) = \begin{bmatrix} 1(-2) - 4(0) & 1(1) - 4(3) \\ 0(-2) - 4(0) & 0(1) - 4(3) \end{bmatrix} \vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix} -2 & -11 \\ 0 & -12 \end{bmatrix} \vec{x}$$

To transform  $\vec{x} = (4, -1)$ , multiply this transformation matrix by  $\vec{x} = (4, -1)$ .

$$T\left(S\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -2 & -11 \\ 0 & -12 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -2(4) - 11(-1) \\ 0(4) - 12(-1) \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$$

7. C. Transform  $\vec{x} = (6, -1)$  with the transformation  $T$ .

$$T(\vec{x}) = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} \vec{x}$$

$$T\left(\begin{bmatrix} 6 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 6 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1(6) - 2(-1) \\ 4(6) - 4(-1) \end{bmatrix}$$



$$T\left(\begin{bmatrix} 6 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 28 \end{bmatrix}$$

The change of basis matrix is given by the column vectors that span  $B$ .

$$C = \begin{bmatrix} 2 & -3 \\ -1 & 0 \end{bmatrix}$$

Now apply the change of basis matrix to the transformed vector to convert the transformed vector into the alternate basis.

$$CT\left(\begin{bmatrix} 6 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 & -3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 28 \end{bmatrix}$$

$$CT\left(\begin{bmatrix} 6 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2(8) - 3(28) \\ -1(8) + 0(28) \end{bmatrix}$$

$$CT\left(\begin{bmatrix} 6 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -68 \\ -8 \end{bmatrix}$$

This is the vector  $\vec{x} = (6, -1)$  after the transformation  $T$ , and converted into the alternate basis. So

$$[\vec{x}]_B = \begin{bmatrix} -68 \\ -8 \end{bmatrix}$$

8. C. The vectors  $\vec{v}_1 = (0, -1, 1)$ ,  $\vec{v}_2 = (2, 0, -3)$ , and  $\vec{v}_3 = (-1, 1, 1)$  form the basis for  $V$ .

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$



Normalize  $\vec{v}_1$ . The length of  $\vec{v}_1$  is

$$||\vec{v}_1|| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{0 + 1 + 1} = \sqrt{2}$$

Then the normalized version of  $\vec{v}_1$  is

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Replace  $\vec{v}_2$  with a vector that's both orthogonal to  $\vec{u}_1$ , and normal.

Name  $\vec{w}_2$  as the vector that connects  $\text{Proj}_{V_1} \vec{v}_2$  to  $\vec{v}_2$ .

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1} \vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} - \left( \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} - \frac{1}{2} (2(0) + 0(-1) - 3(1)) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$



$$\vec{w}_2 = \begin{bmatrix} 2 \\ -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Normalize  $\vec{w}_2$ . The length of  $\vec{w}_2$  is

$$||\vec{w}_2|| = \sqrt{2^2 + \left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{4 + \frac{9}{4} + \frac{9}{4}}$$

$$||\vec{w}_2|| = \sqrt{\frac{34}{4}}$$

$$||\vec{w}_2|| = \frac{\sqrt{34}}{2}$$

Then the normalized version of  $\vec{w}_2$  is  $\vec{u}_2$ :

$$\vec{u}_2 = \frac{2}{\sqrt{34}} \begin{bmatrix} 2 \\ -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{2} \cdot \frac{2}{\sqrt{34}} \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{34}} \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$$



Replace  $\vec{v}_3$  with a vector that's both orthogonal to  $\vec{u}_1$  and  $\vec{u}_2$ , and normal. Name  $\vec{w}_3$  as the vector that connects  $\text{Proj}_{V_2} \vec{v}_3$  to  $\vec{v}_3$ .

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_2} \vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - [(\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2]$$

$$\vec{w}_3 = \vec{v}_3 - \left[ \left( \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 \right]$$

$$\vec{w}_3 = \vec{v}_3 - \left[ \frac{1}{2}(-1(0) + 1(-1) + 1(1)) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 \right]$$

$$\vec{w}_3 = \vec{v}_3 - \left[ \frac{1}{2}(0) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 \right]$$

$$\vec{w}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \left[ 0 + \left( \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{34}} \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix} \right) \frac{1}{\sqrt{34}} \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix} \right]$$

$$\vec{w}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{34} \left( \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix} \right) \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{34}(-1(4) + 1(-3) + 1(-3)) \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \frac{5}{17} \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$$



$$\vec{w}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{20}{17} \\ -\frac{15}{17} \\ -\frac{15}{17} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} \frac{3}{17} \\ \frac{2}{17} \\ \frac{2}{17} \end{bmatrix}$$

$$\vec{w}_3 = \frac{1}{17} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

Normalize  $\vec{w}_3$ . The length of  $\vec{w}_3$  is

$$\|\vec{w}_3\| = \sqrt{\left(\frac{3}{17}\right)^2 + \left(\frac{2}{17}\right)^2 + \left(\frac{2}{17}\right)^2}$$

$$\|\vec{w}_3\| = \sqrt{\frac{9}{289} + \frac{4}{289} + \frac{4}{289}}$$

$$\|\vec{w}_3\| = \sqrt{\frac{17}{289}}$$

$$\|\vec{w}_3\| = \frac{\sqrt{17}}{17}$$

Then the normalized version of  $\vec{w}_3$  is  $\vec{u}_3$ :





$$\vec{u}_3 = \frac{17}{\sqrt{17}} \begin{bmatrix} \frac{3}{17} \\ \frac{2}{17} \\ \frac{2}{17} \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{17} \cdot \frac{17}{\sqrt{17}} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{17}} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

The vectors  $\vec{u}_1$ ,  $\vec{u}_2$ , and  $\vec{u}_3$  form an orthonormal basis for  $V$ .

$$V = \text{Span}\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{34}} \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}, \frac{1}{\sqrt{17}} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}\right)$$

9. Put the matrix  $A$  into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

To find the complementary solution, augment  $\text{rref}(A)$  with the zero vector to get a system of equations.

$$x_1 = 0$$

$$x_2 = 0$$



Then the only vector that satisfies the null space is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The complementary solution is

$$\vec{x}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To find the particular solution, augment  $A$  with  $\vec{b} = (b_1, b_2, b_3)$ , then put it in reduced row-echelon form.

$$\left[ \begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ -2 & -3 & b_3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & 2b_1 + b_3 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & 2b_1 + b_3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_1 + b_2 + b_3 \end{array} \right]$$

From the third row, the system is constrained.

$$b_1 + b_2 + b_3 = 0$$

$$b_1 = -b_2 - b_3$$

Choose values of  $b_1$ ,  $b_2$ , and  $b_3$  that make this equation true, like  $b_2 = 1$ ,  $b_3 = 1$ , and  $b_1 = -2$ . Then the matrix is



$$\left[ \begin{array}{cc|c} 1 & 0 & 2(-2) - 1 \\ 0 & 1 & 1 - (-2) \\ 0 & 0 & -2 + 1 + 1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right]$$

This gives a system of equations

$$x_1 = -5$$

$$x_2 = 3$$

So the particular solution is

$$\vec{x}_p = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

The general solution is the sum of the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Of course, adding the zero vector doesn't change the value of the general solution, so the general solution is

$$\vec{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

10. First put  $A$  into reduced row-echelon form.



$$\begin{aligned}
 A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ -1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & -2 & 2 & 5 \\ 0 & 1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & 5 \end{bmatrix} \\
 &\rightarrow \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &\rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

In  $\text{rref}(A)$ , the pivot columns are the first, second, and fourth columns, which means  $C(A)$  is given by the span of the first, second, and fourth columns of  $A$ .

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}\right)$$

From  $\text{rref}(A)$ , we get the system of equations

$$x_1 - 2x_3 = 0$$

$$x_2 - x_3 = 0$$

$$x_4 = 0$$

Solve the system for the pivot variables.

$$x_1 = 2x_3$$

$$x_2 = x_3$$



$$x_4 = 0$$

Then the solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

which means the null space is given as

$$N(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

Find  $A^T$ ,

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 4 & 2 \end{bmatrix}$$

then put  $A^T$  into reduced row-echelon form.

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \\ 1 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & 5 & 2 \end{bmatrix}$$



$$\rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{9}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{9}{2} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

In  $\text{rref}(A^T)$ , the pivot columns are the first, second, and third columns, which means  $C(A^T)$  is given by the span of the first, second, and third columns of  $A^T$ .

$$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}\right)$$

From  $\text{rref}(A^T)$ , we get the system of equations

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

Then the solution to the system is



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which means the left null space is given as

$$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because  $A$  is an  $m \times n = 3 \times 4$  matrix, the row space and null space are defined in  $\mathbb{R}^n = \mathbb{R}^4$ , and the column space and left null space are defined in  $\mathbb{R}^m = \mathbb{R}^3$ .

The dimension of the column space and row space is the rank of  $A$ ,  $r = 3$ . The dimension of the null space is  $n - r = 4 - 3 = 1$ , and the dimension of the left null space is  $m - r = 3 - 3 = 0$ .

In summary, the four fundamental subspaces are

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}\right) \quad \mathbb{R}^3 \quad \text{Dim} = 3$$

$$N(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) \quad \mathbb{R}^4 \quad \text{Dim} = 1$$

$$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}\right) \quad \mathbb{R}^4 \quad \text{Dim} = 3$$



$$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbb{R}^3$$

$$\text{Dim} = 0$$

11. The subspace  $V$  is a plane in  $\mathbb{R}^4$ , spanned by the three vectors  $\vec{v}_1 = (2, -2, 1, 1)$ ,  $\vec{v}_2 = (-1, 0, 0, 2)$ , and  $\vec{v}_3 = (0, 1, -2, 0)$ . Its orthogonal complement  $V^\perp$  is the set of vectors which are orthogonal to both all three.

$$V^\perp = \left\{ \vec{x} \in \mathbb{R}^4 \mid \vec{x} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix} = 0, \vec{x} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = 0, \vec{x} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} = 0 \right\}$$

Let  $\vec{x} = (x_1, x_2, x_3, x_4)$  to get three equations from these dot products.

$$2x_1 - 2x_2 + x_3 + x_4 = 0$$

$$-x_1 + 2x_4 = 0$$

$$x_2 - 2x_3 = 0$$

Put these equations into an augmented matrix,

$$\left[ \begin{array}{cccc|c} 2 & -2 & 1 & 1 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{array} \right]$$

then put it into reduced row-echelon form.





$$\begin{bmatrix} -1 & 0 & 0 & 2 & | & 0 \\ 2 & -2 & 1 & 1 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 2 & -2 & 1 & 1 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & -2 & 1 & 5 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \\ 0 & -2 & 1 & 5 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \\ 0 & 0 & -3 & 5 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \\ 0 & 0 & 1 & -\frac{5}{3} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & 1 & 0 & -\frac{10}{3} & | & 0 \\ 0 & 0 & 1 & -\frac{5}{3} & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 - 2x_4 = 0$$

$$x_2 - \frac{10}{3}x_4 = 0$$

$$x_3 - \frac{5}{3}x_4 = 0$$

Solve the system for the pivot variables,  $x_1$ ,  $x_2$ , and  $x_3$ .

$$x_1 = 2x_4$$



$$x_2 = \frac{10}{3}x_4$$

$$x_3 = \frac{5}{3}x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 2 \\ \frac{10}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

The orthogonal complement is

$$V^\perp = \text{Span}\left(\begin{bmatrix} 2 \\ \frac{10}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}\right)$$

12. Starting with

$$A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

first, find the determinant  $|\lambda I_n - A|$ .



$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 3 & -6 & 8 \\ 0 & \lambda & -6 \\ 0 & 0 & \lambda - 2 \end{bmatrix} \right|$$

Then let's work along the first column, since it includes two 0 values, to find the determinant of this resulting matrix.

$$(\lambda - 3) \begin{vmatrix} \lambda & -6 \\ 0 & \lambda - 2 \end{vmatrix} - 0 \begin{vmatrix} -6 & 8 \\ 0 & \lambda - 2 \end{vmatrix} + 0 \begin{vmatrix} -6 & 8 \\ \lambda & -6 \end{vmatrix}$$

The last two determinants cancel, leaving us with just

$$(\lambda - 3) \begin{vmatrix} \lambda & -6 \\ 0 & \lambda - 2 \end{vmatrix}$$

$$(\lambda - 3)\lambda(\lambda - 2) - (-6)(0)$$

$$\lambda(\lambda - 3)(\lambda - 2)$$

Remember that we're trying to satisfy  $|\lambda I_n - A| = 0$ , so we can set this characteristic polynomial equal to 0 to get the characteristic equation, and then we'll solve for  $\lambda$ .

$$\lambda(\lambda - 3)(\lambda - 2) = 0$$



$$\lambda = 0 \text{ or } \lambda = 2 \text{ or } \lambda = 3$$

With these three eigenvalues, we'll have three eigenspaces, given by  $E_\lambda = N(\lambda I_n - A)$ . Given

$$E_\lambda = N \left( \begin{bmatrix} \lambda - 3 & -6 & 8 \\ 0 & \lambda & -6 \\ 0 & 0 & \lambda - 2 \end{bmatrix} \right)$$

we get

$$E_0 = N \left( \begin{bmatrix} 0 - 3 & -6 & 8 \\ 0 & 0 & -6 \\ 0 & 0 & 0 - 2 \end{bmatrix} \right)$$

$$E_0 = N \left( \begin{bmatrix} -3 & -6 & 8 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix} \right)$$

and

$$E_2 = N \left( \begin{bmatrix} 2 - 3 & -6 & 8 \\ 0 & 2 & -6 \\ 0 & 0 & 2 - 2 \end{bmatrix} \right)$$

$$E_2 = N \left( \begin{bmatrix} -1 & -6 & 8 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

and



$$E_3 = N \left( \begin{bmatrix} 3-3 & -6 & 8 \\ 0 & 3 & -6 \\ 0 & 0 & 3-2 \end{bmatrix} \right)$$

$$E_3 = N \left( \begin{bmatrix} 0 & -6 & 8 \\ 0 & 3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

Therefore, the eigenvectors in the eigenspace  $E_0$  will satisfy

$$\begin{bmatrix} -3 & -6 & 8 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -6 & 8 & | & 0 \\ 0 & 0 & -6 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -\frac{8}{3} & | & 0 \\ 0 & 0 & -6 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -\frac{8}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 + 2v_2 = 0$$

$$v_3 = 0$$

or

$$v_1 = -2v_2$$



$$v_3 = 0$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Which means that  $E_0$  is defined by

$$E_0 = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right)$$

And the eigenvectors in the eigenspace  $E_2$  will satisfy

$$\begin{bmatrix} -1 & -6 & 8 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -6 & 8 & | & 0 \\ 0 & 2 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 6 & -8 & | & 0 \\ 0 & 2 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 2 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 + 10v_3 = 0$$

$$v_2 - 3v_3 = 0$$



or

$$v_1 = -10v_3$$

$$v_2 = 3v_3$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}$$

Which means that  $E_2$  is defined by

$$E_2 = \text{Span}\left(\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}\right)$$

And the eigenvectors in the eigenspace  $E_3$  will satisfy

$$\begin{bmatrix} 0 & -6 & 8 \\ 0 & 3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 0 & -6 & 8 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$



This gives

$$v_2 = 0$$

$$v_3 = 0$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that  $E_3$  is defined by

$$E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

Let's put these results together. For the eigenvalues  $\lambda = 0$ ,  $\lambda = 2$ , and  $\lambda = 3$ , respectively, we got

$$E_0 = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right), E_2 = \text{Span}\left(\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}\right), \text{ and } E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

Each of these spans represents a line in  $\mathbb{R}^3$ . So for any vector  $\vec{v}$  along any of these lines, when you apply the transformation  $T$  to the vector  $\vec{v}$ ,  $T(\vec{v})$  will be a vector along the same line, it might just be scaled up or scaled down. Specifically,

- since  $\lambda = 0$  in the eigenspace  $E_0$ , any vector  $\vec{v}$  in  $E_0$ , under the transformation  $T$ , will be scaled down to the zero vector, meaning that  $T(\vec{v}) = \lambda \vec{v} = 0 \vec{v} = \vec{0}$ ,





- since  $\lambda = 2$  in the eigenspace  $E_2$ , any vector  $\vec{v}$  in  $E_2$ , under the transformation  $T$ , will be scaled by 2, meaning that  $T(\vec{v}) = \lambda \vec{v} = 2\vec{v}$ , and
- since  $\lambda = 3$  in the eigenspace  $E_3$ , any vector  $\vec{v}$  in  $E_3$ , under the transformation  $T$ , will be scaled by 3, meaning that  $T(\vec{v}) = \lambda \vec{v} = 3\vec{v}$ .



