Linear Algebra Workbook Solutions

Dot products and cross products

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DOT PRODUCTS

■ 1. Find the dot product.

$$\overrightarrow{a} = (-2,5)$$

$$\overrightarrow{b} = (3,4)$$

Solution:

To find the dot product of the two vectors, we multiply corresponding components, and then add the products. So the dot product of $\overrightarrow{a} = (-2,5)$ and $\overrightarrow{b} = (3,4)$ is

$$\overrightarrow{a} \cdot \overrightarrow{b} = (-2)(3) + (5)(4)$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = -6 + 20$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = 14$$

■ 2. Find the dot product.

$$\overrightarrow{x} = (1, -2,0)$$

$$\overrightarrow{y} = (5, -1, -3)$$

Solution:

To find the dot product of the two vectors, we multiply corresponding components, and then add the products. So the dot product of $\vec{x}=(1,-2,0)$ and $\vec{y}=(5,-1,-3)$ is

$$\vec{x}\cdot\vec{y}=(1)(5)+(-2)(-1)+(0)(-3)$$

$$\vec{x}\cdot\vec{y}=5+2+0$$

$$\vec{x}\cdot\vec{y}=7$$

3. Use the dot product to find the length of the vector $\vec{u}=(-5,2,-4,-2)$.

Solution:

We'll get the square of the length if we dot the vector with itself. Use the formula $||\vec{u}||^2=\vec{u}\cdot\vec{u}$.

$$||\vec{u}||^2=(-5)(-5)+(2)(2)+(-4)(-4)+(-2)(-2)$$

$$||\vec{u}||^2=25+4+16+4$$

$$||\vec{u}||^2=49$$

Take the square root of both sides. We can ignore the negative value of the root, since we're looking for the length of the vector, and length will always be positive.

$$\sqrt{||\vec{u}||^2} = \sqrt{49}$$

$$||\vec{u}|| = 7$$

The length of $\vec{u} = (-5,2,-4,-2)$ is 7.

4. Simplify the expression if $\vec{x} = (-2,4)$, $\vec{y} = (0,-1)$, and $\vec{z} = (4,7)$.

$$4\vec{x} \cdot (3\vec{y} - \vec{z})$$

Solution:

To simplify $4\vec{x} \cdot (3\vec{y} - \vec{z})$, start by finding $4\vec{x}$.

$$\vec{x} = (-2,4)$$

$$4\vec{x} = 4(-2,4)$$

$$4\vec{x} = (-8,16)$$

Find $3\vec{y}$.

$$\vec{y} = (0,-1)$$

$$3\vec{y} = 3(0,-1)$$

$$3\vec{y} = (0,-3)$$

Then the difference $3\vec{y} - \vec{z}$ is

$$3\vec{y} - \vec{z} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} - \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$3\vec{y} - \vec{z} = \begin{bmatrix} 0 - 4 \\ -3 - 7 \end{bmatrix}$$

$$3\vec{y} - \vec{z} = \begin{bmatrix} -4 \\ -10 \end{bmatrix}$$

Then the dot product is

$$4\vec{x} \cdot (3\vec{y} - \vec{z})$$

$$\begin{bmatrix} -8 & 16 \end{bmatrix} \begin{bmatrix} -4 \\ -10 \end{bmatrix}$$

$$(-8)(-4) + (16)(-10)$$

$$32 - 160$$

$$-128$$

■ 5. Use the dot product to find $-\vec{a} \cdot (5\vec{b} + 3\vec{c})$.

$$\vec{a} = (-2,0,4)$$

$$\vec{b} = (1,5,3)$$

$$\vec{c} = (-1, -4,0)$$

Solution:

To simplify $-\vec{a}(5\vec{b}+3\vec{c})$, start by finding $-\vec{a}$.

$$\vec{a}=(-2,0,4)$$

$$-\vec{a}=-1(-2,0,4)$$

$$-\vec{a}=(2,0,-4)$$

Find $5\vec{b}$.

$$\vec{b}=(1,5,3)$$

$$5\vec{b}=5(1,5,3)$$

$$5\vec{b}=(5,25,15)$$

Find $3\vec{c}$.

$$\vec{c}=(-1,-4,0)$$

$$3\vec{c}=3(-1,-4,0)$$

$$3\vec{c}=(-3,-12,0)$$

Then the sum $5\vec{b}+3\vec{c}$ is

$$5\vec{b}+3\vec{c}=\begin{bmatrix}5\\25\\15\end{bmatrix}+\begin{bmatrix}-3\\-12\\0\end{bmatrix}$$

$$5\vec{b}+3\vec{c}=\begin{bmatrix}5-3\\25-12\\15+0\end{bmatrix}$$

$$5\vec{b}+3\vec{c}=\begin{bmatrix}2\\13\\15\end{bmatrix}$$

Then the dot product is

$$-\vec{a}\cdot(5\vec{b}+3\vec{c})$$

$$\begin{bmatrix}2 & 0 & -4\end{bmatrix}\begin{bmatrix}2\\13\\15\end{bmatrix}$$

$$(2)(2)+(0)(13)+(-4)(15)$$

$$4+0-60$$

$$-56$$

■ 6. Use the dot product to find $\vec{w}(2\vec{x}+\vec{y})-3\vec{y}(\vec{w}+4\vec{x}-\vec{z})$.

$$\vec{x}=(4,-3,0,7)$$

$$\vec{y}=(-1,5,2,-1)$$

$$\vec{z}=(0,6,-1,9)$$

$$\vec{w}=(1,0,5,0)$$

Solution:

To simplify $\vec{w}(2\vec{x}+\vec{y})-3\vec{y}(\vec{w}+4\vec{x}-\vec{z})$, start by simplifying $\vec{w}(2\vec{x}+\vec{y})$ first. Find $2\vec{x}$.

$$\vec{x}=(4,-3,0,7)$$

$$2\vec{x}=2(4,-3,0,7)$$

$$2\vec{x}=(8,-6,0,14)$$

Then the sum $2\vec{x}+\vec{y}$ is

$$2\vec{x}+\vec{y}=\begin{bmatrix}8\\-6\\0\\14\end{bmatrix}+\begin{bmatrix}-1\\5\\2\\-1\end{bmatrix}$$

$$2\vec{x}+\vec{y}=\begin{bmatrix}8-1\\-6+5\\0+2\\14-1\end{bmatrix}$$

$$2\vec{x}+\vec{y}=\begin{bmatrix}7\\-1\\2\\13\end{bmatrix}$$

The dot product $\vec{w}(2\vec{x}+\vec{y})$ is

$$\begin{bmatrix}1 & 0 & 5 & 0\end{bmatrix}\begin{bmatrix}7\\-1\\2\\13\end{bmatrix}$$

$$(1)(7) + (0)(-1) + (5)(2) + (0)(13)$$

$$7 + 0 + 10 + 0$$

$$17$$

Then to simplify $3\vec{y}(\vec{w} + 4\vec{x} - \vec{z})$, start by finding $3\vec{y}$.

$$\vec{y} = (-1,5,2,-1)$$

$$3\vec{y} = 3(-1,5,2,-1)$$

$$3\vec{y} = (-3,15,6,-3)$$

Find $4\vec{x}$.

$$\vec{x} = (4,-3,0,7)$$

$$4\vec{x} = 4(4,-3,0,7)$$

$$4\vec{x} = (16,-12,0,28)$$

Then $\vec{w} + 4\vec{x} - \vec{z}$ is

$$\vec{w} + 4\vec{x} - \vec{z} = \begin{bmatrix} 1 \\ 0 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 16 \\ -12 \\ 0 \\ 28 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \\ -1 \\ 9 \end{bmatrix}$$

$$\vec{w} + 4\vec{x} - \vec{z} = \begin{bmatrix} 1 + 16 - 0 \\ 0 - 12 - 6 \\ 5 + 0 - (-1) \\ 0 + 28 - 9 \end{bmatrix}$$

$$\vec{w} + 4\vec{x} - \vec{z} = \begin{bmatrix} 17 \\ -18 \\ 6 \\ 19 \end{bmatrix}$$

Then the dot product $3\vec{y}(\vec{w} + 4\vec{x} - \vec{z})$ is

$$\begin{bmatrix} -3 & 15 & 6 & -3 \end{bmatrix} \begin{bmatrix} 17 \\ -18 \\ 6 \\ 19 \end{bmatrix}$$

$$(-3)(17) + (15)(-18) + 6(6) + (-3)(19)$$

$$-51 - 270 + 36 - 57$$

$$-342$$

Then $\vec{w}(2\vec{x} + \vec{y}) - 3\vec{y}(\vec{w} + 4\vec{x} - \vec{z})$ is

$$17 - (-342)$$

$$359$$

CAUCHY-SCHWARZ INEQUALITY

1. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (-1,2)$$

$$\vec{v} = (-5,10)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \cdot ||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-1)(-5) + (2)(10)| = \sqrt{(-1)^2 + 2^2}\sqrt{(-5)^2 + 10^2}$$

$$|5 + 20| = \sqrt{1 + 4}\sqrt{25 + 100}$$

$$|25| = \sqrt{5}\sqrt{125}$$

$$25 = \sqrt{625}$$

$$25 = 25$$

Since the two sides of the Cauchy-Schwarz inequality are equivalent, $\vec{u}=(-1,2)$ and $\vec{v}=(-5,10)$ are linearly dependent.

■ 2. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u}=(-5,2)$$

$$\vec{v}=(3,-7)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u}\cdot\vec{v}|=||\vec{u}||\cdot||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-5)(3)+(2)(-7)|=\sqrt{(-5)^2+2^2}\sqrt{(3)^2+(-7)^2}$$

$$|-15-14|=\sqrt{25+4}\sqrt{9+49}$$

$$|-29|=\sqrt{29}\sqrt{58}$$

$$29=\sqrt{1{,}682}$$

$$29\neq 41.01$$

Since the two sides of the Cauchy-Schwarz inequality are not equivalent, $\vec{u}=(-5,2)$ and $\vec{v}=(3,-7)$ are linearly independent.

■ 3. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u}=(-2,4,0)$$

$$\vec{v}=(1,-5,3)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u}\cdot\vec{v}|=||\vec{u}||\cdot||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-2)(1)+(4)(-5)+(0)(3)|=\sqrt{(-2)^2+4^2+0^2}\sqrt{1^2+(-5)^2+3^2}$$

$$|-2-20+0|=\sqrt{4+16+0}\sqrt{1+25+9}$$

$$|-22|=\sqrt{20}\sqrt{35}$$

$$22=\sqrt{700}$$

$$22\neq 26.46$$

Since the two sides of the Cauchy-Schwarz inequality are not equivalent, $\vec{u}=(-2,4,0)$ and $\vec{v}=(1,-5,3)$ are linearly independent.

4. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u}=(6,3,6)$$

$$\vec{v}=(-2,-1,-2)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u}\cdot\vec{v}|=||\vec{u}||\cdot||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(6)(-2)+(3)(-1)+(6)(-2)|=\sqrt{6^2+3^2+6^2}\sqrt{(-2)^2+(-1)^2+(-2)^2}$$

$$|-12-3-12|=\sqrt{36+9+36}\sqrt{4+1+4}$$

$$|-27|=\sqrt{81}\sqrt{9}$$

$$27=9\cdot 3$$

$$27=27$$

Since the two sides of the Cauchy-Schwarz inequality are equivalent, $\vec{u}=(6,3,6)$ and $\vec{v}=(-2,-1,-2)$ are linearly dependent.

5. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u}=(-13,5,7)$$

$$\vec{v}=(1,-1,-1)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u}\cdot\vec{v}|=||\vec{u}||\cdot||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-13)(1)+(5)(-1)+(7)(-1)|=\sqrt{(-13)^2+5^2+7^2}\sqrt{1^2+(-1)^2+(-1)^2}$$

$$|-13-5-7|=\sqrt{169+25+49}\sqrt{1+1+1}$$

$$|-25|=\sqrt{243}\sqrt{3}$$

$$25=\sqrt{729}$$

$$25\neq 27$$

Since the two sides of the Cauchy-Schwarz inequality are not equivalent, $\vec{u} = (-13,5,7)$ and $\vec{v} = (1,-1,-1)$ are linearly independent.

■ 6. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (-2,0,2)$$

$$\vec{v} = (8,0,-8)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \cdot ||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-2)(8) + (0)(0) + (2)(-8)| = \sqrt{(-2)^2 + 0^2 + 2^2}\sqrt{8^2 + 0^2 + (-8)^2}$$

$$|-16 + 0 - 16| = \sqrt{4 + 0 + 4}\sqrt{64 + 0 + 64}$$

$$|-32| = \sqrt{8}\sqrt{128}$$

$$32 = \sqrt{1,024}$$

$$32 = 32$$

Since the two sides of the Cauchy-Schwarz inequality are equivalent, $\vec{u} = (-2,0,2)$ and $\vec{v} = (8,0,-8)$ are linearly dependent.

VECTOR TRIANGLE INEQUALITY

1. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u}=(\sqrt{3},3) \text{ and } \vec{v}=(2\sqrt{3},0)$$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u}+\vec{v}||\le||\vec{u}||+||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2}\le\sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{(\sqrt{3}+2\sqrt{3})^2+(3+0)^2}\le\sqrt{(\sqrt{3})^2+3^2}+\sqrt{(2\sqrt{3})^2+0^2}$$

$$\sqrt{(3\sqrt{3})^2+3^2}\le\sqrt{3+9}+\sqrt{12+0}$$

$$\sqrt{27+9}\le\sqrt{12}+\sqrt{12}$$

$$\sqrt{36}\le2\sqrt{12}$$

$$6<6.93$$

Because the left side is less than the right side, the vector set is linearly independent.

■ 2. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ span $\mathbb{R}^2$.

$\vec{u} = (5,-7)$ and $\vec{v} = (-4,-3)$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2} + \sqrt{v_1^2+v_2^2}$$

$$\sqrt{(5-4)^2+(-7-3)^2} \le \sqrt{5^2+(-7)^2} + \sqrt{(-4)^2+(-3)^2}$$

$$\sqrt{1^2+(-10)^2} \le \sqrt{25+49} + \sqrt{16+9}$$

$$\sqrt{1+100} \le \sqrt{74} + \sqrt{25}$$

$$\sqrt{101} \le 8.60+5$$

$$10.05 < 13.60$$

Since $10.05 < 13.60$, we can conclude that $\vec{u}$ and $\vec{v}$ are not collinear, and that they therefore span $\mathbb{R}^2$.

■ 3. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u} = (-2,5) \text{ and } \vec{v} = (2,-5)$$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u} + \vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2} \le \sqrt{u_1^2 + u_2^2} + \sqrt{v_1^2 + v_2^2}$$

$$\sqrt{(-2 + 2)^2 + (5 - 5)^2} \le \sqrt{(-2)^2 + 5^2} + \sqrt{2^2 + (-5)^2}$$

$$\sqrt{0^2 + 0^2} \le \sqrt{4 + 25} + \sqrt{4 + 25}$$

$$0 \le \sqrt{29} + \sqrt{29}$$

$$0 \le 2\sqrt{29}$$

Since $||\vec{u}+\vec{v}||$ is 0, the vector set is linearly dependent.

■ 4. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u}=(-3,12,-15) \text{ and } \vec{v}=(-1,4,-5)$$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}||+||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2+(u_3+v_3)^2} \le \sqrt{u_1^2+u_2^2+u_3^2}+\sqrt{v_1^2+v_2^2+v_3^2}$$

$$\sqrt{(-3-1)^2+(12+4)^2+(-15-5)^2} \le \sqrt{(-3)^2+12^2+(-15)^2}+\sqrt{(-1)^2+4^2+(-5)^2}$$

$$\sqrt{(-4)^2+16^2+(-20)^2} \le \sqrt{9+144+225}+\sqrt{1+16+25}$$

$$\sqrt{16+256+400} \le \sqrt{378}+\sqrt{42}$$

$$\sqrt{672} \le 19.44+6.48$$

$$25.92=25.92$$

Because the sides are equal, the vector set is linearly dependent.

■ 5. Use the vector triangle inequality to say whether $\overrightarrow{u}$ and $\overrightarrow{v}$ are linearly independent.

$$\overrightarrow{u} = (1,2,0) \text{ and } \overrightarrow{v} = (-5,1,-6)$$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\overrightarrow{u} + \overrightarrow{v}|| \le ||\overrightarrow{u}|| + ||\overrightarrow{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2+(u_3+v_3)^2} \le \sqrt{u_1^2+u_2^2+u_3^2} + \sqrt{v_1^2+v_2^2+v_3^2}$$

$$\sqrt{(1-5)^2+(2+1)^2+(0-6)^2} \le \sqrt{1^2+2^2+0^2} + \sqrt{(-5)^2+1^2+(-6)^2}$$

$$\sqrt{(-4)^2+3^2+(-6)^2} \le \sqrt{1+4+0} + \sqrt{25+1+36}$$

$$\sqrt{16+9+36} \le \sqrt{5} + \sqrt{62}$$

$$\sqrt{61} \le 2.24 + 7.87$$

$$7.81 \le 10.11$$

Because the left side is less than the right side, the vector set is linearly independent.

6. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u}=(2,-5,4) \text{ and } \vec{v}=(6,-15,12)$$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u}+\vec{v}|| \leq ||\vec{u}||+||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2+(u_3+v_3)^2} \leq \sqrt{u_1^2+u_2^2+u_3^2}+\sqrt{v_1^2+v_2^2+v_3^2}$$

$$\sqrt{(2+6)^2+(-5-15)^2+(4+12)^2} \leq \sqrt{2^2+(-5)^2+4^2}+\sqrt{6^2+(-15)^2+12^2}$$

$$\sqrt{8^2+(-20)^2+16^2} \leq \sqrt{4+25+16}+\sqrt{36+225+144}$$

$$\sqrt{64+400+256} \leq \sqrt{45}+\sqrt{405}$$

$$\sqrt{720} \leq 6.71+20.12$$

$$26.83 = 26.83$$

Because the sides are equal, the vector set is linearly dependent.

ANGLE BETWEEN VECTORS

■ 1. Say whether or not the vectors are orthogonal.

$$\vec{a} = (-1,3)$$

$$\vec{b} = (6,2)$$

Solution:

Test the orthogonality of the vectors $\vec{a} = (-1,3)$ and $\vec{b} = (6,2)$ by calculating their dot product.

$$\vec{a} \cdot \vec{b} = (-1)(6) + (3)(2)$$

$$\vec{a} \cdot \vec{b} = -6 + 6$$

$$\vec{a} \cdot \vec{b} = 0$$

Because the dot product is 0, $\vec{a}$ and $\vec{b}$ are orthogonal to one another.

■ 2. Say whether or not the vectors are orthogonal.

$$\vec{u} = 2i - j + 3k$$

$$\vec{v} = -i - 3j + 2k$$

Solution:

Test the orthogonality of the vectors $\vec{u} = 2i - j + 3k$ and $\vec{v} = -i - 3j + 2k$ by calculating their dot product.

$$\vec{u} \cdot \vec{v} = (2)(-1) + (-1)(-3) + (3)(2)$$

$$\vec{u} \cdot \vec{v} = -2 + 3 + 6$$

$$\vec{u} \cdot \vec{v} = 7$$

Since the dot product is not 0, the vectors are not orthogonal to one another.

3. Find the angle between the vectors.

$$\vec{x} = (0,2)$$

$$\vec{y} = (1,1)$$

Solution:

Find the lengths of both vectors.

$$||\vec{x}|| = \sqrt{x_1^2 + x_2^2} = \sqrt{0 + 2^2} = \sqrt{0 + 4} = \sqrt{4} = 2$$

$$||\vec{y}|| = \sqrt{y_1^2 + y_2^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Then find the dot product of the vectors.

$$\vec{x} \cdot \vec{y} = (0)(1) + (2)(1)$$

$$\vec{x} \cdot \vec{y} = 0 + 2$$

$$\vec{x} \cdot \vec{y} = 2$$

Plug everything into the formula for the angle between the vectors.

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \cdot ||\vec{y}|| \cos\theta$$

$$2 = 2\sqrt{2}\cos\theta$$

$$\frac{2}{2\sqrt{2}} = \cos\theta$$

$$\frac{1}{\sqrt{2}} = \cos\theta$$

Take the inverse cosine of each side to solve for θ.

$$\theta = \arccos\left(\frac{1}{\sqrt{2}}\right)$$

Use a calculator to find $\theta = 45^\circ$.

■ 4. Find the angle between the vectors.

$$\vec{a} = (-5,7,3)$$

$$\vec{b} = (1,2,-3)$$

Solution:

Find the lengths of both vectors.

$$||\overrightarrow{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{(-5)^2 + 7^2 + 3^2} = \sqrt{25 + 49 + 9} = \sqrt{83}$$

$$||\overrightarrow{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

Then find the dot product of the vectors.

$$\overrightarrow{a} \cdot \overrightarrow{b} = (-5)(1) + 7(2) + (3)(-3)$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = -5 + 14 - 9$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = 0$$

Plug everything into the formula for the angle between vectors.

$$\overrightarrow{a} \cdot \overrightarrow{b} = ||\overrightarrow{a}|| \cdot ||\overrightarrow{b}|| \cos\theta$$

$$0 = \sqrt{14}\sqrt{83}\cos\theta$$

$$\frac{0}{\sqrt{1{,}162}} = \cos\theta$$

$$0 = \cos\theta$$

Since the dot product is 0, the vectors are orthogonal, which means $\theta = 90^\circ$.

■ 5. Find the angle between the vectors.

$$\vec{a} = (-1, 3, -4)$$

$$\vec{b} = (2, 1, 0)$$

Solution:

Find the lengths of both vectors.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{(-1)^2 + 3^2 + (-4)^2} = \sqrt{1 + 9 + 16} = \sqrt{26}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{4 + 1 + 0} = \sqrt{5}$$

Then find the dot product of the vectors.

$$\vec{a} \cdot \vec{b} = (-1)(2) + (3)(1) + (-4)(0)$$

$$\vec{a} \cdot \vec{b} = -2 + 3 + 0$$

$$\vec{a} \cdot \vec{b} = 1$$

Plug everything into the formula for the angle between vectors.

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cos\theta$$

$$1 = \sqrt{26}\sqrt{5}\cos\theta$$

$$1 = \sqrt{130}\cos\theta$$

$$\frac{1}{\sqrt{130}} = \cos\theta$$

Take the inverse cosine of each side to solve for θ.

$$\theta = \arccos\left(\frac{1}{\sqrt{130}}\right)$$

Use a calculator to find $\theta \approx 84.97^\circ$.

■ 6. Find the angle between the vectors.

$$\vec{a} = (1, -2, 5)$$

$$\vec{b} = (8, 6, 3)$$

Solution:

Find the lengths of both vectors.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + (-2)^2 + 5^2} = \sqrt{1 + 4 + 25} = \sqrt{30}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{8^2 + 6^2 + 3^2} = \sqrt{64 + 36 + 9} = \sqrt{109}$$

Then find the dot product of the vectors.

$$\vec{a} \cdot \vec{b} = (1)(8) + (-2)(6) + (5)(3)$$

$$\vec{a} \cdot \vec{b} = 8 - 12 + 15$$

$$\vec{a} \cdot \vec{b} = 11$$

Plug everything into the formula for the angle between vectors.

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cos\theta$$

$$11 = \sqrt{30}\sqrt{109}\cos\theta$$

$$11 = \sqrt{3{,}270}\cos\theta$$

$$\frac{11}{\sqrt{3{,}270}} = \cos\theta$$

Take the inverse cosine of each side to solve for θ.

$$\theta = \arccos\left(\frac{11}{\sqrt{3{,}270}}\right)$$

Use a calculator to find $\theta \approx 78.91^\circ$.

EQUATION OF A PLANE, AND NORMAL VECTORS

1. What is the normal vector to the plane?

$$-2x + 5y - 7z = 0$$

Solution:

Given a plane $Ax + By + Cz = D$, the normal vector to the plane is

$$\overrightarrow{n} = (A, B, C)$$

So from the plane $-2x + 5y - 7z = 0$, pull out the coefficients on x, y, and z to get the components of the normal vector.

$$\overrightarrow{n} = (-2,5, -7)$$

2. What is the normal vector to the plane?

$$10y - 5z + 6 = 0$$

Solution:

Given a plane $Ax + By + Cz = D$, the normal vector to the plane is

$$\overrightarrow{n} = (A, B, C)$$

Rewrite the plane in standard form.

$$10y - 5z + 6 = 0$$

$$0x + 10y - 5z = -6$$

So from the plane $10y - 5z = -6$, we can simply pull out the coefficients on x, y, and z to get the components of the normal vector.

$$\vec{n} = (0,10,-5)$$

■ 3. Find the equation of a plane with normal vector $\vec{n} = (-1,0,4)$ that passes through $(1,-3,0)$.

Solution:

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-(x - 1) + 0(y - (-3)) + 4(z - 0) = 0$$

Now we'll simplify and get the equation of the plane into standard form.

$$-(x - 1) + 0(y + 3) + 4z = 0$$

$$-x + 1 + 4z = 0$$

$$-x + 4z = -1$$

4. Find the equation of a plane with normal vector $\overrightarrow{n} = (4, -7, 3)$ that passes through $(-2, 1, 6)$.

Solution:

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$4(x - (-2)) - 7(y - 1) + 3(z - 6) = 0$$

Now we'll simplify and get the equation of the plane into standard form.

$$4(x + 2) - 7(y - 1) + 3(z - 6) = 0$$

$$4x + 8 - 7y + 7 + 3z - 18 = 0$$

$$4x - 7y + 3z - 3 = 0$$

$$4x - 7y + 3z = 3$$

5. Find the equation of a plane with normal vector $\overrightarrow{n} = -3i + 4j - z$ that passes through $(-2, 0, -7)$.

Solution:

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$-3(x-(-2))+4(y-0)-(z-(-7))=0$$

Now we'll simplify and get the equation of the plane into standard form.

$$-3(x+2)+4y-(z+7)=0$$

$$-3x-6+4y-z-7=0$$

$$-3x+4y-z-13=0$$

$$-3x+4y-z=13$$

■ 6. Find the equation of the plane passing through P and perpendicular to $\overrightarrow{PQ}$.

$$P(1,-5,4)$$

$$Q(0,3,-1)$$

Solution:

First find the normal vector to the plane.

$$\overrightarrow{PQ}=(0-1,3-(-5),-1-4)$$

$$\overrightarrow{PQ} = (-1,8,-5)$$

Plugging this normal vector and the point on the plane into the plane equation gives

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$-(x-1)+8(y-(-5))-5(z-4)=0$$

Now we'll simplify and get the equation of the plane into standard form.

$$-(x-1)+8(y+5)-5(z-4)=0$$

$$-x+1+8y+40-5z+20=0$$

$$-x+8y-5z+61=0$$

$$-x+8y-5z=-61$$

CROSS PRODUCTS

■ 1. Find the cross product of $\overrightarrow{a}=(1,-3,-1)$ and $\overrightarrow{b}=(5,6,-2)$.

Solution:

The cross product of $\overrightarrow{a}=(1,-3,-1)$ and $\overrightarrow{b}=(5,6,-2)$ is given by

$$\overrightarrow{a}\times\overrightarrow{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -3 & -1\\ 5 & 6 & -2\end{vmatrix}$$

$$\overrightarrow{a}\times\overrightarrow{b}=\mathbf{i}\begin{vmatrix}-3 & -1\\ 6 & -2\end{vmatrix}-\mathbf{j}\begin{vmatrix}1 & -1\\ 5 & -2\end{vmatrix}+\mathbf{k}\begin{vmatrix}1 & -3\\ 5 & 6\end{vmatrix}$$

Evaluating the determinants gives

$$\overrightarrow{a}\times\overrightarrow{b}=\mathbf{i}((-3)(-2)-(-1)(6))-\mathbf{j}((1)(-2)-(-1)(5))+\mathbf{k}((1)(6)-(-3)(5))$$

$$\overrightarrow{a}\times\overrightarrow{b}=\mathbf{i}(6+6)-\mathbf{j}(-2+5)+\mathbf{k}(6+15)$$

$$\overrightarrow{a}\times\overrightarrow{b}=12\mathbf{i}-3\mathbf{j}+21\mathbf{k}$$

$$\overrightarrow{a}\times\overrightarrow{b}=(12,-3,21)$$

■ 2. Find a vector orthogonal to both $\overrightarrow{a}=(-3,-5,2)$ and $\overrightarrow{b}=(-2,4,-7)$.

Solution:

We need to find the cross product of the two vectors.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -5 & 2 \\ -2 & 4 & -7 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} -5 & 2 \\ 4 & -7 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -3 & 2 \\ -2 & -7 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -3 & -5 \\ -2 & 4 \end{vmatrix}$$

Evaluating the determinants gives

$$\vec{a} \times \vec{b} = \mathbf{i}((-5)(-7) - (2)(4)) - \mathbf{j}((-3)(-7) - (2)(-2)) + \mathbf{k}((-3)(4) - (-5)(-2))$$

$$\vec{a} \times \vec{b} = \mathbf{i}(35 - 8) - \mathbf{j}(21 + 4) + \mathbf{k}(-12 - 10)$$

$$\vec{a} \times \vec{b} = 27\mathbf{i} - 25\mathbf{j} - 22\mathbf{k}$$

So the vector $\vec{a} \times \vec{b} = (27, -25, -22)$ is orthogonal to both $\vec{a} = (-3, -5, 2)$ and $\vec{b} = (-2, 4, -7)$.

■ 3. Find the length of the cross product of $\vec{a} = (-1, -2, 0)$ and $\vec{b} = (1, 1, -2)$.

Solution:

First, find the cross product of the two vectors.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 0 \\ 1 & 1 & -2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix}$$

Evaluating the determinants gives

$$\vec{a} \times \vec{b} = \mathbf{i}((-2)(-2) - (0)(1)) - \mathbf{j}((-1)(-2) - (0)(1)) + \mathbf{k}((-1)(1) - (-2)(1))$$

$$\vec{a} \times \vec{b} = \mathbf{i}(4 - 0) - \mathbf{j}(2 - 0) + \mathbf{k}(-1 + 2)$$

$$\vec{a} \times \vec{b} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\vec{a} \times \vec{b} = (4, -2, 1)$$

Then the length of the cross product is

$$||\vec{a} \times \vec{b}|| = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{16 + 4 + 1} = \sqrt{21}$$

■ 4. Find the length of the cross product of $\vec{a} = (6, -3, 3)$ and $\vec{b} = (3, 0, 3)$ when the angle between $\vec{a}$ and $\vec{b}$ is $\theta = 30^\circ$.

Solution:

First, find the length of each vector individually.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{6^2 + (-3)^2 + 3^2} = \sqrt{36 + 9 + 9} = \sqrt{54} = 3\sqrt{6}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{3^2 + 0^2 + 3^2} = \sqrt{9 + 0 + 9} = \sqrt{18} = 3\sqrt{2}$$

Then the length of the cross product is given by

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||\ ||\vec{b}|| \sin\theta$$

$$||\vec{a} \times \vec{b}|| = 3\sqrt{6} \cdot 3\sqrt{2} \cdot \sin(30^\circ)$$

$$||\vec{a} \times \vec{b}|| = 9\sqrt{12} \cdot \frac{1}{2}$$

$$||\vec{a} \times \vec{b}|| = 9 \cdot 2\sqrt{3} \cdot \frac{1}{2}$$

$$||\vec{a} \times \vec{b}|| = 9\sqrt{3}$$

■ 5. Find the length of the cross product of the vectors $\vec{a} = (2, -5, 3)$ and $\vec{b} = (4, 6, -1)$, and find the sine of the angle between them.

Solution:

First, find the cross product of the two vectors.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 3 \\ 4 & 6 & -1 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} -5 & 3 \\ 6 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -5 \\ 4 & 6 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i}((-5)(-1) - (3)(6)) - \mathbf{j}((2)(-1) - (3)(4)) + \mathbf{k}((2)(6) - (-5)(4))$$

$$\vec{a} \times \vec{b} = \mathbf{i}(5 - 18) - \mathbf{j}(-2 - 12) + \mathbf{k}(12 + 20)$$

$$\vec{a} \times \vec{b} = -13\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}$$

Then the length of the cross product is

$$||\vec{a} \times \vec{b}|| = \sqrt{(-13)^2 + 14^2 + 32^2} = \sqrt{169 + 196 + 1{,}024} = \sqrt{1{,}389}$$

Find the length of each vector individually.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{2^2 + (-5)^2 + 3^2} = \sqrt{4 + 25 + 9} = \sqrt{38}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{4^2 + 6^2 + (-1)^2} = \sqrt{16 + 36 + 1} = \sqrt{53}$$

The sine of the angle between the vectors will be

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||\, ||\vec{b}|| \sin\theta$$

$$\sqrt{1{,}389} = \sqrt{38}\sqrt{53}\sin\theta$$

$$\sin\theta = \frac{\sqrt{1{,}389}}{\sqrt{38}\sqrt{53}}$$

$$\sin\theta \approx 0.8305$$

■ 6. Find the angle between the vectors $\vec{a} = (2, -2, 1)$ and $\vec{b} = (1, 0, 1)$, and find the length of their cross product.

Solution:

First, find the length of each vector individually.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{1 + 0 + 1} = \sqrt{2}$$

The angle between the vectors will be given by

$$\vec{a} \cdot \vec{b} = ||\vec{a}||\ ||\vec{b}|| \cos\theta$$

$$\begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 3\sqrt{2}\cos\theta$$

$$2(1) - 2(0) + 1(1) = 3\sqrt{2}\cos\theta$$

$$2 + 0 + 1 = 3\sqrt{2}\cos\theta$$

$$\frac{3}{3\sqrt{2}} = \cos\theta$$

So the angle is

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta = 45^\circ$$

Then the length of the cross product is given by

$$||\overrightarrow{a} \times \overrightarrow{b}|| = ||\overrightarrow{a}|| \ ||\overrightarrow{b}|| \sin\theta$$

$$||\overrightarrow{a} \times \overrightarrow{b}|| = 3\sqrt{2}\sin 45^\circ$$

$$||\overrightarrow{a} \times \overrightarrow{b}|| = 3\sqrt{2} \cdot \frac{1}{\sqrt{2}}$$

$$||\overrightarrow{a} \times \overrightarrow{b}|| = 3$$

DOT AND CROSS PRODUCTS AS OPPOSITE IDEAS

■ 1. Find the maximum value of the dot product, if $||\vec{u}||=4$ and $||\vec{v}||=5$.

Solution:

The dot product will be maximized when the vectors point in the same direction, where the angle between them is 0°.

$$\vec{u}\cdot\vec{v}=||\vec{u}||\ ||\vec{v}||\cos 0^\circ$$

$$\vec{u}\cdot\vec{v}=||\vec{u}||\ ||\vec{v}||(1)$$

$$\vec{u}\cdot\vec{v}=||\vec{u}||\ ||\vec{v}||$$

$$\vec{u}\cdot\vec{v}=4\cdot 5$$

$$\vec{u}\cdot\vec{v}=20$$

■ 2. Find the minimum value of the dot product of two vectors, if $||\vec{u}||=\sqrt{56}$ and $||\vec{v}||=\sqrt{126}$.

Solution:

The dot product will be minimized when the vectors point in the opposite directions, where the angle between them is 180°.

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}|| \cos 180^\circ$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}||(-1)$$

$$\vec{u} \cdot \vec{v} = -||\vec{u}|| \ ||\vec{v}||$$

$$\vec{u} \cdot \vec{v} = -\sqrt{56}\sqrt{126}$$

$$\vec{u} \cdot \vec{v} = -84$$

■ 3. Find the maximum value of the length of the cross product of $\vec{u}$ and $\vec{v}$, if $||\vec{u}|| = \sqrt{50}$ and $||\vec{v}|| = \sqrt{128}$.

Solution:

The length of the cross product will be maximized when the vectors are orthogonal, so the angle between them is 90°.

$$||\vec{u} \times \vec{v}|| = ||\vec{u}|| \ ||\vec{v}|| \sin 90^\circ$$

$$||\vec{u} \times \vec{v}|| = ||\vec{u}|| \ ||\vec{v}||(1)$$

$$||\vec{u} \times \vec{v}|| = ||\vec{u}|| \ ||\vec{v}||$$

$$||\vec{u} \times \vec{v}|| = \sqrt{50}\sqrt{128}$$

$$||\vec{u} \times \vec{v}|| = 80$$

■ 4. Find the dot product and the length of the cross product of $\vec{u}=(2,1)$ and $\vec{v}=(-6,-3)$. Then interpret the results based on what the dot and cross products indicate.

Solution:

The vector $\vec{v}=(-6,-3)$ is $-3\vec{u}$, which means the vectors $\vec{u}=(2,1)$ and $\vec{v}=(-6,-3)$ point in opposite directions, which means the angle between them is $\theta=180^\circ$.

First, find the length of each vector individually.

$$||\vec{u}||=\sqrt{2^2+1^2}=\sqrt{4+1}=\sqrt{5}$$

$$||\vec{v}||=\sqrt{(-6)^2+(-3)^2}=\sqrt{36+9}=\sqrt{45}=3\sqrt{5}$$

So the dot product is

$$\vec{u}\cdot\vec{v}=||\vec{u}||\ ||\vec{v}||\cos 180^\circ$$

$$\vec{u}\cdot\vec{v}=(\sqrt{5})(3\sqrt{5})(-1)$$

$$\vec{u}\cdot\vec{v}=-15$$

And the cross product is

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin 180^\circ$$

$$||\vec{u}\times\vec{v}||=(\sqrt{5})(3\sqrt{5})(0)$$

$$||\vec{u}\times\vec{v}||=0$$

Because the length of the cross product is 0, we know that the vectors are collinear. For collinear vectors, the dot product is just the product of the lengths of the vectors, which we see in $\vec{u} \cdot \vec{v} = -15$. The fact that the dot product is negative tells us that the vectors point in exactly opposite directions along the same line, where the dot product will have its minimum value.

■ 5. Find the dot product and the length of the cross product of $\vec{u} = (2, -3, -1)$ and $\vec{v} = (4, -6, -2)$. Then interpret the results based on what the dot and cross products indicate.

Solution:

First, find the length of each vector individually.

$$||\vec{u}|| = \sqrt{2^2 + (-3)^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$||\vec{v}|| = \sqrt{4^2 + (-6)^2 + (-2)^2} = \sqrt{16 + 36 + 4} = \sqrt{56} = 2\sqrt{14}$$

So the dot product is

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 2 & -3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = 2(4) - 3(-6) - 1(-2)$$

$$\vec{u} \cdot \vec{v} = 8 + 18 + 2$$

$$\vec{u} \cdot \vec{v} = 28$$

And the cross product is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 4 & -6 & -2 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \mathbf{i} \begin{vmatrix} -3 & -1 \\ -6 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -3 \\ 4 & -6 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \mathbf{i}((-3)(-2) - (-1)(-6)) - \mathbf{j}((2)(-2) - (-1)(4)) + \mathbf{k}((2)(-6) - (-3)(4))$$

$$\vec{u} \times \vec{v} = \mathbf{i}(6 - 6) - \mathbf{j}(-4 + 4) + \mathbf{k}(-12 + 12)$$

$$\vec{u} \times \vec{v} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(0)$$

$$\vec{u} \times \vec{v} = (0,0,0)$$

So the length of the cross product is

$$||\vec{u} \times \vec{v}|| = \sqrt{0^2 + 0^2 + 0^2} = 0$$

Because the length of the cross product is 0, we know that the vectors are collinear. For collinear vectors, the dot product is just the product of the lengths of the vectors, which we see in $\vec{u} \cdot \vec{v} = 28$. The fact that the dot product is positive tells us that the vectors point in the same direction, where the dot product will have its maximum value.

6. Find the dot product and the length of the cross product of $\vec{u}=(-2,4,3)$ and $\vec{v}=(2,1,0)$. Then interpret the results based on what the dot and cross products indicate.

Solution:

First, find the length of each vector individually.

$$||\vec{u}||=\sqrt{(-2)^2+4^2+3^2}=\sqrt{4+16+9}=\sqrt{29}$$

$$||\vec{v}||=\sqrt{2^2+1^2+0^2}=\sqrt{4+1+0}=\sqrt{5}$$

So the dot product is

$$\vec{u}\cdot\vec{v}=[-2\quad 4\quad 3]\cdot\begin{bmatrix}2\\1\\0\end{bmatrix}$$

$$\vec{u}\cdot\vec{v}=-2(2)+4(1)+3(0)$$

$$\vec{u}\cdot\vec{v}=-4+4+0$$

$$\vec{u}\cdot\vec{v}=0$$

The dot product is 0 when the vectors are orthogonal, so the angle between them is 90°.

The length of the cross product is

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin 90^\circ$$

$$||\overrightarrow{u}\times\overrightarrow{v}||=||\overrightarrow{u}||\ ||\overrightarrow{v}||(1)$$

$$||\overrightarrow{u}\times\overrightarrow{v}||=\sqrt{29}\sqrt{5}$$

$$||\overrightarrow{u}\times\overrightarrow{v}||=\sqrt{145}$$

Since the vectors are orthogonal, the length of the cross product is maximized.

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