Functions and transformations

If you've taken Algebra or Calculus, you're familiar with the idea of a **function**, which is a rule that maps one value to another.

For instance, the function $f(x) = x + 1$ maps x to $x + 1$. It tells us that, if we put any value x into the function f, the function will give back $x + 1$. In other words, the function will always return an output value that's related to the input value we gave it.

We can also write the function $f(x) = x + 1$ as $f: \ x \mapsto x + 1$, where the arrow with the line on the back literally means "maps to," telling us that f will map every x to $x + 1$.

Functions vs. transformations

We can also use functions to map vectors. For instance, the function

$$f\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = \begin{bmatrix}3v_1 - v_2\\-2v_2\end{bmatrix}$$

tells us that, for every vector $\vec{v} = (v_1, v_2)$ that we put into f, the function will give us back a new vector, $\vec{v} = (3v_1 - v_2, -2v_2)$. When a function maps vectors, we call it a **vector-valued function**.

While we usually use functions to map coordinate points, if we're going to map vectors from one space to another, we usually switch over from the language of "functions," to "**transformations**."

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In other words, even though functions and transformations perform the same kind of mapping operation, if we want to map vectors, we should really say that the mapping is done by a transformation instead of by a function. So instead of writing

$$f\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}3v_1-v_2\\-2v_2\end{bmatrix}$$

to express the transformation of a vector $\vec{v}=(v_1,v_2)$, it's more appropriate to write

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}3v_1-v_2\\-2v_2\end{bmatrix}$$

In the same way that it's most common to use f to indicate a function, it's most common to use T to represent a transformation.

Domain, codomain, and range

Where before we used the notation $f:\ x\mapsto x+1$ to describe the mapping done by the function, we can use a regular arrow like $T:\ A\to B$ to indicate that the transformation T is mapping vectors from the set (or space) A onto vectors in the set (or space) B.

We also want to always consider the space of what we're mapping from and what we're mapping to. For instance, with $T:\ A\to B$, let's say we're mapping from real numbers to real numbers, where both vector sets A and B are defined by real numbers. We could write

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$$T: \mathbb{R} \to \mathbb{R}$$

More specifically, if T is mapping from the two-dimensional real plane to the two-dimensional real plane, we could write

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

Keep in mind that, in Linear Algebra, we'll sometimes be mapping "across dimensions," for instance, from two dimensions to three dimensions, or vice versa.

In any transformation, the **domain** is what we're mapping from, and the **codomain** is what we're mapping to. So if $T: \mathbb{R}^2 \to \mathbb{R}^3$, then the domain would be the two-dimensional plane $\mathbb{R}^2$, and the codomain would be three-dimensional space $\mathbb{R}^3$. On the other hand, if $T: \mathbb{R}^3 \to \mathbb{R}^2$, then the domain would be $\mathbb{R}^3$ and the codomain would be $\mathbb{R}^2$.

The **range** is within the codomain. It's the specific set of points that the mapping actually maps to inside the codomain. In other words, T might be mapping us into $\mathbb{R}^3$ in general, but T might not be mapping to every single point in $\mathbb{R}^3$. Whatever set of vectors in $\mathbb{R}^3$ are actually getting mapped to will make up the range of the T.

Example

The transformation T maps every vector in $\mathbb{R}^4$ to the zero vector $\vec{v} = (0,0)$ in $\mathbb{R}^2$. What are the domain, codomain, and range of T?

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Because T is mapping vectors in $\mathbb{R}^4$ to vectors in $\mathbb{R}^2$, we can express T as $T: \ \mathbb{R}^4 \to \mathbb{R}^2$, and say that the domain of the transformation is $\mathbb{R}^4$ and its codomain is $\mathbb{R}^2$.

If every vector in $\mathbb{R}^2$ was being mapped to by T, we would say that the range of T is $\mathbb{R}^2$. But the transformation is mapping every vector in $\mathbb{R}^4$ to only the zero vector $\vec{v} = (0,0)$ in $\mathbb{R}^2$. Therefore, the range of T is just the zero vector, $\vec{v} = (0,0)$.

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