Projection onto an orthonormal basis

Finding the projection of a vector onto a subspace is also much easier when the subspace is defined by an orthonormal basis.

Remember previously that we defined the projection of a vector $\vec{x}$ onto a subspace V as

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

where A is the matrix made from the column vectors that define V. But if we can define the subspace V with an orthonormal basis, then the projection can be defined as just

$$\text{Proj}_V\vec{x} = AA^T\vec{x}$$

The reason is because the value $(A^TA)^{-1}$ from the middle of $\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$ actually simplifies to the identity matrix when A is a matrix made of orthonormal column vectors. So the projection formula collapses as

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_V\vec{x} = AIA^T\vec{x}$$

$$\text{Proj}_V\vec{x} = AA^T\vec{x}$$

So you can see how defining the subspace with an orthonormal vector set would make calculating the projection of a vector onto the subspace much easier.

Example

Find the projection of $\vec{x}=(5,6,-1)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{bmatrix},\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{2}}{2}\\ \frac{1}{2}\end{bmatrix}\right)$$

We already confirmed in the example in the previous section that $V=\{\vec{v}_1,\vec{v}_2\}$ was an orthonormal vector set. So the projection of $\vec{x}=(5,6,-1)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{2}\\ 0 & \frac{\sqrt{2}}{2}\\ -\frac{1}{\sqrt{2}} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}5\\ 6\\ -1\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{1}{2}\right) & \frac{1}{\sqrt{2}}(0)+\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) & \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{1}{2}\right)\\ 0\left(\frac{1}{\sqrt{2}}\right)+\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) & 0(0)+\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right) & 0\left(-\frac{1}{\sqrt{2}}\right)+\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ -\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{1}{2}\right) & -\frac{1}{\sqrt{2}}(0)+\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) & -\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{1}{2}\right)\end{bmatrix}\begin{bmatrix}5\\ 6\\ -1\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{2}+\frac{1}{4} & 0+\frac{\sqrt{2}}{4} & -\frac{1}{2}+\frac{1}{4}\\ 0+\frac{\sqrt{2}}{4} & 0+\frac{1}{2} & 0+\frac{\sqrt{2}}{4}\\ -\frac{1}{2}+\frac{1}{4} & 0+\frac{\sqrt{2}}{4} & \frac{1}{2}+\frac{1}{4}\end{bmatrix}\begin{bmatrix}5\\ 6\\ -1\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{3}{4} & \frac{\sqrt{2}}{4} & -\frac{1}{4}\\ \frac{\sqrt{2}}{4} & \frac{1}{2} & \frac{\sqrt{2}}{4}\\ -\frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{3}{4}\end{bmatrix}\begin{bmatrix}5\\ 6\\ -1\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{3}{4}(5)+\frac{\sqrt{2}}{4}(6)-\frac{1}{4}(-1)\\ \frac{\sqrt{2}}{4}(5)+\frac{1}{2}(6)+\frac{\sqrt{2}}{4}(-1)\\ -\frac{1}{4}(5)+\frac{\sqrt{2}}{4}(6)+\frac{3}{4}(-1)\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{15}{4}+\frac{3\sqrt{2}}{2}+\frac{1}{4}\\ \frac{5\sqrt{2}}{4}+3-\frac{\sqrt{2}}{4}\\ -\frac{5}{4}+\frac{3\sqrt{2}}{2}-\frac{3}{4}\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}4+\frac{3\sqrt{2}}{2}\\ 3+\sqrt{2}\\ -2+\frac{3\sqrt{2}}{2}\end{bmatrix}$$

So the vector in the subspace V which is the shadow of $\vec{x} = (5,6,-1)$ on V is

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 4 + \frac{3\sqrt{2}}{2} \\ 3 + \sqrt{2} \\ -2 + \frac{3\sqrt{2}}{2} \end{bmatrix}$$

In this last example, notice how we took one vector away from the set $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and only used the first two vectors, $V = \{\vec{v}_1, \vec{v}_2\}$. That was intentional.

The orthonormal set $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\mathbb{R}^3$. Since $\vec{x} = (5,6,-1)$ is a vector in $\mathbb{R}^3$, finding the projection of $\vec{x} = (5,6,-1)$ onto the subspace $\mathbb{R}^3$ would have simply given us $\vec{x} = (5,6,-1)$ again. In other words, AA^T would give the identity matrix I_3, so

$$\text{Proj}_V\vec{x} = AA^T\vec{x}$$

would become

$$\text{Proj}_V\vec{x} = I\vec{x}$$

$$\text{Proj}_V\vec{x} = \vec{x}$$

Which shows us that the projection of $\vec{x}$ onto V is just $\vec{x}$ itself. But because we instead changed the subspace to be represented by just $V = \{\vec{v}_1, \vec{v}_2\}$, that means the subspace is now a plane floating in $\mathbb{R}^3$. And that allows us to see the shadow of $\vec{x}=(5,6,-1)$ on the plane, thereby finding the projection of $\vec{x}$ onto V as

$$\text{Proj}_V\vec{x}=\begin{bmatrix}4+\frac{3\sqrt{2}}{2}\\ 3+\sqrt{2}\\ -2+\frac{3\sqrt{2}}{2}\end{bmatrix}$$