Null and column spaces of the transpose

Earlier we learned how to find the null and column spaces of a matrix. The null space of a matrix A was the set of vectors $\vec{x}$ that satisfied $A\vec{x} = \vec{O}$.

$$N(A) = \{\vec{x} \mid A\vec{x} = \vec{O}\}$$

The column space of A, $C(A)$, was the span of all the column vectors in A.

$$C(A) = \text{Span}(\vec{a}_1, \ \vec{a}_2, \ \vec{a}_3, \ \dots \ \vec{a}_n)$$

In this lesson, we want to talk about the null and column spaces of A^T. In other words, if we transpose A to get A^T, what are the null and column spaces of A^T, and how do they relate to the null and column spaces of A?

Spaces of the transpose

If we start with an $m \times n$ matrix A, and if the rank of the matrix is r, then we already know its null and column spaces:

Subspace	**Symbol**	**Space**	**Dimension**
Column space of A	$C(A)$	$\mathbb{R}^m$	$\text{Dim}(C(A)) = r$
Null space of A	$N(A)$	$\mathbb{R}^n$	$\text{Dim}(N(A)) = n - r$

If we then transpose A, that means we're swapping all the rows and columns of A to get A^T. Specifically, we could say that all the rows of A become the columns of A^T. Therefore, we could describe the column space of A^T as both the columns of A^T, but also as the rows of A, since those spaces are equivalent. For that reason, we call the column space of A^T the **row space**.

Like the column space, the **row space** of any $m \times n$ matrix A is all the linear combinations of the rows of A. Of course, the span of the rows of A is equivalent to the span of the columns of A^T, which again, is why we refer to the column space of A^T as the row space of A.

Subspace	Symbol	Space	Dimension
Column space of A	$C(A)$	$\mathbb{R}^m$	Dim$(C(A)) = r$
Null space of A	$N(A)$	$\mathbb{R}^n$	Dim$(N(A)) = n - r$
Row space of A (column space of A^T)	$C(A^T)$	$\mathbb{R}^n$	Dim$(C(A^T)) = r$

The null space of the transpose A^T is the set of vectors $\vec{x}$ that satisfy $A^T\vec{x} = \vec{O}$. If we take the transpose of both sides of this null space equation, we get

$$(A^T\vec{x})^T = (\vec{O})^T$$

We learned earlier that the transpose of a product is the product of the individual transposes, but in the reverse order. So on the left side of this transpose equation, we apply the transpose to both A and $\vec{x}$, but we flip the order of the multiplication, and the equation simplifies to

$$\vec{x}^T(A^T)^T = \vec{O}^T$$

$$\vec{x}^T A = \vec{O}^T$$

So we can say that the null space of the transpose A^T is the set of vectors $\vec{x}^T$ that satisfy $\vec{x}^T A = \vec{O}^T$. We call the null space of the transpose the **left null space**, simply because $\vec{x}^T$ is to the left of A in the equation $\vec{x}^T A = \vec{O}^T$.

Subspace	**Symbol**	**Space**	**Dimension**
Column space of A	$C(A)$	$\mathbb{R}^m$	Dim$(C(A)) = r$
Null space of A	$N(A)$	$\mathbb{R}^n$	Dim$(N(A)) = n - r$
Row space of A (column space of A^T)	$C(A^T)$	$\mathbb{R}^n$	Dim$(C(A^T)) = r$
Left null space of A (null space of A^T)	$N(A^T)$	$\mathbb{R}^m$	Dim$(N(A^T)) = m - r$

Space and dimension

We've been building this table of subspaces, and along the way we've been including the space that contains the subspace, as well as the dimension of each space, where the dimension is always in terms of the number of rows and column in the original matrix, m and n, and the rank of the matrix, r. Now we want to explain those a little further.

Remember first that the null and column spaces are subspaces, which sit inside some vector space $\mathbb{R}^i$, for some positive integer i. If A is an $m \times n$ matrix (it has m rows and n columns), then

- The column space of A and null space of A^T are subspaces of $\mathbb{R}^m$
- The null space of A and column space of A^T are subspaces of $\mathbb{R}^n$

Next remember that the dimension (the number of vectors needed to form the basis) of the column space $C(A)$ is given by the number of pivot columns in the reduced row-echelon form of A. We also call this the rank, so the dimension of $C(A)$ will always be equal to the rank of A. The dimension of the column space of A^T will always be the same value; it'll be equal to the rank of A. So if r is the rank of A, then

$$\text{Dim}(C(A)) = \text{Dim}(C(A^T)) = r$$

Similarly, the number of vectors needed to form the basis of the null space of A (the dimension of $N(A)$), will be the difference of the number of columns in A and the rank of A. And the number of vectors needed to form the basis of the null space of the transpose of A (the dimension of $N(A^T)$), will be the difference of the number of rows in A and the rank of A.

- $\text{Dim}(N(A)) = n - r$
- $\text{Dim}(N(A^T)) = m - r$

Let's work through an example so that we can see how to find each of these values for the transpose matrix.

Example

Find the null and column subspaces of the transpose M^T, identify their spaces $\mathbb{R}^i$, and name the dimension of the subspaces of M^T.

$$M = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix}$$

The transpose of M will be

$$M^T = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & 4 & -2 & 7 \end{bmatrix}$$

To find the null space, we'll augment the matrix,

$$\begin{bmatrix} 2 & -1 & 3 & 0 & | & 0 \\ 0 & 4 & -2 & 7 & | & 0 \end{bmatrix}$$

and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 0 & | & 0 \\ 0 & 4 & -2 & 7 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{4} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{5}{4} & \frac{7}{8} & | & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{4} & | & 0 \end{bmatrix}$$

Because we have pivot entries in the first two columns, we'll pull a system of equations from the matrix,

$$x_1 + 0x_2 + \frac{5}{4}x_3 + \frac{7}{8}x_4 = 0$$

$$0x_1 + x_2 - \frac{1}{2}x_3 + \frac{7}{4}x_4 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = -\frac{5}{4}x_3 - \frac{7}{8}x_4$$

$$x_2 = \frac{1}{2}x_3 - \frac{7}{4}x_4$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{5}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{7}{8} \\ -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of the transpose, which we call the left null space, is given by

$$N(M^T) = \text{Span}\left(\begin{bmatrix} -\frac{5}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{7}{8} \\ -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix} \right)$$

The space of the null space of the transpose is always $\mathbb{R}^m$, where m is the number of rows in the original matrix, M. The original matrix has 4 rows, so the null space of the transpose $N(M^T)$ is a subspace of $\mathbb{R}^4$.

The column space of the transpose M^T, which is the same as the row space of M, is simply given by the columns in M^T that contain pivot entries when M^T is in reduced row-echelon form. So the column space of M^T is

$$C(M^T) = \text{Span}\left(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}-1\\4\end{bmatrix}\right)$$

The space of the column space of the transpose is always $\mathbb{R}^n$, where n is the number of columns in the original matrix, M. The original matrix has 2 columns, so the column space of the transpose $C(M^T)$ is a subspace of $\mathbb{R}^2$.

Because there are two vectors that form the basis of $N(M^T)$, the dimension of $N(M^T)$ is 2. This matches the formula in our chart, $\text{Dim}(N(M^T)) = m - r = 4 - 2 = 2$. In other words, the dimension of the null space of the transpose will always be the difference of the number of rows in the original matrix and the rank of the original matrix.

Because there are two vectors that form the basis of $C(M^T)$, the dimension of $C(M^T)$ is 2. This matches the rank of the original matrix M. It will always be true that the dimension of the column space of both M and M^T will be equal to the rank of M, and therefore be equivalent to each other.

Let's summarize what we found.

$$N(M^T) = \text{Span}\left(\begin{bmatrix}-\frac{5}{4}\\ \frac{1}{2}\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}-\frac{7}{8}\\ -\frac{7}{4}\\ 0\\ 1\end{bmatrix}\right) \text{ in } \mathbb{R}^4 \qquad \text{Dim}(N(M^T)) = 2$$

$$C(M^T) = \text{Span}\left(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}-1\\4\end{bmatrix}\right) \text{ in } \mathbb{R}^2 \qquad \text{Dim}(C(M^T)) = 2$$