Linear Algebra and Geometry [1] 1

Systems of equations, matrices, vectors, and geometry

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A short table of contents

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An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

C1 Systems of linear equations

S1 Introduction to the course

1 Introduction to the course. Extra material: this list with all the movies and problems.

S2 Some basic concepts

You will learn: some basic concepts that will be used in this course. Most of them are known from high-school courses in mathematics, some of them are new; the latter will appear later in the course and will be treated more in depth then.

2 Coordinate systems and coordinates in the plane and in the 3-space.

3 Slope-intercept equations of straight lines in the plane.

4 Normal equations of planes in the 3-space.

5 Vectors.

6 Scalars.

7 Vector addition and vector scaling.

8 Linear combinations.

Example: Let $\vec{u} = (1, 0, 2, -3)$, $\vec{v} = (2, 1, 0, 0)$, $\vec{w} = (0, 0, 1, 0)$ be three vectors in $\mathbb{R}^4$, and $\alpha = 2$, $\beta = 3$ and $\gamma = -1$. Compute the linear combination $\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w}$.

Extra material: notes with solved problem.

9 Matrices.

10 Linear transformations.

11 Matrix—vector multiplication.

12 Rules for computations with real numbers.

13 Pythagorean Theorem and distance between points.

14 Sine, cosine, and pythagorean identity.

15 Cosine Rule.

S3 Systems of linear equations; building up your geometrical intuition

You will learn: some basic concepts about linear equations and systems of linear equations; geometry behind systems of linear equations.

16 Different ways of looking at equations.

17 Solution set.

18 Linear and non-linear equations.

Example: Which of the following equations are linear. Explain:

$2x+3y+5z-7t=0,\quad \sqrt{3}x-\pi y=7,\quad \sqrt{x^2+1}=7,\quad \sin x-\cos x=0,\quad (\sin\frac{\pi}{6})x-\pi^2 y-\sqrt{7}z=0,$
$2x+\sin y=5,\quad x+2t=17,\quad \ln x-7=0,\quad (\ln 2)x-\sqrt{\pi}y+\frac{1}{z}=0,\quad xy=7,\quad 2x+6y=5x-7y+z.$

19 Systems of linear equations.

20 Solution sets of systems of linear equations; inconsistent systems, homogenous systems, trivial solutions.

21 An example of a 2×2 system of linear equations, a graphical solution.

Example: solve the following system of linear equations using the graphical method:

$$\begin{cases} 2x+3y=-1 \\ 4x+y=3. \end{cases}$$

This is the example which will be used several times during this course and during the continuation course (Linear Algebra and Geometry 2) for illustration of important concepts and theorems.

22 Possible solution sets of 2×2 systems of linear equations.

23 Possible solution sets of 3×2 systems of linear equations; overdetermined systems.

24 Possible solution sets of 3×3 systems of linear equations.

25 Possible solution sets of 2×3 systems of linear equations: underdetermined systems.

26 Possible solution sets of $m\times n$ systems of linear equations.

S4 Solving systems of linear equations; Gaussian elimination

You will learn: solve systems of linear equations using Gaussian elimination (and back-substitution) and Gauss–Jordan elimination in cases of systems with unique solutions, inconsistent systems, and systems with infinitely many solutions (parameter solutions).

27 Our earlier problem revisited; an algebraical solution.

Example: solve the following system of linear equations using algebraical methods:

$$\begin{cases} 2x+3y=-1 \\ 4x+y=3. \end{cases}$$

Extra material: notes with solved problem.

28 Three elementary operations.

29 What is Gauss—Jordan elimination and Gaussian elimination?

30 Gauss—Jordan elimination, a 2-by-2 system with unique solution.

Example: Solve with help of Gauss–Jordan elimination:

$$\begin{cases} 2x+3y=-1 \\ 4x+y=3. \end{cases}$$

31 The same example solved with Gaussian elimination and back-substitution.

Example: Solve with help of Gaussian elimination and back-substitution:

$$\begin{cases} 2x+3y=-1 \\ 4x+y=3. \end{cases}$$

32 The same example solved with matrix operations; coefficient matrix and augmented matrix.

Example: Solve with help of matrix operations:

$$\begin{cases} 2x+3y=-1 \\ 4x+y=3. \end{cases}$$

33 How to write the augmented matrix for a given system of equations, Problem 1.

Problem 1: Write augmented matrices for the following systems of linear equations:

$$\begin{cases} 2x_1+2x_2+3x_3+x_4=8 \\ 3x_1+x_2+x_3+2x_4=6 \\ 3x_1-x_2-x_3+3x_4=3 \\ 3x_1+2x_2-2x_3+2x_4=5 \end{cases} \qquad \begin{cases} 2x+y-w=1 \\ 2y-z+3w=-1 \\ x+z=1 \\ -x+z+4w=1. \end{cases}$$

Extra material: notes with solved Problem 1. (Both systems are solved in the article at the end of this section; you can try solve them now if you want, and compare your solution with mine.)

34 How to write system of equations corresponding to a given augmented matrix, Problem 2.

Problem 2: Write systems of equations corresponding to the following augmented matrices:

$$\left[\begin{array}{ccc|c} 5 & 1 & 1 & 2 \\ 2 & 0 & 3 & 1 \\ 1 & 2 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 1 & -1 & -7 & 6 \\ 0 & -1 & 4 & 1 & 1 \\ 4 & 2 & 1 & 8 & 0 \end{array}\right]$$

Extra material: notes with solved Problem 2.

35 Gaussian elimination, Problem 3.

Problem 3: Solve the system of equations using its augmented matrix (both Gaussian and Gauss–Jordan elimination):

$$\begin{cases} x+2y-3z=6 \\ 2x-y+4z=1 \\ x-y+z=3 \end{cases}$$

Extra material: notes with solved Problem 3.

36 Gaussian elimination, Problem 4.

Problem 4: Solve the system of equations using its augmented matrix (both Gaussian and Gauss–Jordan elimination):

$$\begin{cases} 2y+z=2 \\ x+2y+2z=3 \\ x+3y+z=2 \end{cases}$$

Extra material: notes with solved Problem 4.

37 Gaussian elimination, Problem 5.

Problem 5: Solve the following systems of equations:

$$\begin{cases} 2x_1+x_2+5x_3=2 \\ 3x_2+2x_3=3 \\ x_1-x_2+2x_3=1 \end{cases} \quad \text{and} \quad \begin{cases} 2y_1+y_2+5y_3=1 \\ 3y_2+2y_3=1 \\ y_1-y_2+2y_3=1 \end{cases}$$

Extra material: notes with solved Problem 5.

38 Gaussian elimination, Problem 6.

Problem 6: Solve the following system of nonlinear equations:

$$\begin{cases} 2x^2 + y^2 - 3z^2 = -8 \\ x^2 - y^2 + 2z^2 = 7 \\ x^2 + 2y^2 - z^2 = 1 \end{cases}$$

Extra material: notes with solved Problem 6.

39 What happens if the system is inconsistent?

Example: Solve with help of Gauss–Jordan elimination:

$$\begin{cases} 2x + 3y = -1 \\ 2x + 3y = 3. \end{cases}$$

40 Gaussian elimination, Problem 7.

Problem 7: Solve the following system of linear equations:

$$\begin{cases} x - y + 4z = 5 \\ 3x + y + z = -2 \\ 5x - y + 9z = 1 \end{cases}$$

Extra material: notes with solved Problem 7.

41 Preparation to the general formulation of the algorithm; REF and RREF matrices.

Example: Gauss–Jordan elimination of the overdetermined system with a unique solution:

$$\begin{cases} -x + y = 0 \\ -5x + 3y = 0 \\ 3x - y = 0. \end{cases}$$

Extra material: notes with solved example.

42 How to read solutions from REF and RREF matrices? Leading variables and free (parameter) variables.

Example: In each example suppose that the augmented matrix for a linear system in the unknowns x, y, and z has been reduced by elementary row operations to the given RREF. Solve the system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Extra material: notes with solved example.

43 General formulation of the algorithm in Gauss–Jordan elimination.

Example: Gauss–Jordan elimination step by step on the following system of linear equations:

$$\begin{cases} - 2x_3 + 7x_5 = 12 \\ 2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 = 28 \\ 2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 = -1 \end{cases}$$

Extra material: notes with solved example.

44 Gauss–Jordan elimination, Problem 8.

Problem 8: Gauss–Jordan elimination step by step on the following system of linear equations:

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

Extra material: notes with solved Problem 8.

45 Gauss–Jordan elimination, Problem 9.

Problem 9: Solve the following systems of equations:

$$\begin{cases} x + y + z = 6 \\ x + 2y + 2z = 9 \\ 2x + 3y + 3z = 15 \end{cases} \quad \text{and} \quad \begin{cases} x + y + z = 2 \\ x + 2y + 2z = 1 \\ 2x + 3y + 3z = 4 \end{cases}$$

Extra material: notes with solved Problem 9.

46 Gaussian elimination, Problem 10.

Problem 10: Solve the following system of linear equations:

$$\begin{cases} x + 2y - z - w = 1 \\ x + y + z + 3w = 2 \\ 3x + 5y - z + w = 3 \end{cases}$$

Extra material: notes with solved Problem 10.

47 Gauss–Jordan elimination, Problem 11.

Problem 11: Solve the system of equations:

$$\begin{cases} x + y + 7z = -7 \\ 2x + 3y + 17z = 11 \\ x + 2y + (a^2+1)z = 6a \end{cases}$$

for all possible values of constant a.

Extra material: notes with solved Problem 11.

48 Gauss–Jordan elimination, Problem 12.

Problem 12: Solve the system of equations:

$$\begin{cases} x + p^2y + z = -p \\ x + y - pz = p^2 \\ y + z = 1 \end{cases}$$

for all possible values of constant p.

Extra material: notes with solved Problem 12.

49 Gauss–Jordan elimination, Problem 13.

Problem 13: Solve the system of equations:

$$\begin{cases} -ax + y + 2z = 3 \\ 2x + (a+2)y + z = 2 \\ (1-a)x + y + z = 2 \end{cases}$$

for all possible values of constant a.

Extra material: notes with solved Problem 13.

Extra material: an article with more solved problems on solving systems of linear equations.

⋆ **Extra problem 1**: Solve the system of equations.

$$\begin{cases} 2x + 3y + z = 0 \\ x - y - z = 0 \\ 3x + 7y + 3z = 0 \\ x - 6y - 4z = 0. \end{cases}$$

⋆ **Extra problem 2**: Find the 4-tuples (x_1, x_2, x_3, x_4) that solve the system of linear equations

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 + x_4 = 8 \\ 3x_1 + x_2 + x_3 + 2x_4 = 6 \\ 3x_1 - x_2 - x_3 + 3x_4 = 3 \\ 3x_1 + 2x_2 - 2x_3 + 2x_4 = 5\,. \end{cases}$$

⋆ **Extra problem 3**: Solve the following system of linear equations.

$$\begin{cases} 2x + y - w = 1 \\ 2y - z + 3w = -1 \\ x + z = 1 \\ -x + z + 4w = 1. \end{cases}$$

⋆ **Extra problem 4**: Solve the following system of linear equations.

$$\begin{cases} 2x + y + z = 1 \\ x - z + 2w = -1 \\ x + 5y - w = 2. \end{cases}$$

S5 Some applications in mathematics and natural sciences

You will learn: how systems of linear equations are used in other branches of mathematics and in natural sciences.

50 Solving systems of linear equations in Linear Algebra and Geometry.

51 Solving systems of linear equations (Calculus) Problem 1.

Problem 1: Find values of a, b and c such that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

Extra material: notes with solved Problem 1.

52 Solving systems of linear equations (Calculus) Problem 2.

Problem 2: Find the cubic polynomial whose graph passes through the points $(-1, -2)$, $(0, -4)$, $(1, 0)$, and $(2, 16)$.

Extra material: notes with solved Problem 2.

53 Solving systems of linear equations (Calculus) Problem 3.

Problem 3: Compute the following integral by using partial fraction decomposition of the integrand.

$$\int \frac{4}{x^3 - x^5}\, dx.$$

Extra material: notes with solved Problem 3.

54 Solving systems of linear equations (Calculus) Problem 4.

Problem 4: Perform partial fraction decomposition of the following rational function.

$$\frac{x^2 + 3x - 2}{(x-1)(x^2+x+1)^2}.$$

Extra material: notes with solved Problem 4.

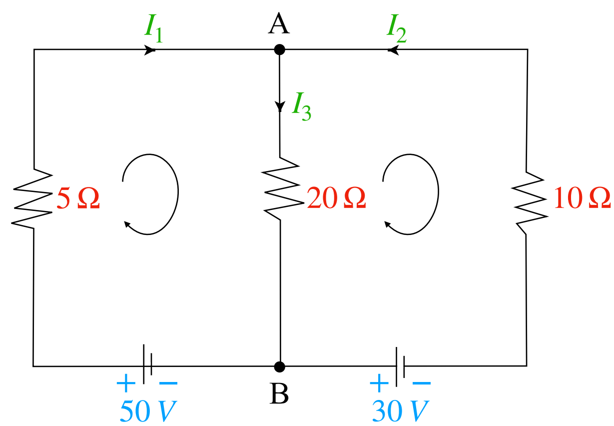
55 Problem 5 (Chemistry).

Problem 5: Balance the chemical equation $\ldots \mathrm{HCl} + \ldots \mathrm{Na_3PO_4} \longrightarrow \ldots \mathrm{H_3PO_4} + \ldots \mathrm{NaCl}$

Extra material: notes with solved Problem 5.

56 Problem 6 (Electrical circuits).

Problem 6: Determine the currents I_1, I_2, and I_3 in the following circuit:



Extra material: notes with solved Problem 6.

C2 Matrices and determinants

S6 Matrices and matrix operations

You will learn: the definition of matrices and their arithmetic operations (matrix addition, matrix subtraction, scalar multiplication, matrix multiplication). Different kinds of matrices (square matrices, triangular matrices, diagonal matrices, zero matrices, identity matrix).

57 Introduction to matrices.

58 Different types of matrices.

59 Matrix addition and subtraction, Problem 1.

Problem 1: Compute $A + B$ and $A - B$ where

$$A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$$

60 Matrix scaling, with geometrical interpretation.

61 Matrix scaling, Problem 2.

Problem 2: Compute αA and βB where $\alpha = 2$, $\beta = -\frac{1}{3}$, and

$$A = \begin{bmatrix} 0 & 3 & 7 \\ -2 & -1 & 5 \\ 8 & -5 & 0 \\ 2 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 & 5 \\ 9 & 4 & 0 \end{bmatrix}$$

Extra material: notes with solved Problem 2.

62 Matrix multiplication, with geometrical interpretation.

63 Matrix multiplication, how to do.

64 Matrix multiplication, Problem 3.

Problem 3: Compute AB and BA where

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

65 Matrix multiplication and systems of equations, Problem 4.

Problem 4: Write the following systems of equations with help of matrix multiplication:

$$\begin{cases} 5x + y + z = 2 \\ 2x \qquad + 3z = 1 \\ x + 2y \qquad = 0 \end{cases} \qquad \begin{cases} x_1 + x_2 - x_3 - 7x_4 = 6 \\ -x_2 + 4x_3 + x_4 = 1 \\ 4x_1 + 2x_2 + x_3 + 8x_4 = 0 \end{cases}$$

66 Transposed matrix, definition and some examples.

Problem 5: Compute the transposed matrices to the matrices A and B from Problem 2.

Extra material: notes with solved Problem 5.

67 Trace of a matrix, definition and an example.

Problem 6: Compute $\mathrm{tr}(A)$ where

$$A = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix}.$$

68 Various matrix operations, Problem 7.

Problem 7: Suppose that A, B, C, D, and E are matrices with the following sizes:

$$A : 5 \times 6, \quad B : 5 \times 6, \quad C : 6 \times 3, \quad D : 5 \times 3, \quad E : 6 \times 5.$$

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.
a) BA, b) $AC+D$, c) $B+EA$, d) $B+AB$, e) $E(B+A)$, f) $(EA)C$, g) $A^T E$, h) $D^T(A+E^T)$.

Extra material: notes with solved Problem 7.

69 Various matrix operations, Problem 8.

Problem 8: Compute the expressions (if possible): a) $D+E$, b) $D-E$, c) $5A$, d) $-9D$, e) $2B-C$, f) $7E+7D$, g) $B-B$, h) $\mathrm{tr}(D)$, i) $\mathrm{tr}(D-E)$, j) $2\mathrm{tr}(4B)$, k) $\mathrm{tr}(A)$, l) AB, m) BA, where:

$$A = \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix}.$$

Extra material: notes with solved Problem 8.

S7 Inverses; Algebraic properties of matrices

You will learn: use matrix algebra; the definition of the inverse of a matrix.

70 Properties of matrix operations, an introduction.

71 Matrix addition has all the good properties.

72 Matrix multiplication has a neutral element for square matrices.

Example: Show that I_2 and I_3 are neutral elements of matrix multiplication for 2×2 and 3×3 matrices respectively.

Extra material: notes with solved example.

73 Matrix multiplication is associative.

Example: Demonstrate associative law for matrix multiplication using following matrices:

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 1 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}.$$

Extra material: notes with solved example.

74 Matrix multiplication is not commutative.

Example: Show that $AP \neq PA$ where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and give a geometrical explanation of this result.

75 Sometimes commutativity happens, Problem 1.

Problem 1: Find all the matrices X satisfying the condition $AX = XA$ for

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}.$$

Extra material: notes with solved Problem 1.

76 Two distributive laws.

Example: Demonstrate the first distributive law for matrices using following matrices:

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 1 & -1 \\ -3 & 0 & 3 \end{bmatrix}.$$

Extra material: notes with solved example.

77 Matrix multiplication does not have the zero-product property.

78 There is no cancellation law for matrix multiplication.

Example: Show using the following matrices that there is no cancellation law for matrices:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

79 Inverse matrices; not all non-zero square matrices have an inverse.

Example: Show that A does not have an inverse:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

80 Inverse matrix for 2-by-2 matrices; non-zero determinant.

Example: Use the formula for computing the inverse of A, where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}.$$

Show that the formula is correct.

Extra material: notes with the proof of the formula.

81 Solving matrix equations, Problem 2.

Problem 2: Solve matrix equation $(AX + B)C = I_2$ where:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Extra material: notes with solved Problem 2.

82 Powers of matrices; powers of diagonal matrices.

83 Computation rules for transposed matrices.

Example: Demonstrate the rule $(AB)^T = B^T A^T$ using following matrices:

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Extra material: notes with solved example.

84 Supplement to Video 83; Inverse of a product.

85 Inverse of a transposed matrix.

86 Various rules, Problem 3.

Problem 3: Assuming that all matrices are $n \times n$ and invertible:

- solve for D:
$$ABC^T DBA^T C = AB^T.$$
- simplify:
$$D^{-1}CBA(BA)^{-1}C^{-1}(C^{-1}D)^{-1}.$$

Extra material: notes with solved Problem 3.

Extra material: an article with more solved problems on matrix arithmetics.

- ⋆ **Extra problem 1**: Simplify the following expressions: $5(3A - B) - 7(A + 2B)$ and $(A - B)^2 - (A + B)^2$, where A and B are matrices of the same size.
- ⋆ **Extra problem 2**: Solve the matrix equation:
$$(5A)^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}.$$
- ⋆ **Extra problem 3**: Solve the matrix equation:
$$(I + 2A)^{-1} = \begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix}.$$
- ⋆ **Extra problem 4**: Show that if a square matrix A satisfies the condition $A^2 + 5A - 2I = 0$ then
$$A^{-1} = \frac{1}{2}(A^2 + 5I).$$

S8 Elementary matrices and a method for finding A inverse

You will learn: how to compute the inverse of a matrix with Gauss–Jordan elimination.

87 Inverse matrices, introduction to the algorithm.

Extra material: notes with the motivation for the algorithm (for 2-by-2 and 3-by-3 matrices).

88 Algorithm for inverse matrices, an example.

Example: Compute the inverse of the matrix A:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 6 & 4 \\ 0 & -2 & 2 \end{bmatrix}.$$

89 Matrix inverse, Problem 1.

Problem 1: Determine if the matrix A is invertible and, if it is invertible, compute its inverse:

$$A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}.$$

Solve the system of linear equations $A\mathbf{x} = \mathbf{b}$ for every RHS $\mathbf{b} = (b_1, b_2, b_3)^T$.

Extra material: notes with solved Problem 1.

90 Matrix inverse, Problem 2.

Problem 2: Determine all the values of c for which the following matrix is invertible:

$$\begin{bmatrix} c & -c & c \\ 1 & c & 1 \\ 0 & 0 & c \end{bmatrix}.$$

Extra material: notes with solved Problem 2.

91 Matrix equations, Problem 3.

Problem 3: Find the matrix A that solves the equation $\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}^T = A + A \begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}$.

Extra material: notes with solved Problem 3.

92 Matrix equations, Problem 4.

Problem 4: Find the matrix A that solves the equation $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} A + 2A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$.

Extra material: notes with solved Problem 4.

93 Matrix equations, Problem 5.

Problem 5: Find all the matrices X satisfying the equation $B - XA = 2XC$, where:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & 3 \\ -1 & -\frac{1}{2} & 0 \end{bmatrix}.$$

Extra material: notes with solved Problem 5.

94 Matrix equations, Problem 6.

Problem 6: Find all the matrices X satisfying the equation $AXB = AB + A^2$, where:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}.$$

Extra material: notes with solved Problem 6.

95 Matrix inverse, Problem 7.

Problem 7: Determine the inverse of matrix A:

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

Extra material: notes with solved Problem 7.

96 Elementary operations and elementary matrices.

97 Inverse elementary operations and their matrices.

98 A really important theorem.

99 Four equivalent statements.

Extra material: notes with the proof of the theorem.

Extra material: an article with more solved problems on inverse matrices.

- ⋆ **Extra problem 1**: Find all the matrices X satisfying the equation

$$X \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}^T - 3X = \begin{bmatrix} 10 & 5 & 0 \\ -5 & 0 & 10 \end{bmatrix}.$$

- ⋆ **Extra problem 2**: Determine all the values of c for which the following matrix is invertible:

$$\begin{bmatrix} c & 2 & 0 \\ 1 & c & 2 \\ 0 & 1 & c \end{bmatrix}.$$

- ⋆ **Extra problem 3**: Find the matrix A that solves the equation

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & -1 \\ 2 & 4 & 0 \end{bmatrix} A + A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & -1 \end{bmatrix}^T.$$

S9 Linear systems and matrices

You will learn: about the link between systems of linear equations and matrix multiplication.

100 Formally about the number of solutions to systems of linear equations.

101 Two more statements in our important theorem.

102 Solution of a linear system using A inverse, Problem 1.

Problem 1: Solve the system using the inverse to the coefficient matrix: $\begin{cases} x + 2y + 3z = 5 \\ 2x + 5y + 3z = 3 \\ x + \quad + 8z = 17 \end{cases}$

Extra material: notes with solved Problem 1.

103 Determining consistency by elimination, Problem 2.

Problem 2: What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations to be consistent?

$$\text{a)} \begin{cases} x + 2y + 3z = b_1 \\ 2x + 5y + 3z = b_2 \\ x + \quad + 8z = b_3 \end{cases} \qquad \text{b)} \begin{cases} x + y + 2z = b_1 \\ x + \quad + z = b_2 \\ 2x + y + 3z = b_3. \end{cases}$$

Extra material: notes with solved Problem 2.

104 Matrix equations, Problem 3.

Problem 3: Find the matrix X that solves the equation: $X \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 0 \\ 6 & -3 & 7 \end{bmatrix}$.

Extra material: notes with solved Problem 3.

S10 Determinants

You will learn: the definition of the determinant; apply the laws of determinant arithmetics, particularly the multiplicative property and the expansion along a row or a column; solving equations involving determinants; the explicit formula for solving of $n \times n$-systems of linear equations (Cramer's rule), the explicit formula for inverse to a non-singular matrix.

105 Why the determinants are important.

106 2-by-2 determinants; notation for n-by-n determinants.

107 Geometrical interpretations of determinants.

108 Geometrically about the determinant of a product.

109 Definition of determinants.

Example: Determinants of three elementary 3-by-3 matrices for multiplying rows by a constant.

110 Conclusion 1: Determinant of matrices with interchanged columns.

Example: Determinants of six elementary 3-by-3 matrices for interchanging rows.

111 Conclusion 2: What happens when one column is a linear combination of the other columns.

Example: Show that

$$\begin{vmatrix} a_{11} & \alpha a_{11} + \beta a_{13} + \gamma a_{14} & a_{13} & a_{14} \\ a_{21} & \alpha a_{21} + \beta a_{23} + \gamma a_{24} & a_{23} & a_{24} \\ a_{31} & \alpha a_{31} + \beta a_{33} + \gamma a_{34} & a_{33} & a_{34} \\ a_{41} & \alpha a_{41} + \beta a_{43} + \gamma a_{44} & a_{43} & a_{44} \end{vmatrix} = 0$$

Extra material: notes with solved Example.

112 Conclusion 3: About adding a multiple of a column to another column.

Example: Show that

$$\begin{vmatrix} a_{11} & a_{12} + \alpha a_{11} & a_{13} & a_{14} \\ a_{21} & a_{22} + \alpha a_{21} & a_{23} & a_{24} \\ a_{31} & a_{32} + \alpha a_{31} & a_{33} & a_{34} \\ a_{41} & a_{42} + \alpha a_{41} & a_{43} & a_{44} \end{vmatrix} = \det A$$

Extra material: notes with solved Example.

113 Conclusion 4: Determinant of kA for any $k \in \mathbb{R}$.

114 Elementary column operations.

Example: Suppose that

$$A = \begin{pmatrix} x & 3 & 1 & t \\ y & 0 & 1 & z \\ z & 2 & 1 & y \\ t & 5 & 1 & x \end{pmatrix}$$

and that the determinant $\det(A)$ is equal to 3. Compute (using only row- and column operations) following determinants. Motivate your answers:

$$\begin{vmatrix} 2x & 9 & -1 & t-15 \\ 2y & 0 & -1 & z \\ 2z & 6 & -1 & y-10 \\ 2t & 15 & -1 & x-25 \end{vmatrix}, \qquad \begin{vmatrix} x & 3x+3 & x+1 & t-4x \\ y & 3y & y+1 & z-4y \\ z & 3z+2 & z+1 & y-4z \\ t & 3t+5 & t+1 & x-4t \end{vmatrix}, \qquad \det(2A).$$

Extra material: notes with solved example.

115 How to compute 2-by-2 determinants from the definition.

116 How to compute 3-by-3 determinants from the definition.

117 Sarrus' rule for 3-by-3 determinants.

Example: Compute the following determinants with Sarrus' method:

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 7 & 1 \\ 0 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 3 & -1 \\ 0 & 5 & 4 \\ 0 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} x & 1 & 0 \\ 1 & x & 0 \\ 0 & 1 & x \end{vmatrix}, \quad \begin{vmatrix} a & a & 1 \\ a & a-1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$

Extra material: notes with solved example.

118 Determinant of transposed matrix; row operations.

Example: Determinants of six elementary 3-by-3 matrices for adding a multiple of a row to another row.

119 Evaluating determinants by cofactor expansion along rows or columns.

Example: Compute the following determinant using cofactor expansion along different rows and columns:

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}, \qquad \begin{vmatrix} 2 & 5 & -2 & 7 \\ 2 & 1 & 1 & 7 \\ 1 & 1 & 1 & 7 \\ 3 & 1 & 2 & -21 \end{vmatrix}$$

Extra material: notes with solved example.

120 Evaluating determinants by row or column reduction.

Example: Compute the determinant by reducing the matrix to a triangular one

$$\begin{vmatrix} 1 & 3 & k \\ 2 & 1 & 3 \\ 4 & 6 & 2 \end{vmatrix}$$

Example: Compute the first determinant from Video 119 by the same method.

Extra material: notes with some computation for the examples above.

121 Determinant of inverse.

122 Properties of determinants, Problem 1.

Problem 1: Compute the determinant of $\frac{1}{5}\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}^T\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}^{-1}$.

Extra material: notes with solved Problem 1.

123 Properties of determinants, Problem 2.

Problem 2: Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Assuming that $\det A = -5$, find:

a) $\det(-4A)$, b) $\det(A^{-1})$, c) $\det(3A^{-1})$, d) $\det((3A)^{-1})$, e) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$.

Extra material: notes with solved Problem 2.

124 Properties of determinants, Problem 3.

Problem 3: Let A be a 5-by-5 matrix and $\det A = -3$, find:

a) $\det(-A)$, b) $\det(A^{-1})$, c) $\det(4A^T)$, d) $\det(A^4)$.

Extra material: notes with solved Problem 3.

125 Determinant equations, Problem 4.

Problem 4: Solve the equation without computing the determinant

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 4 & 6-x & 4 & 4 \\ 6 & 6 & 6 & x \end{vmatrix} = 0.$$

Extra material: notes with solved Problem 4.

126 Determinant equations, Problem 5.

Problem 5: Solve the equation

$$\begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{vmatrix} = 0.$$

Extra material: notes with solved Problem 5.

127 Determinant equations, Problem 6.

Problem 6: Solve the equation

$$\begin{vmatrix} 1 & x & 1 & x \\ x & 1 & 1 & x \\ 2x & 0 & 1 & 2 \\ x & 1 & x & 1 \end{vmatrix} = 0.$$

128 Determinant equations, Problem 7.

Problem 7: Solve the equation

$$\begin{vmatrix} x & 1 & 2 & -1 \\ x & 1 & 1 & x \\ 2x & 0 & 1 & 0 \\ 1 & x & 1 & x \end{vmatrix} = 0.$$

129 Invertible matrices, determinant test with a proof, Problem 8.

Problem 8: Determine the values of x for which matrix A is invertible.

$$A = \begin{pmatrix} 1 & x & x^2 & x^3 \\ 2 & 0 & 1 & x \\ x-1 & 1 & 1 & 0 \\ x & 0 & 2 & x \end{pmatrix}$$

Extra material: notes with solved Problem 8.

130 Cramer's rule, a proof, an example, and a geometrical interpretation.

Example: solve the following system of linear equations using Cramer's rule:

$$\begin{cases} 2x + 3y = -1 \\ 4x + y = 3. \end{cases}$$

Extra material: notes with proof of Cramer's rule.

131 Cramer's rule, Problem 9.

Problem 9: Solve the system of equations (from Video 35) using Cramer's rule:

$$\begin{cases} x + 2y - 3z = 6 \\ 2x - y + 4z = 1 \\ x - y + z = 3 \end{cases}$$

Extra material: notes with solved Problem 9.

132 Inverse matrix, an explicit formula.

Example 1: Determine A inverse for:

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 6 \end{pmatrix}$$

Example 2: For matrix A determine: all the minors M_{ij}, all the cofactors C_{ij}, the adjoint matrix $\mathrm{adj}(A)$, the determinant of A, and the inverse of A.

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 2 & 4 & 4 \end{pmatrix}$$

Extra material: notes with solved Example 2.

133 Invertible matrices, Problem 10.

Problem 10: Determine the values of a for which matrix A is invertible, and determine the inverse A^{-1} for these values of a. Two different solutions: Jacobi's method (Gauss–Jordan elimination) and the explicit formula for the inverse:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ a & -2a & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

Extra material: notes with solved Problem 10.

134 Problem 11, a large determinant.

Problem 11: Compute the following determinant D for $n \geqslant 2$:

$$D = \begin{vmatrix} x & a & a & \dots & a & a \\ 0 & x & 0 & \dots & 0 & 0 \\ 0 & 0 & x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x & 0 \\ b & b & b & \dots & b & x \end{vmatrix}.$$

Determine all the real solutions to the equation $D = 0$ where a and b are constants and x is the unknown.

135 Problem 12, another large determinant.

Problem 12: Show that:

$$D_n = \begin{vmatrix} 5 & 3 & 0 & \dots & 0 & 0 \\ 2 & 5 & 3 & \dots & 0 & 0 \\ 0 & 2 & 5 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 5 & 3 \\ 0 & 0 & 0 & \dots & 2 & 5 \end{vmatrix} = 3^{n+1} - 2^{n+1}$$

where D_n is an $n \times n$-determinant.

Extra material: notes with solved Problem 12.

136 Problem 13: a trigonometric determinant.

Problem 13: Compute

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$$

Extra material: notes with solved Problem 13.

137 Problem 14: Vandermonde determinant.

Problem 14: Let

$$T = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}, \quad T(t) = \begin{vmatrix} 1 & a+t & (a+t)^2 \\ 1 & b+t & (b+t)^2 \\ 1 & c+t & (c+t)^2 \end{vmatrix}, \quad V(t) = \begin{vmatrix} 1 & t & t^2 & t^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{vmatrix}$$

Show that $T = (b-a)(c-a)(c-b)$. Show that the value of $T(t)$ is not dependent on t and show a geometrical interpretation of this fact. Also show that $f(t) = V(t)$ (with distinct x_1, x_2, x_3) is a cubic polynomial, that the coefficient at t^3 is nonzero, and find three points on the graph of $f(t)$.

Extra material: notes with solved Problem 14.

Extra material: an article with more solved problems on determinants.

- ⋆ **Extra problem 1**: Compute the determinant

$$\begin{vmatrix} 5 & 3 & 0 & 3 \\ 6 & 4 & 1 & 4 \\ 3 & 3 & 1 & 1 \\ 1 & 3 & 1 & -1 \end{vmatrix}.$$

- ⋆ **Extra problem 2**: Evaluate the following determinant

$$\begin{vmatrix} 10 & 5 & 1 & -2 \\ -3 & 1 & 0 & 1 \\ 4 & -3 & 2 & 1 \\ 1 & 2 & 0 & -2 \end{vmatrix}.$$

- ⋆ **Extra problem 3**: Determine the values of b for which the following matrix is invertible:

$$A = \begin{pmatrix} b & 1 & 2 & 3 \\ 3 & b & 1 & 2 \\ 2 & 3 & b & 1 \\ 1 & 2 & 3 & b \end{pmatrix}$$

- ⋆ **Extra problem 4**: Solve the equation

$$\begin{vmatrix} x & -1 & -1 & 1 \\ 1 & x & -1 & -1 \\ -1 & -1 & x & 1 \\ -1 & -1 & 1 & x \end{vmatrix} = 0$$

- ⋆ **Extra problem 5**: Determine the values of a for which matrix A is invertible, and determine the inverse A^{-1} for these values of a. Use any method you want to.

$$A = \begin{pmatrix} 1 & -1 & a \\ 0 & 2a & 3 \\ a & 0 & 2 \end{pmatrix}$$

C3 Vectors and their products

S11 Vectors in 2-space, 3-space, and n-space

You will learn: apply and graphically illustrate the arithmetic operations for vectors in the plane; apply the arithmetic operations for vectors in $\mathbb{R}^n$.

138 Vectors, a repetition.

139 Computation rules for vector addition and scaling.

Extra material: notes with proofs of the commutative law for vector addition.

140 Computations with vectors, Problem 1.

Problem 1: Let $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (4, 1)$. Compute and illustrate in the coordinate system: $\mathbf{u} + \mathbf{v}$, $2\mathbf{u} - \mathbf{v}$, $-\mathbf{u} - \mathbf{v}$, $\mathbf{u} + 2\mathbf{v}$.

Extra material: notes with solved Problem 1.

141 Computations with vectors, Problem 2.

Problem 2: Given three points in the 3-space: $A = (1, 2, 3)$, $B = (-1, 0, 4)$ and $C = (-5, -3, 0)$. Compute the coordinates of vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$. Add the three vectors and explain the geometrical meaning behind the result.

Extra material: notes with solved Problem 2.

142 Computations with vectors, Problem 3.

Problem 3: Given the point $A = (1, 2, 3, 4)$ in the 4-space and vector $\overrightarrow{AB} = (-1, 5, -4, 7)$. Determine the coordinates of point B.

Extra material: notes with solved Problem 3.

143 Parallel vectors, Problem 4.

Problem 4: For what value(s) of t, if any, is the given vector parallel to $\mathbf{u} = (5, -2)$?
a) $(15t, -6)$, b) $(15t, 6t)$, c) $(1, t^2)$.

Extra material: notes with solved Problem 4.

144 Parallel vectors, Problem 5.

Problem 5: Which of the following vectors in $\mathbb{R}^6$ are parallel to $\mathbf{u} = (2, 0, -1, 4, -2, 3)$?
a) $(-4, 0, 2, -8, 4, -6)$, b) $(4, 0, -2, 8, 4, 6)$, c) $(0, 0, 0, 0, 0, 0)$.

Extra material: notes with solved Problem 5.

145 Linear combinations, Problem 6.

Problem 6: Show that vector $(-1, 3)^T$ is a linear combination of the vectors $(2, 4)^T$ and $(3, 1)^T$.

146 Linear combinations, Problem 7.

Problem 7: Show that vectors $(1, 1)^T$ and $(-1, 2)^T$ span the entire $\mathbb{R}^2$.

147 Linear combinations, linear independence, Problem 8.

Problem 8: Show that vector $(-10, 2, 7)^T$ is a linear combination of $(-1, 0, 2)^T$, $(2, 2, -2)^T$ and $(1, -2, 1)^T$.

Extra material: notes with solved Problem 8.

148 Linear combinations, linear dependence, Problem 9.

Problem 9: Show that vector $(1, 1, 2)^T$ is not a linear combination of $(1, 2, 3)^T$, $(4, 5, 6)^T$ and $(5, 7, 9)^T$ and that vector $(3, 3, 3)^T$ is a linear combination of these vectors.

Extra material: notes with solved Problem 9.

149 Area, Problem 10.

Problem 10: Compute the area of the parallelogram with vertices in points $(-2, -1)$, $(2, 0)$, $(4, 2)$ and $(0, 1)$.

What is the area of the triangle with vertices in points $(-2,-1)$, $(2,0)$ and $(4,2)$?

Extra material: notes with solved Problem 10.

150 Midpoint of a line segment, Problem 11.

Problem 11: Let P be the point $(2,4,6)$. If the point $(5,1,-12)$ is the midpoint of the line segment connecting P and Q, what is Q?

Extra material: notes with solved Problem 11.

S12 Distance and norm in $\mathbb{R}^n$.

You will learn: compute the distance between points in $\mathbb{R}^n$ and norms of vectors in $\mathbb{R}^n$, normalize vectors.

151 Norm of a vector, Problem 1.

Example: Compute the norm of $\mathbf{u} = (1,2,3,-4)$.

Problem 1: Compute the norm of $\mathbf{v}$ where
a) $\mathbf{v} = (3,-5)$, b) $\mathbf{v} = (3,3,1)$, c) $\mathbf{v} = (0,1,-1,2,6)$.

Extra material: notes with solved Problem 1.

152 Properties of the norm.

Extra material: notes with proof of the third property.

153 Distance between points, Problem 2.

Example: Compute the (generalised) distance between points $P = (1,-1,2,5)$ and $Q = (2,3,-2,4)$.

Problem 2: Compute lengths of the sides in the triangle with vertices in $(-2,-1)$, $(2,0)$ and $(4,2)$.

Extra material: notes with solved Problem 2.

154 Unit vectors, how to normalize a vector.

Example 1: Normalize vector $\mathbf{v} = (3,4)$; make an illustration.

Example 2: Normalize vector $\mathbf{v} = (2,4)$; make an illustration.

155 Unit vectors in given direction, Problem 3.

Problem 3: Determine the unit vector in the direction of $\mathbf{v}$ and the unit vector in the opposite direction, where: a) $\mathbf{v} = (3,-5)$, b) $\mathbf{v} = (3,3,1)$, c) $\mathbf{v} = (0,1,-1,2,6)$. (The same as in Video 151.)

Extra material: notes with solved Problem 3.

S13 Dot product, orthogonality, and orthogonal projections

You will learn: definition of dot product and the way you can use it for computing angles between geometrical vectors.

156 Different products for vectors.

157 Perpendicular straight lines and orthogonal vectors.

158 Orthogonal projections.

159 Definition of dot product for geometrical vectors.

160 How to compute dot product, an example.

Example: Compute $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = (-1,0,5)$ and $\mathbf{v} = (0,7,-2)$.

Extra material: notes with the proof of the computation method for dot product.

161 Dot product for vectors in $\mathbb{R}^n$; orthogonality and angles.

162 Properties of dot product.

Extra material: notes with the proof of some of the properties.

163 Angles between vectors, Problem 1.

Problem 1: Find the cosine of the angles between the vectors $\mathbf{u}$ and $\mathbf{v}$, and then state whether the angle is acute, obtuse or right: a) $\mathbf{u} = (3, 3, 3)$ and $\mathbf{v} = (1, 0, 4)$, b) $\mathbf{u} = (0, -2, -1, 1)$ and $\mathbf{v} = (-3, 2, 4, 4)$.

Extra material: notes with solved Problem 1.

164 Angles between vectors, Problem 2.

Problem 2: Find the radian measure of the angle θ (with $0 \leqslant \theta \leqslant \pi$) between $\mathbf{u}$ and $\mathbf{v}$:
a) $(1, -7)$ and $(21, 3)$, b) $(0, 2)$ and $(3, -3)$, c) $(-1, 1, 0)$ and $(0, -1, 1)$, d) $(1, -1, 0)$ and $(1, 0, 0)$.

Extra material: notes with solved Problem 2.

165 How to find vector orthogonal to a given vector in the plane or in the 3-space.

166 Orthogonal projections and decompositions.

167 Orthogonal projections and decompositions, Problem 3.

Problem 3: Find $\|\text{proj}_{\mathbf{u}}\mathbf{v}\|$ where a) $\mathbf{v} = (2, 4),\ \mathbf{u} = (1, 1)$, b) $\mathbf{v} = (1, -1, 0),\ \mathbf{u} = (2, 0, 1)$.

Extra material: notes with solved Problem 3.

168 Orthogonal projections and decompositions, Problem 4.

Problem 4: Find the vector component of $\mathbf{v}$ along $\mathbf{u}$ and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$, where $\mathbf{v} = (1, 1, 1),\ \mathbf{u} = (0, 2, -1)$.

Extra material: notes with solved Problem 4.

169 Orthogonal sets, Problem 5.

Problem 5: Determine whether the vectors form an orthogonal set:

a) $\mathbf{v} = (1, 2),\ \mathbf{u} = (-2, 1)$

b) $\mathbf{v} = (2, 4),\ \mathbf{u} = (-1, 2)$

c) $\mathbf{v} = (1, 3, -1),\ \mathbf{u} = (-2, 2, 4),\ \mathbf{w} = (14, -2, 8)$

d) $\mathbf{v} = (5, -1, 2),\ \mathbf{u} = (-1, 2, 3),\ \mathbf{w} = (4, -1, 2)$.

Extra material: notes with solved Problem 5.

170 A very, very important Problem 6.

Problem 6: Show that if vector $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then it is also orthogonal to $s\mathbf{u} + t\mathbf{v}$ for any constants s, $t \in \mathbb{R}$ (i.e. to any linear combination of vectors $\mathbf{u}$ and $\mathbf{v}$). What does it mean geometrically in $\mathbb{R}^2$ and in $\mathbb{R}^3$?

Extra material: notes with solved Problem 6.

S14 Cross product, parallelograms and parallelepipedes.

You will learn: definition of cross product and interpretation of 3×3 determinants as the volume of a parallelepiped in the 3-space.

171 Cross product, an introduction.

172 Cross product, how it is defined.

Example 1: Compute the cross product of $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 1, 0)$.

Example 2: Compute the cross product of $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (1, -1, 1)$.

Extra material: notes with derivation of the formula for cross product of two given vectors.

173 Three properties of cross product.

Extra material: notes with the proofs of some properties of cross product.

174 The length of the cross product of two vectors.

Extra material: notes with the proof that the length of the cross product of two vectors is equal to the area of the parallelogram spanned on these vectors.

175 More properties of cross product.

Extra material: notes with the proofs of some properties of cross product.

176 Cross product: Problem 1.

Problem 1: Compute the area of the triangle with vertices in $P = (1, 1, 2)$, $Q = (0, -3, 4)$ and $R = (2, 3, 5)$.

Extra material: notes with solved Problem 1.

177 Cross product in the plane.

178 Scalar triple product and volume.

179 Scalar triple product, Problem 2.

Problem 2: Find the volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, where $\mathbf{u} = (0, 2, -2)$, $\mathbf{v} = (1, 2, 0)$, $\mathbf{w} = (-2, 3, 1)$.

Extra material: notes with solved Problem 2.

180 Collinearity in the plane and coplanarity in the 3-space.

181 Determinant test for vectors, Problem 3.

Problem 3: Determine whether $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in the same plane when positioned so that their initial points coincide. Double check your result with GeoGebra.
a) $\mathbf{u} = (0, 1, -1)$, $\mathbf{v} = (2, 2, 0)$, $\mathbf{w} = (4, 1, 2)$, b) $\mathbf{u} = (2, 1, 1)$, $\mathbf{v} = (-1, 3, 4)$, $\mathbf{w} = (3, 5, 6)$.

Extra material: notes with solved Problem 3.

C4 Analytical geometry of lines and planes

S15 Lines in $\mathbb{R}^2$

You will learn: several ways of describing lines in the plane (slope-intercept equation, intercept form, point-vector equation, parametric equation) and how to compute other kinds of equations given one of the equations named above.

182 Lines in the plane, an introduction.

183 Slope-intercept and intercept form.

Problem 1: Write the intercept equation for the line $y = 5x + 6$.

Write the slope-intercept equation for the line $\frac{x}{2} + \frac{y}{3} = 1$.

Extra material: notes with solved Problem 1.

184 Normal equation.

Problem 2: Write normal equation for the line orthogonal to $\mathbf{n} = (-2, 3)$, passing through the point $(6, 2)$. Write also both slope-intercept and intercept equation for this line. Draw the line.

Extra material: notes with solved Problem 2.

185 Parametric equations.

Problem 3: Write a parametric equation for the line from Video 184. Write another parametric equation for the line in the slide, to show that parametric equations are not unique.

Extra material: notes with solved Problem 3.

186 Determinant equation.

Problem 4: Write the determinant equation for our line from the slide; use $(x_1, y_1) = (-3, -2)$ and $(x_2, y_2) = (1, -1)$; show that the new equation is equivalent to the previous ones.

Extra material: notes with solved Problem 4.

187 Lines in the plane, Problem 5.

Problem 5: Write any equation of the line through $(2,3)$, parallel to the vector $\mathbf{v} = (1,2)$.

Write any equation of the line through $(2,3)$, orthogonal to the vector $\mathbf{v} = (1,2)$.

Extra material: notes with solved Problem 5.

S16 Planes in $\mathbb{R}^3$

You will learn: several ways of describing planes in the 3-spaces (normal equation, intercept form, parametric equation) and how to compute other kinds of equations given one of the equations named above.

188 Planes in the 3-space, an introduction.

189 Normal and intercept equation.

Problem 1: Write a normal and intercept equations for the plane orthogonal to $\mathbf{n} = (-2,3,-5)$ and passing through $P = (0,1,-3)$. Control your answer by plotting the plane with GeoGebra.

Extra material: notes with solved Problem 1.

190 Parametric equations.

Problem 2: Find parametric equation for the plane in the 3-space that contains the points $P = (1,1,1)$, $Q = (-1,0,1)$ and $R = (5,6,7)$. Check if the following points belong to this plane: $A = (-3,-1,1)$, $B = (5,0,1)$, $C = (1,2,3)$.

Extra material: notes with solved Problem 2.

191 Parametric to normal.

Problem 3: Find a normal equation for the plane from Video 190. Do you see that it is now easier to check whether a point belongs to this plane?

Extra material: notes with solved Problem 3.

192 Normal to parametric; how to find a point and two parallel vectors for a given plane.

Problem 4: Find a parametric equation for the plane $x - y + 6z - 1 = 0$. Check if the following points belong to this plane: $P = (2,1,0)$, $Q = (3,2,1)$, $R = (5,10,1)$.

Extra material: notes with solved Problem 4.

193 Determinant equation.

Problem 5: Write the determinant equation for the plane from Video 190 and check that the equation describes the same plane.

Extra material: notes with solved Problem 5.

194 Planes: Problem 6.

Problem 6: Find an equation on parameter-free form for the plane that contains the points $P : (-1,0,1)$, $Q : (3,2,2)$ and $R : (1,1,1)$.

Extra material: notes with solved Problem 6.

S17 Lines in $\mathbb{R}^3$

You will learn: several ways of describing lines in the 3-space (point-vector equation, parametric equation, standard equation) and how to compute other kinds of equations given one of the equations named above.

195 Lines in the 3 space, an introduction.

196 Lines in the 3 space, Problem 1.

Problem 1: Write a parametric equation and standard equation for the line through $(-6,2,5)$, parallel to $\mathbf{v} = (2,1,4)$.

Extra material: notes with solved Problem 1.

197 Lines in the 3 space, Problem 2.

Problem 2: Write a parametric equation and standard equation for the intersection line between the planes $x + y + z = 1$ and $y - z = 1$.

Extra material: notes with solved Problem 2.

198 Lines in the 3 space, Problem 3.

Problem 3: Write a parametric equation and standard equation for the line through points $(-1, 1, 0)$ and $(0, 3, -2)$.

Extra material: notes with solved Problem 3.

S18 Geometry of linear systems; incidence between lines and planes

You will learn: determine the equations for a line and a plane and how to use these for computing intersections by solving systems of equations.

199 Incidence 1: points and planes.

Example 1: (Solved in Video 35 och in Video 131, with two different methods): Planes with equations $x + 2y - 3z = 6$, $2x - y + 4z = 1$ and $x - y + z = 3$ intersect in one point.

Example 2: Planes with equations $x + 2y + z = 1$, $2x + y - z = 0$ and $3x + 3y = 1$ intersect along the line $(x, y, z) = (-1/3,\ 2/3,\ 0) + t(1,\ -1,\ 1),\ t \in \mathbb{R}$.

200 Incidence 2: planes and lines.

Example: Determine whether the line $(x, y, z) = (1 - t,\ 1 + t,\ 2 + t),\quad t \in \mathbb{R}$, and the plane $2x + y - z = 0$ intersect. If they do, find their intersection point.

Problem 1: Show that the line $\begin{cases} x = 1 + t \\ y = -2t \\ z = 3 + 3t \end{cases}$ $t \in \mathbb{R}$, lies in the plane with equation $3x + 3y + z - 6 = 0$.

Extra material: notes with solved Problem 1.

201 Incidence 3: points and lines.

Problem 2: Show that the points $A = (1, -2, 5)$ and $B = (3, -2, 11)$ belong to the line

$$l\ :\ \frac{x-1}{-1} = \frac{z-5}{-3},\ \ y = -2.$$

Write a parametric equation of l and determine which values of the parameter represent A and B.

Problem 3: Show that the lines $\begin{cases} x = t \\ y = -2t \\ z = 3t \end{cases}$ $t \in \mathbb{R}$, and $\begin{cases} x = -1 + s \\ y = 2 - s \\ z = -3 + 4s \end{cases}$ $s \in \mathbb{R}$, intersect.

Determine the intersection point.

Extra material: notes with solved Problems 2 and 3.

202 Parallel and orthogonal objects.

203 Parallel planes, Problem 4.

Problem 4: Determine whether the following planes are parallel:

$$2x + 3y - 5z + 30 = 0 \quad \text{and} \quad \begin{cases} x = -5 + t \\ y = 2 + 5s + t \\ z = 1 + 3s + t \end{cases} \quad s, t \in \mathbb{R}.$$

Extra material: notes with solved Problem 4.

204 Parallel planes: Problem 5.

Problem 5: Determine whether the following planes are parallel: $3x+y+2z-17=0$ and the plane through $P=(0,1,0)$, parallel to the vectors $\mathbf{u}=(-1,3,0)$ and $\mathbf{v}=(3,1,-5)$.

Extra material: notes with solved Problem 5.

205 Orthogonal planes: Problem 6.

Problem 6: Find an equation on parameter-free form and a parametric equation for the plane that contains the point $P=(2,3,-6)$ and is orthogonal to planes $x+y+z-5=0$ and $x-y+2=0$.

Extra material: notes with solved Problem 6.

206 Planes and lines, Problem 7.

Problem 7: Find an equation for the line λ which includes the point $P_1:(3,-5,4)$, is parallel with the plane $\pi_2: x+2y+3z+4=0$, and is perpendicular to vectors parallel with the line $\lambda_3:(x,y,z)=(-7+2t,\ 5-2t,\ 3-t),\ t\in\mathbb{R}$.

Extra material: notes with solved Problem 7.

207 Planes and lines, Problem 8.

Problem 8: Find an equation on parameter-free form for the plane π which contains the point $P_1:(3,-4,2)$, is parallel with the line $\lambda_2:(x,y,z)=(3+5t,\ 2t,\ 4-3t)$, and is orthogonal to the plane $\pi_3: x+3y+z+2=0$.

Extra material: notes with solved Problem 8.

208 Planes, Problem 9.

Problem 9: Find out whether the planes π_1 och π_2 defined by

$$\begin{cases} \pi_1:(x,y,z)=(1+r+3s,\ 2-2r-s,\ 3+r+2s), & r,s\in\mathbb{R} \\ \pi_2: 3x-y-5z+28=0 \end{cases}$$

intersect or not.

Extra material: notes with solved Problem 9.

209 Planes, lines, and systems of equations, Problem 10

Problem 10:

a) Find a homogenous linear system of two equations in three unknowns whose solution space consists of those vectors in $\mathbb{R}^3$ that are orthogonal to $\mathbf{a}=(1,1,-1)$ and $\mathbf{b}=(2,3,0)$.

b) What kind of geometric object is the solution space?

c) Find a general solution to the system obtained in part a), and confirm that Theorem from the previous slide holds.

Extra material: notes with solved Problem 10.

S19 Distance between points, lines, and planes

You will learn: determine the equations for a line and a plane and how to use these for computing distances.

210 Distances between sets, generally.

211 Distance between points and planes.

212 Distance between points and planes, Problem 1.

Problem 1: Compute the distance between the plane with equation $x+2y-4z=1$ and the line with equation $(x,y,z)=(3,3,3)+t(-2,1,0),\ t\in\mathbb{R}$.

Extra material: notes with solved Problem 1.

213 Distance between points and planes, Problem 2.

Problem 2: Find the point on the plane $\pi: 2x-y+2z=2$ that is closest to the point $P:(1,2,2)$.

(This is the same problem as Extra Problem 8 in resources to Video 219, but I present another solution in Video 213, without formulas, but with help of geometrical reasoning.)

Extra material: notes with solved Problem 2.

214 Distance between points and lines.

215 Distance between points and lines, Problem 3.

Problem 3: Compute (with two different methods) the distance between point $Q = (2, 3, 8)$ and the line with equation $(x, y, z) = (2, 2, 6) + t(1, 1, 0),\ t \in \mathbb{R}$.

Extra material: notes with solved Problem 3.

216 Distance between points and lines, Problem 4.

Problem 4: Find the distance between the lines

$$\lambda_1 : \ \frac{x-2}{4} = \frac{y+1}{2} = \frac{z-5}{-2} \qquad \text{and} \qquad \lambda_2 : \ (x, y, z) = (1-2t,\ 2-t,\ 3+t),\ \ t \in \mathbb{R}.$$

Extra material: notes with solved Problem 4.

217 Distance between (skew) lines.

218 Distance between (skew) lines, Problem 5.

Problem 5: Find the distance between the lines

$$\lambda_1 : \ \frac{x-2}{2} = \frac{y-3}{-2},\ z = -2 \qquad \text{and} \qquad \lambda_2 : \ (x, y, z) = (-t,\ -t,\ t),\ \ t \in \mathbb{R}.$$

Extra material: notes with solved Problem 5.

219 Distance between (skew) lines, Problem 6.

Problem 6: Find the distance between the lines

$$\lambda_1 : \ (x, y, z) = (7, 3, 4) + t(-2, 1, 0),\ \ t \in \mathbb{R} \qquad \text{and} \qquad \lambda_2 : \ (x, y, z) = (1, 0, 1) + s(0, -1, 1),\ \ s \in \mathbb{R}.$$

Extra material: notes with solved Problem 6.

Extra material: an article with more solved problems on planes, lines, and distances.

- ⋆ **Extra problem 1**: Find an equation on parameter-free form for the plane that contains the points $P : (-1, 0, 1)$, $Q : (3, 2, 2)$ and $R : (1, 1, 1)$.
- ⋆ **Extra problem 2**: Find an equation on parameter-free form for the plane containing the points $(1, 0, 3)$, $(-1, 1, 2)$ and $(3, 2, 1)$.
- ⋆ **Extra problem 3**: Find parametric equation of the plane in $\mathbb{R}^3$ that contains the points $P(2, -3, 1)$, $Q(1, 2, 0)$ and $R(-1, 1, 1)$.
- ⋆ **Extra problem 4**: Determine whether there exists a plane which contains the four points $(0, 0, 0)$, $(2, 1, -2)$, $(3, 0, 2)$ and $(0, 1, 4)$.
- ⋆ **Extra problem 5**: Find a parametric equation of the line ℓ which contains the points $(1, -3, -1)$ and $(4, -2, 1)$. Determine whether the line ℓ passes through the point $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$.
- ⋆ **Extra problem 6**: The two lines described below intersect. Find their point of intersection.

$$\begin{cases} \ell_1 : (1+2t,\ -7+5t,\ 5-t), & t \in \mathbb{R} \\ \ell_2 : (5+s,\ s,\ 2-s), & s \in \mathbb{R}. \end{cases}$$

- ⋆ **Extra problem 7**: Find, if possible, a point P on the line $(x, y, z) = (-1+t,\ 3+t,\ 1+2t)$, where $t \in \mathbb{R}$, such that the line which passes through the points P and $Q(0, 1, 2)$ is orthogonal to the line which passes through the points P and $R(-1, 3, 0)$.
- ⋆ **Extra problem 8**: Find the point on the plane $\pi : 2x - y + 2z = 2$ that is closest to the point $P : (1, 2, 2)$.

S20 Some words about the next course

You will learn: about the content of the second course.

220 Linear Algebra and Geometry 1, Wrap-up.

221 Linear Algebra and Geometry 2, some words about it.

222 Final words.