Topic: Solving Ax=b

Question: Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}3 & -6 & 6\\ -3 & 7 & -9\\ -6 & 8 & 0\end{bmatrix}$$

$$\vec{b}=(1,-1,-2)$$

Answer choices:

A $\quad \vec{x}=\begin{bmatrix}\frac{5}{3}\\ 3\\ 0\end{bmatrix}+c_1\begin{bmatrix}4\\ 0\\ 1\end{bmatrix}+c_2\begin{bmatrix}0\\ 3\\ 1\end{bmatrix}$

B $\quad \vec{x}=\begin{bmatrix}\frac{1}{3}\\ 3\\ 0\end{bmatrix}+c_1\begin{bmatrix}4\\ 3\\ 1\end{bmatrix}$

C $\quad \vec{x}=\begin{bmatrix}\frac{5}{3}\\ 0\\ 0\end{bmatrix}+c_1\begin{bmatrix}4\\ 0\\ 1\end{bmatrix}+c_2\begin{bmatrix}0\\ 3\\ 1\end{bmatrix}$

D $\quad \vec{x}=\begin{bmatrix}\frac{1}{3}\\ 0\\ 0\end{bmatrix}+c_1\begin{bmatrix}4\\ 3\\ 1\end{bmatrix}$

Solution: D

To find the general solution to $A\vec{x}=\vec{b}$, we need to find the solutions in the null space, and then the particular solution. Let's start with the solutions in the null space, which we'll find by solving $A\vec{x}=\vec{O}$. To get those null space solutions, we'll augment the matrix.

$$\begin{bmatrix} 3 & -6 & 6 & | & 0 \\ -3 & 7 & -9 & | & 0 \\ -6 & 8 & 0 & | & 0 \end{bmatrix}$$

We want to put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & | & 0 \\ -3 & 7 & -9 & | & 0 \\ -6 & 8 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ -6 & 8 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & -4 & 12 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Let's parse out a system of equations.

$$x_1 - 4x_3 = 0$$

$$x_2 - 3x_3 = 0$$

Solve for the pivot variables in terms of the free variable.

$$x_1 = 4x_3$$

$$x_2 = 3x_3$$

The vector that satisfies the null space is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

We could therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

Now we need to find the particular solution that satisfies $A\vec{x}_p = \vec{b}$. So instead of augmenting the matrix with the zero vector, we augment the matrix with $\vec{b} = (b_1, b_2, b_3)$.

$$\begin{bmatrix} 3 & -6 & 6 & | & b_1 \\ -3 & 7 & -9 & | & b_2 \\ -6 & 8 & 0 & | & b_3 \end{bmatrix}$$

Now we'll put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & | & \frac{1}{3}b_1 \\ -3 & 7 & -9 & | & b_2 \\ -6 & 8 & 0 & | & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 & | & \frac{1}{3}b_1 \\ 0 & 1 & -3 & | & 3(\frac{1}{3}b_1) + b_2 \\ -6 & 8 & 0 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 & | & \frac{1}{3}b_1 \\ 0 & 1 & -3 & | & b_1 + b_2 \\ 0 & -4 & 12 & | & 6(\frac{1}{3}b_1) + b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 & | & \frac{1}{3}b_1 \\ 0 & 1 & -3 & | & b_1 + b_2 \\ 0 & 0 & 0 & | & 4(b_1 + b_2) + 2b_1 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & | & 2(b_1+b_2)+\frac{1}{3}b_1 \\ 0 & 1 & -3 & | & b_1+b_2 \\ 0 & 0 & 0 & | & 6b_1+4b_2+b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & \frac{7}{3}b_1+2b_2 \\ 0 & 1 & -3 & | & b_1+b_2 \\ 0 & 0 & 0 & | & 6b_1+4b_2+b_3 \end{bmatrix}$$

Substitute the values from $\overrightarrow{b} = (1, -1, -2)$.

$$\begin{bmatrix} 1 & 0 & -4 & | & \frac{7}{3}(1)+2(-1) \\ 0 & 1 & -3 & | & 1+(-1) \\ 0 & 0 & 0 & | & 6(1)+4(-1)+(-2) \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & \frac{1}{3} \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Rewrite the matrix as a system of equations.

$$x_1 - 4x_3 = \frac{1}{3}$$

$$x_2 - 3x_3 = 0$$

Now, because x_3 is a free variable, set $x_3 = 0$.

$$x_1 - 4(0) = \frac{1}{3}$$

$$x_2 - 3(0) = 0$$

The system becomes

$$x_1 = \frac{1}{3}$$

$$x_2 = 0$$

So the particular solution is

$$\overrightarrow{x_p} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}$$

We'll get the general solution by adding the particular and complementary solutions.

$$\overrightarrow{x} = \overrightarrow{x_p} + \overrightarrow{x_n}$$

$$\overrightarrow{x} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

Topic: Solving Ax=b

Question: Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}1 & -1 & 3 & -1 & 3\\ -1 & 0 & 0 & 2 & 1\\ 0 & 1 & -3 & -1 & -4\end{bmatrix}$$

$$\vec{b}=(1,-2,1)$$

Answer choices:

A $\vec{x}=\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}+c_1\begin{bmatrix}0\\3\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\1\\0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\4\\0\\0\\1\end{bmatrix}$

B $\vec{x}=\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}+c_1\begin{bmatrix}1\\0\\0\\2\\1\end{bmatrix}+c_2\begin{bmatrix}0\\1\\3\\1\\4\end{bmatrix}$

C $\vec{x}=\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+c_1\begin{bmatrix}0\\3\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\1\\0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\4\\0\\0\\1\end{bmatrix}$

D $\quad \vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 4 \end{bmatrix}$

Solution: A

To find the general solution to $A\vec{x} = \vec{b}$, we need to find the solutions in the null space, and then the particular solution. Let's start with the solutions in the null space, which we'll find by solving $A\vec{x} = \vec{O}$. To get those null space solutions, we'll augment the matrix.

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & 0 \\ -1 & 0 & 0 & 2 & 1 & | & 0 \\ 0 & 1 & -3 & -1 & -4 & | & 0 \end{bmatrix}$$

We want to put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & 0 \\ 0 & -1 & 3 & 1 & 4 & | & 0 \\ 0 & 1 & -3 & -1 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & 0 \\ 0 & 1 & -3 & -1 & -4 & | & 0 \\ 0 & 1 & -3 & -1 & -4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & 0 \\ 0 & 1 & -3 & -1 & -4 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & -1 & | & 0 \\ 0 & 1 & -3 & -1 & -4 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Let's parse out a system of equations.

$$x_1 - 2x_4 - x_5 = 0$$

$$x_2 - 3x_3 - x_4 - 4x_5 = 0$$

Solve for the pivot variables in terms of the free variables.

$$x_1 = 2x_4 + x_5$$

$$x_2 = 3x_3 + x_4 + 4x_5$$

The vectors that satisfy the null space are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We could therefore write the complementary solution as

$$\vec{x_n} = c_1 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Now we need to find the particular solution that satisfies $A\vec{x}_p = \vec{b}$. So instead of augmenting the matrix with the zero vector, we augment the matrix with $\vec{b} = (b_1, b_2, b_3)$.

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & b_1 \\ -1 & 0 & 0 & 2 & 1 & | & b_2 \\ 0 & 1 & -3 & -1 & -4 & | & b_3 \end{bmatrix}$$

Now we'll put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & b_1 \\ 0 & -1 & 3 & 1 & 4 & | & b_2+b_1 \\ 0 & 1 & -3 & -1 & -4 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & b_1 \\ 0 & 1 & -3 & -1 & -4 & | & -b_2-b_1 \\ 0 & 1 & -3 & -1 & -4 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & b_1 \\ 0 & 1 & -3 & -1 & -4 & | & -b_2-b_1 \\ 0 & 0 & 0 & 0 & 0 & | & b_3+b_2+b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & -1 & | & -b_2 \\ 0 & 1 & -3 & -1 & -4 & | & -b_2-b_1 \\ 0 & 0 & 0 & 0 & 0 & | & b_3+b_2+b_1 \end{bmatrix}$$

Substitute the values from $\overrightarrow{b}=(1,-2,1)$.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -1 & | & -(-2) \\ 0 & 1 & -3 & -1 & -4 & | & -(-2)-1 \\ 0 & 0 & 0 & 0 & 0 & | & 1-2+1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & -1 & | & 2 \\ 0 & 1 & -3 & -1 & -4 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Rewrite the matrix as a system of equations.

$$x_1-2x_4-x_5=2$$

$$x_2-3x_3-x_4-4x_5=1$$

Now, because x_3, x_4, and x_5 are free variables, set $x_3=0$, $x_4=0$, and $x_5=0$.

$$x_1-2(0)-0=2$$

$$x_2-3(0)-0-4(0)=1$$

The system becomes

$$x_1=2$$

$$x_2 = 1$$

So the particular solution is

$$\overrightarrow{x_p} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We'll get the general solution by adding the particular and complementary solutions.

$$\overrightarrow{x} = \overrightarrow{x_p} + \overrightarrow{x_n}$$

$$\overrightarrow{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Topic: Solving Ax=b

Question: Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}2 & 3 & 4 & -4\\ 2 & 3 & 8 & -10\\ 6 & 9 & 16 & -18\end{bmatrix}$$

$$\vec{b}=(1,-1,1)$$

Answer choices:

A $\vec{x}=\begin{bmatrix}1\\ 0\\ -\frac{1}{4}\\ 0\end{bmatrix}+c_1\begin{bmatrix}\frac{3}{2}\\ 0\\ 0\\ 0\end{bmatrix}+c_2\begin{bmatrix}-1\\ 0\\ 0\\ \frac{3}{2}\end{bmatrix}$

B $\vec{x}=\begin{bmatrix}\frac{3}{2}\\ 0\\ -\frac{1}{2}\\ 0\end{bmatrix}+c_1\begin{bmatrix}-\frac{3}{2}\\ 1\\ 0\\ 0\end{bmatrix}+c_2\begin{bmatrix}-1\\ 0\\ \frac{3}{2}\\ 1\end{bmatrix}$

C $\vec{x}=\begin{bmatrix}1\\ 0\\ -\frac{1}{4}\\ 0\end{bmatrix}$

D $\quad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{4} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 1 \\ \frac{3}{2} \end{bmatrix}$

Solution: B

To find the general solution to $A\vec{x} = \vec{b}$, we need to find the solutions in the null space, and then the particular solution. Let's start with the solutions in the null space, which we'll find by solving $A\vec{x} = \vec{O}$. To get those null space solutions, we'll augment the matrix.

$$\begin{bmatrix} 2 & 3 & 4 & -4 & | & 0 \\ 2 & 3 & 8 & -10 & | & 0 \\ 6 & 9 & 16 & -18 & | & 0 \end{bmatrix}$$

We want to put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & 0 \\ 2 & 3 & 8 & -10 & | & 0 \\ 6 & 9 & 16 & -18 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & 0 \\ 0 & 0 & 4 & -6 & | & 0 \\ 6 & 9 & 16 & -18 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & 0 \\ 0 & 0 & 4 & -6 & | & 0 \\ 0 & 0 & 4 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & | & 0 \\ 0 & 0 & 4 & -6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Let’s parse out a system of equations.

$$x_1 + \frac{3}{2}x_2 + x_4 = 0$$

$$x_3 - \frac{3}{2}x_4 = 0$$

Solve for the pivot variables in terms of the free variables.

$$x_1 = -\frac{3}{2}x_2 - x_4$$

$$x_3 = \frac{3}{2}x_4$$

The vectors that satisfy the null space are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

We could therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

Now we need to find the particular solution that satisfies $A\vec{x}_p = \vec{b}$. So instead of augmenting the matrix with the zero vector, we augment the matrix with $\vec{b} = (b_1, b_2, b_3)$.

$$\begin{bmatrix} 2 & 3 & 4 & -4 & | & b_1 \\ 2 & 3 & 8 & -10 & | & b_2 \\ 6 & 9 & 16 & -18 & | & b_3 \end{bmatrix}$$

Now we'll again put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 2 & 3 & 8 & -10 & | & b_2 \\ 6 & 9 & 16 & -18 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 0 & 0 & 4 & -6 & | & -2(\frac{1}{2}b_1) + b_2 \\ 6 & 9 & 16 & -18 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 0 & 0 & 4 & -6 & | & -b_1 + b_2 \\ 0 & 0 & 4 & -6 & | & -6(\frac{1}{2}b_1) + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 0 & 0 & 4 & -6 & | & -b_1 + b_2 \\ 0 & 0 & 0 & 0 & | & -3b_1 + b_3 - (-b_1 + b_2) \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 0 & 0 & 1 & -\frac{3}{2} & | & \frac{1}{4}(-b_1 + b_2) \\ 0 & 0 & 0 & 0 & | & -2b_1 + b_3 - b_2 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & -2 \cdot \frac{1}{4}(-b_1 + b_2) + \frac{1}{2}b_1 \\ 0 & 0 & 1 & -\frac{3}{2} & | & \frac{1}{4}(-b_1 + b_2) \\ 0 & 0 & 0 & 0 & | & -2b_1 + b_3 - b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & b_1 - \frac{1}{2}b_2 \\ 0 & 0 & 1 & -\frac{3}{2} & | & \frac{1}{4}(-b_1 + b_2) \\ 0 & 0 & 0 & 0 & | & -2b_1 + b_3 - b_2 \end{bmatrix}$$

Substitute the values from $\overrightarrow{b} = (1, -1, 1)$.

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & 1 - \frac{1}{2}(-1) \\ 0 & 0 & 1 & -\frac{3}{2} & | & \frac{1}{4}(-1 + (-1)) \\ 0 & 0 & 0 & 0 & | & -2(1) + 1 - (-1) \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & | & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Rewrite the matrix as a system of equations.

$$x_1 + \frac{3}{2}x_2 + x_4 = \frac{3}{2}$$

$$x_3 - \frac{3}{2}x_4 = -\frac{1}{2}$$

Now, because x_2 and x_4 are free variables, set $x_2 = 0$ and $x_4 = 0$.

$$x_1 + \frac{3}{2}(0) + 0 = \frac{3}{2}$$

$$x_3 - \frac{3}{2}(0) = -\frac{1}{2}$$

The system becomes

$$x_1 = \frac{3}{2}$$

$$x_3 = -\frac{1}{2}$$

So the particular solution is

$$\overrightarrow{x_p} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

We'll get the general solution by adding the particular and complementary solutions.

$$\overrightarrow{x} = \overrightarrow{x_p} + \overrightarrow{x_n}$$

$$\overrightarrow{x} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$