**Topic**: Projection onto an orthonormal basis

**Question**: Find the projection of $\overrightarrow{x} = (-15, -75, 25)$ onto the subspace $V$.

$$V = \text{Span}\left(\begin{bmatrix} \frac{5}{\sqrt{50}} \\ -\frac{3}{\sqrt{50}} \\ \frac{4}{\sqrt{50}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}\right)$$

**Answer choices**:

A $\text{Proj}_V \overrightarrow{x} = \begin{bmatrix} \frac{140}{3} \\ -\frac{92}{3} \\ \frac{98}{3} \end{bmatrix}$

B $\text{Proj}_V \overrightarrow{x} = \begin{bmatrix} \frac{80}{3} \\ -\frac{50}{3} \\ \frac{50}{3} \end{bmatrix}$

C $\text{Proj}_V \overrightarrow{x} = \begin{bmatrix} \frac{80}{3} \\ -\frac{92}{3} \\ \frac{56}{3} \end{bmatrix}$

D $\text{Proj}_V \overrightarrow{x} = \begin{bmatrix} -\frac{80}{3} \\ \frac{50}{3} \\ -\frac{50}{3} \end{bmatrix}$

**Solution**: B

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2=\left(\frac{5}{\sqrt{50}}\right)^2+\left(-\frac{3}{\sqrt{50}}\right)^2+\left(\frac{4}{\sqrt{50}}\right)^2=\frac{25}{50}+\frac{9}{50}+\frac{16}{50}=\frac{50}{50}=1$$

$$||\vec{v}_2||^2=\left(\frac{1}{\sqrt{6}}\right)^2+\left(-\frac{1}{\sqrt{6}}\right)^2+\left(-\frac{2}{\sqrt{6}}\right)^2=\frac{1}{6}+\frac{1}{6}+\frac{4}{6}=\frac{6}{6}=1$$

The length of both vectors is $1$, and the dot product of the vectors is

$$\vec{v}_1\cdot\vec{v}_2=\frac{5}{\sqrt{50}}\left(\frac{1}{\sqrt{6}}\right)+\left(-\frac{3}{\sqrt{50}}\right)\left(-\frac{1}{\sqrt{6}}\right)+\frac{4}{\sqrt{50}}\left(-\frac{2}{\sqrt{6}}\right)$$

$$=\frac{5}{\sqrt{300}}+\frac{3}{\sqrt{300}}-\frac{8}{\sqrt{300}}=0$$

Because the vectors are orthogonal to one another, and because they both have the length of $1$, the set is orthonormal. So the projection of $\vec{x}=(-15,-75,25)$ onto $V$ is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{5}{\sqrt{50}} & \frac{1}{\sqrt{6}}\\ -\frac{3}{\sqrt{50}} & -\frac{1}{\sqrt{6}}\\ \frac{4}{\sqrt{50}} & -\frac{2}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{5}{\sqrt{50}} & -\frac{3}{\sqrt{50}} & \frac{4}{\sqrt{50}}\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{5}{\sqrt{50}}\left(\frac{5}{\sqrt{50}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & \frac{5}{\sqrt{50}}\left(-\frac{3}{\sqrt{50}}\right)+\frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}\right) & \frac{5}{\sqrt{50}}\left(\frac{4}{\sqrt{50}}\right)+\frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right)\\ -\frac{3}{\sqrt{50}}\left(\frac{5}{\sqrt{50}}\right)-\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & -\frac{3}{\sqrt{50}}\left(-\frac{3}{\sqrt{50}}\right)-\frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}\right) & -\frac{3}{\sqrt{50}}\left(\frac{4}{\sqrt{50}}\right)-\frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right)\\ \frac{4}{\sqrt{50}}\left(\frac{5}{\sqrt{50}}\right)-\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & \frac{4}{\sqrt{50}}\left(-\frac{3}{\sqrt{50}}\right)-\frac{2}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}\right) & \frac{4}{\sqrt{50}}\left(\frac{4}{\sqrt{50}}\right)-\frac{2}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{25}{50}+\frac{1}{6} & -\frac{15}{50}-\frac{1}{6} & \frac{20}{50}-\frac{2}{6}\\ -\frac{15}{50}-\frac{1}{6} & \frac{9}{50}+\frac{1}{6} & -\frac{12}{50}+\frac{2}{6}\\ \frac{20}{50}-\frac{2}{6} & -\frac{12}{50}+\frac{2}{6} & \frac{16}{50}+\frac{4}{6}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{2}+\frac{1}{6} & -\frac{3}{10}-\frac{1}{6} & \frac{2}{5}-\frac{1}{3}\\ -\frac{3}{10}-\frac{1}{6} & \frac{9}{50}+\frac{1}{6} & -\frac{6}{25}+\frac{1}{3}\\ \frac{2}{5}-\frac{1}{3} & -\frac{6}{25}+\frac{1}{3} & \frac{8}{25}+\frac{2}{3}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{2}{3} & -\frac{7}{15} & \frac{1}{15}\\ -\frac{7}{15} & \frac{26}{75} & \frac{7}{75}\\ \frac{1}{15} & \frac{7}{75} & \frac{74}{75}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(-15,-75,25)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{2}{3} & -\frac{7}{15} & \frac{1}{15}\\ -\frac{7}{15} & \frac{26}{75} & \frac{7}{75}\\ \frac{1}{15} & \frac{7}{75} & \frac{74}{75}\end{bmatrix}\begin{bmatrix}-15\\ -75\\ 25\end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{2}{3}(-15) - \frac{7}{15}(-75) + \frac{1}{15}(25) \\ -\frac{7}{15}(-15) + \frac{26}{75}(-75) + \frac{7}{75}(25) \\ \frac{1}{15}(-15) + \frac{7}{75}(-75) + \frac{74}{75}(25) \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} -10 + 35 + \frac{5}{3} \\ 7 - 26 + \frac{7}{3} \\ -1 - 7 + \frac{74}{3} \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{80}{3} \\ -\frac{50}{3} \\ \frac{50}{3} \end{bmatrix}$$

**Topic**: Projection onto an orthonormal basis

**Question**: Find the projection of $\vec{x}=(-1,2,-2)$ onto the subspace $V$.

$$V=\text{Span}\left(\begin{bmatrix}\frac{2}{3}\\ -\frac{1}{3}\\ \frac{2}{3}\end{bmatrix},\begin{bmatrix}\frac{2}{3}\\ \frac{2}{3}\\ -\frac{1}{3}\end{bmatrix}\right)$$

**Answer choices**:

A $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{9}\\ \frac{16}{9}\\ -\frac{16}{9}\end{bmatrix}$

B $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{3}\\ \frac{16}{3}\\ -\frac{8}{3}\end{bmatrix}$

C $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{3}\\ \frac{16}{3}\\ -\frac{14}{3}\end{bmatrix}$

D $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{9}\\ \frac{16}{9}\\ -\frac{20}{9}\end{bmatrix}$

**Solution**: D

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = \frac{9}{9} = 1$$

$$||\vec{v}_2||^2 = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = \frac{9}{9} = 1$$

The length of both vectors is $1$, and the dot product of the vectors is

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{2}{3}\left(\frac{2}{3}\right) - \frac{1}{3}\left(\frac{2}{3}\right) + \frac{2}{3}\left(-\frac{1}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$$

Because the vectors are orthogonal to one another, and because they both have the length of $1$, the set is orthonormal. So the projection of $\vec{x} = (-1,2,-2)$ onto $V$ is

$$\text{Proj}_V\vec{x} = AA^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{2}{3}\left(\frac{2}{3}\right) + \frac{2}{3}\left(\frac{2}{3}\right) & \frac{2}{3}\left(-\frac{1}{3}\right) + \frac{2}{3}\left(\frac{2}{3}\right) & \frac{2}{3}\left(\frac{2}{3}\right) + \frac{2}{3}\left(-\frac{1}{3}\right) \\ -\frac{1}{3}\left(\frac{2}{3}\right) + \frac{2}{3}\left(\frac{2}{3}\right) & -\frac{1}{3}\left(-\frac{1}{3}\right) + \frac{2}{3}\left(\frac{2}{3}\right) & -\frac{1}{3}\left(\frac{2}{3}\right) + \frac{2}{3}\left(-\frac{1}{3}\right) \\ \frac{2}{3}\left(\frac{2}{3}\right) - \frac{1}{3}\left(\frac{2}{3}\right) & \frac{2}{3}\left(-\frac{1}{3}\right) - \frac{1}{3}\left(\frac{2}{3}\right) & \frac{2}{3}\left(\frac{2}{3}\right) - \frac{1}{3}\left(-\frac{1}{3}\right) \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{4}{9}+\frac{4}{9} & -\frac{2}{9}+\frac{4}{9} & \frac{4}{9}-\frac{2}{9} \\ -\frac{2}{9}+\frac{4}{9} & \frac{1}{9}+\frac{4}{9} & -\frac{2}{9}-\frac{2}{9} \\ \frac{4}{9}-\frac{2}{9} & -\frac{2}{9}-\frac{2}{9} & \frac{4}{9}+\frac{1}{9} \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{8}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{5}{9} & -\frac{4}{9} \\ \frac{2}{9} & -\frac{4}{9} & \frac{5}{9} \end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x} = (-1,2,-2)$ gives

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{8}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{5}{9} & -\frac{4}{9} \\ \frac{2}{9} & -\frac{4}{9} & \frac{5}{9} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{8}{9}(-1)+\frac{2}{9}(2)+\frac{2}{9}(-2) \\ \frac{2}{9}(-1)+\frac{5}{9}(2)-\frac{4}{9}(-2) \\ \frac{2}{9}(-1)-\frac{4}{9}(2)+\frac{5}{9}(-2) \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -\frac{8}{9}+\frac{4}{9}-\frac{4}{9} \\ -\frac{2}{9}+\frac{10}{9}+\frac{8}{9} \\ -\frac{2}{9}-\frac{8}{9}-\frac{10}{9} \end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{9}\\ \frac{16}{9}\\ -\frac{20}{9}\end{bmatrix}$$

**Topic**: Projection onto an orthonormal basis

**Question**: Find the projection of $\vec{x} = (3, -2, 4)$ onto the subspace $V$.

$$V = \text{Span}\left(\begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}\right)$$

**Answer choices**:

A $\quad \text{Proj}_V \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

B $\quad \text{Proj}_V \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

C $\quad \text{Proj}_V \vec{x} = \begin{bmatrix} 3 \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

D $\quad \text{Proj}_V \vec{x} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$

**Solution**: A

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = \left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$$

$$||\vec{v}_2||^2 = \left(\frac{2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = \frac{4}{6} + \frac{1}{6} + \frac{1}{6} = \frac{6}{6} = 1$$

The length of both vectors is $1$, and the dot product of the vectors is

$$\vec{v}_1 \cdot \vec{v}_2 = \left(-\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right)$$

$$= -\frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$$

Because the vectors are orthogonal to one another, and because they both have the length of $1$, the set is orthonormal. So the projection of $\vec{x} = (3, -2, 4)$ onto $V$ is

$$\text{Proj}_V \vec{x} = AA^T\vec{x}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{6}}\left(\frac{2}{\sqrt{6}}\right) & -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) \\ \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{2}{\sqrt{6}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) \\ \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{2}{\sqrt{6}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{1}{3}+\frac{4}{6} & -\frac{1}{3}+\frac{1}{3} & -\frac{1}{3}+\frac{1}{3} \\ -\frac{1}{3}+\frac{1}{3} & \frac{1}{3}+\frac{1}{6} & \frac{1}{3}+\frac{1}{6} \\ -\frac{1}{3}+\frac{1}{3} & \frac{1}{3}+\frac{1}{6} & \frac{1}{3}+\frac{1}{6} \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x} = (3,-2,4)$ gives

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1(3)+0(-2)+0(4) \\ 0(3)+\frac{1}{2}(-2)+\frac{1}{2}(4) \\ 0(3)+\frac{1}{2}(-2)+\frac{1}{2}(4) \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} 3+0+0 \\ 0-1+2 \\ 0-1+2 \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$