Transposes of products, sums, and inverses

Now that we understand the idea of a transpose as a matrix with swapped rows and columns, we want to expand our understanding to matrix operations.

More specifically, in this lesson we'll look at how we define the transpose of a matrix product, and then the transpose of the sum of matrices.

Transpose of a matrix product

We have a special rule when it comes to transposing the product of matrices. Let's say that we have two matrices X and Y, $m \times n$ and $n \times m$, respectively, and we multiply X by Y to get the matrix product XY. The transpose of XY will be $(XY)^T$.

This transpose of the product is actually equivalent to the product of the individual transposes, but in the opposite order. In other words, the individual transposes are X^T and Y^T, $n \times m$ and $m \times n$, respectively, and instead of multiplying X by Y, we'd use the opposite order and multiply Y^T by X^T. So

$$(XY)^T = Y^T X^T$$

And this extends to the product of any number of matrices. So for instance, for the product of three matrices X, Y, and Z, the transpose of the matrix XYZ would be

$$(XYZ)^T = Z^T Y^T X^T$$

Let’s do an example to prove to ourselves that the transpose “flips the order” of the product.

Example

Show that $(XY)^T = Y^T X^T$.

$$X = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

Let’s first calculate the transpose of the product by finding the matrix XY, and then taking its transpose. Multiplying the matrices gives

$$XY = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

$$XY = \begin{bmatrix} 2(-6) + 1(2) & 2(0) + 1(-1) \\ -3(-6) + 0(2) & -3(0) + 0(-1) \end{bmatrix}$$

$$XY = \begin{bmatrix} -12 + 2 & 0 - 1 \\ 18 + 0 & 0 + 0 \end{bmatrix}$$

$$XY = \begin{bmatrix} -10 & -1 \\ 18 & 0 \end{bmatrix}$$

Then the transpose of XY is what we get when we swap the rows and columns:

$$(XY)^T = \begin{bmatrix} -10 & 18 \\ -1 & 0 \end{bmatrix}$$

Now let's try the method where we transpose the matrices individually, and then multiply those transposes in the reverse order. The transpose X^T is

$$X^T = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$$

The transpose Y^T is

$$Y^T = \begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix}$$

So the product $Y^T X^T$ is

$$Y^T X^T = \begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$$

$$Y^T X^T = \begin{bmatrix} -6(2) + 2(1) & -6(-3) + 2(0) \\ 0(2) - 1(1) & 0(-3) - 1(0) \end{bmatrix}$$

$$Y^T X^T = \begin{bmatrix} -12 + 2 & 18 + 0 \\ 0 - 1 & 0 - 0 \end{bmatrix}$$

$$Y^T X^T = \begin{bmatrix} -10 & 18 \\ -1 & 0 \end{bmatrix}$$

By calculating both values, we've shown they're equivalent, which means that $(XY)^T = Y^T X^T$.

Transpose of a matrix sum

We also know that, given two matrices X and Y, which must be the same size, the transpose of their sum $(X+Y)^T$ is equivalent to the sum of their individual transposes.

$$(X+Y)^T = X^T + Y^T$$

Which means that, just like with the matrix product, we can calculate the transpose two ways.

We can either find the sum of the matrices, and then transpose the sum, or we can take the transpose of each matrix individually, and then sum the individual transposes. And this extends to the sum of any number of matrices.

Example

Show that $(X+Y)^T = X^T + Y^T$.

$$X = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

Let's first calculate the transpose of the sum by finding the sum $X+Y$, and then taking its transpose. Adding the matrices gives

$$X+Y=\begin{bmatrix}2 & 1\\ -3 & 0\end{bmatrix}+\begin{bmatrix}-6 & 0\\ 2 & -1\end{bmatrix}$$

$$X+Y=\begin{bmatrix}2+(-6) & 1+0\\ -3+2 & 0+(-1)\end{bmatrix}$$

$$X+Y=\begin{bmatrix}2-6 & 1+0\\ -3+2 & 0-1\end{bmatrix}$$

$$X+Y=\begin{bmatrix}-4 & 1\\ -1 & -1\end{bmatrix}$$

Then the transpose of $X+Y$ is what we get when we swap the rows and columns:

$$(X+Y)^T=\begin{bmatrix}-4 & -1\\ 1 & -1\end{bmatrix}$$

Now let's try the method where we transpose the matrices individually, and then add those transposes. The transpose X^T is

$$X^T=\begin{bmatrix}2 & -3\\ 1 & 0\end{bmatrix}$$

The transpose Y^T is

$$Y^T=\begin{bmatrix}-6 & 2\\ 0 & -1\end{bmatrix}$$

So the sum $X^T + Y^T$ is

$$X^T + Y^T = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix}$$

$$X^T + Y^T = \begin{bmatrix} 2+(-6) & -3+2 \\ 1+0 & 0+(-1) \end{bmatrix}$$

$$X^T + Y^T = \begin{bmatrix} 2-6 & -3+2 \\ 1+0 & 0-1 \end{bmatrix}$$

$$X^T + Y^T = \begin{bmatrix} -4 & -1 \\ 1 & -1 \end{bmatrix}$$

By calculating both values, we've shown they're equivalent, which means that $(X+Y)^T = X^T + Y^T$.

Transpose of a matrix inverse

And when it comes to a square matrix X and its inverse X^{-1}, their transposes are also inverses of one another. In other words, X^T is the inverse of $(X^{-1})^T$. Which means we can also say that the inverse matrix of the transpose is equivalent to the transpose of the inverse matrix.

$$(X^T)^{-1} = (X^{-1})^T$$

Let's do an example to prove that we get the same result, whether we take the inverse and then transpose it, or transpose the matrix and then take the inverse.

Example

Show that $(X^T)^{-1} = (X^{-1})^T$.

$$X = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

Let's see what we get when we transpose X, and then find the inverse of the transpose. The transpose is

$$X^T = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$$

Then we'll find the inverse of the transpose. Augment the matrix with I_2, find the pivot in the first column, then zero out the rest of the first column.

$$\begin{bmatrix} 2 & -3 & | & 1 & 0 \\ 1 & 0 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & | & \frac{1}{2} & 0 \\ 1 & 0 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & | & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & | & -\frac{1}{2} & 1 \end{bmatrix}$$

Find the pivot in the second column, then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -\frac{3}{2} & | & \frac{1}{2} & 0 \\ 0 & 1 & | & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 & 1 \\ 0 & 1 & | & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

So the inverse of the transpose is

$$(X^T)^{-1} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Now let's see what we get when we find the inverse and then transpose it. To find the inverse, first augment X with I_2, find the pivot in the first column, then zero out the rest of the first column.

$$\begin{bmatrix} 2 & 1 & | & 1 & 0 \\ -3 & 0 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ -3 & 0 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & | & \frac{3}{2} & 1 \end{bmatrix}$$

Find the pivot in the second column, then zero out the rest of the second column.

$$\begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ 0 & 1 & | & 1 & \frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 & -\frac{1}{3} \\ 0 & 1 & | & 1 & \frac{2}{3} \end{bmatrix}$$

So the inverse is

$$X^{-1} = \begin{bmatrix} 0 & -\frac{1}{3} \\ 1 & \frac{2}{3} \end{bmatrix}$$

Then the transpose of the inverse is

$$(X^{-1})^T = \begin{bmatrix} 0 & 1 \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

By calculating both values, we've shown they're equivalent, which means that $(X^T)^{-1} = (X^{-1})^T$.