Multiplying matrices by vectors

We already learned how to multiply matrices. As a reminder, matrix multiplication is not commutative, so if A and B are matrices, then $AB \neq BA$. Order matters.

Furthermore, we learned that the number of rows in the second matrix had to equal the number of columns in the first matrix in order for the product to be defined.

Vectors as matrices

And we know that vectors can be written as column matrices (with one column and any finite number of rows) or as row matrices (with one row and any finite number of columns).

That being said, hopefully it won't surprise you too much to say that we can multiply matrices by vectors. After all, we've already been writing vectors as column matrices. So if before we multiplied matrix A by matrix B,

$$AB = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(1) + 0(0) - 2(-2) + 3(0) \\ 0(1) + 4(0) + 3(-2) - 1(0) \end{bmatrix}$$

$$AB = \begin{bmatrix} 1+0+4+0 \\ 0+0-6-0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

we could instead multiply matrix A by vector $\overrightarrow{b}$:

$$A\overrightarrow{b} = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

$$A\overrightarrow{b} = \begin{bmatrix} 1(1)+0(0)-2(-2)+3(0) \\ 0(1)+4(0)+3(-2)-1(0) \end{bmatrix}$$

$$A\overrightarrow{b} = \begin{bmatrix} 1+0+4+0 \\ 0+0-6-0 \end{bmatrix}$$

$$A\overrightarrow{b} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

Both operations (multiplying a matrix by a matrix, and multiplying a matrix by a vector) are valid operations in which we get the same result. And of course, that's because a vector is like a one column matrix or a one row matrix.

We'll need to be really comfortable finding matrix-vector products, so let's do an example.

Example

Find the matrix-vector product, $M\vec{v}$.

$$M = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 0 \\ -6 & 1 & -2 \end{bmatrix}$$

$$\vec{v} = (-2,1,3)$$

To find $M\vec{v}$, we'll multiply the matrix M by the column vector $\vec{v}$. We know the product is defined, since the matrix has 3 columns and the vector has 3 rows.

$$M\vec{v} = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 0 \\ -6 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$$M\vec{v} = \begin{bmatrix} 4(-2) - 2(1) + 1(3) \\ 0(-2) + 3(1) + 0(3) \\ -6(-2) + 1(1) - 2(3) \end{bmatrix}$$

$$M\vec{v} = \begin{bmatrix} -8 - 2 + 3 \\ 0 + 3 + 0 \\ 12 + 1 - 6 \end{bmatrix}$$

$$M\vec{v} = \begin{bmatrix} -7 \\ 3 \\ 7 \end{bmatrix}$$

Notice how, in this last example, multiplying the matrix by a column vector resulted in another column vector.

This will always be the case. If we multiply an $m \times n$ matrix by an n-row column vector, the result will be an m-row column vector.

On the other hand, if we multiply an m-column row vector by an $m \times n$ matrix, the result will be an n-column row vector. Let's show that by flipping around the last example.

Example

Find the matrix-vector product, $\vec{v}M$.

$$M = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 0 \\ -6 & 1 & -2 \end{bmatrix}$$

$$\vec{v} = (-2,1,3)$$

To find $\vec{v}M$, we'll multiply the row vector $\vec{v}$ by the matrix M. We know the product is defined, since the vector has 3 columns and the matrix has 3 rows.

$$\vec{v}M = \begin{bmatrix} -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 0 \\ -6 & 1 & -2 \end{bmatrix}$$

$$\vec{v}M = \begin{bmatrix} -2(4) + 1(0) + 3(-6) & -2(-2) + 1(3) + 3(1) & -2(1) + 1(0) + 3(-2) \end{bmatrix}$$

$$\vec{v}M = \begin{bmatrix} -8 + 0 - 18 & 4 + 3 + 3 & -2 + 0 - 6 \end{bmatrix}$$

$$\vec{v}M = \begin{bmatrix} -26 & 10 & -8 \end{bmatrix}$$