Angle between vectors

The angle between two vectors, θ, is given by a relationship between the dot product of the vectors and the lengths of the vectors.

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos\theta$$

In order to use this formula to find the angle between the vectors, we need both vectors $\vec{u}$ and $\vec{v}$ to be non-zero. That's because, if either vector is the zero vector, we'll be left with the equation $0 = 0\cos\theta$, or just $0 = 0$, and won't be able to solve for the angle between them.

And this makes sense. Because the zero vector has no length and no direction, it wouldn't make sense to look for an angle between the zero vector and another vector. If we're trying to solve for the angle between two vectors, it only makes sense to do so if both of them are non-zero.

Example

Find the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (2, -1), \ \vec{v} = (-1, 4)$$

First, let's find the length of both $\vec{u}$ and $\vec{v}$.

$$||\vec{u}|| = \sqrt{u_1^2 + u_2^2} = \sqrt{2^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$$

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2} = \sqrt{(-1)^2 + 4^2} = \sqrt{1+16} = \sqrt{17}$$

Now we can plug everything into the formula for the angle between vectors.

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}||\cos\theta$$

$$(2, -1) \cdot (-1,4) = \sqrt{5}\sqrt{17}\cos\theta$$

Calculate the dot product, and simplify.

$$(2)(-1) + (-1)(4) = \sqrt{85}\cos\theta$$

$$-2 - 4 = \sqrt{85}\cos\theta$$

$$-6 = \sqrt{85}\cos\theta$$

$$-\frac{6}{\sqrt{85}} = \cos\theta$$

Take the inverse cosine of each side of the equation to solve for θ.

$$\arccos\left(-\frac{6}{\sqrt{85}}\right) = \arccos(\cos\theta)$$

$$\theta = \arccos\left(-\frac{6}{\sqrt{85}}\right)$$

If we use a calculator to find this arccosine value, we find that the angle between $\vec{u}$ and $\vec{v}$ is $\theta \approx 130.62^\circ$.

Perpendicular vectors

If the vectors are perpendicular, then $\theta = 90^\circ$ (in degrees) or $\theta = \pi/2$ (in radians). If we substitute $\theta = 90^\circ$ into the formula for the angle between vectors, we get an interesting result.

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}||\cos\theta$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}||\cos(90^\circ)$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}||(0)$$

$$\vec{u} \cdot \vec{v} = 0$$

We're left with just the dot product of $\vec{u}$ and $\vec{v}$. Which means that, if we calculate a dot product of two non-zero vectors and find that the dot product is 0, we know right away that those vectors are perpendicular.

Notice how we said the two vectors had to be non-zero. Again, if either vector is the zero vector, we'll get 0 for the dot product every time. In other words, if the dot product of two vectors is 0, it means one of two things must be true:

1. at least one of the vectors is the zero vector, or

2. the vectors are perpendicular.

Orthogonality

When we talk about two vectors being perpendicular to one another, it's easy for us to understand this in two dimensions. We picture two vectors that lie at a 90° angle from one another, and we call them perpendicular.

But this concept gets a little murky when we start talking about three dimensions, or even n dimensions. After all, if two vectors are both defined in n dimensions, does it really make much sense to say they're "perpendicular?"

Our intuitive understanding of "perpendicular" really starts to break down in anything higher than two dimensions, so for three or more dimensions, we use a new word to define the idea of "perpendicular," and we'll use the word "orthogonal." If the dot product of two vectors is 0, we say that they're **orthogonal** to one another.

And where before we said that our two vectors both had to be non-zero in order for them to be perpendicular, we no longer exclude the zero vector from the definition once we switch our language from perpendicularity to orthogonality.

So any two vectors whose dot product is 0 are orthogonal (regardless of whether one or both of them is the zero vector). Which means

1. the zero vector is orthogonal to every non-zero vector, and

2. the zero vector is orthogonal to itself.

And of course, just like with perpendicularity, any two non-zero vectors whose dot product is 0 are orthogonal to one another.

Example

Say whether or not the vectors are orthogonal.

$$\overrightarrow{u} = (2, -1, 0), \ \overrightarrow{v} = (0, 0, 3)$$

If the vectors are orthogonal to one another, then their dot product will be 0. So let's take the dot product of $\overrightarrow{u}$ and $\overrightarrow{v}$.

$$\overrightarrow{u} \cdot \overrightarrow{v} = (2, -1, 0) \cdot (0, 0, 3)$$

$$\overrightarrow{u} \cdot \overrightarrow{v} = (2)(0) + (-1)(0) + (0)(3)$$

$$\overrightarrow{u} \cdot \overrightarrow{v} = 0 + 0 + 0$$

$$\overrightarrow{u} \cdot \overrightarrow{v} = 0$$

Because the dot product is 0, $\overrightarrow{u}$ and $\overrightarrow{v}$ are orthogonal to one another.