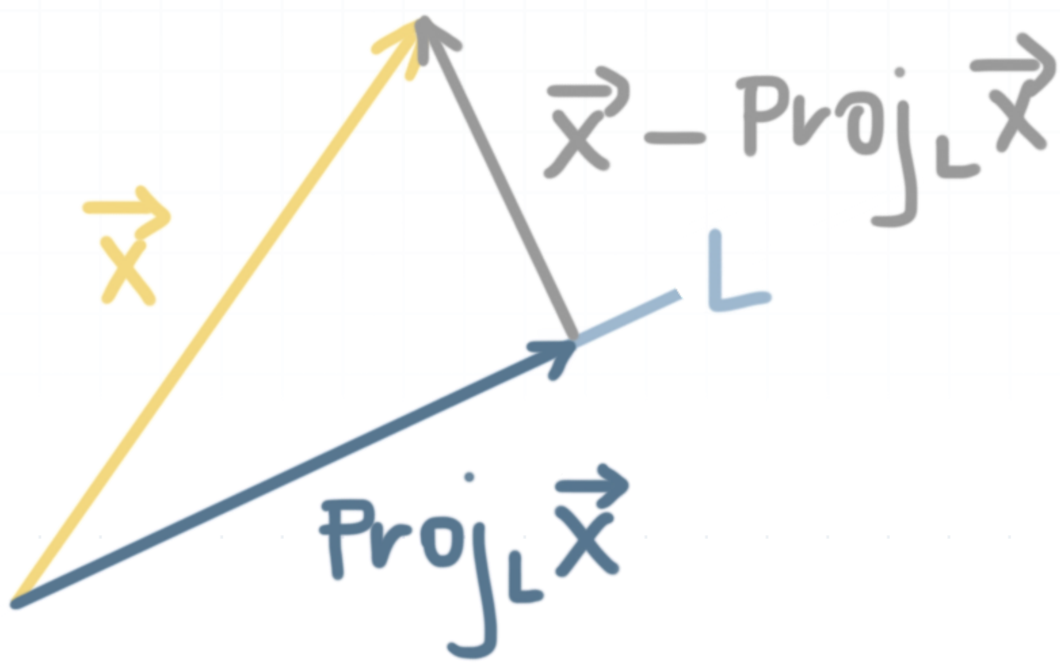
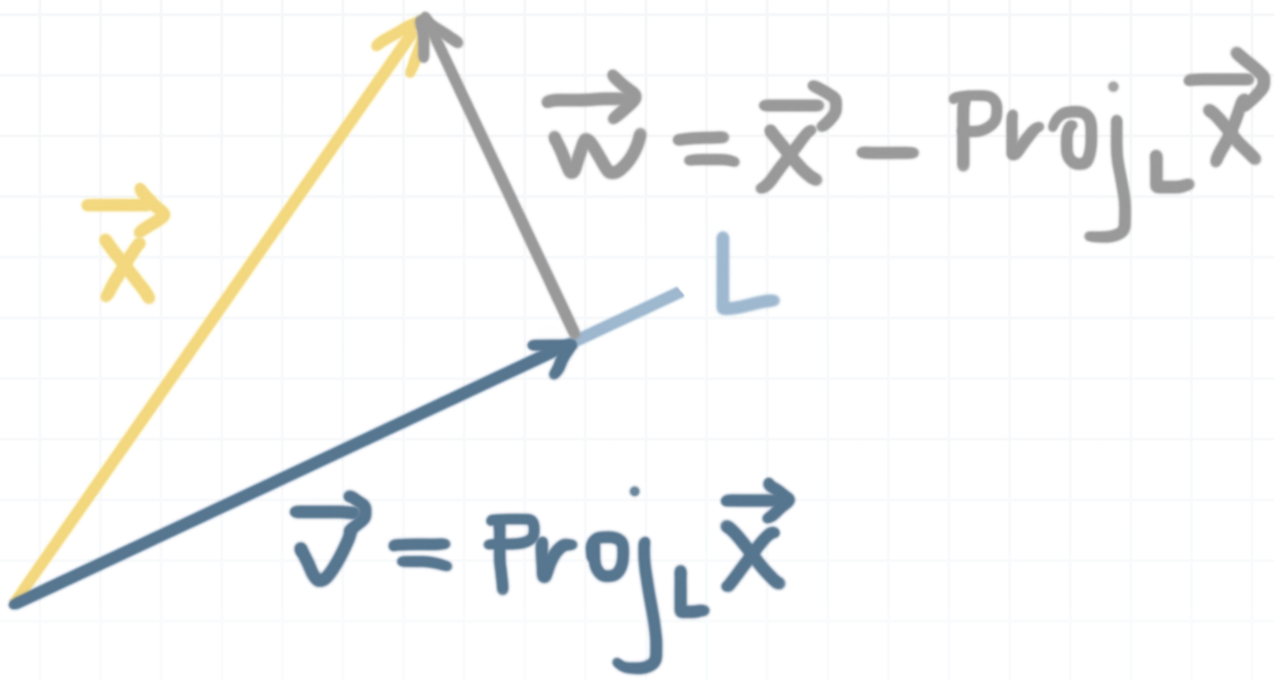
# Projection onto the subspace

We learned earlier how to find the projection of a vector onto a line, but we can also project a vector onto a subspace. So let's start by generalizing the definition of a projection onto a line, so that we can see how it might apply to a subspace.

If you project a vector $\vec{x}$ onto a line $L$, then the projection vector that sits on $L$ is $\text{Proj}_L\vec{x}$, and the vector from $L$ to the terminal point of $\vec{x}$, which is orthogonal to $L$, is $\vec{x}-\text{Proj}_L\vec{x}$.




But let's say we call $\text{Proj}_L\vec{x}$ the vector $\vec{v}$, and we call $\vec{x}-\text{Proj}_L\vec{x}$ the vector $\vec{w}$. Then we could say that $\vec{x}-\vec{v}=\vec{w}$.




Then we can rewrite $\vec{x}-\vec{v}=\vec{w}$ as

$$\vec{x}=\vec{v}+\vec{w}$$

Because $\vec{w}$ is orthogonal to $L$, in the same way that $\vec{v}$ is a member of $L$, $\vec{w}$ is a member of the orthogonal complement of $L$, or $L^\perp$.

And this doesn't just work for the projection of a vector onto a line. It also works for projecting a vector onto any subspace. And just like with the projection onto a line, if $\vec{x}$ is projected onto a subspace $V$, and the projection is $\text{Proj}_V\vec{x}$, then the vector $\vec{x}$ is closer to the vector $\text{Proj}_V\vec{x}$ than it is to any other vector in $V$.

## The projection is a linear transformation

If $V$ is a subspace of $\mathbb{R}^n$, then the projection of $\vec{x}$ onto a subspace $V$ is a linear transformation that can be written as the matrix-vector product

$$\text{Proj}_V\vec{x}=A(A^TA)^{-1}A^T\vec{x}$$

where $A$ is a matrix of column vectors that form the basis for the subspace $V$.

There are a couple of important things we want to realize about this formula. First, notice that the product $A(A^TA)^{-1}A^T$ will eventually simplify to one matrix. Second, it's important to say that we cannot distribute the inverse across the matrix product inside the parentheses. In other words, we **cannot** rewrite the formula as

$$\text{Proj}_V\vec{x} = A(A)^{-1}(A^T)^{-1}A^T\vec{x}$$

The only time we'd be allowed to do this is if $A$ is a square, invertible matrix. But if $A$ is a square, invertible matrix, then by definition $A$ defines all of $\mathbb{R}^n$, and the projection of a vector $\vec{x}$ onto $V$ will just be itself, $\vec{x}$. So we only need the projection formula when $A$ is not square and not invertible. But if $A$ is not square and not invertible, such that it makes sense to use the projection formula, then we can't split up the $(A^TA)^{-1}$.

Let's do an example so that we can see how to use it to find the projection of a vector $\vec{x}$ onto the subspace $V$.

### Example

If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace $V$.

$$V = \text{Span}\left(\begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\-2\end{bmatrix}\right)$$

Because the vectors that span $V$ are linearly independent, the matrix $A$ of the basis vectors that define $V$ is

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix}$$

The transpose $A^T$ is then

$$A^T = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

Our next step is to find $A^TA$.

$$A^TA = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) - 2(-2) + 0(0) & 1(2) - 2(1) + 0(-2) \\ 2(1) + 1(-2) - 2(0) & 2(2) + 1(1) - 2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 4 + 0 & 2 - 2 + 0 \\ 2 - 2 - 0 & 4 + 1 + 4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$$

Then we need to find the inverse of $A^TA$.

$$[A^TA \mid I_2] = \begin{bmatrix} 5 & 0 & | & 1 & 0 \\ 0 & 9 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & 0 & | & \frac{1}{5} & 0 \\ 0 & 9 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & 0 & | & \frac{1}{5} & 0 \\ 0 & 1 & | & 0 & \frac{1}{9} \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$$

Now the projection of $\vec{x}$ onto the subspace $V$ will be

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix} \vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5}(1) + 0(2) & \frac{1}{5}(-2) + 0(1) & \frac{1}{5}(0) + 0(-2) \\ 0(1) + \frac{1}{9}(2) & 0(-2) + \frac{1}{9}(1) & 0(0) + \frac{1}{9}(-2) \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5} + 0 & -\frac{2}{5} + 0 & 0 + 0 \\ 0 + \frac{2}{9} & 0 + \frac{1}{9} & 0 - \frac{2}{9} \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \end{bmatrix} \vec{x}$$

Next, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x}=\begin{bmatrix}1\left(\frac{1}{5}\right)+2\left(\frac{2}{9}\right) & 1\left(-\frac{2}{5}\right)+2\left(\frac{1}{9}\right) & 1(0)+2\left(-\frac{2}{9}\right)\\ -2\left(\frac{1}{5}\right)+1\left(\frac{2}{9}\right) & -2\left(-\frac{2}{5}\right)+1\left(\frac{1}{9}\right) & -2(0)+1\left(-\frac{2}{9}\right)\\ 0\left(\frac{1}{5}\right)-2\left(\frac{2}{9}\right) & 0\left(-\frac{2}{5}\right)-2\left(\frac{1}{9}\right) & 0(0)-2\left(-\frac{2}{9}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{5}+\frac{4}{9} & -\frac{2}{5}+\frac{2}{9} & 0-\frac{4}{9}\\ -\frac{2}{5}+\frac{2}{9} & \frac{4}{5}+\frac{1}{9} & 0-\frac{2}{9}\\ 0-\frac{4}{9} & 0-\frac{2}{9} & 0+\frac{4}{9}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{45}+\frac{20}{45} & -\frac{18}{45}+\frac{10}{45} & -\frac{4}{9}\\ -\frac{18}{45}+\frac{10}{45} & \frac{36}{45}+\frac{5}{45} & -\frac{2}{9}\\ -\frac{4}{9} & -\frac{2}{9} & \frac{4}{9}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{29}{45} & -\frac{8}{45} & -\frac{4}{9}\\ -\frac{8}{45} & \frac{41}{45} & -\frac{2}{9}\\ -\frac{4}{9} & -\frac{2}{9} & \frac{4}{9}\end{bmatrix}\vec{x}$$

To simplify the matrix, let's factor out a $1/9$.

$$\text{Proj}_V\vec{x}=\frac{1}{9}\begin{bmatrix}\frac{29}{5} & -\frac{8}{5} & -4\\ -\frac{8}{5} & \frac{41}{5} & -2\\ -4 & -2 & 4\end{bmatrix}\vec{x}$$

This result tells us that we can find the projection of any $\vec{x}$ onto the subspace $V$ by multiplying the transformation matrix

$$\frac{1}{9}\begin{bmatrix} \frac{29}{5} & -\frac{8}{5} & -4 \\ -\frac{8}{5} & \frac{41}{5} & -2 \\ -4 & -2 & 4 \end{bmatrix}$$

by the vector $\vec{x}$.