The product of a matrix and its transpose

Now that we know more about a given matrix A and its transpose matrix A^T, we can learn more about the relationship between them.

The product is invertible

For instance, assuming that the columns of A are linearly independent, A^TA is an invertible matrix. That's because we can call A any $m \times n$ matrix, and then A^T is an $n \times m$ matrix. That means A^TA will be a square $n \times n$ matrix.

So if we can then put A^TA into reduced row-echelon form, and we end up with the identity matrix I_n, that means all of the columns in the square $n \times n$ product matrix A^TA are linearly independent, and A^TA is invertible.

Example

Show that A^TA is invertible.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & -1 \end{bmatrix}$$

We've been given A, so first we'll find A^T by swapping all the rows and columns in A.

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Then we'll find the product A^TA.

$$A^TA = \begin{bmatrix}1 & 2 & -1\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\2 & 0\\-1 & -1\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1(1)+2(2)-1(-1) & 1(1)+2(0)-1(-1)\\1(1)+0(2)-1(-1) & 1(1)+0(0)-1(-1)\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1+4+1 & 1+0+1\\1+0+1 & 1+0+1\end{bmatrix}$$

$$A^TA = \begin{bmatrix}6 & 2\\2 & 2\end{bmatrix}$$

To see whether or not A^TA is invertible, we'll put the product matrix into reduced row-echelon form. Find the pivot entry in the first column, and then zero out the rest of the first column.

$$A^TA = \begin{bmatrix}1 & \frac{1}{3}\\2 & 2\end{bmatrix} \to \begin{bmatrix}1 & \frac{1}{3}\\0 & \frac{4}{3}\end{bmatrix}$$

Find the pivot entry in the second column, and then zero out the rest of the second column.

$$A^TA = \begin{bmatrix}1 & \frac{1}{3}\\0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Because the matrix A^TA in reduced row-echelon form is the identity matrix I_2, that tells us that A^TA is invertible. And that's what we expected to find, since we know that any A^TA is invertible, as long as the columns of A are linearly independent.