Topic: Orthogonal complements

Question: Find the orthogonal complement of W, $W^{\perp}$.

$$W = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ -5 \end{bmatrix}\right)$$

Answer choices:

A $\quad W^{\perp} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}\right)$

B $\quad W^{\perp} = \text{Span}\left(\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}\right)$

C $\quad W^{\perp} = \text{Span}\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$

D $\quad W^{\perp} = \text{Span}\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$

Solution: A

The subspace W is a plane in $\mathbb{R}^4$, spanned by the two vectors $\vec{w}_1=(-1,0,-2,4)$ and $\vec{w}_2=(2,0,3,-5)$. Therefore, its orthogonal complement $W^{\perp}$ is the set of vectors which are orthogonal to both $\vec{w}_1=(-1,0,-2,4)$ and $\vec{w}_2=(2,0,3,-5)$.

$$W^{\perp}=\{\vec{x}\in\mathbb{R}^4\ |\ \vec{x}\cdot\begin{bmatrix}-1\\0\\-2\\4\end{bmatrix}=0 \quad \text{and} \quad \vec{x}\cdot\begin{bmatrix}2\\0\\3\\-5\end{bmatrix}=0\}$$

If we let $\vec{x}=(x_1,x_2,x_3,x_4)$, we get two equations from these dot products.

$$-x_1-2x_3+4x_4=0$$

$$2x_1+3x_3-5x_4=0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix}-1 & 0 & -2 & 4 & | & 0\\ 2 & 0 & 3 & -5 & | & 0\end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix}1 & 0 & 2 & -4 & | & 0\\ 2 & 0 & 3 & -5 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 2 & -4 & | & 0\\ 0 & 0 & -1 & 3 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 2 & -4 & | & 0\\ 0 & 0 & 1 & -3 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & 2 & | & 0\\ 0 & 0 & 1 & -3 & | & 0\end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + 2x_4 = 0$$

$$x_3 - 3x_4 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = -2x_4$$

$$x_3 = 3x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement $W^{\perp}$ is

$$W^{\perp} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right)$$

Topic: Orthogonal complements

Question: Rewrite the orthogonal complement of V, $V^{\perp}$, if V is a vector set in $\mathbb{R}^3$.

$$V = \begin{bmatrix} -2y+z \\ y \\ z \end{bmatrix}$$

Answer choices:

A $\quad V^{\perp} = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$

B $\quad V^{\perp} = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

C $\quad V^{\perp} = \text{Span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right)$

D $\quad V^{\perp} = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right)$

Solution: C

We can rewrite V as

$$V = \{y \cdot \begin{bmatrix}-2\\1\\0\end{bmatrix} + z \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} \ | \ y, z \in \mathbb{R}^3\}$$

The subspace V is a plane in $\mathbb{R}^3$, spanned by the two vectors $\overrightarrow{v}_1 = (-2,1,0)$ and $\overrightarrow{v}_2 = (1,0,1)$. Therefore, its orthogonal complement $V^\perp$ is the set of vectors which are orthogonal to both $\overrightarrow{v}_1 = (-2,1,0)$ and $\overrightarrow{v}_2 = (1,0,1)$.

$$V^\perp = \{\overrightarrow{x} \in \mathbb{R}^3 \ | \ \overrightarrow{x} \cdot \begin{bmatrix}-2\\1\\0\end{bmatrix} = 0 \quad \text{and} \quad \overrightarrow{x} \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} = 0\}$$

If we let $\overrightarrow{x} = (x_1, x_2, x_3)$, we get two equations from these dot products.

$$-2x_1 + x_2 = 0$$

$$x_1 + x_3 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix}-2 & 1 & 0 & | & 0\\1 & 0 & 1 & | & 0\end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix}1 & -\frac{1}{2} & 0 & | & 0\\1 & 0 & 1 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -\frac{1}{2} & 0 & | & 0\\0 & \frac{1}{2} & 1 & | & 0\end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + x_3 = 0$$

$$x_2 + 2x_3 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = -x_3$$

$$x_2 = -2x_3$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$V^\perp = \text{Span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right)$$

Topic: Orthogonal complements

Question: Describe the orthogonal complement of V, $V^{\perp}$.

$$V = \text{Span}\left(\begin{bmatrix}1\\-2\\3\\5\end{bmatrix},\begin{bmatrix}0\\4\\-8\\8\end{bmatrix},\begin{bmatrix}1\\3\\-5\\-1\end{bmatrix}\right)$$

Answer choices:

A $\quad V^{\perp} = \text{Span}\left(\begin{bmatrix}9\\-14\\-8\\1\end{bmatrix}\right)$

B $\quad V^{\perp} = \text{Span}\left(\begin{bmatrix}-1\\14\\8\\1\end{bmatrix}\right)$

C $\quad V^{\perp} = \text{Span}\left(\begin{bmatrix}9\\-14\\-8\\0\end{bmatrix}\right)$

D $\quad V^{\perp} = \text{Span}\left(\begin{bmatrix}1\\-14\\-8\\0\end{bmatrix}\right)$

Solution: B

The subspace V is a plane in $\mathbb{R}^4$, spanned by the three vectors $\vec{v}_1 = (1, -2, 3, 5)$, $\vec{v}_2 = (0, 4, -8, 8)$, and $\vec{v}_3 = (1, 3, -5, -1)$. Therefore, its orthogonal complement $V^{\perp}$ is the set of vectors which are orthogonal to $\vec{v}_1 = (1, -2, 3, 5)$, $\vec{v}_2 = (0, 4, -8, 8)$, and $\vec{v}_3 = (1, 3, -5, -1)$.

$$V^{\perp} = \{\vec{x} \in \mathbb{R}^4 \mid \vec{x} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \\ 5 \end{bmatrix} = 0 \text{ , } \vec{x} \cdot \begin{bmatrix} 0 \\ 4 \\ -8 \\ 8 \end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix} 1 \\ 3 \\ -5 \\ -1 \end{bmatrix} = 0\}$$

If we let $\vec{x} = (x_1, x_2, x_3, x_4)$, we get three equations from these dot products.

$$x_1 - 2x_2 + 3x_3 + 5x_4 = 0$$

$$4x_2 - 8x_3 + 8x_4 = 0$$

$$x_1 + 3x_2 - 5x_3 - x_4 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix} 1 & -2 & 3 & 5 & | & 0 \\ 0 & 4 & -8 & 8 & | & 0 \\ 1 & 3 & -5 & -1 & | & 0 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 3 & 5 & | & 0 \\ 0 & 4 & -8 & 8 & | & 0 \\ 0 & 5 & -8 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 5 & | & 0 \\ 0 & 1 & -2 & 2 & | & 0 \\ 0 & 5 & -8 & -6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & 5 & | & 0 \\ 0 & 1 & -2 & 2 & | & 0 \\ 0 & 0 & 2 & -16 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 9 & | & 0 \\ 0 & 1 & -2 & 2 & | & 0 \\ 0 & 0 & 2 & -16 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 9 & | & 0 \\ 0 & 1 & -2 & 2 & | & 0 \\ 0 & 0 & 1 & -8 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 9 & | & 0 \\ 0 & 1 & 0 & -14 & | & 0 \\ 0 & 0 & 1 & -8 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & -14 & | & 0 \\ 0 & 0 & 1 & -8 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + x_4 = 0$$

$$x_2 - 14x_4 = 0$$

$$x_3 - 8x_4 = 0$$

which we can solve for the pivot variables.

$$x_1 = -x_4$$

$$x_2 = 14x_4$$

$$x_3 = 8x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 14 \\ 8 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$V^{\perp} = \text{Span}\left(\begin{bmatrix} -1 \\ 14 \\ 8 \\ 1 \end{bmatrix} \right)$$