Inverse transformations are linear

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The inverse of an invertible linear transformation T is also itself a linear transformation. Which means that the inverse transformation T^{-1} is closed under addition and closed under scalar multiplication.

$$T^{-1}(\overrightarrow{u}+\overrightarrow{v})=T^{-1}(\overrightarrow{u})+T^{-1}(\overrightarrow{v})$$

$$T^{-1}(c\overrightarrow{u})=cT^{-1}(\overrightarrow{u})$$

In other words, as long as the original transformation T

1. is a linear transformation itself, and

2. is invertible (its inverse is defined, you can find its inverse),

then the inverse of T, T^{-1}, is also a linear transformation.

Inverse transformations as matrix-vector products

Remember that any linear transformation can be represented as a matrix-vector product. Normally, we rewrite the linear transformation $T:\mathbb{R}^n\to\mathbb{R}^n$ as

$T(\overrightarrow{x})=A\overrightarrow{x}$, where A is a square $n\times n$ matrix

But because the inverse transformation T^{-1} is a linear transformation as well, we can also write the inverse as a matrix-vector product.

$$T^{-1}(\vec{x}) = A^{-1}\vec{x}$$

The matrix A^{-1} is the inverse of the matrix A that we used to define the transformation T.

In other words, given a linear transformation T and its inverse T^{-1}, if we want to express both of them as matrix-vector products, we know that the matrices we use to do so will be inverses of one another.

The reason this is true is because we already know that taking the composition of the inverse transformation T^{-1} with T will always give us the identity matrix I_n, where n is the dimension of the domain and codomain, $\mathbb{R}^n$.

$$(T^{-1} \circ T)(\vec{x}) = I_n\vec{x}$$

Going the other way (applying the transformation to the inverse transformation), gives the same result:

$$(T \circ T^{-1})(\vec{x}) = I_n\vec{x}$$

The only way these can be true is if the matrices that are part of the matrix-vector products are inverses of one another.

$$(T^{-1} \circ T)(\vec{x}) = I_n\vec{x} \qquad \to \qquad (T^{-1} \circ T)(\vec{x}) = A^{-1}A\vec{x}$$

$$(T \circ T^{-1})(\vec{x}) = I_n\vec{x} \qquad \to \qquad (T^{-1} \circ T)(\vec{x}) = AA^{-1}\vec{x}$$

Remember that matrix multiplication is not commutative, which means that, given two matrices A and B, AB is not equal to BA. We can't change the order of the matrix multiplication and still get the same answer. But based on these compositions of T with T^{-1} and vice versa, we're saying

$A^{-1}A$ and AA^{-1} must both equal I_n, which means they must equal each other. The only way that $A^{-1}A$ and AA^{-1} can both equal I_n is if A and A^{-1} are inverses of one another.

This is why, when we represent the inverse transformation T^{-1} as a matrix-vector product, we know that the matrix we use must be the inverse of the matrix we used to represent T. So if we represent T as $T(\vec{x}) = A\vec{x}$ with the matrix A, that means T^{-1} can only be represented as $T^{-1}(\vec{x}) = A^{-1}\vec{x}$ with the inverse of A, A^{-1}.

Finding the matrix inverse

We've said that the inverse transformation T^{-1} can be represented as the matrix-vector product $T^{-1}(\vec{x}) = A^{-1}\vec{x}$. Here's how we can find A^{-1}.

If we start with A, but augment it with the identity matrix, then all we have to do to find A^{-1} is work on the augmented matrix until A is in reduced row-echelon form.

In other words, given the matrix A, we'll start with the augmented matrix

$$\left[A \mid I\right]$$

Through the process of putting A in reduced row-echelon form, I will be transformed into A^{-1}, and we'll end up with

$$\left[I \mid A^{-1}\right]$$

This seems like a magical process, but there's a very simple reason why it works. Remember earlier in the course that we learned how one row operation could be expressed as an elimination matrix E. And that if we performed lots of row operations, through matrix multiplication of E_1, E_2, E_3, etc., we could find one consolidated elimination matrix.

That's exactly what we're doing here. We're performing row operations on A to change it into I. All those row operations could be expressed as the elimination matrix E. And we're saying that if we multiply E by A, that we'll get the identity matrix, so $EA = I$. But as you know, $A^{-1}A = I$, which means $E = A^{-1}$.

Let's try an example to see how the augmented matrix flips from A into A^{-1}.

Example

Find A^{-1}.

$$A = \begin{bmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{bmatrix}$$

Because A is a 3×3 matrix, its associated identity matrix is I_3. So we'll augment A with I_3.

$$[A_{3\times3} \mid I_3]$$

$$\begin{bmatrix} -2 & 4 & 0 & | & 1 & 0 & 0 \\ 1 & -1 & 4 & | & 0 & 1 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

Now we need to put A into reduced row-echelon form. We start by switching R_2 and R_1, to get a 1 into the first entry of the first row.

$$\begin{bmatrix} 1 & -1 & 4 & | & 0 & 1 & 0 \\ -2 & 4 & 0 & | & 1 & 0 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

Now we'll zero out the rest of the first column.

$$\begin{bmatrix} 1 & -1 & 4 & | & 0 & 1 & 0 \\ 0 & 2 & 8 & | & 1 & 2 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -1 & 4 & | & 0 & 1 & 0 \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & 8 & | & \frac{1}{2} & 2 & 0 \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 8 & | & \frac{1}{2} & 2 & 0 \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 0 & -28 & | & -3 & -6 & 1 \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & 8 & | & \frac{1}{2} & 2 & 0 \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & | & \frac{3}{28} & \frac{3}{14} & -\frac{1}{28} \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{5}{14} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & | & \frac{3}{28} & \frac{3}{14} & -\frac{1}{28} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -\frac{5}{14} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & | & \frac{1}{14} & \frac{1}{7} & \frac{1}{7} \\ 0 & 0 & 1 & | & \frac{3}{28} & \frac{3}{14} & -\frac{1}{28} \end{bmatrix}$$

Now that A has been put into reduced row-echelon form on the left side of the augmented matrix, the identity matrix on the right side has been turned into the inverse matrix A^{-1}. So we can say

$$A^{-1} = \begin{bmatrix} -\frac{5}{14} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{14} & \frac{1}{7} & \frac{1}{7} \\ \frac{3}{28} & \frac{3}{14} & -\frac{1}{28} \end{bmatrix}$$