Adding and scaling linear transformations

Now that we know about linear transformations, we want to think about operations with transformations. So in this lesson we're talking about how to add and scale linear transformations.

Sums of transformations

Given two transformations of $\vec{x}$, $S(\vec{x}):\mathbb{R}^n\to\mathbb{R}^m$ and $T(\vec{x}):\mathbb{R}^n\to\mathbb{R}^m$, remember that you can represent any linear transformation as a matrix-vector product. So rewrite $S(\vec{x})$ as $A\vec{x}$, and rewrite $T(\vec{x})$ as $B\vec{x}$, where A and B are $m\times n$ matrices. Then the sum of the transformations is

$$(S+T)(\vec{x})=S(\vec{x})+T(\vec{x})$$

$$(S+T)(\vec{x})=A\vec{x}+B\vec{x}$$

$$(S+T)(\vec{x})=(A+B)\vec{x}$$

The $(A+B)$ is simply a matrix addition problem, and then $(A+B)\vec{x}$ tells us to multiply the resulting $A+B$ matrix by the vector $\vec{x}$.

Example

Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x})=\begin{bmatrix}2 & 1\\ -3 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To add the transformations, we first need to recognize that these transformations are written as matrix-vector products. If we call the matrix in $S(\vec{x})$ the matrix A,

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

and we call the matrix in $T(\vec{x})$ the matrix B,

$$B = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

then the sum of the transformations is simply the sum of these matrices. First, find the sum of the matrices.

$$A + B = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 + (-6) & 1 + 0 \\ -3 + 2 & 0 + (-1) \end{bmatrix}$$

$$A + B = \begin{bmatrix} -4 & 1 \\ -1 & -1 \end{bmatrix}$$

So the sum of the transformations would be

$$S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} -4 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Scaled transformations

We can also multiply transformations by scalars. The transformation $T(\vec{x})$, multiplied by a scalar c, would be written as $cT(\vec{x})$. But if we write $T(\vec{x})$ as its matrix-vector product, $B\vec{x}$, then the scaled transformation is $cB\vec{x}$.

$$cT(\vec{x}) = c(B\vec{x}) = (cB)\vec{x}$$

Let's do an example where we apply a scalar to a transformation.

Example

Find the product of a scalar $c = -4$ and the transformation $T(\vec{x})$.

$$T(\vec{x}) = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The transformation T is given as a matrix-vector product. If we call the matrix that's in the transformation T the matrix B, then multiplying the transformation by the scalar $c = -4$ gives

$$cT(\vec{x}) = -4 \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

First find cB.

$$cB = -4\begin{bmatrix}-6 & 0\\ 2 & -1\end{bmatrix}$$

$$cB = \begin{bmatrix}-4(-6) & -4(0)\\ -4(2) & -4(-1)\end{bmatrix}$$

$$cB = \begin{bmatrix}24 & 0\\ -8 & 4\end{bmatrix}$$

So the scaled transformation would be

$$cT(\vec{x}) = \begin{bmatrix}24 & 0\\ -8 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$