Topic: Transformation matrix for a basis

Question: Use the transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ to transform $[\vec{x}]_B = (5,4,-2)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right)$$

Answer choices:

A $\quad [T(\vec{x})]_B = \begin{bmatrix} -20 \\ 9 \\ 4 \end{bmatrix}$

B $\quad [T(\vec{x})]_B = \begin{bmatrix} -15 \\ -28 \\ -2 \end{bmatrix}$

C $\quad [T(\vec{x})]_B = \begin{bmatrix} -15 \\ -36 \\ 12 \end{bmatrix}$

D $\quad [T(\vec{x})]_B = \begin{bmatrix} -20 \\ 78 \\ -93 \end{bmatrix}$

Solution: C

In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C that transforms vectors from the standard basis into vectors in the alternate basis B is made of the column vectors that span B, $\vec{v}_1 = (1, -1, 1)$, $\vec{v}_2 = (0, 1, -1)$, and $\vec{v}_3 = (2, 1, -2)$, so

$$C = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ -1 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & -1 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 1 & -1 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 0 & -1 & -4 & | & -1 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 0 & 0 & -1 & | & 0 & 1 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & -1 & -1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1 & 4 & 3 \\ 0 & 0 & 1 & | & 0 & -1 & -1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 0 & | & 1 & 2 & 2 \\ 0 & 1 & 0 & | & 1 & 4 & 3 \\ 0 & 0 & 1 & | & 0 & -1 & -1 \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

With A, C, and C^{-1}, we can find $M = C^{-1}AC$.

$$M = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} -2 & -2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} -2(1) - 2(-1) + 1(1) & -2(0) - 2(1) + 1(-1) & -2(2) - 2(1) + 1(-2) \\ 1(1) + 0(-1) - 2(1) & 1(0) + 0(1) - 2(-1) & 1(2) + 0(1) - 2(-2) \\ 0(1) + 1(-1) + 0(1) & 0(0) + 1(1) + 0(-1) & 0(2) + 1(1) + 0(-2) \end{bmatrix}$$

$$M=\begin{bmatrix}1&2&2\\1&4&3\\0&-1&-1\end{bmatrix}\begin{bmatrix}-2+2+1&0-2-1&-4-2-2\\1+0-2&0+0+2&2+0+4\\0-1+0&0+1+0&0+1+0\end{bmatrix}$$

$$M=\begin{bmatrix}1&2&2\\1&4&3\\0&-1&-1\end{bmatrix}\begin{bmatrix}1&-3&-8\\-1&2&6\\-1&1&1\end{bmatrix}$$

$$M=\begin{bmatrix}1(1)+2(-1)+2(-1)&1(-3)+2(2)+2(1)&1(-8)+2(6)+2(1)\\1(1)+4(-1)+3(-1)&1(-3)+4(2)+3(1)&1(-8)+4(6)+3(1)\\0(1)-1(-1)-1(-1)&0(-3)-1(2)-1(1)&0(-8)-1(6)-1(1)\end{bmatrix}$$

$$M=\begin{bmatrix}1-2-2&-3+4+2&-8+12+2\\1-4-3&-3+8+3&-8+24+3\\0+1+1&0-2-1&0-6-1\end{bmatrix}$$

$$M=\begin{bmatrix}-3&3&6\\-6&8&19\\2&-3&-7\end{bmatrix}$$

We’ve been asked to transform $[\vec{x}]_B=(5,4,-2)$, so we’ll multiply M by this vector.

$$[T(\vec{x})]_B=M[\vec{x}]_B$$

$$[T(\vec{x})]_B=\begin{bmatrix}-3&3&6\\-6&8&19\\2&-3&-7\end{bmatrix}\begin{bmatrix}5\\4\\-2\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}-3(5)+3(4)+6(-2)\\-6(5)+8(4)+19(-2)\\2(5)-3(4)-7(-2)\end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -15+12-12 \\ -30+32-38 \\ 10-12+14 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -15 \\ -36 \\ 12 \end{bmatrix}$$

Topic: Transformation matrix for a basis

Question: Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $\vec{x} = (2, -2)$ to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -2 & -5 \\ 3 & 1 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ -2 \end{bmatrix} \right)$$

Answer choices:

A $\quad [T(\vec{x})]_B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

B $\quad [T(\vec{x})]_B = \begin{bmatrix} -8 \\ -3 \end{bmatrix}$

C $\quad [T(\vec{x})]_B = \begin{bmatrix} -12 \\ -2 \end{bmatrix}$

D $\quad [T(\vec{x})]_B = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

Solution: D

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (2,1)$ and $\vec{v}_2 = (-6,-2)$, so

$$C = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \left[\begin{array}{cc|cc} 2 & -6 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

$$[C \mid I] = \left[\begin{array}{cc|cc} 1 & -3 & \frac{1}{2} & 0 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

$$[C \mid I] = \left[\begin{array}{cc|cc} 1 & -3 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right]$$

$$[C \mid I] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 3 \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right]$$

So,

$$C^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Find the transformation $T(\vec{x})$ where $T(\vec{x}) = A\vec{x}$.

$$T(\vec{x}) = \begin{bmatrix} -2 & -5 \\ 3 & 1 \end{bmatrix} \vec{x}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & -5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -2(2) - 5(-2) \\ 3(2) + 1(-2) \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -4 + 10 \\ 6 - 2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Then $[T(\vec{x})]_B = C^{-1}T(\vec{x})$.

$$[T(\vec{x})]_B = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -1(6) + 3(4) \\ -\frac{1}{2}(6) + 1(4) \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -6 + 12 \\ -3 + 4 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Topic: Transformation matrix for a basis

Question: Use the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (2,3)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -4 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right)$$

Answer choices:

A $\quad [T(\vec{x})]_B = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

B $\quad [T(\vec{x})]_B = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

C $\quad [T(\vec{x})]_B = \begin{bmatrix} \frac{7}{4} \\ \frac{7}{2} \end{bmatrix}$

D $\quad [T(\vec{x})]_B = \begin{bmatrix} -1 \\ 12 \end{bmatrix}$

Solution: A

In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (-4,4)$ and $\vec{v}_2 = (4,0)$, so

$$C = \begin{bmatrix} -4 & 4 \\ 4 & 0 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C\ |\ I] = \begin{bmatrix} -4 & 4 & | & 1 & 0 \\ 4 & 0 & | & 0 & 1 \end{bmatrix}$$

$$[C\ |\ I] = \begin{bmatrix} 1 & -1 & | & -\frac{1}{4} & 0 \\ 4 & 0 & | & 0 & 1 \end{bmatrix}$$

$$[C\ |\ I] = \begin{bmatrix} 1 & -1 & | & -\frac{1}{4} & 0 \\ 0 & 4 & | & 1 & 1 \end{bmatrix}$$

$$[C\ |\ I] = \begin{bmatrix} 1 & -1 & | & -\frac{1}{4} & 0 \\ 0 & 1 & | & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & | & 0 & \frac{1}{4} \\ 0 & 1 & | & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

With A, C, and C^{-1}, we can find $M = C^{-1}AC$.

$$M = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4 & 4 \\ 4 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} -1(-4) + 3(4) & -1(4) + 3(0) \\ 2(-4) + 1(4) & 2(4) + 1(0) \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 4 + 12 & -4 + 0 \\ -8 + 4 & 8 + 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 16 & -4 \\ -4 & 8 \end{bmatrix}$$

$$M = \begin{bmatrix} 0(16) + \frac{1}{4}(-4) & 0(-4) + \frac{1}{4}(8) \\ \frac{1}{4}(16) + \frac{1}{4}(-4) & \frac{1}{4}(-4) + \frac{1}{4}(8) \end{bmatrix}$$

$$M=\begin{bmatrix}0-1 & 0+2\\ 4-1 & -1+2\end{bmatrix}$$

$$M=\begin{bmatrix}-1 & 2\\ 3 & 1\end{bmatrix}$$

We’ve been asked to transform $[\vec{x}]_B=(2,3)$, so we’ll multiply M by this vector.

$$[T(\vec{x})]_B=M[\vec{x}]_B$$

$$[T(\vec{x})]_B=\begin{bmatrix}-1 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}2\\ 3\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}-1(2)+2(3)\\ 3(2)+1(3)\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}-2+6\\ 6+3\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}4\\ 9\end{bmatrix}$$