

Linear transformations as rotations

In the previous lesson, we looked at an example of a linear transformation that included a reflection and a stretch. We can apply the same process for other kinds of transformations, like compressions, or for rotations.

But we can also use a linear transformation to rotate a vector by a certain angle, either in degrees or in radians.

The rotation matrix

Instead of using the appropriate identity matrix like we did for reflecting, stretching, and compressing, we'll use a matrix specifically for rotations. But the matrix will still always match the dimension of the space in which we're transforming.

If we're rotating in \mathbb{R}^2 , we're rotating counterclockwise around the origin through the angle θ , and the transforming rotation matrix will be

$$\text{Rot}_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

And the transformation to rotate any vector \vec{x} in \mathbb{R}^2 will be

$$\text{Rot}_\theta(\vec{x}) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If we're rotating in \mathbb{R}^3 , the transforming rotation matrix will be different depending on which axis we're rotating around.



$$\text{Rot}_{\theta \text{ around } x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } y} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } z} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the transformation to rotate any vector \vec{x} in \mathbb{R}^3 will be

$$\text{Rot}_{\theta \text{ around } x}(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } y}(\vec{x}) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } z}(\vec{x}) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So if we know the angle by which we're trying to rotate a vector, we can plug the angle into the rotation matrix, and then multiply the rotation matrix by the vector we want to transform.

We also want to know that rotations follow these properties:

$$\text{Rot}_{\theta}(\vec{u} + \vec{v}) = \text{Rot}_{\theta}(\vec{u}) + \text{Rot}_{\theta}(\vec{v})$$



$$\text{Rot}_\theta(c\vec{u}) = c\text{Rot}_\theta(\vec{u})$$

Let's do an example of a rotation in \mathbb{R}^2 .

Example

Rotate \vec{x} by an angle of $\theta = 135^\circ$.

$$\vec{x} = (3, 2)$$

The transformation to rotate any vector \vec{x} in \mathbb{R}^2 by 135° will be

$$\text{Rot}_{135^\circ}(\vec{x}) = \begin{bmatrix} \cos(135^\circ) & -\sin(135^\circ) \\ \sin(135^\circ) & \cos(135^\circ) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

First, we'll simplify the rotation matrix. We can get the sine and cosine values at $\theta = 135^\circ$ from the unit circle.

$$\begin{bmatrix} \cos(135^\circ) & -\sin(135^\circ) \\ \sin(135^\circ) & \cos(135^\circ) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Then the transformation to rotate any vector \vec{x} in \mathbb{R}^2 by 135° will be rewritten as

$$\text{Rot}_{135^\circ}(\vec{x}) = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Now we'll apply this specific rotation matrix to $\vec{x} = (3,2)$.

$$\text{Rot}_{135^\circ} \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\text{Rot}_{135^\circ} \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -\frac{3\sqrt{2}}{2} & -\frac{2\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} & -\frac{2\sqrt{2}}{2} \end{bmatrix}$$

$$\text{Rot}_{135^\circ} \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -\frac{5\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

If the original vector is \vec{x} , and we call the transformed vector \vec{x}_1 , then we can sketch \vec{x} and \vec{x}_1 and the 135° angle between them.



