Linear Algebra Workbook

Transformations

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MATH

FUNCTIONS AND TRANSFORMATIONS

■ 1. The transformation T maps every vector in $\mathbb{R}^4$ to $\vec{O}=(0,0,0)$. What are the domain, codomain, and range of T?

■ 2. The transformation T maps every vector in $\mathbb{R}^3$ to every vector in $\mathbb{R}^2$. What are the domain, codomain, and range of T?

■ 3. The transformation T maps every vector in $\mathbb{R}^3$ to the zero vector $\vec{O}$ in $\mathbb{R}^3$. What are the domain, codomain, and range of T?

■ 4. The transformation T maps $\vec{a}=(-1,0,3)$ to $\vec{b}=(-2,1,-2)$. What are the domain, codomain, and range of T?

■ 5. The transformation T maps $\vec{a}=(-2,0)$ to every vector in $\mathbb{R}^4$. What are the domain, codomain, and range of T?

■ 6. The transformation T maps $\vec{a}=(-2,-3,1)$ to the zero vector $\vec{O}$ in $\mathbb{R}^3$. What are the domain, codomain, and range of T?

TRANSFORMATION MATRICES AND THE IMAGE OF THE SUBSET

1. Find the resulting vector $\overrightarrow{b}$ after $\overrightarrow{a}=(1,6)$ undergoes a transformation by matrix M.

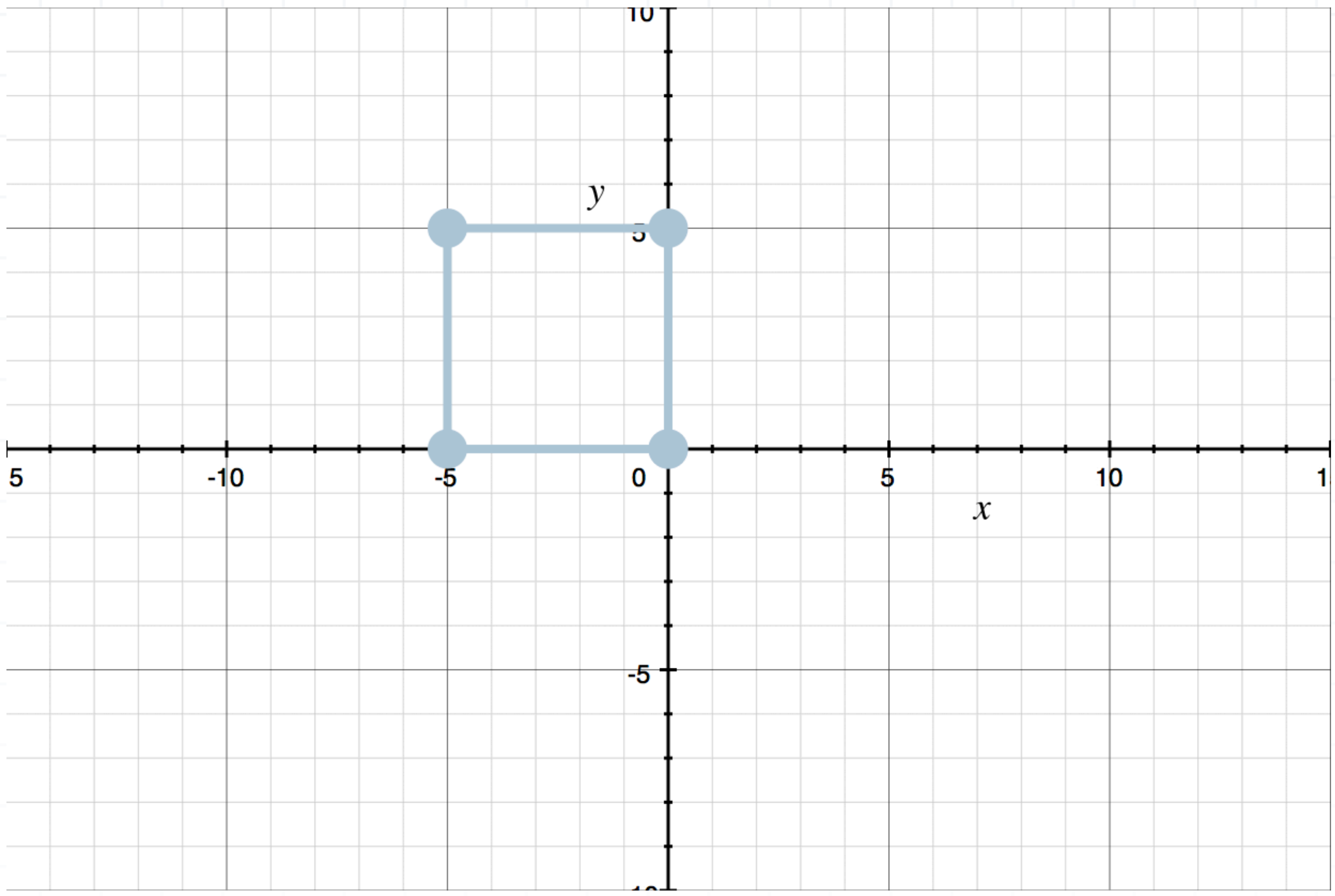
$$M=\begin{bmatrix}-7 & 1\\ 0 & -2\end{bmatrix}$$

2. Sketch triangle $\triangle ABC$ with vertices $(2,3)$, $(-3,-1)$, and $(1,-4)$, and the transformation of $\triangle ABC$ after it's transformed by matrix L.

$$L=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$

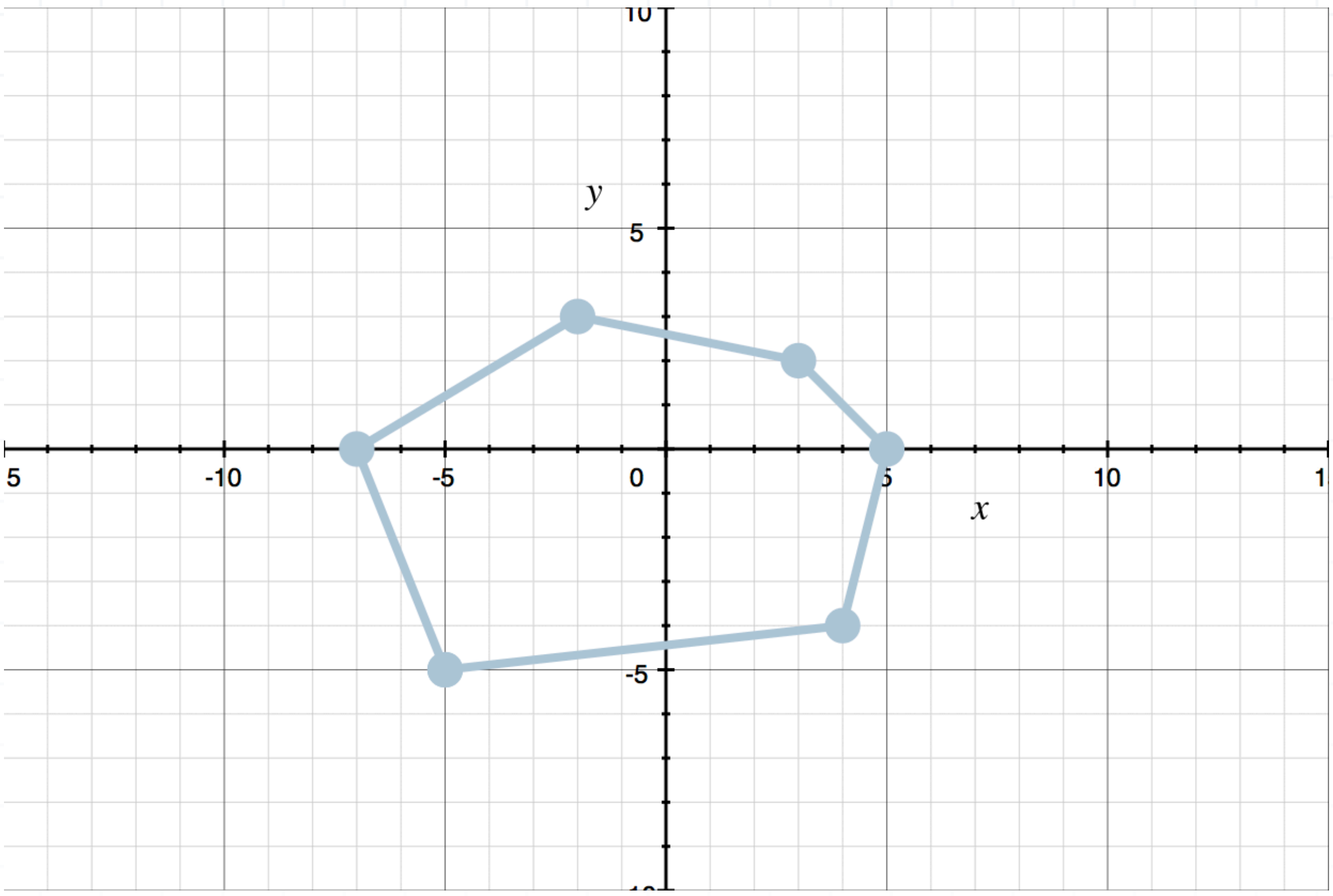
3. Sketch the transformation of the square in the graph after it's transformed by matrix Z.

$$Z=\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}$$



4. Sketch the transformation of the hexagon after it's transformed by matrix Y.

$$Y = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$



■ 5. What happens to the unit vector $\overrightarrow{a}=(1,0)$ after the transformation given by matrix K.

$$K=\begin{bmatrix}3 & -5\\ -1 & 0\end{bmatrix}$$

■ 6. What happens to the unit vector $\overrightarrow{b}=(0,1)$ after the transformation given by matrix K.

$$K=\begin{bmatrix}3 & -5\\ -1 & 0\end{bmatrix}$$

PREIMAGE, IMAGE, AND THE KERNEL

■ 1. Find the preimage A_1 of the subset B_1 under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 2. Find the preimage A_1 of the subset B_1 under the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -1 & -3 & 2 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 3. Find the kernel of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

$$T(\vec{x}) = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 4. Find the kernel of the transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$.

$$T(\vec{x}) = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 5. Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$.

$$B_1 = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 6. Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$.

$$B_1 = \left\{ \begin{bmatrix} -4 \\ 4 \\ -12 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 & -6 \\ 1 & -3 & 0 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

LINEAR TRANSFORMATIONS AS MATRIX-VECTOR PRODUCTS

■ 1. Use a matrix-vector product to reflect the square with vertices $(3,-1)$, $(1,-1)$, $(1,1)$, and $(3,1)$ over the y-axis. What are the vertices of the reflected square? Graph the resulting figure.

■ 2. Use a matrix-vector product to transform the triangle with vertices $(-2,-3)$, $(4,2)$, and $(2,-5)$. The transformation T should include a reflection over the x-axis and horizontal stretch by a factor of 5. Graph the resulting figure.

■ 3. Use a matrix-vector product to reflect the triangle with vertices $(3,3)$, $(1,-2)$, and $(-3,3)$ over the line $y=x$. What are the vertices of the reflected triangle? Graph the resulting figure.

■ 4. Use a matrix-vector product to triple the width of the rectangle that has vertices $(-2,-1)$, $(-2,-5)$, $(5,-1)$, and $(5,-5)$, and then compress it vertically by a factor of 2. What are the vertices of the transformed rectangle? Graph the resulting figure.

■ 5. Use a matrix-vector product to reflect the parallelogram with vertices $(-3,1)$, $(0,4)$, $(7,4)$, and $(4,1)$ over the x-axis, and then over the y-axis. What are the vertices of the reflected parallelogram? Graph the resulting figure.

■ 6. Use a matrix-vector product to reflect the triangle with vertices $(2,3)$, $(-5,-4)$, and $(-4,5)$ over the y-axis, and then stretch it horizontally by a factor of 4. What are the vertices of the transformed triangle? Graph the resulting figure.

LINEAR TRANSFORMATIONS AS ROTATIONS

■ 1. Find the rotation of $\vec{x}=(2,-4)$ by an angle of $\theta=120^\circ$.

■ 2. Find the rotation of $\vec{x}=(1,-5)$ by an angle of $\theta=60^\circ$.

■ 3. Find the rotation of $\vec{x}=(-7,4)$ by an angle of $\theta=180^\circ$.

■ 4. Find the rotation of $\vec{x}=(-4,1,3)$ by an angle of $\theta=90^\circ$ about the x-axis.

■ 5. Find the rotation of $\vec{x}=(-2/\sqrt{2},2,0)$ by an angle of $\theta=315^\circ$ about the y-axis.

■ 6. Find the rotation of $\vec{x}=(-2,0,3)$ by an angle of $\theta=150^\circ$ about the z-axis.

ADDING AND SCALING LINEAR TRANSFORMATIONS

■ 1. Find the product of a scalar $c = 5$ and the transformation $T(\vec{x})$.

$$T(\vec{x}) = \begin{bmatrix} 0 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 2. Find the product of a scalar $c = -2$ and the transformation $T(\vec{x})$.

$$T(\vec{x}) = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -5 & 7 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 3. Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 4. Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} 0 & -4 & 1 \\ 1 & -1 & 3 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 5 & -3 & 3 \\ 2 & 0 & -1 \\ 1 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 5. Find the sum of the transformation $S(\vec{x})$ and the product of a scalar $c = -1/2$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} -5 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 6. Find the product of $c = 1/3$ with the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} -5 & 4 & -3 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -1 & 2 & 0 \\ 6 & -4 & 5 \\ 9 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

PROJECTIONS AS LINEAR TRANSFORMATIONS

1. Find the projection of $\vec{v}$ onto L.

$$L = \left\{ c \begin{bmatrix} 4 \\ 2 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

2. Find the projection of $\vec{v}$ onto M.

$$M = \left\{ c \begin{bmatrix} -4 \\ 3 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

3. Find the projection of $\vec{v}$ onto L and the vector complement of $\vec{v}$ orthogonal to L.

$$L = \left\{ c \begin{bmatrix} -3 \\ 1 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

4. Find the projection of $\vec{v}$ onto L and the vector complement of $\vec{v}$ orthogonal to L.

$$L = \left\{ c \begin{bmatrix} -2 \\ 0 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

5. Find the projection of $\vec{v}$ onto L.

$$L = \left\{ c \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

6. Find the projection of $\vec{v}$ onto L and the vector complement of $\vec{v}$ orthogonal to L.

$$L = \left\{ c \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

COMPOSITIONS OF LINEAR TRANSFORMATIONS

■ 1. If $S: \mathbb{R}^2 \to \mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$, then what is $T(S(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -x_2 + 3x_1 \\ x_1 + 2x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

■ 2. If $S: \mathbb{R}^2 \to \mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$, then what is $S(T(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -2x_1 + x_2 \\ -x_1 - x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -x_1 + 3x_2 \\ -2x_1 + 2x_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

■ 3. If $S: \mathbb{R}^3 \to \mathbb{R}^3$ and $T: \mathbb{R}^3 \to \mathbb{R}^3$, then what is $T(S(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -2x_1 + x_2 - x_3 \\ -x_2 + x_3 \\ x_1 + 2x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -3x_3 \\ x_1 + x_2 - 2x_3 \\ -x_1 - x_2 + x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

■ 4. If $S: \mathbb{R}^2 \to \mathbb{R}^3$ and $T: \mathbb{R}^3 \to \mathbb{R}^2$, then what is $S(T(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -x_1 \\ x_1 - 3x_2 \\ 2x_2 - 3x_1 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 2x_1 - x_3 \\ x_2 - x_1 + x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

■ 5. If $S: \mathbb{R}^2 \to \mathbb{R}^3$ and $T: \mathbb{R}^3 \to \mathbb{R}^2$, then what is $T(S(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -x_1 + x_2 \\ 2x_2 - 3x_1 \\ x_1 + 2x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_1 + x_2 - x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

■ 6. If $S : \mathbb{R}^3 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$, then what are $T(S(\vec{x}))$ and $S(T(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -x_3 + 2x_2 \\ x_1 - x_3 \\ x_1 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -2x_1 + x_2 + 2x_3 \\ 3x_1 \\ x_1 - 2x_2 + x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$