Linear Algebra and Geometry 1

Systems of equations, matrices, vectors, and geometry

Linear transformations

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MiTM
Man in The Middle AB

An introduction to linear operations

Linear operations preserve linear combinations

Differentiation is a linear operation

$$(f+g)' = f' + g'$$

$$(\alpha f)' = \alpha f'$$

$$(\alpha f + \beta g)' = \alpha f' + \beta g'$$

Integration is linear

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$$

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

Sine is not a linear function

$$\sin(x+y) \neq \sin x + \sin y$$

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$

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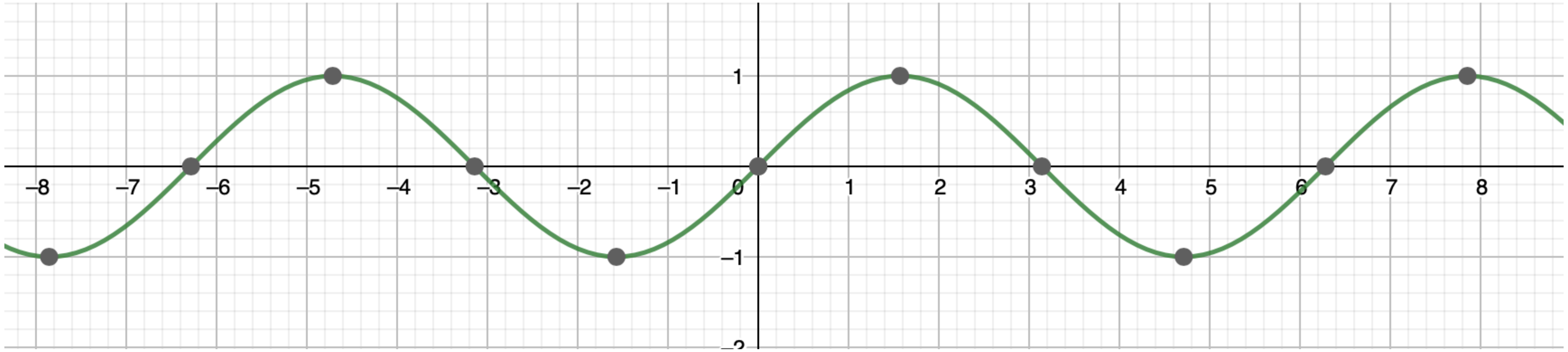

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$

Logarithm is not a linear function

$$\ln(x+y) \neq \ln x + \ln y$$

$$\ln(xy) = \ln x + \ln y$$

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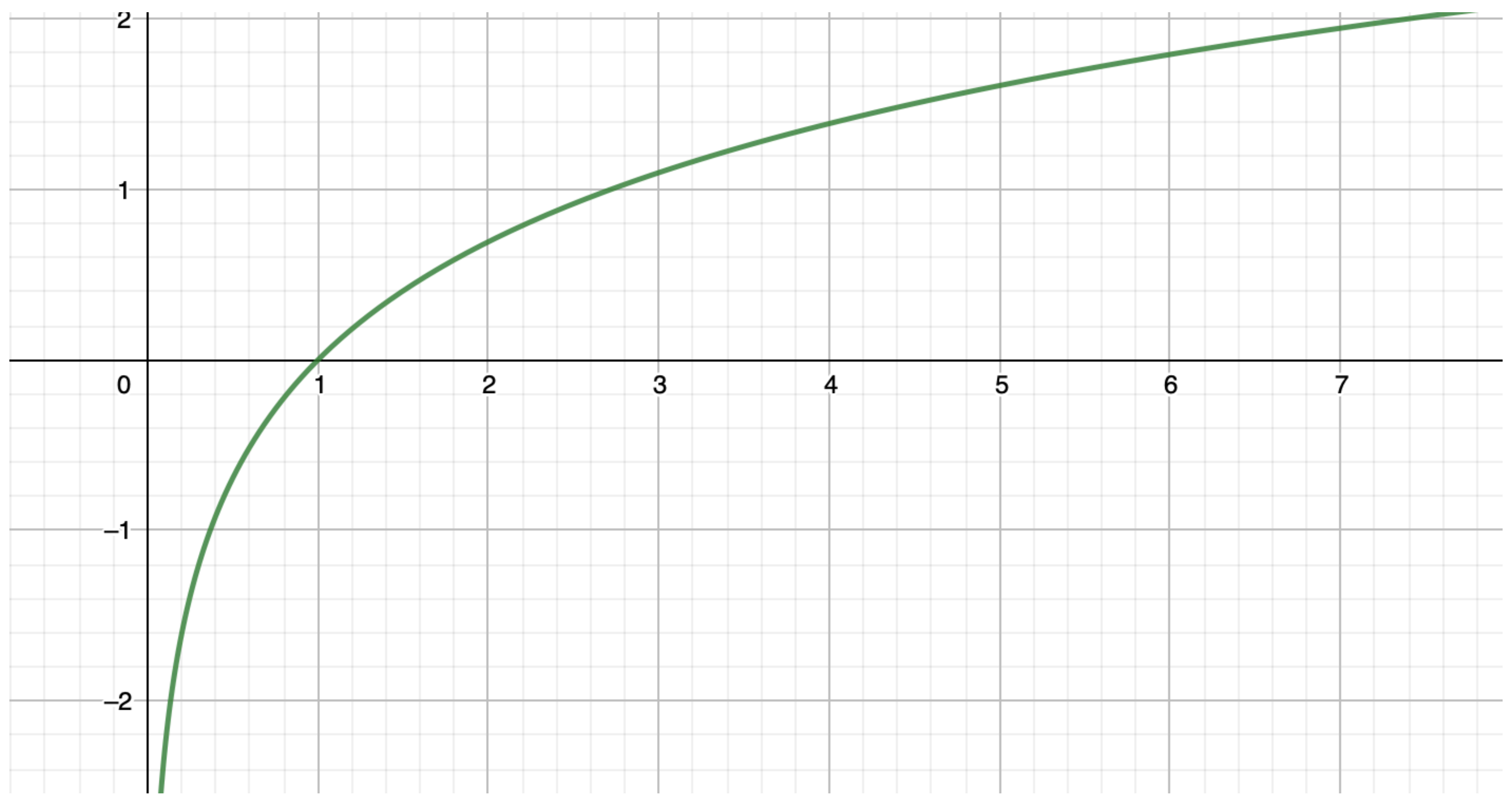

$$\ln(xy) = \ln x + \ln y$$

Exponential function is not linear

$$a^{x+y} \neq a^x + a^y$$

$$a^{x+y} = a^x \cdot a^y$$

Exponential function is not linear

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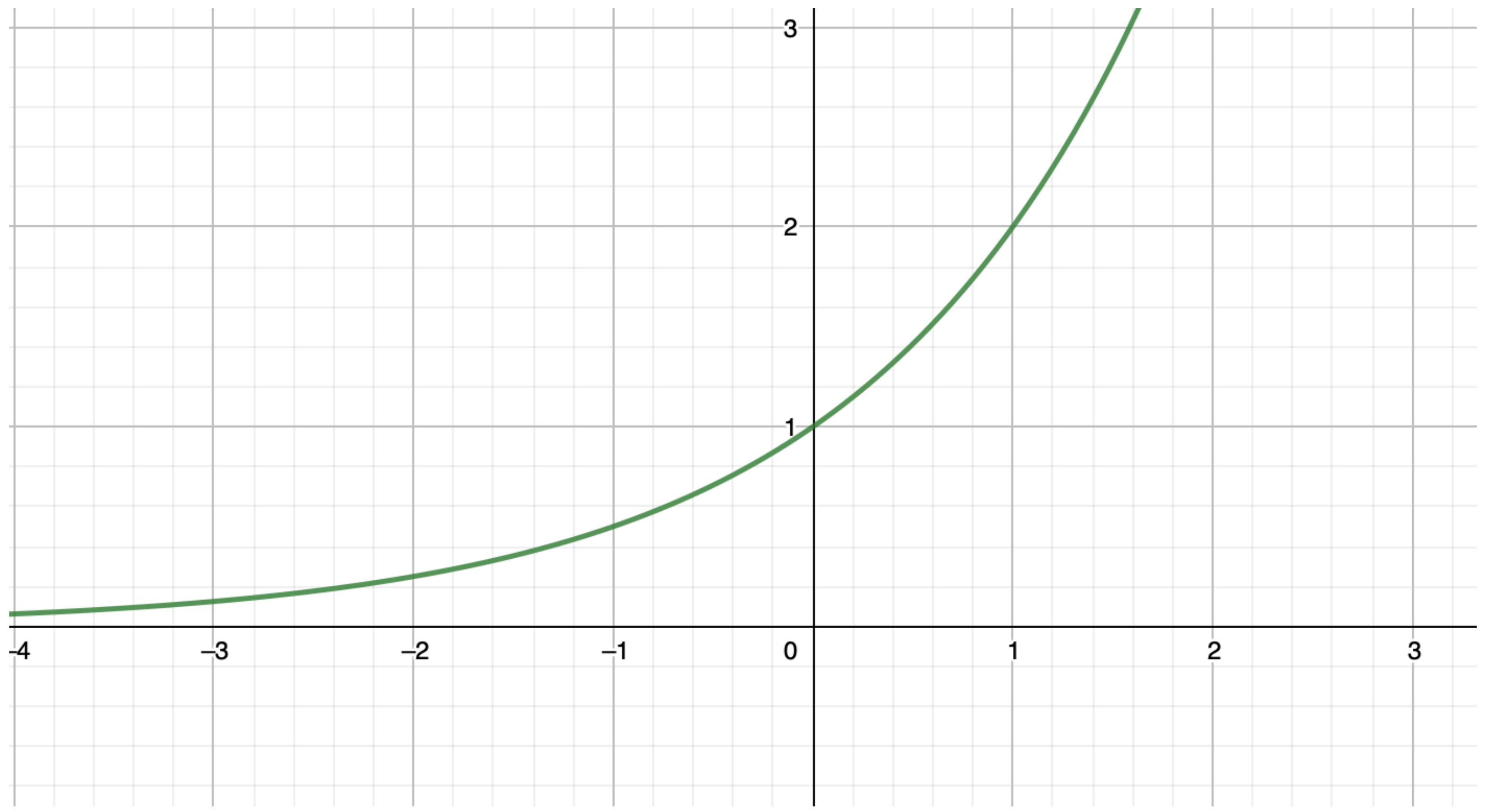

$$a^{x+y} = a^x \cdot a^y$$

Power functions are not linear

$$(x+y)^2 \neq x^2 + y^2$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

Power functions are not linear

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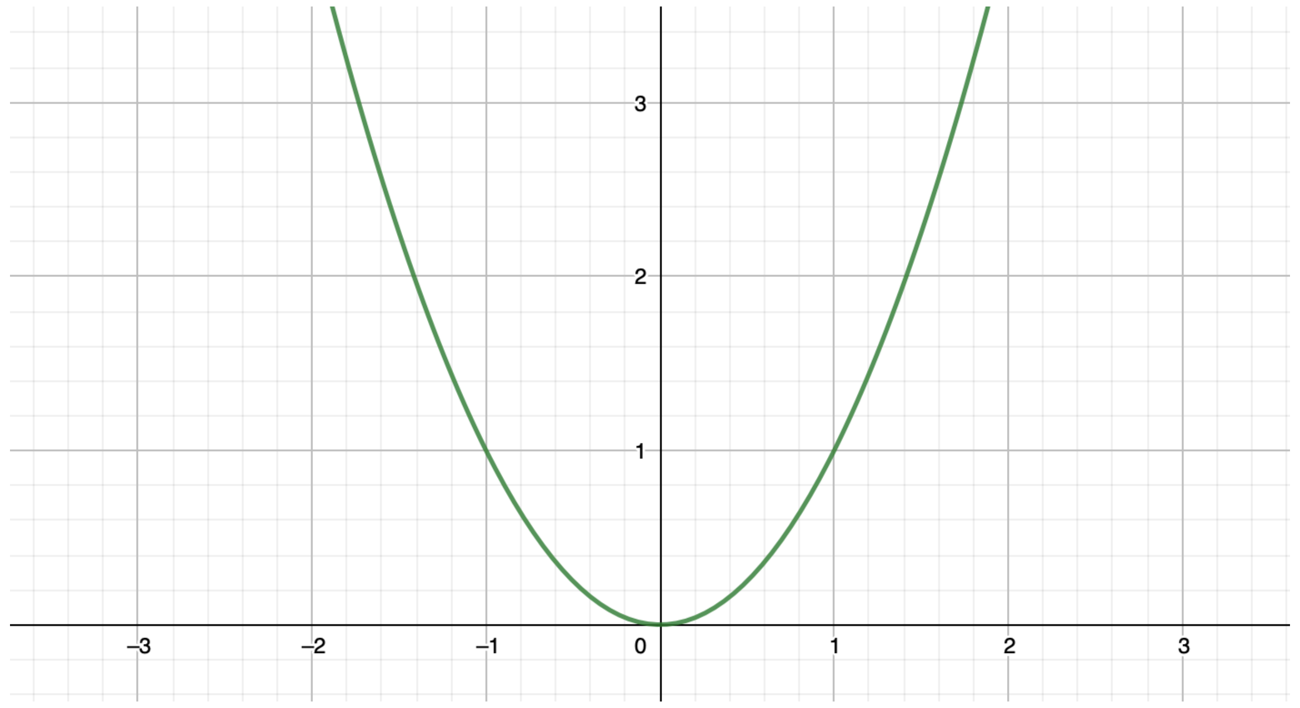

$$(x+y)^2 = x^2 + 2xy + y^2$$

Square root is not a linear function

$$\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$$

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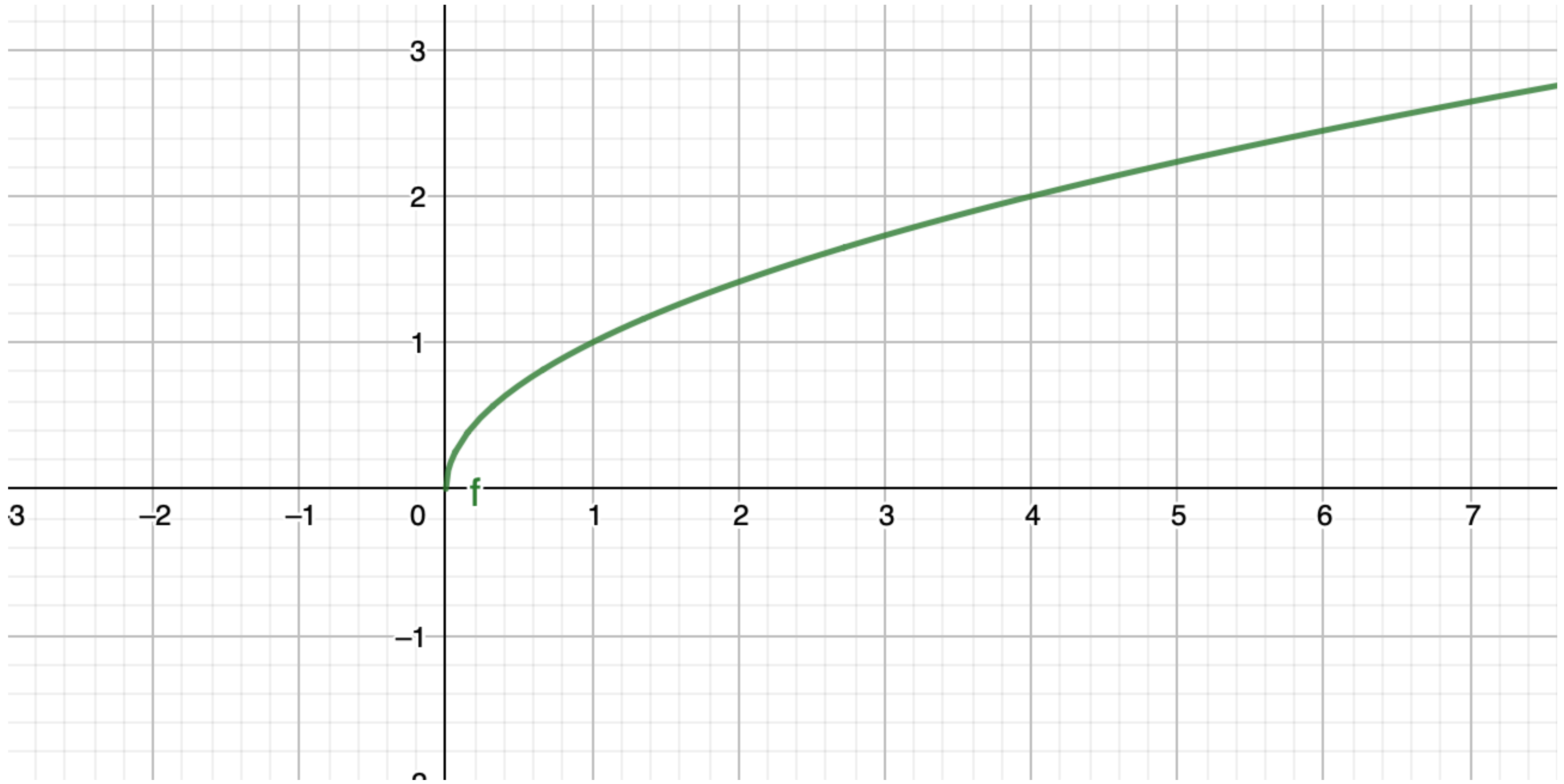

Functions, generally

Domain, codomain, range

$$f : X \to Y$$

$D_f \subset X$ all the possible arguments

Y

$V_f \subset Y$ $V_f = \{\, y;\ y = f(x) \text{ for some } x \in D_f \,\}$

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$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$$

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$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$$

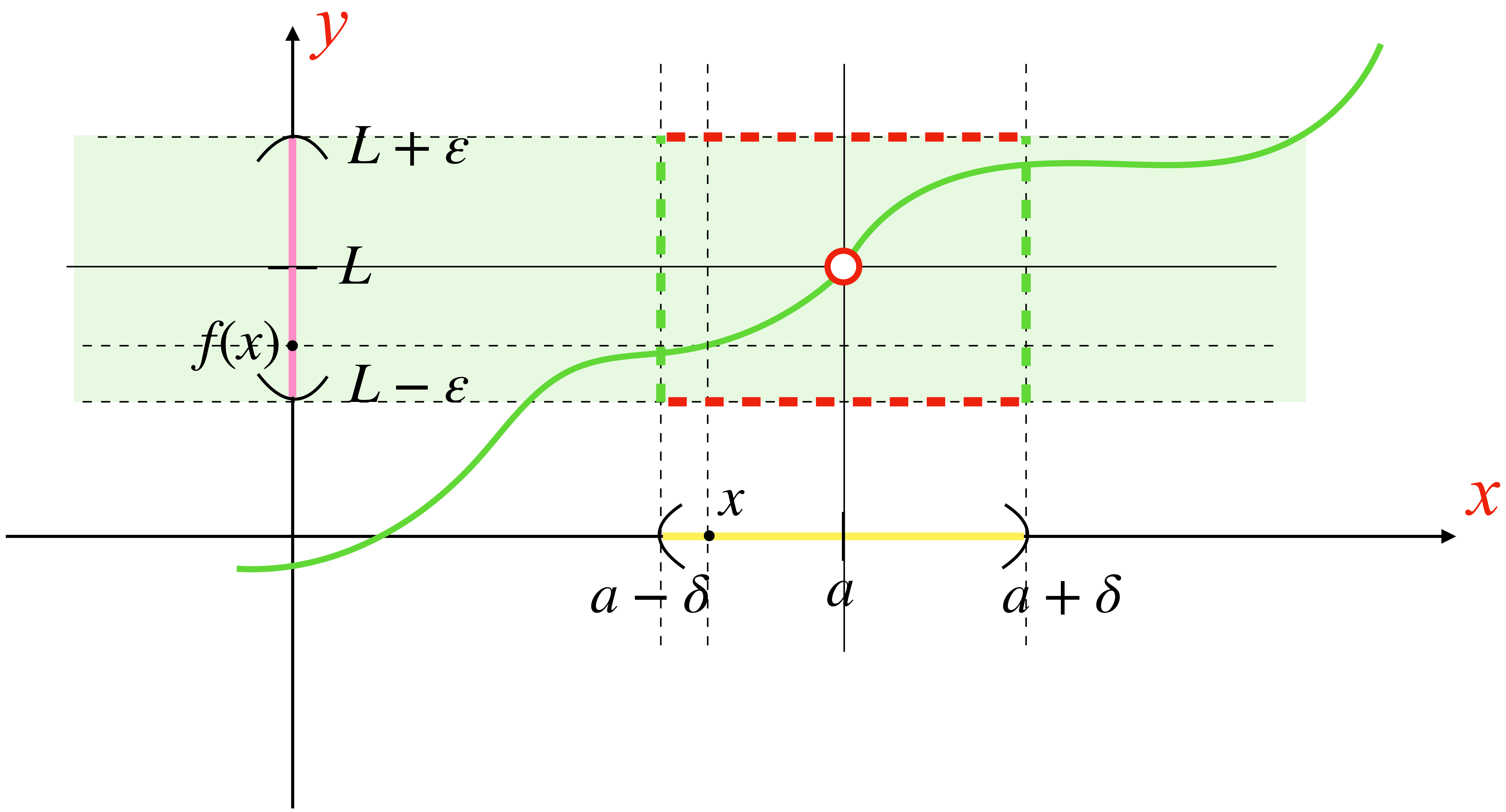
$D_f = \mathbb{R} \setminus \{0\} = V_f$

Functions in Calculus 1

Continuous functions

$$\lim_{x \to a} f(x) = L$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D_f \quad 0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$



The values $f(x)$ can get arbitrarily close (ε-close) to L if only the arguments x are close enough (δ-close) the point a

$$x \to a \quad \Rightarrow \quad |f(x) - L| \to 0$$

$$\lim_{x \to a} f(x) = f(a)$$

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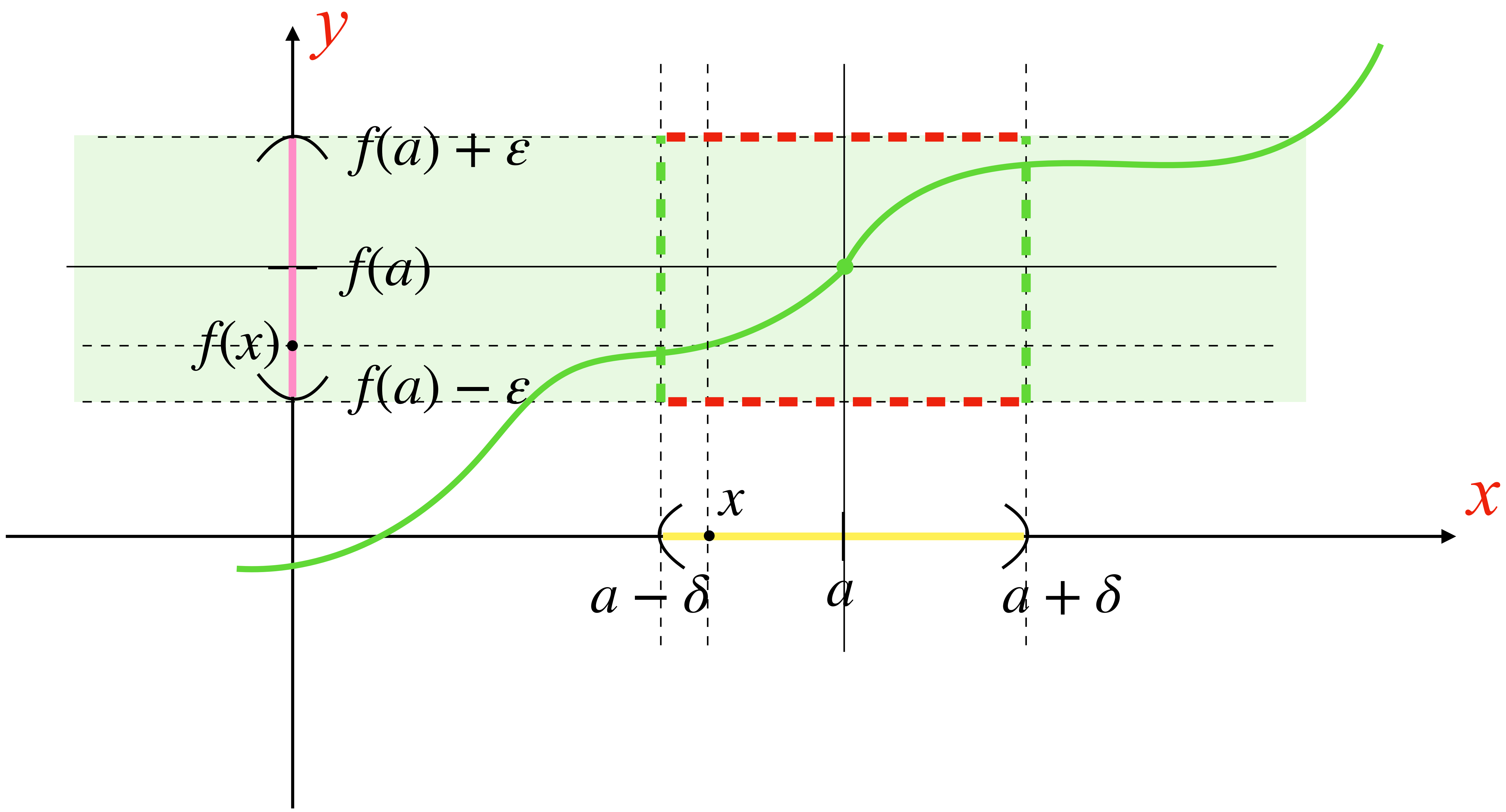

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Functions in Linear Algebra

Linear transformations

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

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The image of a linear combination of two vectors
is the same linear combination of the images of the vectors.

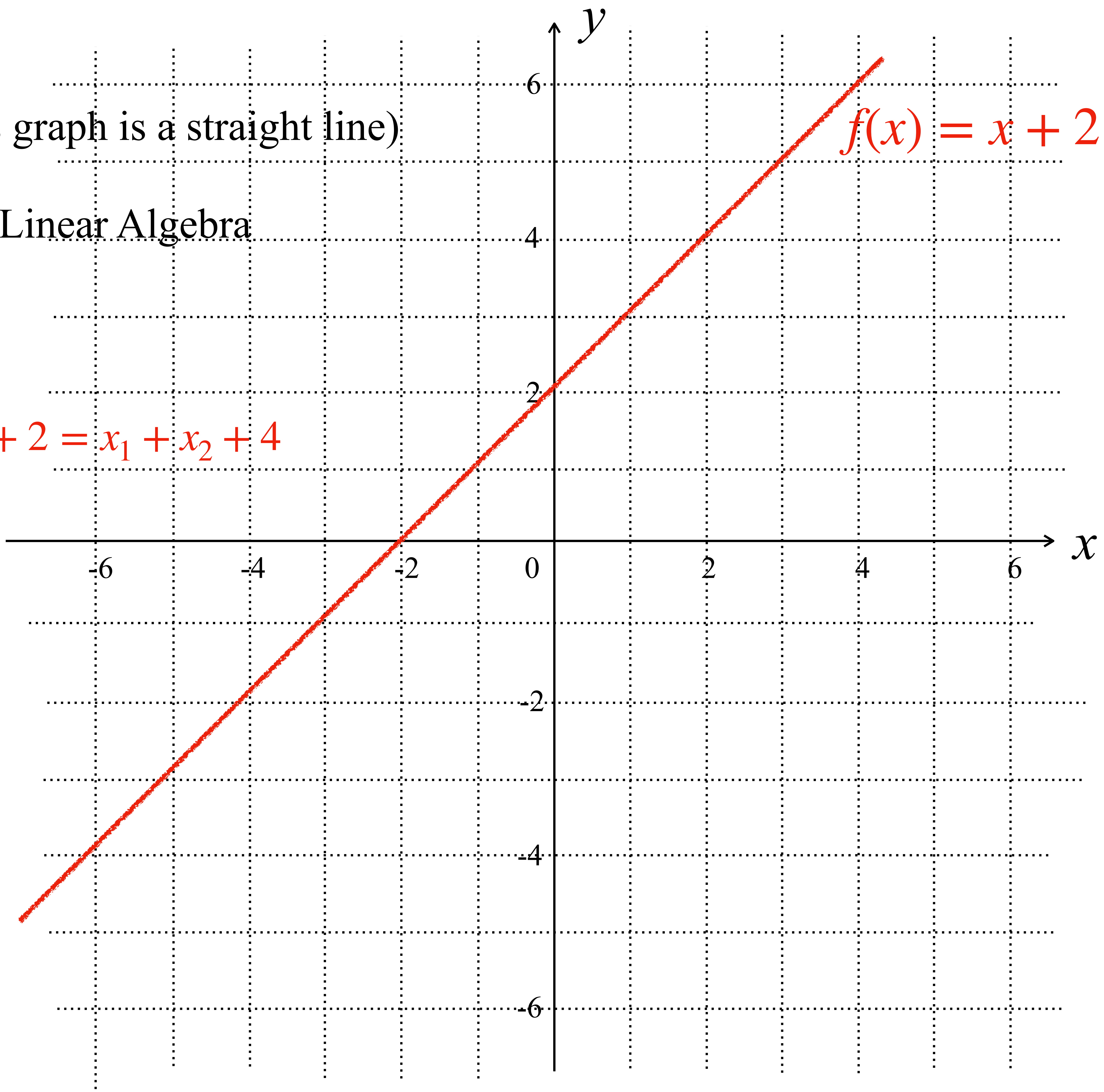
$T : \mathbb{R} \to \mathbb{R}$

Linear in Calculus (its graph is a straight line)

Not linear in Linear Algebra

$f(x_1 + x_2) = x_1 + x_2 + 2$

$f(x_1) + f(x_2) = x_1 + 2 + x_2 + 2 = x_1 + x_2 + 4$



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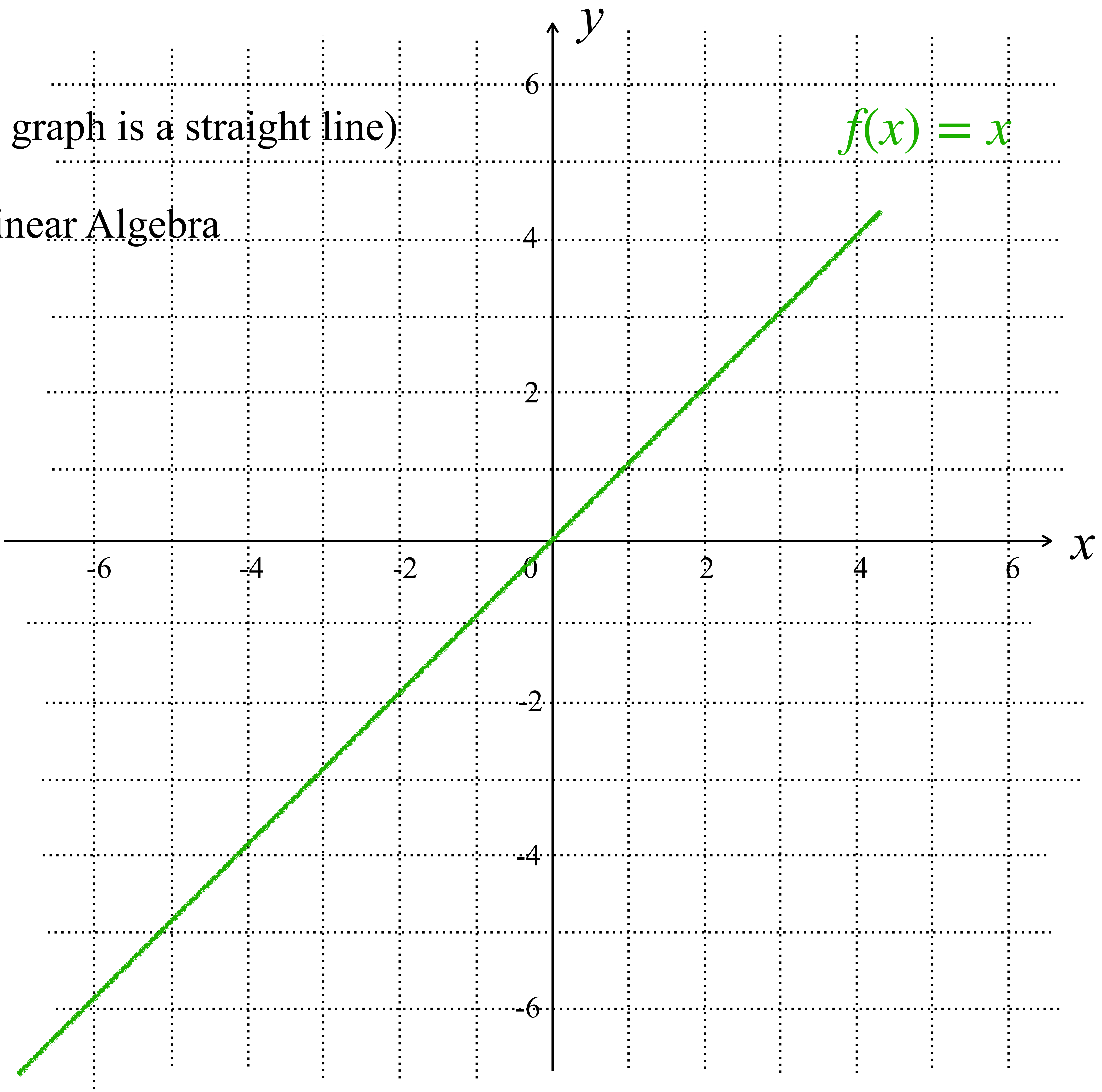

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Linear in Calculus (its graph is a straight line)

Linear in Linear Algebra

$f(x_1 + x_2) = 3(x_1 + x_2) = 3x_1 + 3x_2$

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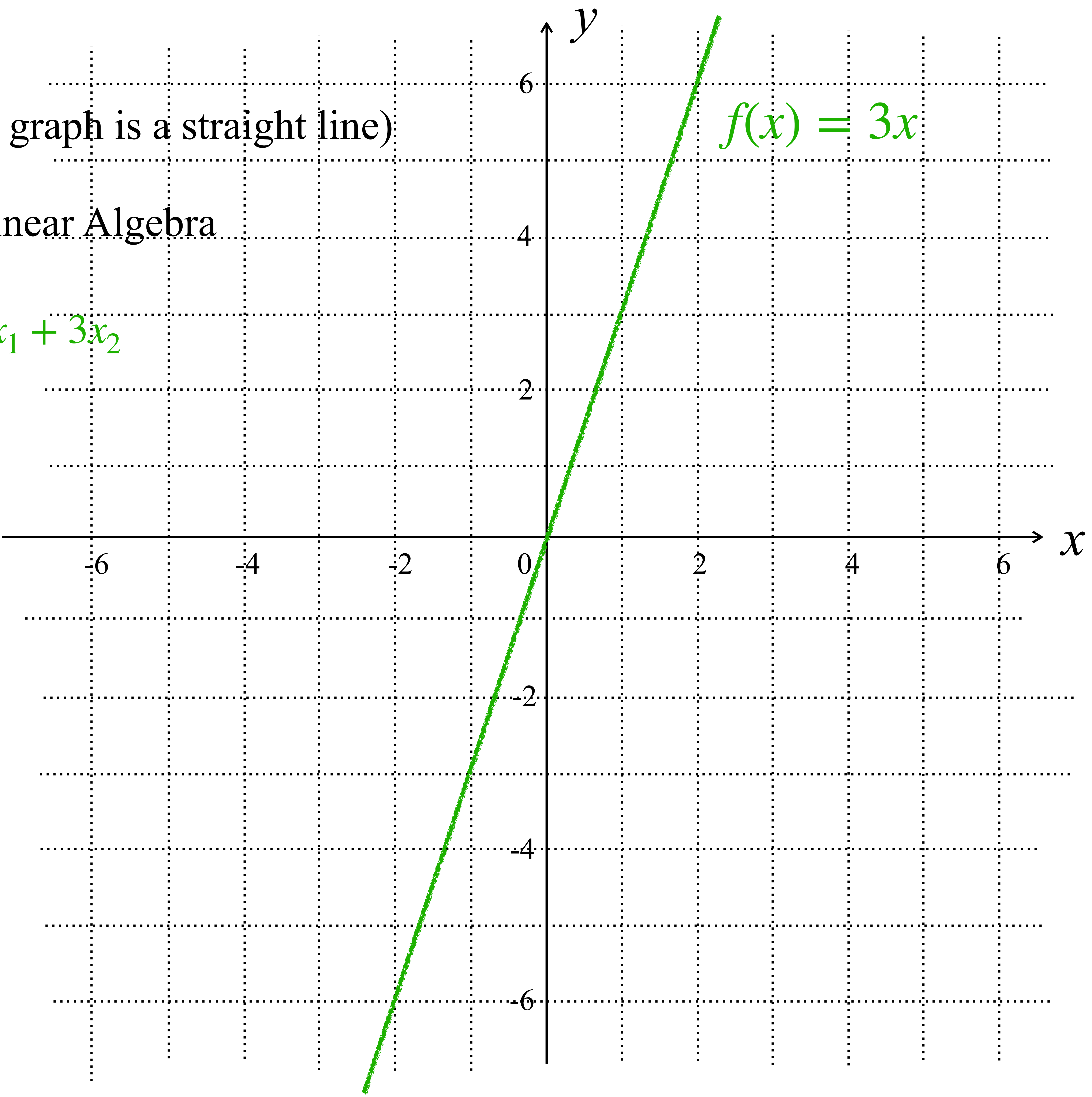
$f(\alpha x) = 3\alpha x = \alpha 3x = \alpha f(x)$



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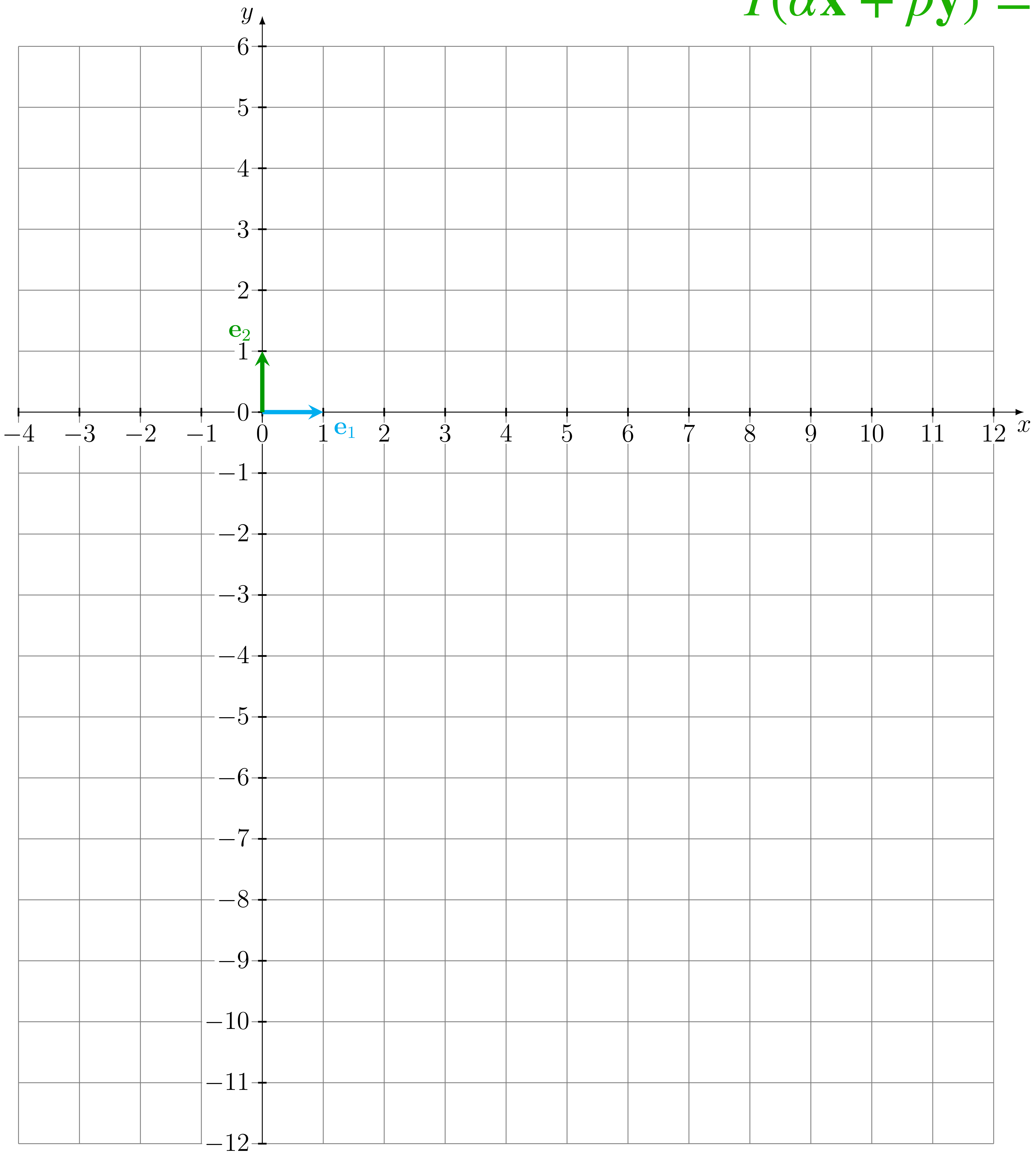

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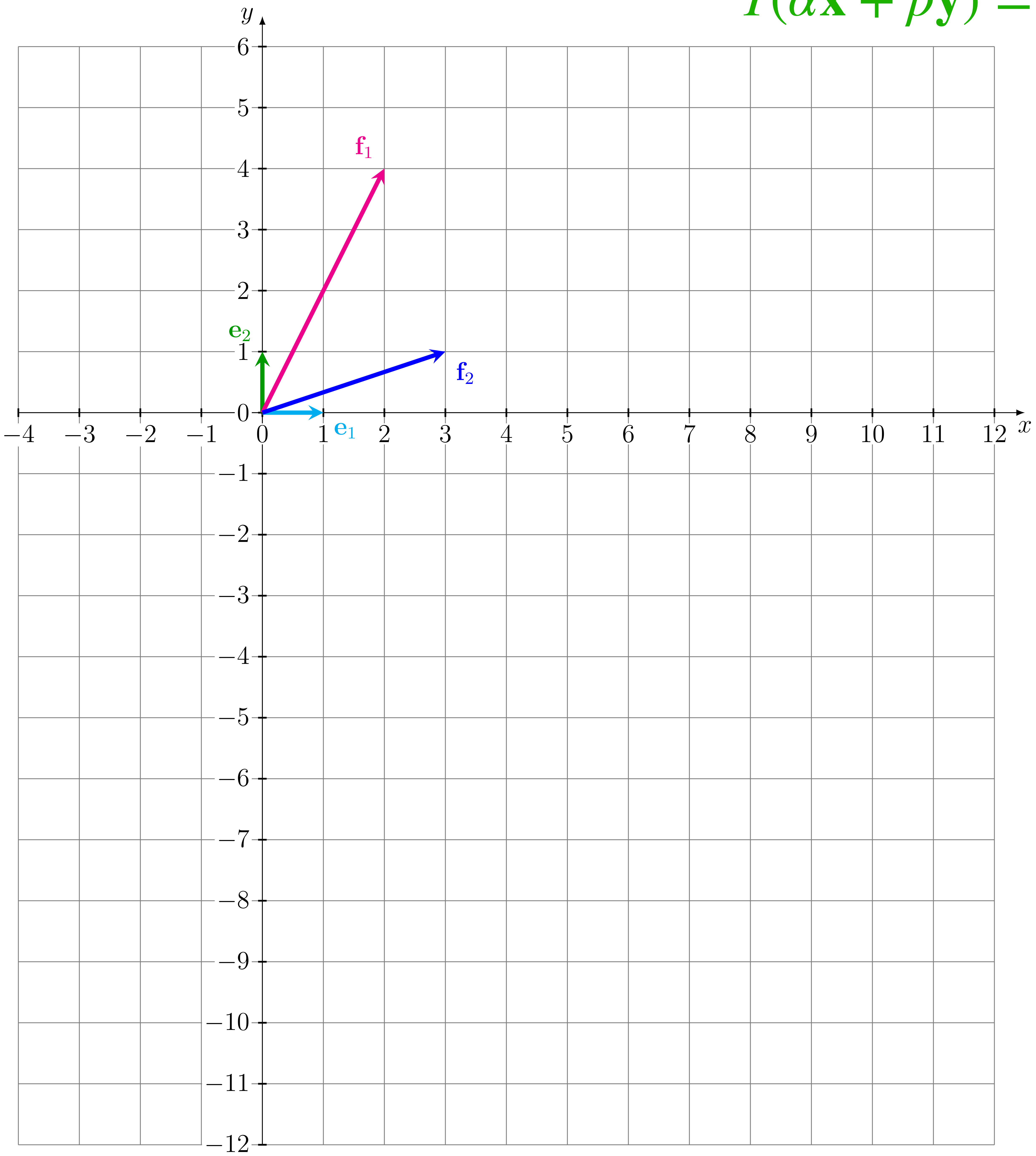

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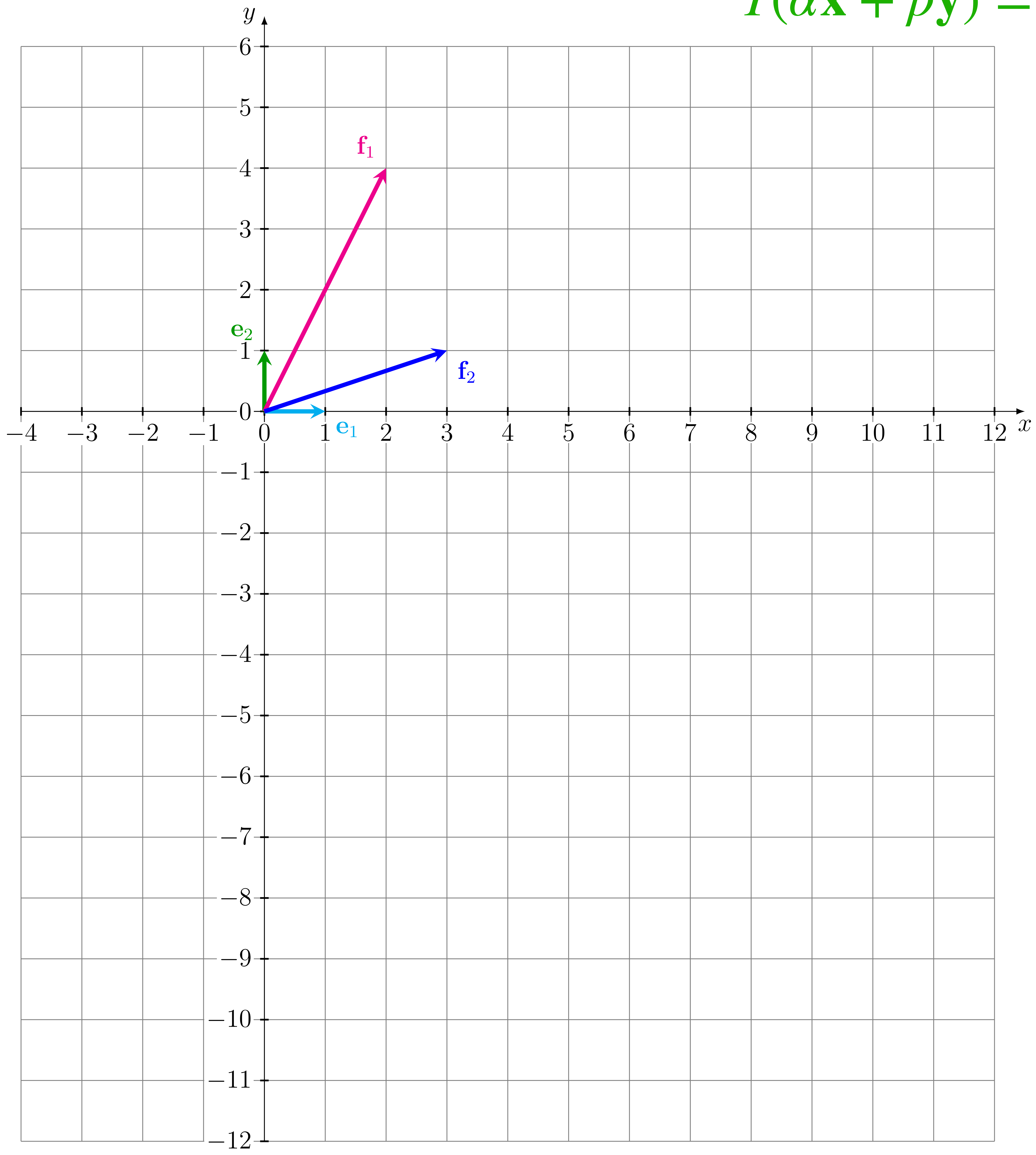

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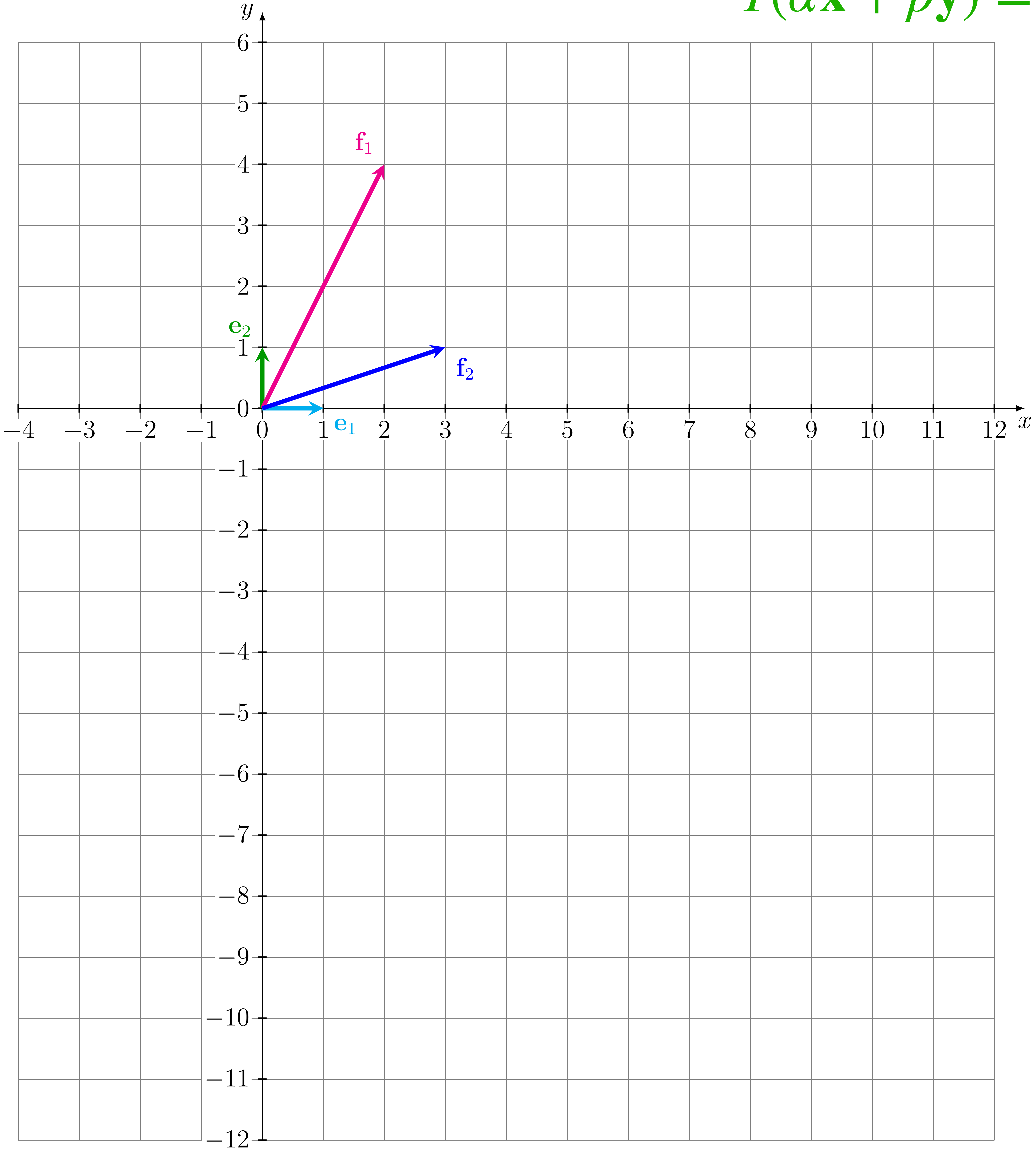

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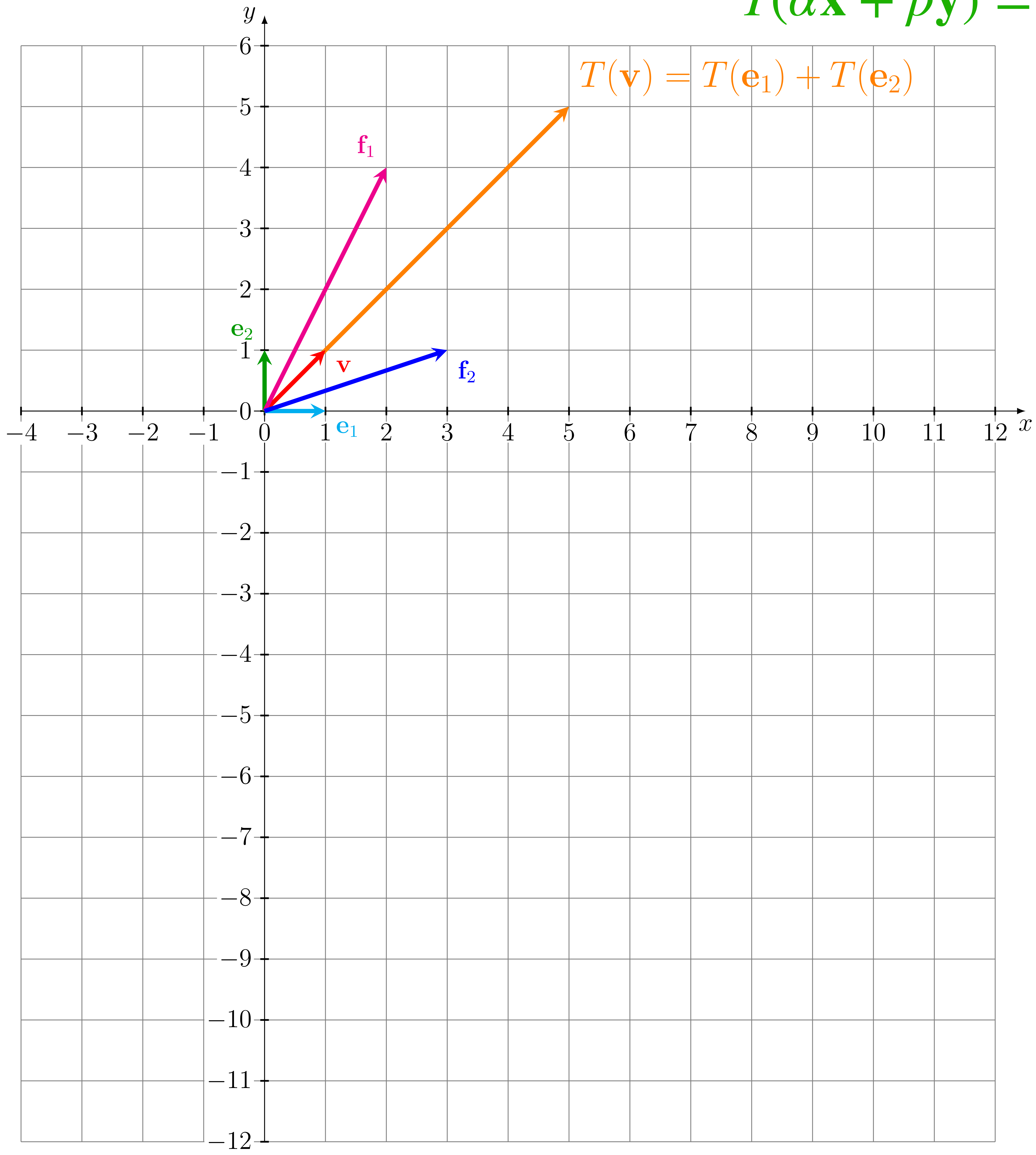

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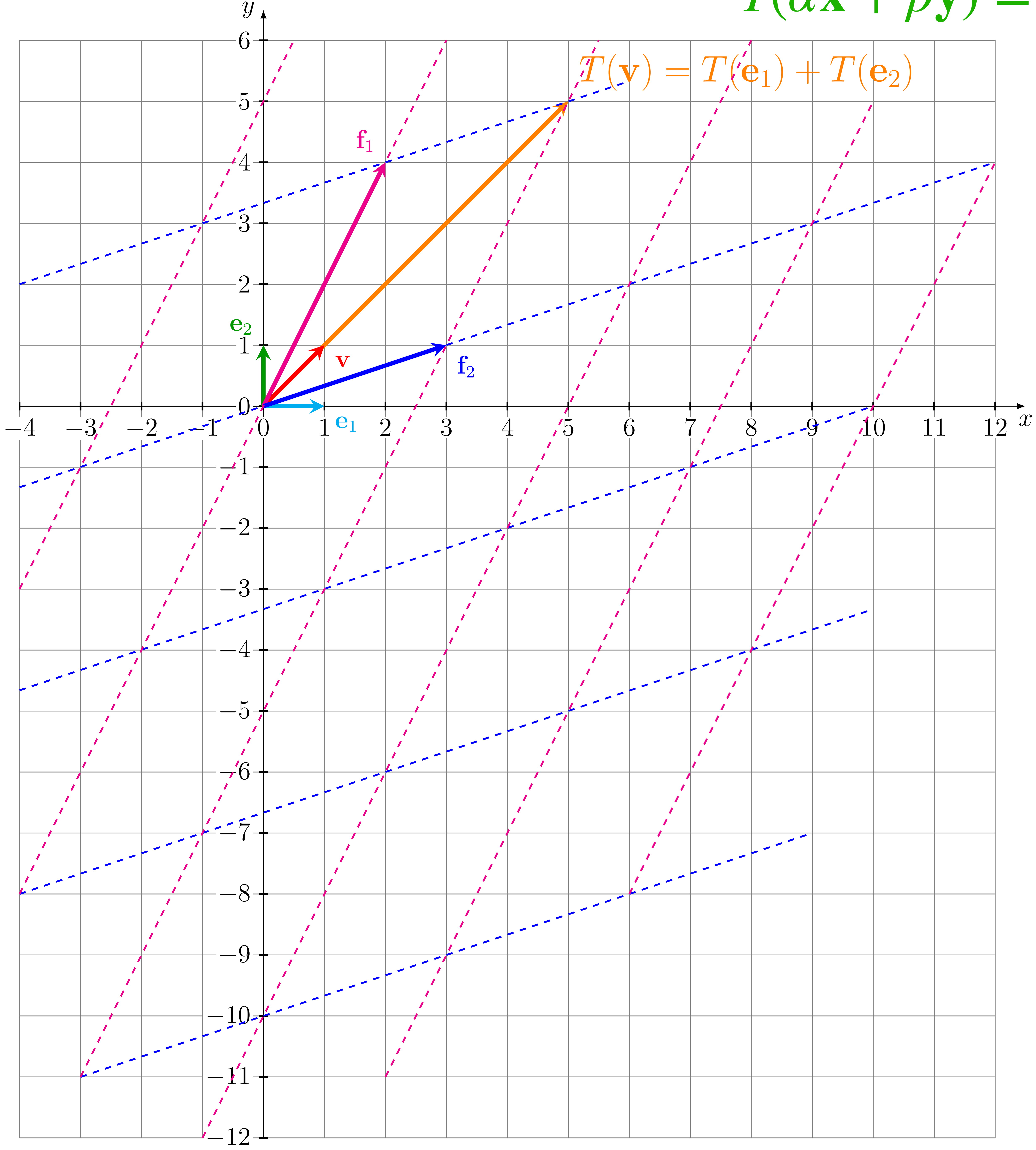

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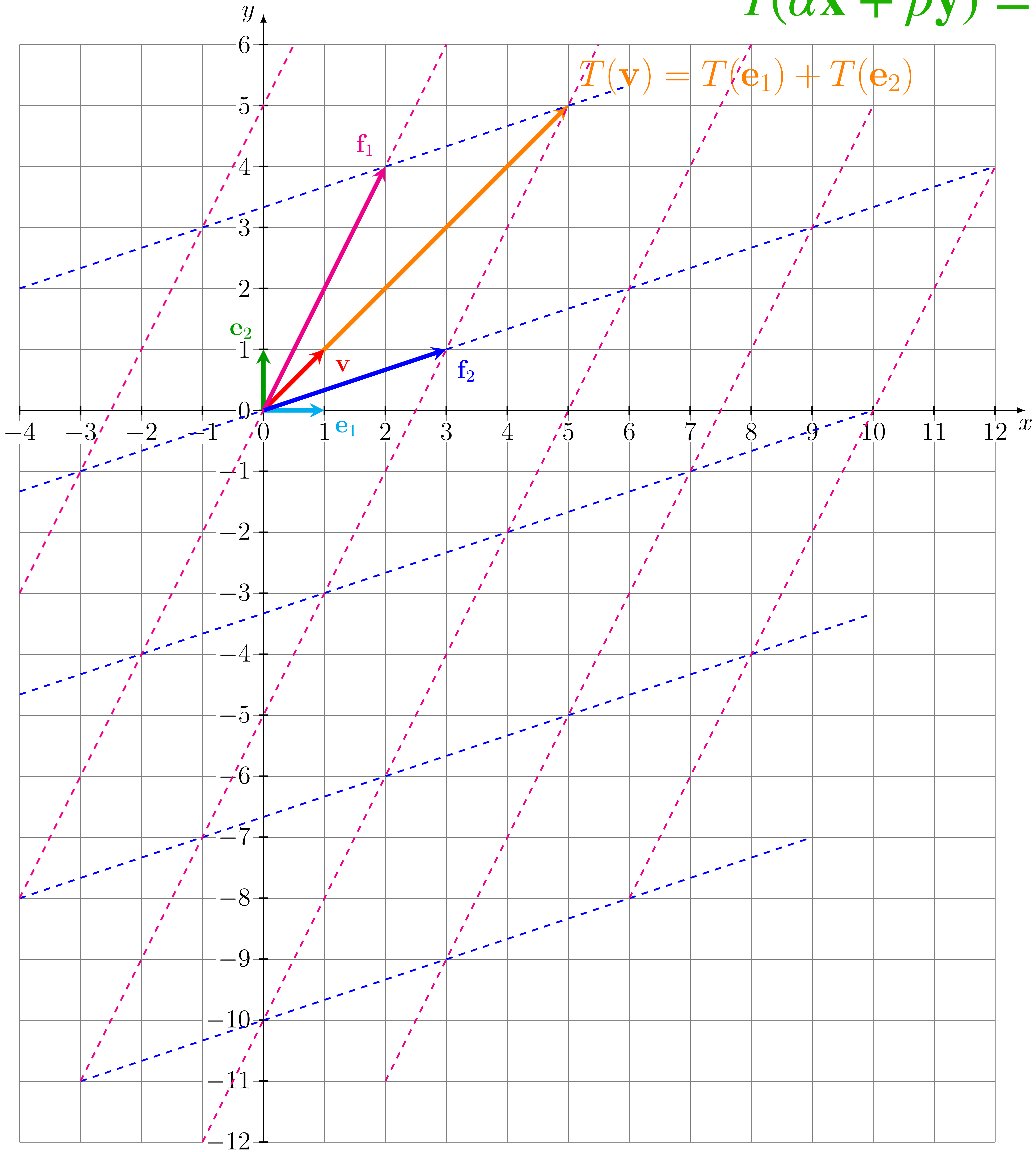
The grid lines transform into new grid lines: parallel and equally spaced.



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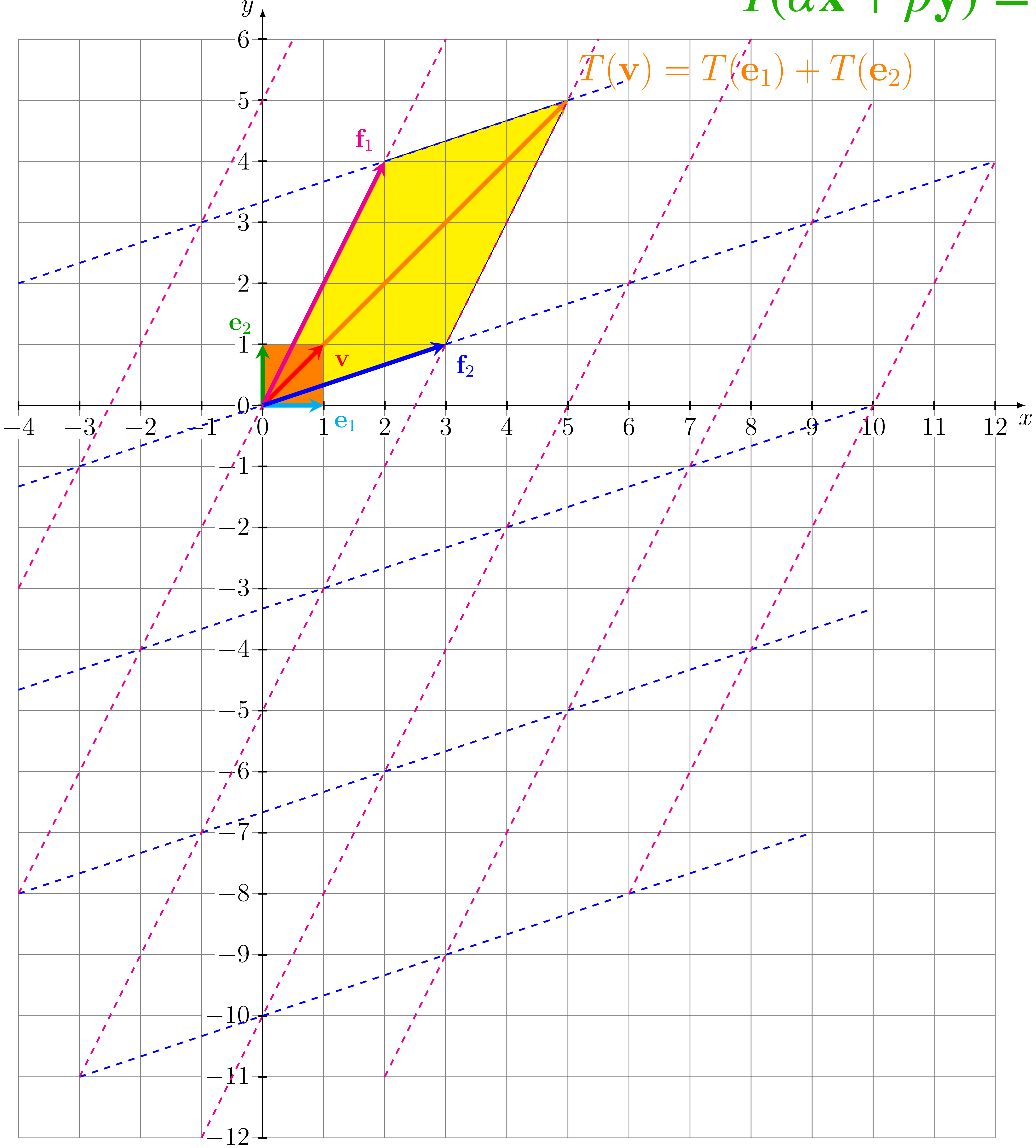

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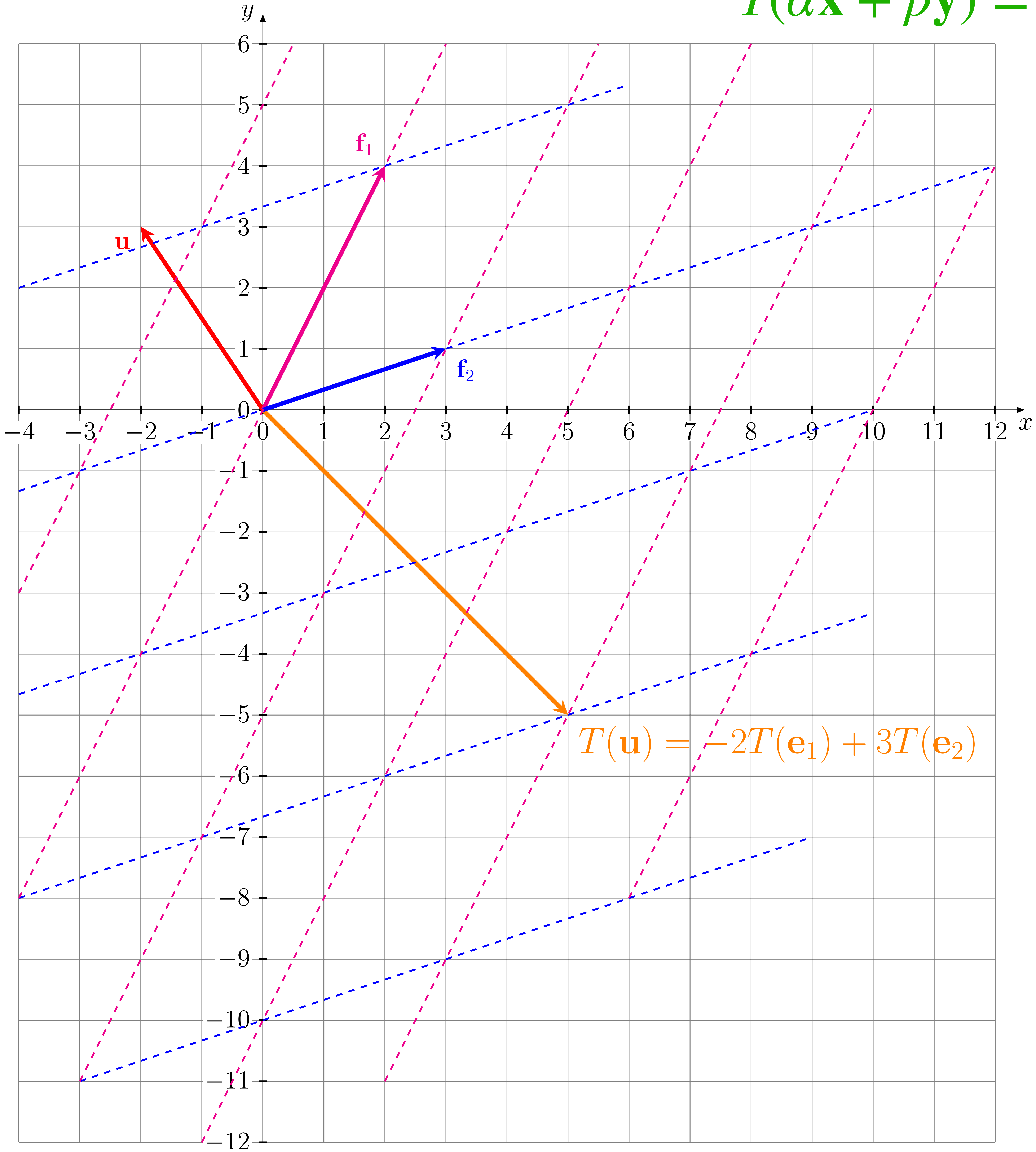

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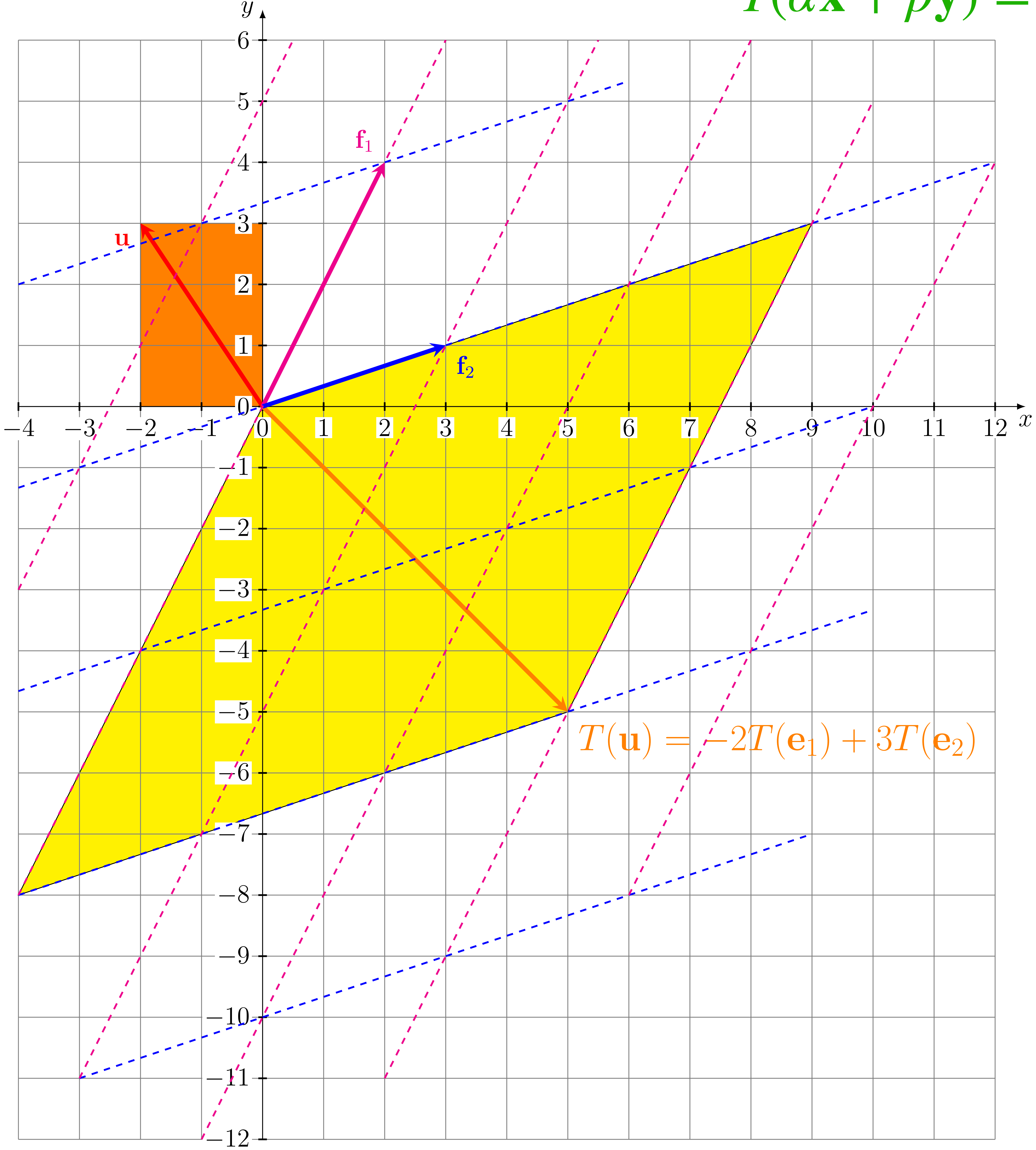

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induction

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