Dot and cross products as opposite ideas

At this point, we know how to calculate dot products and cross products, and we know we can use them to find angles between vectors, test for linear independence, etc.

Now we want to look at what dot products and cross products tell us in general, and how they really represent totally opposite ideas.

What the dot product measures

The **dot product** measures how much two vectors move in the same direction. We can see this somewhat if we look at the formula for the angle between two vectors.

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||\cos\theta$$

When two vectors are perpendicular, the angle between them is 90° (in degrees, or $\pi/2$ in radians). At that angle, $\cos\theta$ is 0, which means we get

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||\cos(90^\circ)$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||(0)$$

$$\vec{u} \cdot \vec{v} = 0$$

When two vectors point in the same direction, the angle between them is 0° (in degrees, or 0 in radians). At that angle, $\cos\theta$ is 1, which means we get

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||\cos(0^\circ)$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}||(1)$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}||$$

When two vectors point in exactly opposite directions, the angle between them is 180° (in degrees, or π in radians). At that angle, $\cos\theta$ is -1, which means we get

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}||\cos(180^\circ)$$

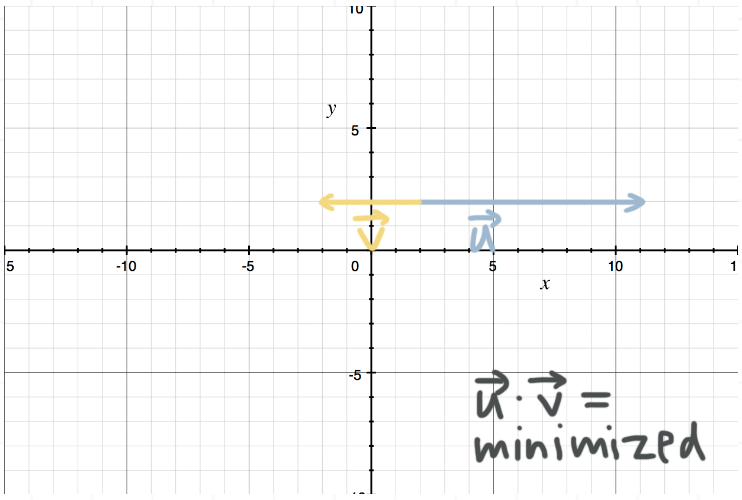
$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}||(-1)$$

$$\vec{u} \cdot \vec{v} = -||\vec{u}|| \ ||\vec{v}||$$

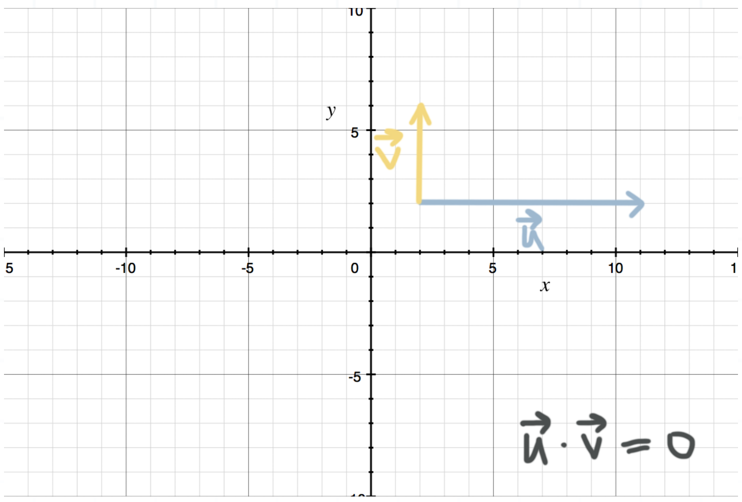
So what we really see here is that the dot product will have its minimum value when vectors point in exactly opposite directions, that the dot product will be 0 when the vectors are orthogonal, and that the dot product will have its maximum value when vectors point in exactly the same direction. When we say that the dot product is maximized, what we really mean is that the dot product, given by $\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}||$, is just the product of the lengths of the vectors.

Here's a summary of these dot product values:

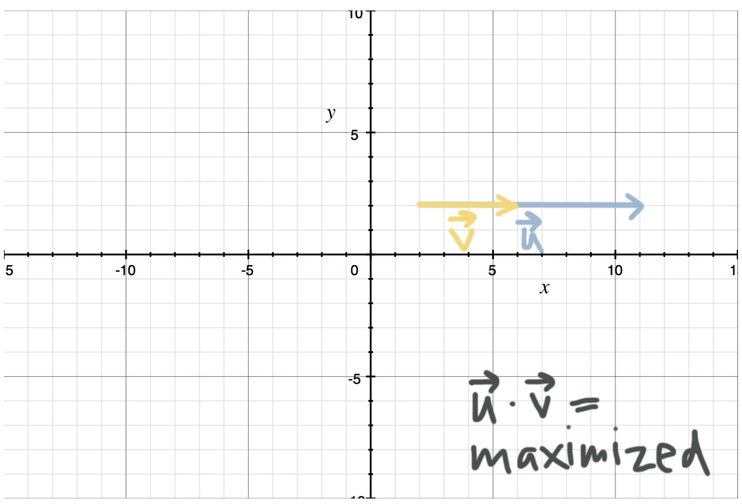
Minimum (and negative) dot product when the vectors point in opposite directions:



Dot product of 0 when the vectors are orthogonal:



Maximum (and positive) dot product when the vectors point in the same direction:



What the cross product measures

The **cross product** measures how much two vectors move in different directions. We can see this somewhat if we look at the formula for the length of the cross product.

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin\theta$$

When two vectors are perpendicular, the angle between them is 90° (in degrees, or $\pi/2$ in radians). At that angle, $\sin\theta$ is 1, which means we get

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin(90^\circ)$$

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||(1)$$

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||$$

When two vectors point in the same direction, the angle between them is 0° (in degrees, or 0 in radians). At that angle, $\sin\theta$ is 0, which means we get

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin(0^\circ)$$

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||(0)$$

$$||\vec{u}\times\vec{v}||=0$$

When two vectors point in opposite directions, the angle between them is 180° (in degrees, or π in radians). At that angle, $\sin\theta$ is 0, which means we get

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin(180^\circ)$$

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||(0)$$

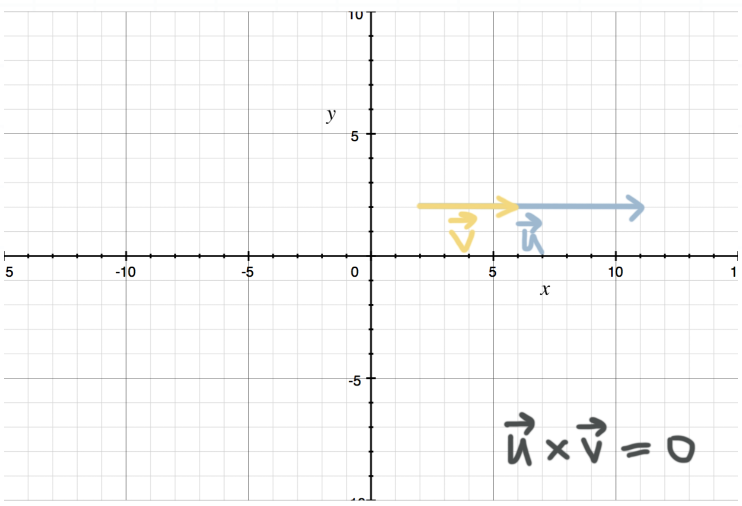
$$||\vec{u}\times\vec{v}||=0$$

So what we really see here is that the length of the cross product is minimized ($||\vec{u} \times \vec{v}|| = 0$) when the vectors are collinear (when they point in the same or opposite directions), and that the length of the cross product is maximized ($||\vec{u} \times \vec{v}|| = ||\vec{u}|| \ ||\vec{v}||$) when the vectors are orthogonal.

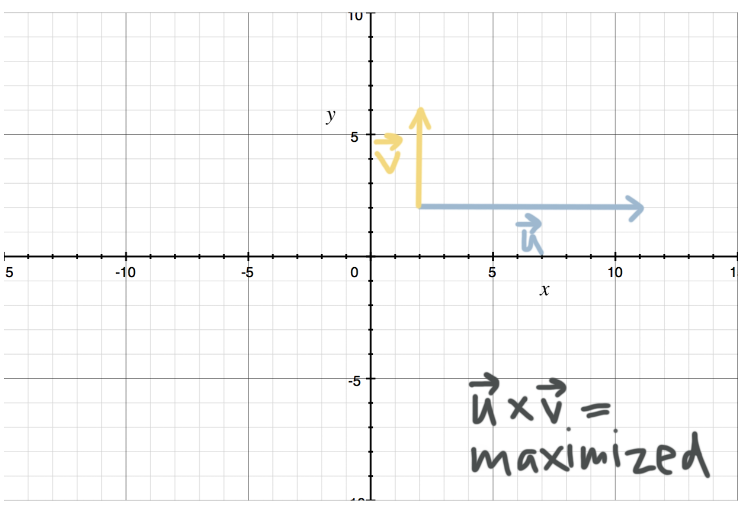
When we say that the length of the cross product is maximized, what we really mean is that the length of the cross product, given by $||\vec{u} \times \vec{v}|| = ||\vec{u}|| \ ||\vec{v}||$, is just the product of the lengths of the vectors.

Here's a summary of these cross product values:

Cross product of 0 when the vectors are collinear:



Maximum cross product when the vectors are orthogonal:



Let's work through an example of where we find the dot and cross products.

Example

Find the dot product and the length of the cross product of $\overrightarrow{u}=(2,0)$ and $\overrightarrow{v}=(-4,0)$. Then interpret the results based on what the dot and cross products indicate.

The vector $\overrightarrow{u}=(2,0)$ points toward the positive direction of the x-axis. The vector $\overrightarrow{v}=(-4,0)$ points toward the negative direction of the x-axis. Which means the angle between them is $\theta=180^\circ$. The length of $\overrightarrow{u}=(2,0)$ is 2, and the length of $\overrightarrow{v}=(-4,0)$ is 4.

So the dot product is

$$\overrightarrow{u}\cdot\overrightarrow{v}=\begin{bmatrix}2 & 0\end{bmatrix}\cdot\begin{bmatrix}-4\\0\end{bmatrix}$$

$$\overrightarrow{u}\cdot\overrightarrow{v}=2(-4)+0(0)$$

$$\overrightarrow{u}\cdot\overrightarrow{v}=-8$$

And the cross product is

$$||\overrightarrow{u}\times\overrightarrow{v}||=||\overrightarrow{u}||\ ||\overrightarrow{v}||\sin\theta$$

$$||\overrightarrow{u}\times\overrightarrow{v}||=(2)(4)\sin(180^\circ)$$

$$||\overrightarrow{u}\times\overrightarrow{v}||=(2)(4)(0)$$

$$||\vec{u} \times \vec{v}|| = 0$$

Because the length of the cross product is 0, we know that the vectors are collinear. For collinear vectors, the dot product is just the product of the lengths of the vectors, which we see in $\vec{u} \cdot \vec{v} = -8$. The fact that the dot product is negative tells us that the vectors point in exactly opposite directions along the same line.

If the vectors were collinear and the dot product was positive, we'd know that the vectors pointed in exactly the same direction along the same line.
