Basis

The concept of a basis is closely connected to the idea of linear independence. Remember that, in general terms, a set of vectors are linearly independent if none of them are redundant, meaning that none of them can be made from linear combinations of the others. The vectors

$$v_1 = \begin{bmatrix}1\\2\end{bmatrix} \text{ and } v_2 = \begin{bmatrix}2\\4\end{bmatrix}$$

are *not* linearly independent (they're linearly dependent), because v_2 is the linear combination $2v_1$. On the other hand,

$$v_1 = \begin{bmatrix}1\\2\end{bmatrix} \text{ and } v_3 = \begin{bmatrix}1\\3\end{bmatrix}$$

are linearly independent, because neither of them can be made with a linear combination of the other.

The difference between span and basis

So we could say, for example, that $\mathbb{R}^2$ is spanned by the vector set V that includes all of these vectors.

$$\mathbb{R}^2 = \text{Span}(V) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right)$$

But this isn't the simplest vector set that we could name to span $\mathbb{R}^2$. We know that the first two vectors are linearly dependent, which means that including $v_2 = (2,4)$ in the set doesn't actually add any new information. We can span $\mathbb{R}^2$ with just v_1 and v_3 alone. So while it's not incorrect to say

$$\mathbb{R}^2 = \text{Span}(V) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right)$$

it's simpler to say

$$\mathbb{R}^2 = \text{Span}(V) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right)$$

Whenever you have a set of linearly independent vectors that span a vector space like $\mathbb{R}^2$, you can say that the vector set forms a **basis** for that vector space $\mathbb{R}^2$. In other words, if you have a basis for a space, it means you have enough vectors to span the space, but not more than you need. So a vector set is a basis for a space if it

1. spans the space, and

2. is linearly independent.

To take another example, this time in three-dimensional space, both W_1 and W_2 span $\mathbb{R}^3$.

$$\mathbb{R}^3 = \text{Span}(W_1) = \text{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}1\\3\\-3\end{bmatrix}, \begin{bmatrix}-2\\1\\4\end{bmatrix}\right)$$

$$\mathbb{R}^3 = \text{Span}(W_2) = \text{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\-3\end{bmatrix}, \begin{bmatrix}-2\\1\\4\end{bmatrix}\right)$$

But only W_2 forms a basis for $\mathbb{R}^3$. That's because the set of vectors in W_2 is linearly independent, whereas the first two vectors in W_1 are linearly dependent, making one of them redundant and unnecessary. So any set of vectors that is linearly dependent cannot be a basis, since a basis has to consist of only linearly independent vectors.

The basis of the subspace

So what we can say then is that, if any subspace Q is given as the span of a set of vectors (which means the subspace is made up of all of the linear combinations of the set of vectors), and if all the vectors in the set are linearly independent, then the set of vectors is a **basis** for the subspace Q.

Said a different way, if a set of vectors forms the basis of a subspace Q, it means the span of those vectors forms a subspace (which means you can "get to" any vector in the subspace using a linear combination of the vectors in the set), and that the vectors in the set are linearly independent.

So just like before, think about the basis of a subspace as the smallest, or minimum, set of vectors that can span the subspace. There are no "redundant" or "unnecessary" vectors in the set.

Example

Show that the span of the vector set K forms a basis for $\mathbb{R}^2$.

$$K = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$$

In order for K to form a basis for $\mathbb{R}^2$,

1. the vectors in K need to span $\mathbb{R}^2$, and

2. the vectors in K need to be linearly independent.

To span $\mathbb{R}^2$, we need to be able to get any vector in $\mathbb{R}^2$ using a linear combination of the vectors in the set. In other words,

$$c_1\begin{bmatrix}1\\4\end{bmatrix}+c_2\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

From this linear combination, we can build a system of equations.

$$c_1=x$$

$$4c_1+3c_2=y$$

We'll substitute $c_1=x$ into the second equation in the system, and then solve that equation for y.

$$4x+3c_2=y$$

$$3c_2=y-4x$$

$$c_2=\frac{1}{3}y-\frac{4}{3}x$$

From this process we can conclude that, given any vector $\overrightarrow{v}=(x,y)$ in $\mathbb{R}^2$, we can "get to it" using the values of c_1 and c_2 given by

$$c_1=x$$

$$c_2 = \frac{1}{3}y - \frac{4}{3}x$$

It doesn't matter which vector we pick in $\mathbb{R}^2$. If we use the values of x and y that we want, and plug them into these equations for c_1 and c_2, we'll get the values of c_1 and c_2 that we need to use in the linear combination in order to arrive at the vector $\vec{v} = (x, y)$. These formulas for c_1 and c_2 won't break, regardless of which (x, y) we pick for the vector, so the vector set K spans $\mathbb{R}^2$.

Then to show that the vectors in K are linearly independent, we'll set $(x, y) = (0,0)$.

$$c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When we do, we get

$$c_1 = 0$$

$$c_2 = \frac{1}{3}(0) - \frac{4}{3}(0), \text{ or } c_2 = 0$$

Because the only values of c_1 and c_2 that give the zero vector are $c_1 = 0$ and $c_2 = 0$, we know that the vectors in K are linearly independent.

Therefore, because the vector set K spans all of $\mathbb{R}^2$, and because the vectors in K are linearly independent, we can say that K forms a basis for $\mathbb{R}^2$.

Standard basis

We've just seen that the specific set K is a basis for $\mathbb{R}^2$. In fact, any set of two linearly independent vectors will form a basis for $\mathbb{R}^2$. In other words, you can pick any two linearly independent vectors in $\mathbb{R}^2$, and as a set, they will form a basis for $\mathbb{R}^2$.

We call the **standard basis** of $\mathbb{R}^2$ the set of unit vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Similarly, any three linearly independent vectors in $\mathbb{R}^3$ will form a basis for $\mathbb{R}^3$. The standard basis of $\mathbb{R}^3$ is the set of unit vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If we extrapolate this pattern, what we see is that any set of n linearly independent vectors will form a basis for $\mathbb{R}^n$.