Topic: Invertibility from the matrix-vector product

Question: Say whether or not the transformation T is surjective or injective.

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad$ T is surjective and injective

B $\quad$ T is surjective, but not injective

C $\quad$ T is not surjective, but it's injective

D $\quad$ T is neither surjective nor injective

Solution: A

If we say that the transformation T is given by the matrix-vector product

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \vec{x}$$

and we name the matrix A, then we always want to start by putting A into reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because we get the identity matrix when we put A into rref, that tells us that the transformation T is both surjective and injective.

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Question: Say whether or not the transformation T is surjective or injective.

$$T(\vec{x}) = \begin{bmatrix} 3 & -4 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \vec{x}$$

Answer choices:

A T is surjective and injective

B T is surjective, but not injective

C T is not surjective, but it's injective

D T is neither surjective nor injective

Solution: B

If we say that the transformation T is given by the matrix-vector product

$$T(\vec{x}) = \begin{bmatrix} 3 & -4 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \vec{x}$$

and we name the matrix A, then we always want to start by putting A into reduced row-echelon form.

$$A = \begin{bmatrix} 3 & -4 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -1 & 2 & -6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 8 \\ 0 & 1 & -2 & 6 \end{bmatrix}$$

If we try to find the null space of A, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

So any linear combination of these two four-dimensional vectors is a vector in the null space. Which means there are an infinite number of vectors that the transformation T maps to the zero vector. And because we're mapping multiple vectors all to the same vector, the transformation is not injective.

To see whether or not the transformation is surjective, we'll look at the column space of A.

$$C(A) = \text{Span}\left(\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-4\\-1\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right)$$

From the reduced row-echelon form of A,

$$A = \begin{bmatrix}1 & 0 & -2 & 8\\0 & 1 & -2 & 6\end{bmatrix}$$

we know that the first two column vectors from A are linearly independent two-dimensional vectors in $\mathbb{R}^2$. Because two linearly independent two-dimensional vectors in $\mathbb{R}^2$ will span all of $\mathbb{R}^2$, we know that the transformation T can map to any vector in $\mathbb{R}^2$, which means that the transformation is surjective.

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Question: Say whether or not the transformation T is surjective or injective.

$$T(\vec{x}) = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \vec{x}$$

Answer choices:

A T is surjective and injective

B T is surjective, but not injective

C T is not surjective, but it's injective

D T is neither surjective nor injective

Solution: C

If we say that the transformation T is given by the matrix-vector product

$$T(\vec{x})=\begin{bmatrix}-1&2&3\\0&0&1\\2&-2&1\\0&4&2\end{bmatrix}\vec{x}$$

and we name the matrix A, then we always want to start by putting A into reduced row-echelon form.

$$A=\begin{bmatrix}-1&2&3\\0&0&1\\2&-2&1\\0&4&2\end{bmatrix}\to\begin{bmatrix}1&-2&-3\\0&0&1\\2&-2&1\\0&4&2\end{bmatrix}\to\begin{bmatrix}1&-2&-3\\0&0&1\\0&2&7\\0&4&2\end{bmatrix}\to\begin{bmatrix}1&-2&-3\\0&2&7\\0&0&1\\0&4&2\end{bmatrix}$$

$$\to\begin{bmatrix}1&-2&-3\\0&1&\frac{7}{2}\\0&0&1\\0&4&2\end{bmatrix}\to\begin{bmatrix}1&0&4\\0&1&\frac{7}{2}\\0&0&1\\0&4&2\end{bmatrix}\to\begin{bmatrix}1&0&4\\0&1&\frac{7}{2}\\0&0&1\\0&0&-12\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&\frac{7}{2}\\0&0&1\\0&0&-12\end{bmatrix}$$

$$\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&-12\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

If we try to find the null space of A, we get

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Because the zero vector is the only vector in the null space, we can say that the transformation T is injective.

To see whether or not the transformation is surjective, we'll look at the column space of A.

$$C(A) = \text{Span}\left(\begin{bmatrix}-1\\0\\2\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-2\\4\end{bmatrix}, \begin{bmatrix}3\\1\\1\\2\end{bmatrix}\right)$$

From the reduced row-echelon form of A,

$$A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

we know that the first three column vectors from A are linearly independent three-dimensional vectors in $\mathbb{R}^4$. Because three linearly independent three-dimensional vectors in $\mathbb{R}^4$ won't span all of $\mathbb{R}^4$, we know that the transformation T can't map to every vector in $\mathbb{R}^4$, which means that the transformation is not surjective.