Cauchy-Schwarz inequality

Now that we understand dot products and some of their properties, and we've learned formulas that are based on the dot product, we can start to explore different applications of the dot product, one of which is a formula called the Cauchy-Schwarz inequality.

The **Cauchy-Schwarz inequality** (also called **Cauchy's inequality**, or **Schwarz's inequality**) tells us that given two nonzero vectors in $\mathbb{R}^n$, the absolute value of the their dot product is always less than or equal to the product of their lengths.

$$|\vec{u}\cdot\vec{v}| \le ||\vec{u}||\ ||\vec{v}||$$

The inequality includes the "less than or equal to" sign, so you might be wondering whether there are specific instances in which the left side of the inequality is less than the right side, and others in which the sides are equivalent. As it turns out, the two sides of the Cauchy-Schwarz inequality are only equal to one another when the two vectors $\vec{u}$ and $\vec{v}$ are collinear, meaning that one vector is a scalar multiple of the other.

$$|\vec{u}\cdot\vec{v}| = ||\vec{u}||\ ||\vec{v}|| \text{ if and only if } \vec{u} = c\vec{v}$$

If this is not the case (if one vector isn't a scalar multiple of the other), then the left side of the Cauchy-Schwarz inequality will be less than its right side:

$$|\vec{u}\cdot\vec{v}| < ||\vec{u}||\ ||\vec{v}||$$

Testing for linear independence

The Cauchy-Schwarz inequality also gives us another way to test for linear independence.

Remember that we just said that Cauchy-Schwarz will only be an equality, $|\vec{u}\cdot\vec{v}|=||\vec{u}||\ ||\vec{v}||$, when the vectors are collinear (when $\vec{u}=c\vec{v}$). We know that collinear vectors are linearly dependent.

So given any $\vec{u}$ and $\vec{v}$,

if $|\vec{u}\cdot\vec{v}|=||\vec{u}||\ ||\vec{v}||$, then $\vec{u}$ and $\vec{v}$ are linearly dependent.

if $|\vec{u}\cdot\vec{v}|<||\vec{u}||\ ||\vec{v}||$, then $\vec{u}$ and $\vec{v}$ are linearly independent.

Of course, knowing whether or not two vectors are linearly independent is important, since only linearly independent vectors can span a subspace.

Let's work through an example where we use the Cauchy-Schwarz inequality as a test for linear independence.

Example

Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$\vec{u}=(3,4)$ and $\vec{v}=(-6,-8)$

Let's first find the value of the left side of the Cauchy-Schwarz inequality.

$$|\vec{u}\cdot\vec{v}|$$

$$|(3)(-6)+(4)(-8)|$$

$$|-18-32|$$

$$|-50|$$

$$50$$

Now find the value of the right side of the Cauchy-Schwarz inequality.

$$||\vec{u}||\ ||\vec{v}||$$

$$\sqrt{u_1^2+u_2^2}\sqrt{v_1^2+v_2^2}$$

$$\sqrt{3^2+4^2}\sqrt{(-6)^2+(-8)^2}$$

$$\sqrt{9+16}\sqrt{36+64}$$

$$\sqrt{25}\sqrt{100}$$

$$5(10)$$

$$50$$

If we plug these two into the Cauchy-Schwarz inequality, we get

$$50=50$$

$$|\vec{u}\cdot\vec{v}|=||\vec{u}||\ ||\vec{v}||$$

which tells us that $\overrightarrow{u}=(3,4)$ and $\overrightarrow{v}=(-6,-8)$ are linearly dependent. We can see the linear dependence also from the fact that each vector is a scalar multiple of the other:

$$\overrightarrow{u}=c\overrightarrow{v}: \qquad (3,4)=(-1/2)(-6,-8)$$

$$\overrightarrow{v}=c\overrightarrow{u}: \qquad (-6,-8)=-2(3,4)$$

Therefore, we could conclude that $\overrightarrow{u}=(3,4)$ and $\overrightarrow{v}=(-6,-8)$ are collinear and do not span $\mathbb{R}^2$.
