Linear Algebra Workbook Solutions

Orthonormal bases and Gram-Schmidt

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ORTHONORMAL BASES

1. Verify that the vector set $V = \{\vec{v}_1, \vec{v}_2\}$ is orthonormal if $\vec{v}_1 = (1,0,0)$ and $\vec{v}_2 = (0,0,-1)$.

Solution:

If the set is orthonormal, each vector has length 1.

$$||\vec{v}_1||^2 = \vec{v}_1 \cdot \vec{v}_1 = 1(1) + 0(0) + 0(0) = 1 + 0 + 0 = 1$$

$$||\vec{v}_2||^2 = \vec{v}_2 \cdot \vec{v}_2 = 0(0) + 0(0) - 1(-1) = 0 + 0 + 1 = 1$$

Both vectors have length 1, so now we'll just confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = 1(0) + 0(0) + 0(-1) = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, $\vec{v}_1$ and $\vec{v}_2$ form an orthonormal set, so V is orthonormal.

2. Determine that the vector set $V = \{\vec{v}_1, \vec{v}_2\}$ is orthonormal.

$$\vec{v}_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

$$\vec{v}_2 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Solution:

If the set is orthonormal, each vector has length 1.

$$||\vec{v}_1||^2 = \vec{v}_1 \cdot \vec{v}_1 = \frac{2}{3}\left(\frac{2}{3}\right) - \frac{1}{3}\left(-\frac{1}{3}\right) - \frac{2}{3}\left(-\frac{2}{3}\right) = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$$

$$||\vec{v}_2||^2 = \vec{v}_2 \cdot \vec{v}_2 = -\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) + 0(0) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + 0 + \frac{1}{2} = 1$$

Both vectors have length 1, so now we'll just confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{2}{3}\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{3}(0) - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right) = -\frac{2}{3\sqrt{2}} - 0 - \frac{2}{3\sqrt{2}} = -\frac{4}{3\sqrt{2}}$$

Because the dot product of the vectors is nonzero, $V = \{\vec{v}_1, \vec{v}_2\}$ is not an orthonormal set.

■ 3. Convert $\vec{x} = (-2,10)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{3}{4} \\ -\frac{\sqrt{7}}{4} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{\sqrt{7}}{4} \\ \frac{3}{4} \end{bmatrix}$$

Solution:

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2 = \left(\frac{3}{4}\right)^2 + \left(-\frac{\sqrt{7}}{4}\right)^2 = \frac{9}{16} + \frac{7}{16} = \frac{16}{16} = 1$$

$$||\vec{v}_2||^2 = \left(\frac{\sqrt{7}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{7}{16} + \frac{9}{16} = \frac{16}{16} = 1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{3}{4}\left(\frac{\sqrt{7}}{4}\right) - \frac{\sqrt{7}}{4}\left(\frac{3}{4}\right) = \frac{3\sqrt{7}}{16} - \frac{3\sqrt{7}}{16} = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x} = (-2,10)$ can be converted to the alternate basis B with dot products.

$$[\vec{x}]_B = \begin{bmatrix} \frac{3}{4}(-2) - \frac{\sqrt{7}}{4}(10) \\ \frac{\sqrt{7}}{4}(-2) + \frac{3}{4}(10) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -\frac{3}{2} - \frac{5\sqrt{7}}{2} \\ -\frac{\sqrt{7}}{2} + \frac{15}{2} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -\frac{3+5\sqrt{7}}{2} \\ -\frac{\sqrt{7}-15}{2} \end{bmatrix}$$

■ 4. Convert $\vec{x} = (-25,10)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$$

Solution:

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2 = \left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$$

$$||\vec{v}_2||^2 = \left(-\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{3}{5}\left(-\frac{4}{5}\right) - \frac{4}{5}\left(-\frac{3}{5}\right) = -\frac{12}{25} + \frac{12}{25} = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x} = (-25,10)$ can be converted to the alternate basis B with dot products.

$$[\vec{x}]_B = \begin{bmatrix} \frac{3}{5}(-25) - \frac{4}{5}(10) \\ -\frac{4}{5}(-25) - \frac{3}{5}(10) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -15 - 8 \\ 20 - 6 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -23 \\ 14 \end{bmatrix}$$

■ 5. Convert $\vec{x} = (-6,3,12)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

Solution:

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2=\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2=\frac{4}{9}+\frac{1}{9}+\frac{4}{9}=\frac{9}{9}=1$$

$$||\vec{v}_2||^2=\left(-\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2=\frac{1}{9}+\frac{4}{9}+\frac{4}{9}=\frac{9}{9}=1$$

$$||\vec{v}_3||^2=\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2=\frac{4}{9}+\frac{4}{9}+\frac{1}{9}=\frac{9}{9}=1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1\cdot\vec{v}_2=\frac{2}{3}\left(-\frac{1}{3}\right)-\frac{1}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right)=-\frac{2}{9}-\frac{2}{9}+\frac{4}{9}=0$$

$$\vec{v}_1\cdot\vec{v}_3=\frac{2}{3}\left(\frac{2}{3}\right)-\frac{1}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(-\frac{1}{3}\right)=\frac{4}{9}-\frac{2}{9}-\frac{2}{9}=0$$

$$\vec{v}_2\cdot\vec{v}_3=-\frac{1}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(-\frac{1}{3}\right)=-\frac{2}{9}+\frac{4}{9}-\frac{2}{9}=0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x}=(-6,3,12)$ can be converted to the alternate basis B with dot products.

$$[\vec{x}]_B=\begin{bmatrix}\frac{2}{3}(-6)-\frac{1}{3}(3)+\frac{2}{3}(12)\\ -\frac{1}{3}(-6)+\frac{2}{3}(3)+\frac{2}{3}(12)\\ \frac{2}{3}(-6)+\frac{2}{3}(3)-\frac{1}{3}(12)\end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -4-1+8 \\ 2+2+8 \\ -4+2-4 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 3 \\ 12 \\ -6 \end{bmatrix}$$

6. Convert $\vec{x} = (2,0,-3)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Solution:

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2 = 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 0 + \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$$

$$||\vec{v}_2||^2 = \left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$$

$$||\vec{v}_3||^2=\left(\frac{2}{\sqrt{6}}\right)^2+\left(\frac{1}{\sqrt{6}}\right)^2+\left(\frac{1}{\sqrt{6}}\right)^2=\frac{4}{6}+\frac{1}{6}+\frac{1}{6}=\frac{6}{6}=1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1\cdot\vec{v}_2=0\left(-\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}\right)=0+\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{6}}=0$$

$$\vec{v}_1\cdot\vec{v}_3=0\left(\frac{2}{\sqrt{6}}\right)+\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}\right)-\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}\right)=0+\frac{1}{\sqrt{12}}-\frac{1}{\sqrt{12}}=0$$

$$\vec{v}_2\cdot\vec{v}_3=\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{6}}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{6}}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{6}}\right)=-\frac{2}{\sqrt{18}}+\frac{1}{\sqrt{18}}+\frac{1}{\sqrt{18}}=0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x}=(2,0,-3)$ can be converted to the alternate basis B with dot products.

$$[\vec{x}]_B=\begin{bmatrix}0(2)+\frac{1}{\sqrt{2}}(0)-\frac{1}{\sqrt{2}}(-3)\\ -\frac{1}{\sqrt{3}}(2)+\frac{1}{\sqrt{3}}(0)+\frac{1}{\sqrt{3}}(-3)\\ \frac{2}{\sqrt{6}}(2)+\frac{1}{\sqrt{6}}(0)+\frac{1}{\sqrt{6}}(-3)\end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 0+0+\frac{3}{\sqrt{2}} \\ -\frac{2}{\sqrt{3}}+0-\frac{3}{\sqrt{3}} \\ \frac{4}{\sqrt{6}}+0-\frac{3}{\sqrt{6}} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{5}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

PROJECTION ONTO AN ORTHONORMAL BASIS

■ 1. Find the projection of $\vec{x}=(-5,0,-2)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\\ 0\end{bmatrix},\begin{bmatrix}0\\ 0\\ -1\end{bmatrix}\right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2=\left(\frac{2}{\sqrt{5}}\right)^2+\left(\frac{1}{\sqrt{5}}\right)^2+0^2=\frac{4}{5}+\frac{1}{5}+0=\frac{5}{5}=1$$

$$||\vec{v}_2||^2=0^2+0^2+(-1)^2=0+0+1=1$$

$$\vec{v}_1\cdot\vec{v}_2=\frac{2}{\sqrt{5}}(0)+\frac{1}{\sqrt{5}}(0)+0(-1)=0+0+0=0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x}=(-5,0,-2)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \vec{x}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{2}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\right) + 0(0) & \frac{2}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right) + 0(0) & \frac{2}{\sqrt{5}}(0) + 0(-1) \\ \frac{1}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\right) + 0(0) & \frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right) + 0(0) & \frac{1}{\sqrt{5}}(0) + 0(-1) \\ 0\left(\frac{2}{\sqrt{5}}\right) - 1(0) & 0\left(\frac{1}{\sqrt{5}}\right) - 1(0) & 0(0) - 1(-1) \end{bmatrix} \vec{x}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{4}{5} + 0 & \frac{2}{5} + 0 & 0 + 0 \\ \frac{2}{5} + 0 & \frac{1}{5} + 0 & 0 + 0 \\ 0 - 0 & 0 - 0 & 0 + 1 \end{bmatrix} \vec{x}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

Applying the projection to $\vec{x} = (-5,0,-2)$ gives

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ -2 \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{4}{5}(-5) + \frac{2}{5}(0) + 0(-2) \\ \frac{2}{5}(-5) + \frac{1}{5}(0) + 0(-2) \\ 0(-5) + 0(0) + 1(-2) \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -4 + 0 + 0 \\ -2 + 0 + 0 \\ 0 + 0 - 2 \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix}$$

■ 2. Find the projection of $\vec{x} = (-66,33,11)$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} -\frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ -\frac{1}{\sqrt{11}} \end{bmatrix}\right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(-\frac{2}{\sqrt{6}}\right)^2 = \frac{1}{6} + \frac{1}{6} + \frac{4}{6} = \frac{6}{6} = 1$$

$$||\vec{v}_2||^2=\left(-\frac{3}{\sqrt{11}}\right)^2+\left(\frac{1}{\sqrt{11}}\right)^2+\left(-\frac{1}{\sqrt{11}}\right)^2=\frac{9}{11}+\frac{1}{11}+\frac{1}{11}=\frac{11}{11}=1$$

$$\vec{v}_1\cdot\vec{v}_2=\frac{1}{\sqrt{6}}\left(-\frac{3}{\sqrt{11}}\right)+\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{11}}\right)-\frac{2}{\sqrt{6}}\left(-\frac{1}{\sqrt{11}}\right)$$

$$=-\frac{3}{\sqrt{66}}+\frac{1}{\sqrt{66}}+\frac{2}{\sqrt{66}}=0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x}=(-66,33,11)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{\sqrt{6}} & -\frac{3}{\sqrt{11}}\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{11}}\\ -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{11}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\\ -\frac{3}{\sqrt{11}} & \frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)-\frac{3}{\sqrt{11}}\left(-\frac{3}{\sqrt{11}}\right) & \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)-\frac{3}{\sqrt{11}}\left(\frac{1}{\sqrt{11}}\right) & \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right)-\frac{3}{\sqrt{11}}\left(-\frac{1}{\sqrt{11}}\right)\\ \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)+\frac{1}{\sqrt{11}}\left(-\frac{3}{\sqrt{11}}\right) & \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)+\frac{1}{\sqrt{11}}\left(\frac{1}{\sqrt{11}}\right) & \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right)+\frac{1}{\sqrt{11}}\left(-\frac{1}{\sqrt{11}}\right)\\ -\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)-\frac{1}{\sqrt{11}}\left(-\frac{3}{\sqrt{11}}\right) & -\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)-\frac{1}{\sqrt{11}}\left(\frac{1}{\sqrt{11}}\right) & -\frac{2}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right)-\frac{1}{\sqrt{11}}\left(-\frac{1}{\sqrt{11}}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{6}+\frac{9}{11} & \frac{1}{6}-\frac{3}{11} & -\frac{1}{3}+\frac{3}{11}\\ \frac{1}{6}-\frac{3}{11} & \frac{1}{6}+\frac{1}{11} & -\frac{1}{3}-\frac{1}{11}\\ -\frac{1}{3}+\frac{3}{11} & -\frac{1}{3}-\frac{1}{11} & \frac{2}{3}+\frac{1}{11}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{65}{66} & -\frac{7}{66} & -\frac{2}{33}\\ -\frac{7}{66} & \frac{17}{66} & -\frac{14}{33}\\ -\frac{2}{33} & -\frac{14}{33} & \frac{25}{33}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(-66,33,11)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{65}{66} & -\frac{7}{66} & -\frac{2}{33}\\ -\frac{7}{66} & \frac{17}{66} & -\frac{14}{33}\\ -\frac{2}{33} & -\frac{14}{33} & \frac{25}{33}\end{bmatrix}\begin{bmatrix}-66\\ 33\\ 11\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{65}{66}(-66)-\frac{7}{66}(33)-\frac{2}{33}(11)\\ -\frac{7}{66}(-66)+\frac{17}{66}(33)-\frac{14}{33}(11)\\ -\frac{2}{33}(-66)-\frac{14}{33}(33)+\frac{25}{33}(11)\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-65-\frac{7}{2}-\frac{2}{3}\\ 7+\frac{17}{2}-\frac{14}{3}\\ 4-14+\frac{25}{3}\end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -\frac{415}{6} \\ \frac{65}{6} \\ -\frac{5}{3} \end{bmatrix}$$

■ 3. Find the projection of $\vec{x} = (-6, -3, 6)$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = \left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$$

$$||\vec{v}_2||^2 = \left(-\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + 0 + \frac{1}{2} = \frac{2}{2} = 1$$

$$\vec{v}_1 \cdot \vec{v}_2 = -\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{3}}(0) + \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x}=(-6,-3,6)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & 0\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) & -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}(0) & -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)\\ \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)+0\left(-\frac{1}{\sqrt{2}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+0(0) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+0\left(-\frac{1}{\sqrt{2}}\right)\\ \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}(0) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{3}+\frac{1}{2} & -\frac{1}{3}-0 & -\frac{1}{3}+\frac{1}{2}\\ -\frac{1}{3}+0 & \frac{1}{3}+0 & \frac{1}{3}+0\\ -\frac{1}{3}+\frac{1}{2} & \frac{1}{3}-0 & \frac{1}{3}+\frac{1}{2}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{5}{6} & -\frac{1}{3} & \frac{1}{6}\\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6}\end{bmatrix}\vec{x}$$

Applying the projection to $\overrightarrow{x}=(-6,-3,6)$ gives

$$\text{Proj}_V\overrightarrow{x}=\begin{bmatrix}\frac{5}{6} & -\frac{1}{3} & \frac{1}{6}\\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6}\end{bmatrix}\begin{bmatrix}-6\\ -3\\ 6\end{bmatrix}$$

$$\text{Proj}_V\overrightarrow{x}=\begin{bmatrix}\frac{5}{6}(-6)-\frac{1}{3}(-3)+\frac{1}{6}(6)\\ -\frac{1}{3}(-6)+\frac{1}{3}(-3)+\frac{1}{3}(6)\\ \frac{1}{6}(-6)+\frac{1}{3}(-3)+\frac{5}{6}(6)\end{bmatrix}$$

$$\text{Proj}_V\overrightarrow{x}=\begin{bmatrix}-5+1+1\\ 2-1+2\\ -1-1+5\end{bmatrix}$$

$$\text{Proj}_V\overrightarrow{x}=\begin{bmatrix}-3\\ 3\\ 3\end{bmatrix}$$

■ 4. Find the projection of $\overrightarrow{x}=(-2,3,5)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}0\\ -1\\ 0\end{bmatrix},\begin{bmatrix}\frac{3}{\sqrt{10}}\\ 0\\ \frac{1}{\sqrt{10}}\end{bmatrix}\right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = 0^2 + (-1)^2 + 0^2 = 0 + 1 + 0 = 1$$

$$||\vec{v}_2||^2 = \left(\frac{3}{\sqrt{10}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{10}}\right)^2 = \frac{9}{10} + 0 + \frac{1}{10} = \frac{10}{10} = 1$$

$$\vec{v}_1 \cdot \vec{v}_2 = 0\left(\frac{3}{\sqrt{10}}\right) - 1(0) + 0\left(\frac{1}{\sqrt{10}}\right) = 0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x} = (-2,3,5)$ onto V is

$$\text{Proj}_V\vec{x} = AA^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 0 & \frac{3}{\sqrt{10}} \\ -1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 0(0) + \frac{3}{\sqrt{10}}\left(\frac{3}{\sqrt{10}}\right) & 0(-1) + \frac{3}{\sqrt{10}}(0) & 0(0) + \frac{3}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}\right) \\ -1(0) + 0\left(\frac{3}{\sqrt{10}}\right) & -1(-1) + 0(0) & -1(0) + 0\left(\frac{1}{\sqrt{10}}\right) \\ 0(0) + \frac{1}{\sqrt{10}}\left(\frac{3}{\sqrt{10}}\right) & 0(-1) + \frac{1}{\sqrt{10}}(0) & 0(0) + \frac{1}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}\right) \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}0+\frac{9}{10} & 0+0 & 0+\frac{3}{10}\\ 0+0 & 1+0 & 0+0\\ 0+\frac{3}{10} & 0+0 & 0+\frac{1}{10}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{10} & 0 & \frac{3}{10}\\ 0 & 1 & 0\\ \frac{3}{10} & 0 & \frac{1}{10}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(-2,3,5)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{10} & 0 & \frac{3}{10}\\ 0 & 1 & 0\\ \frac{3}{10} & 0 & \frac{1}{10}\end{bmatrix}\begin{bmatrix}-2\\ 3\\ 5\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{10}(-2)+0(3)+\frac{3}{10}(5)\\ 0(-2)+1(3)+0(5)\\ \frac{3}{10}(-2)+0(3)+\frac{1}{10}(5)\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{18}{10}+0+\frac{15}{10}\\ 0+3+0\\ -\frac{6}{10}+0+\frac{5}{10}\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{3}{10}\\ 3\\ -\frac{1}{10}\end{bmatrix}$$

■ 5. Find the projection of $\vec{x} = (0, -13, 4)$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}\frac{3}{\sqrt{13}}\\ \frac{2}{\sqrt{13}}\\ 0\end{bmatrix}, \begin{bmatrix}\frac{2}{\sqrt{17}}\\ -\frac{3}{\sqrt{17}}\\ \frac{2}{\sqrt{17}}\end{bmatrix}\right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = \left(\frac{3}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 + 0^2 = \frac{9}{13} + \frac{4}{13} + 0 = \frac{13}{13} = 1$$

$$||\vec{v}_2||^2 = \left(\frac{2}{\sqrt{17}}\right)^2 + \left(-\frac{3}{\sqrt{17}}\right)^2 + \left(\frac{2}{\sqrt{17}}\right)^2 = \frac{4}{17} + \frac{9}{17} + \frac{4}{17} = \frac{17}{17} = 1$$

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{3}{\sqrt{13}}\left(\frac{2}{\sqrt{17}}\right) + \frac{2}{\sqrt{13}}\left(-\frac{3}{\sqrt{17}}\right) + 0\left(\frac{2}{\sqrt{17}}\right)$$

$$= \frac{6}{\sqrt{221}} - \frac{6}{\sqrt{221}} + 0 = 0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x}=(0,-13,4)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{17}}\\ \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{17}}\\ 0 & \frac{2}{\sqrt{17}}\end{bmatrix}\begin{bmatrix}\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0\\ \frac{2}{\sqrt{17}} & -\frac{3}{\sqrt{17}} & \frac{2}{\sqrt{17}}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{3}{\sqrt{13}}\left(\frac{3}{\sqrt{13}}\right)+\frac{2}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right) & \frac{3}{\sqrt{13}}\left(\frac{2}{\sqrt{13}}\right)+\frac{2}{\sqrt{17}}\left(-\frac{3}{\sqrt{17}}\right) & \frac{3}{\sqrt{13}}(0)+\frac{2}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right)\\ \frac{2}{\sqrt{13}}\left(\frac{3}{\sqrt{13}}\right)-\frac{3}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right) & \frac{2}{\sqrt{13}}\left(\frac{2}{\sqrt{13}}\right)-\frac{3}{\sqrt{17}}\left(-\frac{3}{\sqrt{17}}\right) & \frac{2}{\sqrt{13}}(0)-\frac{3}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right)\\ 0\left(\frac{3}{\sqrt{13}}\right)+\frac{2}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right) & 0\left(\frac{2}{\sqrt{13}}\right)+\frac{2}{\sqrt{17}}\left(-\frac{3}{\sqrt{17}}\right) & 0(0)+\frac{2}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{13}+\frac{4}{17} & \frac{6}{13}-\frac{6}{17} & 0+\frac{4}{17}\\ \frac{6}{13}-\frac{6}{17} & \frac{4}{13}+\frac{9}{17} & 0-\frac{6}{17}\\ 0+\frac{4}{17} & 0-\frac{6}{17} & 0+\frac{4}{17}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{205}{221} & \frac{24}{221} & \frac{4}{17}\\ \frac{24}{221} & \frac{185}{221} & -\frac{6}{17}\\ \frac{4}{17} & -\frac{6}{17} & \frac{4}{17}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(0,-13,4)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{205}{221} & \frac{24}{221} & \frac{4}{17}\\ \frac{24}{221} & \frac{185}{221} & -\frac{6}{17}\\ \frac{4}{17} & -\frac{6}{17} & \frac{4}{17}\end{bmatrix}\begin{bmatrix}0\\ -13\\ 4\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{205}{221}(0)+\frac{24}{221}(-13)+\frac{4}{17}(4)\\ \frac{24}{221}(0)+\frac{185}{221}(-13)-\frac{6}{17}(4)\\ \frac{4}{17}(0)-\frac{6}{17}(-13)+\frac{4}{17}(4)\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}0-\frac{24}{17}+\frac{16}{17}\\ 0-\frac{185}{17}-\frac{24}{17}\\ 0+\frac{78}{17}+\frac{16}{17}\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{17}\\ -\frac{209}{17}\\ \frac{94}{17}\end{bmatrix}$$

6. Find the projection of $\vec{x}=(-3,10,-10)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{3}{\sqrt{19}}\\ -\frac{3}{\sqrt{19}}\\ \frac{1}{\sqrt{19}}\end{bmatrix},\begin{bmatrix}0\\ \frac{1}{\sqrt{10}}\\ \frac{3}{\sqrt{10}}\end{bmatrix}\right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = \left(\frac{3}{\sqrt{19}}\right)^2 + \left(-\frac{3}{\sqrt{19}}\right)^2 + \left(\frac{1}{\sqrt{19}}\right)^2 = \frac{9}{19} + \frac{9}{19} + \frac{1}{19} = \frac{19}{19} = 1$$

$$||\vec{v}_2||^2 = 0^2 + \left(\frac{1}{\sqrt{10}}\right)^2 + \left(\frac{3}{\sqrt{10}}\right)^2 = 0 + \frac{1}{10} + \frac{9}{10} = \frac{10}{10} = 1$$

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{3}{\sqrt{19}}(0) - \frac{3}{\sqrt{19}}\left(\frac{1}{\sqrt{10}}\right) + \frac{1}{\sqrt{19}}\left(\frac{3}{\sqrt{10}}\right) = 0 - \frac{3}{\sqrt{190}} + \frac{3}{\sqrt{190}} = 0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x} = (-3,10,-10)$ onto V is

$$\text{Proj}_V\vec{x} = AA^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{3}{\sqrt{19}} & 0 \\ -\frac{3}{\sqrt{19}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{19}} & \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{19}} & -\frac{3}{\sqrt{19}} & \frac{1}{\sqrt{19}} \\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{3}{\sqrt{19}}\left(\frac{3}{\sqrt{19}}\right)+0(0) & \frac{3}{\sqrt{19}}\left(-\frac{3}{\sqrt{19}}\right)+0\left(\frac{1}{\sqrt{10}}\right) & \frac{3}{\sqrt{19}}\left(\frac{1}{\sqrt{19}}\right)+0\left(\frac{3}{\sqrt{10}}\right)\\ -\frac{3}{\sqrt{19}}\left(\frac{3}{\sqrt{19}}\right)+\frac{1}{\sqrt{10}}(0) & -\frac{3}{\sqrt{19}}\left(-\frac{3}{\sqrt{19}}\right)+\frac{1}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}\right) & -\frac{3}{\sqrt{19}}\left(\frac{1}{\sqrt{19}}\right)+\frac{1}{\sqrt{10}}\left(\frac{3}{\sqrt{10}}\right)\\ \frac{1}{\sqrt{19}}\left(\frac{3}{\sqrt{19}}\right)+\frac{3}{\sqrt{10}}(0) & \frac{1}{\sqrt{19}}\left(-\frac{3}{\sqrt{19}}\right)+\frac{3}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}\right) & \frac{1}{\sqrt{19}}\left(\frac{1}{\sqrt{19}}\right)+\frac{3}{\sqrt{10}}\left(\frac{3}{\sqrt{10}}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{19}+0 & -\frac{9}{19}+0 & \frac{3}{19}+0\\ -\frac{9}{19}+0 & \frac{9}{19}+\frac{1}{10} & -\frac{3}{19}+\frac{3}{10}\\ \frac{3}{19}+0 & -\frac{3}{19}+\frac{3}{10} & \frac{1}{19}+\frac{9}{10}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{19} & -\frac{9}{19} & \frac{3}{19}\\ -\frac{9}{19} & \frac{109}{190} & \frac{27}{190}\\ \frac{3}{19} & \frac{27}{190} & \frac{181}{190}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(-3,10,-10)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{19} & -\frac{9}{19} & \frac{3}{19}\\ -\frac{9}{19} & \frac{109}{190} & \frac{27}{190}\\ \frac{3}{19} & \frac{27}{190} & \frac{181}{190}\end{bmatrix}\begin{bmatrix}-3\\10\\-10\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{19}(-3)-\frac{9}{19}(10)+\frac{3}{19}(-10)\\ -\frac{9}{19}(-3)+\frac{109}{190}(10)+\frac{27}{190}(-10)\\ \frac{3}{19}(-3)+\frac{27}{190}(10)+\frac{181}{190}(-10)\end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} -\frac{27}{19} - \frac{90}{19} - \frac{30}{19} \\ \frac{27}{19} + \frac{109}{19} - \frac{27}{19} \\ -\frac{9}{19} + \frac{27}{19} - \frac{181}{19} \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} -\frac{147}{19} \\ \frac{109}{19} \\ -\frac{163}{19} \end{bmatrix}$$

GRAM-SCHMIDT PROCESS FOR CHANGE OF BASIS

■ 1. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}0\\-4\\3\end{bmatrix}, \begin{bmatrix}-2\\3\\-1\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (0, -4, 3)$ and $\vec{v}_2 = (-2, 3, -1)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{0^2 + (-4)^2 + 3^2} = \sqrt{0 + 16 + 9} = \sqrt{25} = 5$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{5}\begin{bmatrix}0\\-4\\3\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$ and $\vec{v}_2$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2)$$

Now all we need to do is replace $\vec{v}_2$ with a vector that's both orthogonal to $\vec{u}_1$, and normal. If we can do that, then the vector set that spans V will be orthonormal. We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \right) \frac{1}{5} \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \frac{1}{25} \left(\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \right) \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \frac{1}{25}((-2)(0) + (3)(-4) + (-1)(3)) \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \frac{1}{25}(-15) \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 + 0 \\ 3 - \frac{12}{5} \\ -1 + \frac{9}{5} \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it.

$$||\vec{w}_2|| = \sqrt{(-2)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{4 + \frac{9}{25} + \frac{16}{25}}$$

$$||\vec{w}_2|| = \sqrt{5}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$ and $\vec{u}_2$ form an orthonormal basis for V.

$$V_2 = \text{Span}\left(\frac{1}{5}\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}\right)$$

$$V_2 = \text{Span}\left(\begin{bmatrix} 0 \\ -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}, \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{3}{5\sqrt{5}} \\ \frac{4}{5\sqrt{5}} \end{bmatrix}\right)$$

■ 2. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (1, -1, 1)$ and $\vec{v}_2 = (-3, 5, 2)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{1 + 1 + 1} = \sqrt{3}$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$ and $\vec{v}_2$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2)$$

Now all we need to do is replace $\vec{v}_2$ with a vector that's both orthogonal to $\vec{u}_1$, and normal. If we can do that, then the vector set that spans V will be orthonormal. We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix}-3\\5\\2\end{bmatrix} - \left(\begin{bmatrix}-3\\5\\2\end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-3\\5\\2\end{bmatrix} - \frac{1}{3}\left(\begin{bmatrix}-3\\5\\2\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\1\end{bmatrix}\right)\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-3\\5\\2\end{bmatrix} - \frac{1}{3}((-3)(1) + (5)(-1) + (2)(1))\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-3\\5\\2\end{bmatrix} - \frac{1}{3}(-6)\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-3\\5\\2\end{bmatrix} + 2\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3+2 \\ 5-2 \\ 2+2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it.

$$||\vec{w}_2|| = \sqrt{(-1)^2 + 3^2 + 4^2}$$

$$||\vec{w}_2|| = \sqrt{1 + 9 + 16}$$

$$||\vec{w}_2|| = \sqrt{26}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{26}} \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$ and $\vec{u}_2$ form an orthonormal basis for V.

$$V_2 = \text{Span}\left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{26}} \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right)$$

$$V_2 = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{26}} \\ \frac{3}{\sqrt{26}} \\ \frac{4}{\sqrt{26}} \end{bmatrix} \right)$$

■ 3. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-2\\1\\-2\end{bmatrix}, \begin{bmatrix}-3\\-1\\4\end{bmatrix}, \begin{bmatrix}2\\-1\\5\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (-2,1,-2)$, $\vec{v}_2 = (-3,-1,4)$, and $\vec{v}_3 = (2,-1,5)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-2)^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix}-2\\1\\-2\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} - \left(\begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \right) \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{9} \left(\begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \right) \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{9}((-3)(-2) + (-1)(1) + (4)(-2)) \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{9}(-3) \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{\left(-\frac{11}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{10}{3}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{\frac{121}{9} + \frac{4}{9} + \frac{100}{9}}$$

$$||\vec{w}_2|| = \sqrt{\frac{225}{9}}$$

$$||\vec{w}_2|| = \sqrt{25}$$

$$||\vec{w}_2|| = 5$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{5}\begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have.

$$\vec{w}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}\right)\frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}\right)\frac{1}{5}\begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \frac{1}{9}\left(\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}\right)\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} - \frac{1}{25}\left(\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}\right)\begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \frac{1}{9}(2(-2) - 1(1) + 5(-2))\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$-\frac{1}{25}\left(2\left(-\frac{11}{3}\right) - 1\left(-\frac{2}{3}\right) + 5\left(\frac{10}{3}\right)\right)\begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + \frac{5}{3}\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 - \frac{10}{3} + \frac{22}{15} \\ -1 + \frac{5}{3} + \frac{4}{15} \\ 5 - \frac{10}{3} - \frac{20}{15} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} \frac{2}{15} \\ \frac{14}{15} \\ \frac{1}{3} \end{bmatrix}$$

So $\vec{w}_3$ is orthogonal to $\vec{u}_2$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{2}{15}\right)^2 + \left(\frac{14}{15}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{4}{225} + \frac{196}{225} + \frac{1}{9}}$$

$$||\vec{w}_3|| = \sqrt{\frac{225}{225}}$$

$$||\vec{w}_3|| = 1$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{1}\begin{bmatrix} \frac{2}{15} \\ \frac{14}{15} \\ \frac{1}{3} \end{bmatrix}$$

$$\vec{u}_3 = \begin{bmatrix} \frac{2}{15} \\ \frac{14}{15} \\ \frac{1}{3} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{3}\begin{bmatrix}-2\\1\\-2\end{bmatrix}, \frac{1}{5}\begin{bmatrix}-\frac{11}{3}\\-\frac{2}{3}\\\frac{10}{3}\end{bmatrix}, \begin{bmatrix}\frac{2}{15}\\\frac{14}{15}\\\frac{1}{3}\end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix}, \begin{bmatrix}-\frac{11}{15}\\-\frac{2}{15}\\\frac{2}{3}\end{bmatrix}, \begin{bmatrix}\frac{2}{15}\\\frac{14}{15}\\\frac{1}{3}\end{bmatrix}\right)$$

■ 4. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-3\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\2\end{bmatrix}, \begin{bmatrix}-5\\5\\0\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (-3,0,0)$, $\vec{v}_2 = (-2,1,2)$, and $\vec{v}_3 = (-5,5,0)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-3)^2 + 0^2 + 0^2} = \sqrt{9} = 3$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix}-3\\0\\0\end{bmatrix}$$

$$\vec{u}_1 = \begin{bmatrix}-1\\0\\0\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\2\end{bmatrix} - \left(\begin{bmatrix}-2\\1\\2\end{bmatrix} \cdot \begin{bmatrix}-1\\0\\0\end{bmatrix}\right)\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\2\end{bmatrix} - ((-2)(-1) + (1)(0) + (2)(0))\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\2\end{bmatrix} - 2\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{0^2 + 1^2 + 2^2}$$

$$||\vec{w}_2|| = \sqrt{0 + 1 + 4}$$

$$||\vec{w}_2|| = \sqrt{5}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have

$$\vec{w}_3=\begin{bmatrix}-5\\5\\0\end{bmatrix}-\left(\begin{bmatrix}-5\\5\\0\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\0\end{bmatrix}\right)\begin{bmatrix}-1\\0\\0\end{bmatrix}-\left(\begin{bmatrix}-5\\5\\0\end{bmatrix}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-5\\5\\0\end{bmatrix}-\left(\begin{bmatrix}-5\\5\\0\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\0\end{bmatrix}\right)\begin{bmatrix}-1\\0\\0\end{bmatrix}-\frac{1}{5}\left(\begin{bmatrix}-5\\5\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\2\end{bmatrix}\right)\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-5\\5\\0\end{bmatrix}-(-5(-1)+5(0)+0(0))\begin{bmatrix}-1\\0\\0\end{bmatrix}-\frac{1}{5}(-5(0)+5(1)+0(2))\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-5\\5\\0\end{bmatrix}-5\begin{bmatrix}-1\\0\\0\end{bmatrix}-\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-5+5-0\\5-0-1\\0-0-2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}0\\4\\-2\end{bmatrix}$$

So $\vec{w}_3$ is orthogonal to $\vec{u}_2$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_3$ is

$$||\vec{w}_3||=\sqrt{0^2+4^2+(-2)^2}$$

$$||\vec{w}_3||=\sqrt{0+16+4}$$

$$||\vec{w}_3||=\sqrt{20}$$

$$||\vec{w}_3|| = 2\sqrt{5}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{2\sqrt{5}}\begin{bmatrix}0\\4\\-2\end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\begin{bmatrix}-1\\0\\0\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix}, \frac{1}{2\sqrt{5}}\begin{bmatrix}0\\4\\-2\end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix}-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}0\\\frac{2}{\sqrt{5}}\\-\frac{1}{\sqrt{5}}\end{bmatrix}\right)$$

■ 5. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}, \begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (-3,0,4,0)$, $\vec{v}_2 = (-1,2,-2,0)$, and $\vec{v}_3 = (5,-1,0,2)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-3)^2 + 0^2 + (-4)^2 + 0^2} = \sqrt{9 + 0 + 16 + 0} = \sqrt{25} = 5$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{5}\begin{bmatrix} -3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have

$$\vec{w}_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ 2 \\ -2 \\ 0 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} -3 \\ 0 \\ 4 \\ 0 \end{bmatrix}\right)\frac{1}{5}\begin{bmatrix} -3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}-\frac{1}{25}\left(\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}\cdot\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}\right)\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}-\frac{1}{25}(-1(-3)+2(0)-2(4)+0(0))\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}-\frac{1}{25}(-5)\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}+\frac{1}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1-\frac{3}{5}\\2+0\\-2+\frac{4}{5}\\0+0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{\left(-\frac{8}{5}\right)^2 + 2^2 + \left(-\frac{6}{5}\right)^2 + 0^2}$$

$$||\vec{w}_2|| = \sqrt{\frac{64}{25} + 4 + \frac{36}{25} + 0}$$

$$||\vec{w}_2|| = \sqrt{8}$$

$$||\vec{w}_2|| = 2\sqrt{2}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{2\sqrt{2}}\begin{bmatrix} -\frac{8}{5} \\ 2 \\ -\frac{6}{5} \\ 0 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u_1} - (\vec{v}_3 \cdot \vec{u}_2)\vec{u_2}$$

Plug in the values we already have.

$$\vec{w}_3=\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}-\left(\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\cdot\frac{1}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}\right)\frac{1}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}-\left(\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\cdot\frac{1}{2\sqrt{2}}\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}\right)\frac{1}{2\sqrt{2}}\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}-\frac{1}{25}\left(\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\cdot\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}\right)\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}-\frac{1}{8}\left(\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\cdot\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}\right)\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}-\frac{1}{25}(5(-3)-1(0)+0(4)+2(0))\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$-\frac{1}{8}\left(5\left(-\frac{8}{5}\right)-1(2)+0\left(-\frac{6}{5}\right)+2(0)\right)\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}-\frac{1}{25}(-15)\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}-\frac{1}{8}(-10)\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}+\frac{3}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}+\frac{5}{4}\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}5-\frac{9}{5}-2\\-1+0+\frac{5}{2}\\0+\frac{12}{5}-\frac{3}{2}\\2+0+0\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}\frac{6}{5}\\\frac{3}{2}\\\frac{9}{10}\\2\end{bmatrix}$$

The length of $\vec{w}_3$ is

$$||\vec{w}_3||=\sqrt{\left(\frac{6}{5}\right)^2+\left(\frac{3}{2}\right)^2+\left(\frac{9}{10}\right)^2+2^2}$$

$$||\vec{w}_3||=\sqrt{\frac{36}{25}+\frac{9}{4}+\frac{81}{100}+4}$$

$$||\vec{w}_3||=\sqrt{\frac{17}{2}}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{\sqrt{\frac{17}{2}}}\begin{bmatrix} \frac{6}{5} \\ \frac{3}{2} \\ \frac{9}{10} \\ 2 \end{bmatrix}$$

$$\vec{u}_3 = \sqrt{\frac{2}{17}}\begin{bmatrix} \frac{6}{5} \\ \frac{3}{2} \\ \frac{9}{10} \\ 2 \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{5}\begin{bmatrix} -3 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \frac{1}{2\sqrt{2}}\begin{bmatrix} -\frac{8}{5} \\ 2 \\ -\frac{6}{5} \\ 0 \end{bmatrix}, \sqrt{\frac{2}{17}}\begin{bmatrix} \frac{6}{5} \\ \frac{3}{2} \\ \frac{9}{10} \\ 2 \end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix} -\frac{3}{5} \\ 0 \\ \frac{4}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{3}{5\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{6}{5}\sqrt{\frac{2}{17}} \\ \frac{3}{2}\sqrt{\frac{2}{17}} \\ \frac{9}{10}\sqrt{\frac{2}{17}} \\ 2\sqrt{\frac{2}{17}} \end{bmatrix}\right)$$

6. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\0\\-1\\-1\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (-2, -2, 2, -2)$, $\vec{v}_2 = (-2, 1, 0, -1)$, and $\vec{v}_3 = (4, 0, -1, -1)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-2)^2 + (-2)^2 + 2^2 + (-2)^2} = \sqrt{4+4+4+4} = \sqrt{16} = 4$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{4}\begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right) \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{16}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right) \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{16}(-2(-2) + 1(-2) + 0(2) - 1(-2)) \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{\frac{9}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4}}$$

$$||\vec{w}_2|| = \sqrt{5}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have.

$$\vec{w}_3 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right) \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} - \frac{1}{16} \left(\begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right) \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} - \frac{1}{5} \left(\begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} - \frac{1}{16}(4(-2) + 0(-2) - 1(2) - 1(-2)) \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$-\frac{1}{5}\left(4\left(-\frac{3}{2}\right) + 0\left(\frac{3}{2}\right) - 1\left(-\frac{1}{2}\right) - 1\left(-\frac{1}{2}\right)\right) \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} - \frac{1}{16}(-8)\begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} - \frac{1}{5}(-5)\begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}$$

The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{5}{2}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{25}{4}}$$

$$||\vec{w}_3|| = \sqrt{9}$$

$$||\vec{w}_3|| = 3$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{3}\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{4}\begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \frac{1}{3}\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{3}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} \end{bmatrix}, \frac{1}{3}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{6} \\ -\frac{1}{6} \\ -\frac{5}{6} \end{bmatrix}\right)$$