Topic: Null space of a matrix

Question: Find the null space of the matrix A.

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 4 \\ 6 & -6 & -12 \end{bmatrix}$$

Answer choices:

A $\quad N(A) = \text{Span}\left(\begin{bmatrix} -6 \\ -4 \\ 1 \end{bmatrix}\right)$

B $\quad N(A) = \text{Span}\left(\begin{bmatrix} -6 \\ -4 \\ 0 \end{bmatrix}\right)$

C $\quad N(A) = \text{Span}\left(\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}\right)$

D $\quad N(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}\right)$

Solution: C

To find the null space of A, we need to find the vector set that satisfies $A\vec{x}=\vec{O}$, so we need to set up a matrix equation.

Because A has three columns, $\vec{x}$ needs to have three rows, so we'll use a 3-row column vector for $\vec{x}$. And multiplying the 3×3 matrix by the 3-row column vector will result in a 3×1 zero-vector, so the matrix equation must be

$$\begin{bmatrix}1 & -1 & -2\\ 0 & -1 & 4\\ 6 & -6 & -12\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

We can write the system as an augmented matrix,

$$\begin{bmatrix}1 & -1 & -2 & | & 0\\ 0 & -1 & 4 & | & 0\\ 6 & -6 & -12 & | & 0\end{bmatrix}$$

and then use Gaussian elimination to put it in reduced row-echelon form.

$$\begin{bmatrix}1 & -1 & -2 & | & 0\\ 0 & -1 & 4 & | & 0\\ 0 & 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & -2 & | & 0\\ 0 & 1 & -4 & | & 0\\ 0 & 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -6 & | & 0 \\ 0 & 1 & -4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From this matrix, we get a system of equations,

$$x_1 - 6x_3 = 0$$

$$x_2 - 4x_3 = 0$$

which we can solve for the pivot variables.

$$x_1 = 6x_3$$

$$x_2 = 4x_3$$

So the solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

Then the null space of A is the span of $\vec{x} = (6,4,1)$.

$$N(A) = \text{Span}\left(\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} \right)$$

Topic: Null space of a matrix

Question: Find the null space of M.

$$M=\begin{bmatrix}1 & 2 & -3 & 5\\ 2 & 4 & -6 & 10\\ -3 & -6 & 9 & -15\\ 4 & 1 & -12 & 6\end{bmatrix}$$

Answer choices:

A $N(M)=\text{Span}\left(\begin{bmatrix}3\\0\\1\\0\end{bmatrix}\right)$

B $N(M)=\text{Span}\left(\begin{bmatrix}3\\0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\-2\\0\\1\end{bmatrix}\right)$

C $N(M)=\text{Span}\left(\begin{bmatrix}-5\\-2\\0\\1\end{bmatrix}\right)$

D $N(M)=\text{Span}\left(\begin{bmatrix}3\\-7\\0\\0\end{bmatrix}\right)$

Solution: B

To find the null space, put the matrix M into reduced row-echelon form.

$$M=\begin{bmatrix}1&2&-3&5\\2&4&-6&10\\-3&-6&9&-15\\4&1&-12&6\end{bmatrix}\to\begin{bmatrix}1&2&-3&5\\0&0&0&0\\-3&-6&9&-15\\4&1&-12&6\end{bmatrix}$$

$$\to\begin{bmatrix}1&2&-3&5\\0&0&0&0\\0&0&0&0\\4&1&-12&6\end{bmatrix}\to\begin{bmatrix}1&2&-3&5\\0&0&0&0\\0&0&0&0\\0&-7&0&-14\end{bmatrix}\to\begin{bmatrix}1&2&-3&5\\0&-7&0&-14\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\to\begin{bmatrix}1&2&-3&5\\0&1&0&2\\0&0&0&0\\0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&-3&1\\0&1&0&2\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Then set up the equation $(\text{rref}(M))\vec{x}=\vec{O}$. Because M has four columns, $\vec{x}$ needs to have four rows, so we'll use a 4-row column vector for $\vec{x}$. And multiplying the 4×4 matrix by the 4-row column vector will result in a 4×1 zero-vector, so the matrix equation must be

$$\begin{bmatrix}1&0&-3&1\\0&1&0&2\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

From this matrix, we get a system of equations,

$$x_1-3x_3+x_4=0$$

$$x_2 + 2x_4 = 0$$

which we can solve for the pivot variables.

$$x_1 = 3x_3 - x_4$$

$$x_2 = -2x_4$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of M is the span of the vectors in this linear combination equation.

$$N(M) = \text{Span}\left(\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Topic: Null space of a matrix

Question: Find the null space of B.

$$B=\begin{bmatrix}2 & 2 & -4 & 10\\ -1 & -1 & 2 & -5\\ 3 & 3 & -6 & 15\end{bmatrix}$$

Answer choices:

A $N(B)=\text{Span}\left(\begin{bmatrix}-1\\ 1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}2\\ 0\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-5\\ 0\\ 0\\ 1\end{bmatrix}\right)$

B $N(B)=\text{Span}\left(\begin{bmatrix}1\\ -1\\ 2\\ -5\end{bmatrix}\right)$

C $N(B)=\text{Span}\left(\begin{bmatrix}-1\\ 0\\ 0\\ 0\end{bmatrix},\begin{bmatrix}2\\ 0\\ 0\\ 0\end{bmatrix},\begin{bmatrix}-5\\ 0\\ 0\\ 0\end{bmatrix}\right)$

D $N(B)=\text{Span}\left(\begin{bmatrix}1\\ 1\\ -2\\ 5\end{bmatrix}\right)$

Solution: A

To find the null space, put the matrix B in reduced row-echelon form.

$$B = \begin{bmatrix} 2 & 2 & -4 & 10 \\ -1 & -1 & 2 & -5 \\ 3 & 3 & -6 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 5 \\ -1 & -1 & 2 & -5 \\ 3 & 3 & -6 & 15 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & -6 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then set up the equation $(\text{rref}(B))\vec{x} = \vec{O}$. Because B has four columns, $\vec{x}$ needs to have four rows, so we'll use a 4-row column vector for $\vec{x}$. And multiplying the 3×4 matrix by the 4-row column vector will result in a 3×1 zero-vector, so the matrix equation must be

$$\begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For this matrix, we get the equation,

$$x_1 + x_2 - 2x_3 + 5x_4 = 0$$

which we can solve for the single pivot variable.

$$x_1 = -x_2 + 2x_3 - 5x_4$$

We can rewrite this as a linear combination

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of B is the span of the vectors in this linear combination equation.

$$N(B) = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$