

Transposes and their determinants

In this lesson, we want to start talking about matrix transposes.

The **transpose** of a matrix is simply the matrix you get when you swap all the rows and columns. In other words, the first row becomes the first column, the second row becomes the second column, and the n th row becomes the n th column.

You can find the transpose of any matrix with any dimensions, and we indicate the transpose of an $m \times n$ matrix A as an $n \times m$ matrix A^T , and we call it “A transpose.”

Example

Find the transpose of A .

$$A = \begin{bmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{bmatrix}$$

The first row of A is $[-2 \ 4 \ 0]$, so this sequence of entries needs to become the first column of A^T .

$$A^T = \begin{bmatrix} -2 & \cdot & \cdot \\ 4 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{bmatrix}$$

The second row of A is $[1 \ -1 \ 4]$, so this sequence of entries needs to become the second column of A^T .



$$A^T = \begin{bmatrix} -2 & 1 & \cdot \\ 4 & -1 & \cdot \\ 0 & 4 & \cdot \end{bmatrix}$$

The third row of A is $[0 \ 6 \ -4]$, so this sequence of entries needs to become the third column of A^T .

$$A^T = \begin{bmatrix} -2 & 1 & 0 \\ 4 & -1 & 6 \\ 0 & 4 & -4 \end{bmatrix}$$

This works for non-square matrices, too. So the transpose of a matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -3 & 0 & 1 & 2 \\ 4 & -2 & 0 & 3 \end{bmatrix}$$

is

$$B^T = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 0 & -2 \\ 3 & 1 & 0 \\ 4 & 2 & 3 \end{bmatrix}$$

It's also important to realize that the transpose of a transpose gets you back to the original matrix. So given matrix A and the transpose of matrix A^T , then $(A^T)^T = A$.

Determinant of the transpose



The determinant of a transpose of a square matrix will always be equal to the determinant of the original matrix. In other words, given an $n \times n$ matrix A , the determinant of A and the determinant of A^T are equivalent.

Example

Show that the determinants of A and A^T are equivalent.

$$A = \begin{bmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{bmatrix}$$

The determinant of A is

$$|A| = \begin{vmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{vmatrix}$$

$$|A| = -2 \begin{vmatrix} -1 & 4 \\ 6 & -4 \end{vmatrix} - 4 \begin{vmatrix} 1 & 4 \\ 0 & -4 \end{vmatrix} + 0 \begin{vmatrix} 1 & -1 \\ 0 & 6 \end{vmatrix}$$

The last term gets zeroed out.

$$|A| = -2((-1)(-4) - (4)(6)) - 4((1)(-4) - (4)(0)) + 0((1)(6) - (-1)(0))$$

$$|A| = -2(4 - 24) - 4(-4 - 0) + 0(6 - 0)$$

$$|A| = -2(-20) - 4(-4) + 0(6)$$

$$|A| = 40 + 16 + 0$$

$$|A| = 56$$



The matrix A^T is

$$A^T = \begin{bmatrix} -2 & 1 & 0 \\ 4 & -1 & 6 \\ 0 & 4 & -4 \end{bmatrix}$$

The determinant of A^T is

$$|A^T| = \begin{vmatrix} -2 & 1 & 0 \\ 4 & -1 & 6 \\ 0 & 4 & -4 \end{vmatrix}$$

$$|A^T| = -2 \begin{vmatrix} -1 & 6 \\ 4 & -4 \end{vmatrix} - 1 \begin{vmatrix} 4 & 6 \\ 0 & -4 \end{vmatrix} + 0 \begin{vmatrix} 4 & -1 \\ 0 & 4 \end{vmatrix}$$

The last term gets zeroed out.

$$|A^T| = -2((-1)(-4) - (6)(4)) - 1((4)(-4) - (6)(0)) + 0((4)(4) - (-1)(0))$$

$$|A^T| = -2(4 - 24) - 1(-16 - 0) + 0(16 - 0)$$

$$|A^T| = -2(-20) - 1(-16) + 0(16)$$

$$|A^T| = 40 + 16 + 0$$

$$|A^T| = 56$$

So we've shown that the determinants of A and A^T are equivalent.

$$|A| = |A^T| = 56$$

