Topic: Linear transformations as matrix-vector products

Question: Use a matrix-vector product to reflect the square with vertices $(-3,2)$, $(4,2)$, $(4,-5)$, and $(-3,-5)$ over the x-axis. What are the vertices of the reflected square?

Answer choices:

A $(-3,-2)$, $(4,-2)$, $(4,5)$, $(-3,5)$

B $(3,2)$, $(4,-2)$, $(-4,-5)$, $(-3,5)$

C $(-3,-2)$, $(-4,2)$, $(4,5)$, $(3,-5)$

D $(3,2)$, $(-4,2)$, $(-4,-5)$, $(3,-5)$

Solution: A

If each point in the square is given by (x, y), a reflection over the x-axis means we'll take the y-coordinate of each point in the square and multiply it by -1. So after the reflection, each transformed point will be $(x, -y)$.

Therefore, if a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original square, then a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

represents the corresponding point in the transformed square. So a transformation T that expresses the reflection for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the square, $(-3,2)$, $(4,2)$, $(4,-5)$, and $(-3,-5)$.

$$T\left(\begin{bmatrix}-3\\2\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}-3\\2\end{bmatrix}=\begin{bmatrix}1(-3)+0(2)\\0(-3)-1(2)\end{bmatrix}=\begin{bmatrix}-3\\-2\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\2\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}1(4)+0(2)\\0(4)-1(2)\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\-5\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\-5\end{bmatrix}=\begin{bmatrix}1(4)+0(-5)\\0(4)-1(-5)\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}$$

$$T\left(\begin{bmatrix}-3\\-5\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}-3\\-5\end{bmatrix}=\begin{bmatrix}1(-3)+0(-5)\\0(-3)-1(-5)\end{bmatrix}=\begin{bmatrix}-3\\5\end{bmatrix}$$

Topic: Linear transformations as matrix-vector products

Question: Use a matrix-vector product to double the width of the rectangle that has vertices $(3,-6)$, $(3,1)$, $(-1,1)$, and $(-1,-6)$. What are the vertices of the stretched rectangle?

Answer choices:

A $(3,-12)$, $(3,2)$, $(-1,2)$, $(-1,-12)$

B $(3,12)$, $(3,-2)$, $(-1,-2)$, $(-1,12)$

C $(-6,-6)$, $(-6,1)$, $(2,1)$, $(2,-6)$

D $(6,-6)$, $(6,1)$, $(-2,1)$, $(-2,-6)$

Solution: D

Doubling the width of the rectangle means we're stretching it horizontally by a factor of 2.

If each point in the rectangle is given by (x, y), then doubling the width means we'll take the x-coordinate of each point in the rectangle and multiply it by 2. So after the stretch, each transformed point will be $(2x, y)$.

Therefore, if a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original rectangle, then a position vector

$$\vec{v} = \begin{bmatrix} 2v_1 \\ v_2 \end{bmatrix}$$

represents the corresponding point in the transformed rectangle. So a transformation T that expresses the stretch for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 2v_1 \\ v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2(1) \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix}2(0)\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the rectangle, $(3,-6)$, $(3,1)$, $(-1,1)$, and $(-1,-6)$.

$$T\left(\begin{bmatrix}3\\-6\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}3\\-6\end{bmatrix}=\begin{bmatrix}2(3)+0(-6)\\0(3)+1(-6)\end{bmatrix}=\begin{bmatrix}6\\-6\end{bmatrix}$$

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}2(3)+0(1)\\0(3)+1(1)\end{bmatrix}=\begin{bmatrix}6\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}2(-1)+0(1)\\0(-1)+1(1)\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}-1\\-6\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}-1\\-6\end{bmatrix}=\begin{bmatrix}2(-1)+0(-6)\\0(-1)+1(-6)\end{bmatrix}=\begin{bmatrix}-2\\-6\end{bmatrix}$$

Topic: Linear transformations as matrix-vector products

Question: Use a matrix-vector product to reflect the parallelogram with vertices $(1,1)$, $(0,-4)$, $(-4,-4)$, and $(-3,1)$ over the y-axis, and then compress it vertically by a factor of 3. What are the vertices of the transformed parallelogram?

Answer choices:

A $(-1,1)$, $(0,-4)$, $(4,-4)$, $(3,1)$

B $(1,-1)$, $(0,4)$, $(-4,4)$, $(-3,-1)$

C $\left(-1,\frac{1}{3}\right)$, $\left(0,-\frac{4}{3}\right)$, $\left(4,-\frac{4}{3}\right)$, $\left(3,\frac{1}{3}\right)$

D $\left(1,-\frac{1}{3}\right)$, $\left(0,\frac{4}{3}\right)$, $\left(-4,\frac{4}{3}\right)$, $\left(-3,-\frac{1}{3}\right)$

Solution: C

If each point in the square is given by (x, y), a reflection over the y-axis means we'll take the x-coordinate of each point in the parallelogram and multiply it by -1. So after the reflection, each transformed point will be $(-x, y)$.

Therefore, if a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original parallelogram, then a position vector

$$\vec{v} = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$$

represents the corresponding point in the transformed parallelogram. So a transformation T that expresses the reflection for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$$

A vertical compression by a factor of 3 means we'll take the y-coordinate of each point in the parallelogram and multiply it by $1/3$. So after the compression (and the reflection), each transformed point will be

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} -v_1 \\ \frac{1}{3}v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\\frac{1}{3}(0)\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-0\\\frac{1}{3}(1)\end{bmatrix}=\begin{bmatrix}0\\\frac{1}{3}\end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the parallelogram, $(1,1)$, $(0,-4)$, $(-4,-4)$, and $(-3,1)$.

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1(1)+0(1)\\0(1)+\frac{1}{3}(1)\end{bmatrix}=\begin{bmatrix}-1\\\frac{1}{3}\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\-4\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}0\\-4\end{bmatrix}=\begin{bmatrix}-1(0)+0(-4)\\0(0)+\frac{1}{3}(-4)\end{bmatrix}=\begin{bmatrix}0\\-\frac{4}{3}\end{bmatrix}$$

$$T\left(\begin{bmatrix}-4\\-4\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}-4\\-4\end{bmatrix}=\begin{bmatrix}-1(-4)+0(-4)\\0(-4)+\frac{1}{3}(-4)\end{bmatrix}=\begin{bmatrix}4\\-\frac{4}{3}\end{bmatrix}$$

$$T\left(\begin{bmatrix}-3\\1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}-3\\1\end{bmatrix}=\begin{bmatrix}-1(-3)+0(1)\\0(-3)+\frac{1}{3}(1)\end{bmatrix}=\begin{bmatrix}3\\\frac{1}{3}\end{bmatrix}$$