Linear Algebra Workbook

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MATH

LINEAR SYSTEMS IN TWO UNKNOWNS

1. Find the unique solution to the system of equations.

$$-x + 2y = 6$$

$$3x = y - 10$$

2. Find the unique solution to the system of equations.

$$-5x + y = 8$$

$$y = 3x - 8$$

3. Find the unique solution to the system of equations.

$$2x - y = 5$$

$$-3x + y = 7$$

4. Find the unique solution to the system of equations.

$$x = 2y - 5$$

$$-3x + 6y = 15$$

■ 5. Find the unique solution to the system of equations using the graphing method.

$$y - 2 = -(x + 1)$$

$$y = x + 1$$

■ 6. Find the unique solution to the system of equations using the substitution method.

$$5y + x = 4$$

$$3y - 3x = 6$$

LINEAR SYSTEMS IN THREE UNKNOWNS

■ 1. Find the unique solution to the system of equations.

$$2x + y - z = 3$$

$$x - y + z = 0$$

$$x - 2y - 3z = 4$$

■ 2. Find the unique solution to the system of equations.

$$3x + y - z = -2$$

$$x - 2y + 3z = 23$$

$$2x + 3y + 2z = 5$$

■ 3. Find the unique solution to the system of equations.

$$5x - 3y + z = -8$$

$$2x + y - 2z = -6$$

$$-3x + 2y + 4z = 19$$

■ 4. Find the unique solution to the system of equations.

$$-2x + 3y - 4z = 10$$

$$4x + 3y + 2z = 4$$

$$x - 6y + 4z = -19$$

5. Find the unique solution to the system of equations.

$$2x - y + z = 9$$

$$4x - 2y + 2z = 18$$

$$-2x + y - z = -9$$

6. Find the unique solution to the system of equations.

$$x + 2y - z = 9$$

$$3x + y - z = 5$$

$$-x - 4y + z = 2$$

MATRIX DIMENSIONS AND ENTRIES

■ 1. Give the dimensions of the matrix.

$$D = \begin{bmatrix} 11 & 9 \\ -4 & 8 \end{bmatrix}$$

■ 2. Give the dimensions of the matrix.

$$A = \begin{bmatrix} 3 & 5 & -2 & 1 & 8 \end{bmatrix}$$

■ 3. Given matrix J, find $J_{4,1}$.

$$J = \begin{bmatrix} 6 \\ 2 \\ 7 \\ 1 \end{bmatrix}$$

■ 4. Given matrix C, find $C_{1,2}$.

$$C = \begin{bmatrix} 3 & 12 \\ 1 & 4 \\ 9 & 5 \\ -3 & 2 \end{bmatrix}$$

5. Given matrix N, state the dimensions and find $N_{1,3}$.

$$N = \begin{bmatrix} 1 & 5 & 9 \\ 14 & -8 & 6 \end{bmatrix}$$

6. Given matrix S, state the dimensions and find $S_{3,4}$.

$$S = \begin{bmatrix} 3 & 6 & -7 & 1 & 0 \\ 0 & 9 & 15 & 3 & 4 \\ 4 & 0 & 2 & 11 & 8 \\ -5 & 8 & 7 & 9 & 2 \end{bmatrix}$$

REPRESENTING SYSTEMS WITH MATRICES

■ 1. Represent the system with a matrix called A.

$$-2x + 5y = 12$$

$$6x - 2y = 4$$

■ 2. Represent the system with a matrix called D.

$$9y - 3x + 12 = 0$$

$$8 - 4x = 11y$$

■ 3. Represent the system with an augmented matrix called H.

$$4a + 7b - 5c + 13d = 6$$

$$3a - 8b = -2c + 1$$

■ 4. Represent the system with a matrix called M.

$$-2x + 4y = 9 - 6z$$

$$7y + 2z - 3 = -3t - 9x$$

5. Represent the system with a matrix called A.

$$3x - 8y + z = 7$$

$$2z = 3y - 2x + 4$$

$$5y = 12 - 9x$$

6. Represent the system with a matrix called K.

$$-4b + 2c = 3 - 7a$$

$$9c = 4 - 2b$$

$$8a - 2c = 5b$$

SIMPLE ROW OPERATIONS

1. Write the new matrix after $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 2 & 6 & -4 & 1 \\ 8 & 2 & 1 & -5 \end{bmatrix}$$

2. Write the new matrix after $R_2 \leftrightarrow R_4$.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 6 & 1 & 5 & -4 \\ -7 & 7 & 0 & 3 \\ 9 & 2 & 8 & 3 \end{bmatrix}$$

3. Write the new matrix after $R_1 \leftrightarrow 3R_2$.

$$\begin{bmatrix} 9 & 2 & -7 \\ 1 & 6 & 4 \end{bmatrix}$$

4. Write the new matrix after $3R_2 \leftrightarrow 3R_4$.

$$\begin{bmatrix} 0 & 11 & 6 \\ 7 & -3 & 9 \\ 8 & 8 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

5. Write the new matrix after $R_1 + 2R_2 \to R_1$.

$$\begin{bmatrix} 6 & 2 & 7 \\ 1 & -5 & 15 \end{bmatrix}$$

6. Write the new matrix after $4R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 13 & 5 & -2 & 9 \\ 8 & 2 & 0 & 6 \\ 4 & 1 & 7 & -3 \end{bmatrix}$$

PIVOT ENTRIES AND ROW-ECHELON FORMS

1. Use row operations to put the matrix into row-echelon form.

$$\begin{bmatrix} 3 & 6 & -7 \\ 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

2. Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -6 & 0 \end{bmatrix}$$

3. Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & -3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$$

4. Use row operations to put the matrix into row-echelon form.

$$\begin{bmatrix} 3 & 2 & 0 & 9 \\ 2 & 4 & -3 & -1 \\ 2 & 12 & -12 & 1 \end{bmatrix}$$

5. Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -3 & 0 \\ 2 & -3 \end{bmatrix}$$

6. Use row operations to put the matrix into row-echelon form.

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ -1 & 3 & -6 & -13 \\ -5 & -2 & 22 & -28 \end{bmatrix}$$

GAUSS-JORDAN ELIMINATION

■ 1. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$x + 2y = -2$$

$$3x + 2y = 6$$

■ 2. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$2x + 4y = 22$$

$$3x + 3y = 15$$

■ 3. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$x - 3y - 6z = 4$$

$$y + 2z = -2$$

$$-4x + 12y + 21z = -4$$

4. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$2y + 4z = 4$$

$$x + 3y + 3z = 5$$

$$2x + 7y + 6z = 10$$

5. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$3x + 12y + 42z = -27$$

$$x + 2y + 8z = -5$$

$$2x + 5y + 16z = -6$$

6. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$4x + 8y + 4z = 20$$

$$4x + 6y = 4$$

$$3x + 3y - z = 1$$

NUMBER OF SOLUTIONS TO THE LINEAR SYSTEM

■ 1. Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$2x - 8y = 18$$

$$-7x + 2y - 5z = -6$$

$$3x + 2z = 1$$

■ 2. Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$-x + 3y - 5z - 8w = 2$$

$$4x - 8y + 4z + 4w = -44$$

$$3x + 5y - 16z + w = 18$$

$$-x + y - 3z - w = 6$$

■ 3. How many solutions does the linear system have?

$$3x - 3y + 5z = -11$$

$$-2x + y - 2z = 5$$

$$x + y - z = 9$$

■ 4. How many solutions does the linear system have?

$$-x + 6y + 4z = -22$$

$$4x - 22y - 2z + 2w = 0$$

$$x - 6y - 5z + 3w = 5$$

$$-3y - 22z = 6$$

■ 5. Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$2x + 2y - 8z = 4$$

$$-3x - 5y + 6z = -4$$

$$5x - y - 38z = 16$$

■ 6. For the linear system below, determine whether it has one solution, no solutions, or infinitely many solutions.

$$x + y - z + 2w = 7$$

$$4x + 2y - 6z + 2w = 16$$

$$-3x + y + 7z + 6w = 3$$

$$-x - y + 4z + 3w = 8$$

MATRIX ADDITION AND SUBTRACTION

1. Add the matrices.

$$\begin{bmatrix} 7 & 6 \\ 17 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ -2 & 5 \end{bmatrix}$$

2. Add the matrices.

$$\begin{bmatrix} 8 & 3 \\ -4 & 7 \\ 6 & 0 \\ 1 & 13 \end{bmatrix} + \begin{bmatrix} 6 & 7 \\ 2 & -3 \\ 9 & 11 \\ 7 & -2 \end{bmatrix}$$

3. Subtract the matrices.

$$\begin{bmatrix} 7 & 9 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 8 \\ 12 & -3 \end{bmatrix}$$

4. Subtract the matrices.

$$\begin{bmatrix} 8 & 11 & 2 & 9 \\ 6 & 3 & 16 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 11 & 7 & -4 \\ 5 & 8 & 1 & 15 \end{bmatrix}$$

■ 5. Solve for M.

$$\begin{bmatrix}6 & 5\\ 9 & -9\end{bmatrix}+\begin{bmatrix}3 & 7\\ 1 & 6\end{bmatrix}=M+\begin{bmatrix}7 & 12\\ -3 & -1\end{bmatrix}-\begin{bmatrix}1 & 8\\ 4 & -7\end{bmatrix}$$

■ 6. Solve for N.

$$\begin{bmatrix}4 & 12\\ 9 & 8\end{bmatrix}-\begin{bmatrix}0 & 3\\ 9 & 9\end{bmatrix}=N-\begin{bmatrix}6 & 3\\ 5 & 11\end{bmatrix}+\begin{bmatrix}7 & -4\\ -18 & 1\end{bmatrix}$$

SCALAR MULTIPLICATION

■ 1. Use scalar multiplication to simplify the expression.

$$\frac{1}{4}\begin{bmatrix} 12 & 8 & 3 \\ 2 & -16 & 0 \\ 1 & 5 & 7 \end{bmatrix}$$

■ 2. Solve for Y.

$$4\begin{bmatrix} 2 & 9 \\ -5 & 0 \end{bmatrix} + Y = 5\begin{bmatrix} 1 & -3 \\ 6 & 8 \end{bmatrix}$$

■ 3. Solve for N.

$$-2\begin{bmatrix} 6 & 5 \\ 0 & 11 \end{bmatrix} = N - 4\begin{bmatrix} 2 & 4 \\ -1 & 9 \end{bmatrix}$$

■ 4. Solve the equation for M.

$$-4M = \begin{bmatrix} -5 & 0 & 4 \\ 1 & -8 & -2 \\ -4 & 12 & 3 \end{bmatrix}$$

■ 5. Use scalar multiplication to simplify the expression.

$$-5A+\frac{1}{3}B$$

$$A=\begin{bmatrix}\frac{2}{5} & -\frac{1}{5}\\ 3 & 0\end{bmatrix}$$

$$B=\begin{bmatrix}-\frac{1}{3} & 0\\ 6 & -2\end{bmatrix}$$

■ 6. Solve the equation for X.

$$2X-\frac{1}{2}\begin{bmatrix}0 & -2 & 6\\ 4 & -1 & 2\\ 8 & 6 & 0\end{bmatrix}=\begin{bmatrix}6 & 3 & 7\\ 0 & -\frac{3}{2} & 1\\ 6 & 5 & 4\end{bmatrix}$$

ZERO MATRICES

1. Add the zero matrix to the given matrix.

$$\begin{bmatrix} 8 & 17 \\ -6 & 0 \end{bmatrix}$$

2. Find the opposite matrix.

$$\begin{bmatrix} 6 & 8 & 0 \\ 2 & -3 & 11 \\ 4 & 12 & 9 \end{bmatrix}$$

3. Multiply the matrix by a scalar of 0.

$$\begin{bmatrix} 14 & -1 & 7 & 5 \\ 3 & 7 & 18 & -4 \end{bmatrix}$$

4. Add the opposite of A to A.

$$A = \begin{bmatrix} 1 & -5 & 7 \\ -3 & 2 & 8 \end{bmatrix}$$

5. Solve the equation for X.

$$X+\begin{bmatrix}-1 & 2 & 5\\ 7 & -4 & 3\\ 1 & -2 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & -5\\ 0 & 7 & 3\\ -4 & 0 & -1\end{bmatrix}+\begin{bmatrix}-1 & 0 & 5\\ 0 & -7 & -3\\ 4 & 0 & 1\end{bmatrix}$$

■ 6. Solve the equation for A.

$$\begin{bmatrix}-1 & 5 & 4\\ -2 & 0 & -3\\ 5 & 7 & -9\end{bmatrix}-A=0\begin{bmatrix}-2 & 3 & 0\\ -1 & 5 & -2\\ -7 & 0 & 4\end{bmatrix}+\begin{bmatrix}2 & 4 & -7\\ 8 & 0 & -5\\ -1 & 4 & 3\end{bmatrix}-\begin{bmatrix}3 & -1 & -11\\ 10 & 0 & -2\\ -6 & -3 & 12\end{bmatrix}$$

MATRIX MULTIPLICATION

1. If matrix A is 3×3 and matrix B is 3×4, say whether AB or BA is defined, and give the dimensions of any product that's defined.

2. Find the product of matrices A and B.

$$A = \begin{bmatrix} 2 & 6 \\ -3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 \\ 5 & -4 \end{bmatrix}$$

3. Find the product of matrices A and B.

$$A = \begin{bmatrix} 5 & -1 \\ 0 & 11 \\ 7 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 1 & 8 \\ -3 & 0 & 4 \end{bmatrix}$$

4. Find the product of matrices A and B.

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 8 \\ 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$$

■ 5. Use the distributive property to find $A(B+C)$.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 1 \\ 3 & -1 \end{bmatrix}$$

■ 6. Find the product of matrices A and B.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & -2 & 8 & 1 \\ 7 & 3 & 5 & 2 \end{bmatrix}$$

IDENTITY MATRICES

■ 1. Write the identity matrix I_4.

■ 2. If we want to find the product IA, where I is the identity matrix and A is 4×2, then what are the dimensions of I?

■ 3. If we want to find the product IA, where I is the identity matrix and A is a 3×4, then what are the dimensions of I?

■ 4. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

■ 5. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

■ 6. If A is a 2×4 matrix, what are the dimensions of the identity matrix that make the equation true?

$$AI = A$$

THE ELIMINATION MATRIX

■ 1. Find a single 2×2 elimination matrix E that accomplishes the given row operations.

1. $(1/3)R_1 \to R_1$
2. $-2R_1 + R_2 \to R_2$

■ 2. Find a single 3×3 elimination matrix E that accomplishes the given row operations.

1. $-3R_1 + R_3 \to R_3$
2. $5R_2 + R_1 \to R_1$
3. $-R_3 \to R_3$

■ 3. Find a single 2×2 elimination matrix E that accomplishes the given row operations.

1. $-R_1 \to R_1$
2. $5R_1 + R_2 \to R_2$
3. $-(1/7)R_2 \to R_2$

4. $R_2 + R_1 \to R_1$

■ 4. Find the single elimination matrix E that puts A into reduced row-echelon form, where E accounts for the given set of row operations.

$$A = \begin{bmatrix} -3 & 6 \\ 1 & 2 \end{bmatrix}$$

1. $-\frac{1}{3}R_1 \to R_1$

2. $-R_1 + R_2 \to R_2$

3. $\frac{1}{4}R_2 \to R_2$

4. $2R_2 + R_1 \to R_1$

■ 5. Find the single elimination matrix E that puts X into reduced row-echelon form, where E accounts for the given set of row operations.

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -4 \\ -2 & -1 & 5 \end{bmatrix}$$

1. $-3R_1 + R_2 \to R_2$

2. $2R_1 + R_3 \to R_3$

3. $R_2 + R_3 \to R_3$

4. $4R_3 + R_2 \to R_2$

■ 6. Find the single elimination matrix E that puts B into reduced row-echelon form, where E accounts for the given set of row operations.

$$B = \begin{bmatrix} 1 & 0 & -5 \\ 3 & 2 & -9 \\ 1 & -2 & -10 \end{bmatrix}$$

1. $-3R_1 + R_2 \to R_2$

2. $-R_1 + R_3 \to R_3$

3. $\frac{1}{2}R_2 \to R_2$

4. $2R_2 + R_3 \to R_3$

5. $-3R_3 + R_2 \to R_2$

6. $5R_3 + R_1 \to R_1$

VECTORS

■ 1. For the matrix A, find the row vectors, the space, $\mathbb{R}^n$, that contains the row vectors, and the dimension of the space they form.

$$A = \begin{bmatrix} -4 & 8 & 6 & 12 & -1 \\ 3 & -2 & 18 & 0 & -3 \\ 12 & -17 & -4 & 1 & 1 \end{bmatrix}$$

■ 2. For the matrix B, find the column vectors, the space, $\mathbb{R}^n$, that contains the column vectors, and the dimension of the space they form.

$$B = \begin{bmatrix} 12 & 0 & 9 \\ 3 & -21 & -1 \\ -7 & 4 & 13 \end{bmatrix}$$

■ 3. Sketch the vectors in standard position.

$$\overrightarrow{w} = (1,2),\ \overrightarrow{x} = (-5,0),\ \overrightarrow{y} = (-3,-4),\ \vec{z} = (0,-1)$$

■ 4. Sketch the vectors in order from tip to tail (where the terminal point of one is the initial point of the next), starting at the origin, and determine the shape they form.

$$\overrightarrow{a}_1 = (1,2) \qquad \overrightarrow{a}_3 = (1,-2) \qquad \overrightarrow{a}_5 = (-2,0)$$

$\vec{a}_2 = (2,0)$ $\qquad$ $\vec{a}_4 = (-1, -2)$ $\qquad$ $\vec{a}_6 = (-1,2)$

■ 5. Find $\vec{b}_1 + \vec{b}_2$, $\vec{b}_1 - \vec{b}_2$, and $2\vec{b}_2$.

$$\vec{b}_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}$$

■ 6. Is the product of $\vec{b}_1$ and $\vec{b}_2$ defined? Why or why not?

$$\vec{b}_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}$$

VECTOR OPERATIONS

1. Find $\vec{u}+\vec{w}$, $\vec{x}-\vec{y}$, and $\vec{v}-(\vec{w}+\vec{u})$.

$\vec{u}=(-3,5)$ $\quad$ $\vec{w}=(5,-13)$ $\quad$ $\vec{y}=(1,4,2)$

$\vec{v}=(2,1)$ $\quad$ $\vec{x}=(4,5,-7)$

2. Sketch $\vec{u}+\vec{w}$, $\vec{x}-\vec{y}$, and $\vec{v}-(\vec{w}+\vec{u})$.

$\vec{u}=(-3,5)$ $\quad$ $\vec{w}=(5,-13)$ $\quad$ $\vec{y}=(1,4,2)$

$\vec{v}=(2,1)$ $\quad$ $\vec{x}=(4,5,-7)$

3. Find $b\vec{x}$, $c\vec{u}+b\vec{u}$, and $(c+b)\vec{u}$. What can you say about the relationship between $c\vec{u}+b\vec{u}$ and $(c+b)\vec{u}$.

$\vec{u}=(-3,5)$ $\quad$ $b=-1$

$\vec{x}=(4,5,-7)$ $\quad$ $c=3$

4. Find $\vec{x}+b\vec{y}-c\vec{x}-\vec{y}$.

$\vec{x}=(4,5,-7)$ $\quad$ $b=-1$

$\vec{y}=(1,4,2)$ $\quad$ $c=3$

■ 5. Sketch the individual vectors from tip to tail.

$\vec{x} = (4,5,-7)$

$\vec{y} = (1,4,2)$

■ 6. Find $\vec{x} \cdot \vec{y}$, $\vec{w} \cdot \vec{w}$, and $b(\vec{u} \cdot \vec{v})$.

$\vec{u} = (-3,5)$ $\quad$ $\vec{w} = (5,-13)$ $\quad$ $\vec{y} = (1,4,2)$

$\vec{v} = (2,1)$ $\quad$ $\vec{x} = (4,5,-7)$ $\quad$ $b = -1$

UNIT VECTORS AND BASIS VECTORS

■ 1. Change each vector to a unit vector.

$$\vec{a} = (3, -4)$$

$$\vec{b} = (12,2)$$

$$\vec{c} = (0,7,1)$$

■ 2. Confirm that the vectors each have length 1.

$$\hat{u}_a = \frac{1}{5}\begin{bmatrix}3\\-4\end{bmatrix}$$

$$\hat{u}_b = \frac{1}{\sqrt{37}}\begin{bmatrix}6\\1\end{bmatrix}$$

$$\hat{u}_c = \frac{1}{5\sqrt{2}}\begin{bmatrix}0\\7\\1\end{bmatrix}$$

■ 3. What are the basis vectors for $\mathbb{R}^4$?

■ 4. Express the vectors as linear combinations of the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$.

$$\hat{u}_a = \frac{1}{5}\begin{bmatrix}3\\-4\end{bmatrix}$$

$$\hat{u}_b = \frac{1}{\sqrt{37}}\begin{bmatrix}6\\1\end{bmatrix}$$

$$\hat{u}_c = \frac{1}{5\sqrt{2}}\begin{bmatrix}0\\7\\1\end{bmatrix}$$

5. Express $\vec{v} = (x, 2x, -1)$ in terms of the standard basis vectors.

6. Sketch the basis vectors $\hat{i}$ and $\hat{j}$ in $\mathbb{R}^2$, and the vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ in $\mathbb{R}^3$.

LINEAR COMBINATIONS AND SPAN

1. Say whether each of the following is a linear combination. If it isn't, say why.

$$-\pi\vec{x} - e\vec{y}$$

$$\vec{x} \cdot \vec{y}$$

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{u} = \frac{1}{\sqrt{2}}((3,0) - (1,1))$$

$$||\vec{b}||$$

2. Do the vectors span $\mathbb{R}^4$?

$$\left\{ \begin{bmatrix} 3 \\ \frac{1}{2} \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -\pi \\ \pi \\ \pi \\ -\pi \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ 8 \\ 22 \\ 9 \end{bmatrix} \right\}$$

3. Do the vectors span $\mathbb{R}^4$?

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

■ 4. Do the vectors span $\mathbb{R}^2$?

$$\left\{\begin{bmatrix}44\\-8\end{bmatrix}, \begin{bmatrix}11\\-2\end{bmatrix}\right\}$$

■ 5. What is the zero vector $\overrightarrow{O}$ in $\mathbb{R}^5$? What is its span?

■ 6. Prove that any vector $\overrightarrow{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ can be reached by a linear combination of $\hat{i}$, $\hat{j}$, and $\hat{k}$.

LINEAR INDEPENDENCE IN TWO DIMENSIONS

■ 1. Are the column vectors of the following matrix linearly independent?

$$A = \begin{bmatrix} 2 & 6 & 7 \\ -1 & 11 & 3 \end{bmatrix}$$

■ 2. Show how one of the vectors could be written as a linear combination of the other two.

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{z} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

■ 3. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

■ 4. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{a} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

■ 5. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$$

■ 6. Use a matrix to say whether the vectors are linearly dependent or linearly independent.

$$\vec{x} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \vec{y} = \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$$

LINEAR INDEPENDENCE IN THREE DIMENSIONS

1. Use a matrix to say whether the vector set is linearly independent.

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$$

2. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$$

3. Use a matrix to say whether the vector set is linearly independent.

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

4. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

5. Is the vector set linearly independent? Why or why not?

$$\vec{u}=\begin{bmatrix}1\\4\\5\end{bmatrix}, \vec{v}=\begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{w}=\begin{bmatrix}3\\6\\8\end{bmatrix}, \vec{x}=\begin{bmatrix}-2\\7\\1\end{bmatrix}$$

6. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{u}=\begin{bmatrix}1\\4\\5\end{bmatrix}, \vec{v}=\begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{w}=\begin{bmatrix}3\\6\\8\end{bmatrix}, \vec{x}=\begin{bmatrix}-2\\7\\1\end{bmatrix}$$

LINEAR SUBSPACES

1. What are the criteria that define a subspace? Which criteria is logically part of another criteria?

2. Sketch the graph of each space.

$$V_a = \left\{(x, y) \in \mathbb{R}^2 \mid x, y \leq -1\right\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid y < x^2\right\}$$

$$V_c = \left\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, y \leq x\right\}$$

3. What space is being described by each of the sets?

$$V_a = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0\right\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0,\ x = y\right\}$$

$$V_c = \left\{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\right\}$$

4. Are these spaces subspaces?

$$V_a = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0\right\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0,\ x = y\right\}$$

$$V_c = \left\{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\right\}$$

■ 5. Show that each space is not a subspace.

$$V_a = \left\{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - 7z = 3\right\}$$

$$V_b = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \leq 0\right\}$$

$$V_c = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0 \right\}$$

■ 6. Prove that the zero vector $\overrightarrow{O} = (0,0,0)$ is a subspace of $\mathbb{R}^3$.

SPANS AS SUBSPACES

1. Sketch the spans together on the same set of axes.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)$$

$$V = \text{Span}\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

2. Show that that spans are subspaces.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)$$

$$V = \text{Span}\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

3. Prove that the span forms a subspace of $\mathbb{R}^3$.

$$\text{Span}\left(\begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix}\right)$$

4. Write the line $y = 3x + 2$ in set notation, and then write it as a single vector, only using x.

■ 5. Write the line $2y + 4x = 0$ in set notation, and then write it as a single vector, only using y.

■ 6. Write the line $y = 3x + 2$ as the linear combination of two vectors. Then plug in $x = -1$, $x = 0$, and $x = 1$, and sketch all three in the same plane.

BASIS

■ 1. What requirements must be met in order for a vector set to form the basis for a space?

■ 2. What's the standard basis for $\mathbb{R}^4$?

■ 3. Say whether or not the vector set V forms a basis for $\mathbb{R}^2$.

$$V = \text{Span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$$

■ 4. Which scalars c_1 and c_2 would you need to form the vector $\overrightarrow{v} = (7, -3)$ as a linear combination of the vectors in the span?

$$V = \text{Span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$$

■ 5. Say whether the span forms a basis for $\mathbb{R}^3$.

$$V = \text{Span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\4\end{bmatrix}\right)$$

■ 6. What scalars would you need to get the vector $\vec{v}=(2,0,-5)$ from a linear combination of the set V?

$$V=\text{Span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\1\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\4\end{bmatrix}\right)$$

DOT PRODUCTS

■ 1. Find the dot product.

$$\vec{a} = (-2,5)$$

$$\vec{b} = (3,4)$$

■ 2. Find the dot product.

$$\vec{x} = (1, -2,0)$$

$$\vec{y} = (5, -1, -3)$$

■ 3. Use the dot product to find the length of the vector $\vec{u} = (-5,2, -4, -2)$.

■ 4. Simplify the expression if $\vec{x} = (-2,4)$, $\vec{y} = (0, -1)$, and $\vec{z} = (4,7)$.

$$4\vec{x} \cdot (3\vec{y} - \vec{z})$$

■ 5. Use the dot product to find $-\vec{a} \cdot (5\vec{b} + 3\vec{c})$.

$$\vec{a} = (-2,0,4)$$

$$\vec{b} = (1,5,3)$$

$\vec{c} = (-1, -4, 0)$

■ 6. Use the dot product to find $\vec{w}(2\vec{x} + \vec{y}) - 3\vec{y}(\vec{w} + 4\vec{x} - \vec{z})$.

$\vec{x} = (4, -3, 0, 7)$

$\vec{y} = (-1, 5, 2, -1)$

$\vec{z} = (0, 6, -1, 9)$

$\vec{w} = (1, 0, 5, 0)$

CAUCHY-SCHWARZ INEQUALITY

■ 1. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (-1,2)$$

$$\vec{v} = (-5,10)$$

■ 2. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (-5,2)$$

$$\vec{v} = (3,-7)$$

■ 3. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (-2,4,0)$$

$$\vec{v} = (1,-5,3)$$

■ 4. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\overrightarrow{u} = (6,3,6)$$

$$\overrightarrow{v} = (-2,-1,-2)$$

5. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\overrightarrow{u} = (-13,5,7)$$

$$\overrightarrow{v} = (1,-1,-1)$$

6. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\overrightarrow{u} = (-2,0,2)$$

$$\overrightarrow{v} = (8,0,-8)$$

VECTOR TRIANGLE INEQUALITY

■ 1. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u}=(\sqrt{3},3) \text{ and } \vec{v}=(2\sqrt{3},0)$$

■ 2. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ span $\mathbb{R}^2$.

$$\vec{u}=(5,-7) \text{ and } \vec{v}=(-4,-3)$$

■ 3. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u}=(-2,5) \text{ and } \vec{v}=(2,-5)$$

■ 4. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u}=(-3,12,-15) \text{ and } \vec{v}=(-1,4,-5)$$

■ 5. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$\vec{u}=(1,2,0)$ and $\vec{v}=(-5,1,-6)$

■ 6. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$\vec{u}=(2,-5,4)$ and $\vec{v}=(6,-15,12)$

ANGLE BETWEEN VECTORS

1. Say whether or not the vectors are orthogonal.

$$\vec{a} = (-1,3)$$

$$\vec{b} = (6,2)$$

2. Say whether or not the vectors are orthogonal.

$$\vec{u} = 2i - j + 3k$$

$$\vec{v} = -i - 3j + 2k$$

3. Find the angle between the vectors.

$$\vec{x} = (0,2)$$

$$\vec{y} = (1,1)$$

4. Find the angle between the vectors.

$$\vec{a} = (-5,7,3)$$

$$\vec{b} = (1,2,-3)$$

5. Find the angle between the vectors.

$$\vec{a} = (-1,3,-4)$$

$$\vec{b} = (2,1,0)$$

6. Find the angle between the vectors.

$$\vec{a} = (1,-2,5)$$

$$\vec{b} = (8,6,3)$$

EQUATION OF A PLANE, AND NORMAL VECTORS

■ 1. What is the normal vector to the plane?

$$-2x+5y-7z=0$$

■ 2. What is the normal vector to the plane?

$$10y-5z+6=0$$

■ 3. Find the equation of a plane with normal vector $\overrightarrow{n}=(-1,0,4)$ that passes through $(1,-3,0)$.

■ 4. Find the equation of a plane with normal vector $\overrightarrow{n}=(4,-7,3)$ that passes through $(-2,1,6)$.

■ 5. Find the equation of a plane with normal vector $\overrightarrow{n}=-3i+4j-z$ that passes through $(-2,0,-7)$.

■ 6. Find the equation of the plane passing through P and perpendicular to $\overrightarrow{PQ}$.

$P(1,-5,4)$

$Q(0,3,-1)$

CROSS PRODUCTS

■ 1. Find the cross product of $\overrightarrow{a}=(1,-3,-1)$ and $\overrightarrow{b}=(5,6,-2)$.

■ 2. Find a vector orthogonal to both $\overrightarrow{a}=(-3,-5,2)$ and $\overrightarrow{b}=(-2,4,-7)$.

■ 3. Find the length of the cross product of $\overrightarrow{a}=(-1,-2,0)$ and $\overrightarrow{b}=(1,1,-2)$.

■ 4. Find the length of the cross product of $\overrightarrow{a}=(6,-3,3)$ and $\overrightarrow{b}=(3,0,3)$ when the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is $\theta=30^\circ$.

■ 5. Find the length of the cross product of the vectors $\overrightarrow{a}=(2,-5,3)$ and $\overrightarrow{b}=(4,6,-1)$, and find the sine of the angle between them.

■ 6. Find the angle between the vectors $\overrightarrow{a}=(2,-2,1)$ and $\overrightarrow{b}=(1,0,1)$, and find the length of their cross product.

DOT AND CROSS PRODUCTS AS OPPOSITE IDEAS

■ 1. Find the maximum value of the dot product, if $||\overrightarrow{u}|| = 4$ and $||\overrightarrow{v}|| = 5$.

■ 2. Find the minimum value of the dot product of two vectors, if $||\overrightarrow{u}|| = \sqrt{56}$ and $||\overrightarrow{v}|| = \sqrt{126}$.

■ 3. Find the maximum value of the length of the cross product of $\overrightarrow{u}$ and $\overrightarrow{v}$, if $||\overrightarrow{u}|| = \sqrt{50}$ and $||\overrightarrow{v}|| = \sqrt{128}$.

■ 4. Find the dot product and the length of the cross product of $\overrightarrow{u} = (2,1)$ and $\overrightarrow{v} = (-6, -3)$. Then interpret the results based on what the dot and cross products indicate.

■ 5. Find the dot product and the length of the cross product of $\overrightarrow{u} = (2, -3, -1)$ and $\overrightarrow{v} = (4, -6, -2)$. Then interpret the results based on what the dot and cross products indicate.

■ 6. Find the dot product and the length of the cross product of $\overrightarrow{u} = (-2,4,3)$ and $\overrightarrow{v} = (2,1,0)$. Then interpret the results based on what the dot and cross products indicate.

MULTIPLYING MATRICES BY VECTORS

1. Find the matrix-vector product, $A\vec{x}$.

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\vec{x} = (4, -1)$$

2. Find the matrix-vector product, $\vec{x}A$.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\vec{x} = (-2,3)$$

3. Find the matrix-vector product, $A\vec{x}$.

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 0 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\vec{x} = (2,0,1)$$

4. Find the matrix-vector product, $\vec{x}A$.

$$A = \begin{bmatrix} 1 & -1 & 0 & -2 \\ -3 & 0 & -2 & 1 \end{bmatrix}$$

$$\vec{x} = (2, -6)$$

■ 5. Find the matrix-vector product, $A\vec{x}$.

$$A = \begin{bmatrix} 4 & 6 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{x} = (3,3)$$

■ 6. Find the matrix-vector product, $\vec{x}A$.

$$A = \begin{bmatrix} 6 & -4 & -4 \\ 1 & -4 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{x} = (-3,1,1)$$

THE NULL SPACE AND AX=O

■ 1. Is $\vec{x}=(1,2)$ in the null space of A?

$$A=\begin{bmatrix}4 & -2\\ 2 & -1\end{bmatrix}$$

■ 2. Is $\vec{x}=(5,-8,-9)$ in the null space of A?

$$A=\begin{bmatrix}6 & 1 & 1\\ 0 & -2 & 3\\ -1 & 0 & 4\end{bmatrix}$$

■ 3. Is $\vec{x}=(1,1,1)$ in the null space of A?

$$A=\begin{bmatrix}2 & -3 & 1\\ 1 & 4 & -5\\ 1 & -6 & 5\end{bmatrix}$$

■ 4. Is $\vec{x}=(4,-2)$ in the null space of A?

$$A=\begin{bmatrix}1 & 2\\ -1 & -2\\ 2 & 4\\ -2 & -4\end{bmatrix}$$

5. Is $\vec{x} = (1,1,2,1)$ in the null space of A?

$$A = \begin{bmatrix} 1 & -7 & 3 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

6. Is $\vec{x} = (-1, -3, 1)$ in the null space of A?

$$A = \begin{bmatrix} -4 & 3 & 5 \\ 3 & 1 & 6 \\ 0 & -2 & -6 \end{bmatrix}$$

NULL SPACE OF A MATRIX

1. Find the null space of A.

$$A = \begin{bmatrix} 4 & -3 \\ 0 & 4 \end{bmatrix}$$

2. Find the null space of A.

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 5 & 1 & -6 \\ 1 & 4 & -5 \end{bmatrix}$$

3. Find the null space of A.

$$A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \\ 9 & -3 \\ 0 & 0 \end{bmatrix}$$

4. Find the null space of A.

$$A = \begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

5. Find the null space of A.

$$A = \begin{bmatrix} 4 & -2 & 1 & 1 \\ -1 & 0 & 3 & -3 \\ 0 & 0 & -4 & 6 \end{bmatrix}$$

6. Find the null space of A.

$$A = \begin{bmatrix} -2 & 0 & 7 \\ 3 & -1 & 4 \\ 0 & 3 & -2 \\ 1 & 4 & -5 \\ 2 & 2 & 1 \end{bmatrix}$$

THE COLUMN SPACE AND AX=B

1. Find the column space of A.

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 5 & 1 & -6 \\ 1 & 4 & -5 \end{bmatrix}$$

2. Find the column space of A.

$$A = \begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

3. Find a basis for the column space of A.

$$A = \begin{bmatrix} 4 & -3 \\ 0 & 4 \end{bmatrix}$$

4. Find a basis for the column space of A.

$$A = \begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

5. Find a basis for the column space of A.

$$A = \begin{bmatrix} 5 & -2 & 6 \\ -3 & 1 & 0 \\ 0 & -1 & -4 \\ 8 & 2 & 2 \end{bmatrix}$$

6. Find a basis for the column space of A.

$$A = \begin{bmatrix} 2 & -4 & 3 & -6 \\ 1 & -2 & 0 & 0 \\ 4 & -8 & 5 & -10 \end{bmatrix}$$

SOLVING AX=B

■ 1. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}2 & -4 & 3 & -6\\ 1 & -2 & 0 & 0\\ 4 & -8 & 5 & -10\end{bmatrix}$$

■ 2. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}3 & 6\\ 6 & 12\\ 1 & 1\\ 2 & 2\end{bmatrix}$$

■ 3. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}1 & -5 & 3\\ -1 & 4 & 0\\ 3 & -16 & 12\end{bmatrix}$$

■ 4. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}-2 & 10 & -6 & 2\\ 1 & -5 & 3 & -1\end{bmatrix}$$

■ 5. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix} 2 & 0 & 0 & 12 \\ -1 & 2 & -1 & 4 \\ 5 & -6 & 3 & 0 \end{bmatrix}$$

■ 6. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix} 1 & 0 & 3 & -5 \\ 4 & -2 & 2 & 0 \\ -1 & 2 & -1 & 1 \\ 3 & 2 & -1 & 5 \end{bmatrix}$$

DIMENSIONALITY, NULLITY, AND RANK

■ 1. Find the nullity of A.

$$A = \begin{bmatrix} 1 & -3 & 2 & -1 \\ 3 & -7 & 0 & 1 \end{bmatrix}$$

■ 2. Find the rank of X.

$$X = \begin{bmatrix} -2 & 3 & 1 \\ -1 & 0 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

■ 3. Find the nullity and the rank of A.

$$A = \begin{bmatrix} -1 & -3 & 2 & 4 & -2 \\ -3 & -5 & -2 & 1 & 4 \\ 0 & 4 & -8 & -11 & 10 \\ 1 & 3 & -2 & -4 & 5 \end{bmatrix}$$

■ 4. Find the nullity of M.

$$M = \begin{bmatrix} -4 & 2 & -2 & 1 \\ -1 & 0 & -3 & 2 \\ 3 & -2 & 5 & 0 \end{bmatrix}$$

■ 5. Find the rank of M.

$$M = \begin{bmatrix} -2 & 0 & -5 & 6 & 2 \\ 1 & -1 & 3 & 0 & 5 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix}$$

■ 6. Find the nullity and the rank of M.

$$M = \begin{bmatrix} -1 & 2 & 0 & 3 \\ -2 & 0 & -1 & 2 \\ 3 & -2 & 0 & -4 \\ 1 & -4 & 2 & 0 \end{bmatrix}$$

FUNCTIONS AND TRANSFORMATIONS

■ 1. The transformation T maps every vector in $\mathbb{R}^4$ to $\vec{O}=(0,0,0)$. What are the domain, codomain, and range of T?

■ 2. The transformation T maps every vector in $\mathbb{R}^3$ to every vector in $\mathbb{R}^2$. What are the domain, codomain, and range of T?

■ 3. The transformation T maps every vector in $\mathbb{R}^3$ to the zero vector $\vec{O}$ in $\mathbb{R}^3$. What are the domain, codomain, and range of T?

■ 4. The transformation T maps $\vec{a}=(-1,0,3)$ to $\vec{b}=(-2,1,-2)$. What are the domain, codomain, and range of T?

■ 5. The transformation T maps $\vec{a}=(-2,0)$ to every vector in $\mathbb{R}^4$. What are the domain, codomain, and range of T?

■ 6. The transformation T maps $\vec{a}=(-2,-3,1)$ to the zero vector $\vec{O}$ in $\mathbb{R}^3$. What are the domain, codomain, and range of T?

TRANSFORMATION MATRICES AND THE IMAGE OF THE SUBSET

■ 1. Find the resulting vector $\overrightarrow{b}$ after $\overrightarrow{a}=(1,6)$ undergoes a transformation by matrix M.

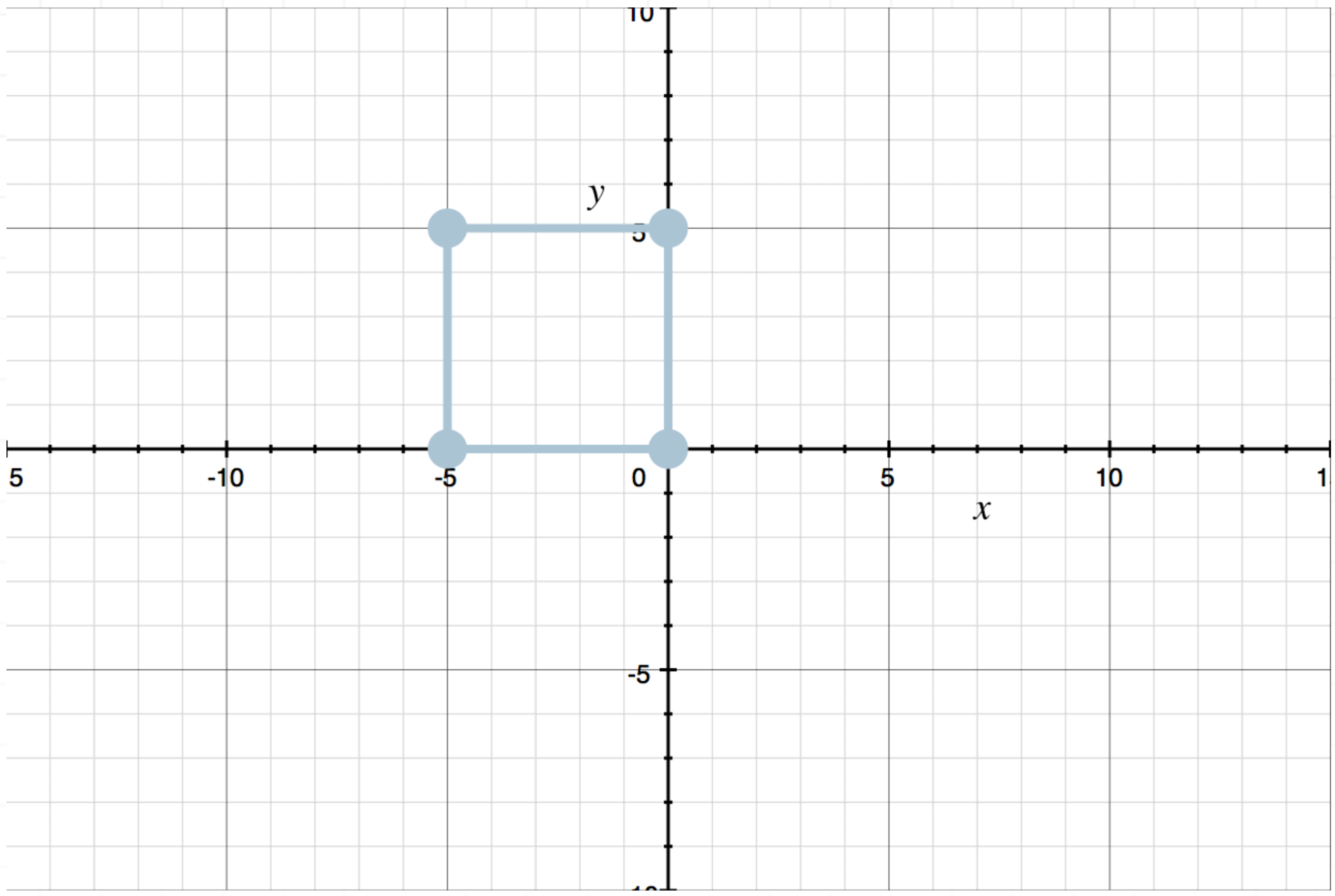
$$M=\begin{bmatrix}-7 & 1\\ 0 & -2\end{bmatrix}$$

■ 2. Sketch triangle $\triangle ABC$ with vertices $(2,3)$, $(-3,-1)$, and $(1,-4)$, and the transformation of $\triangle ABC$ after it's transformed by matrix L.

$$L=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$

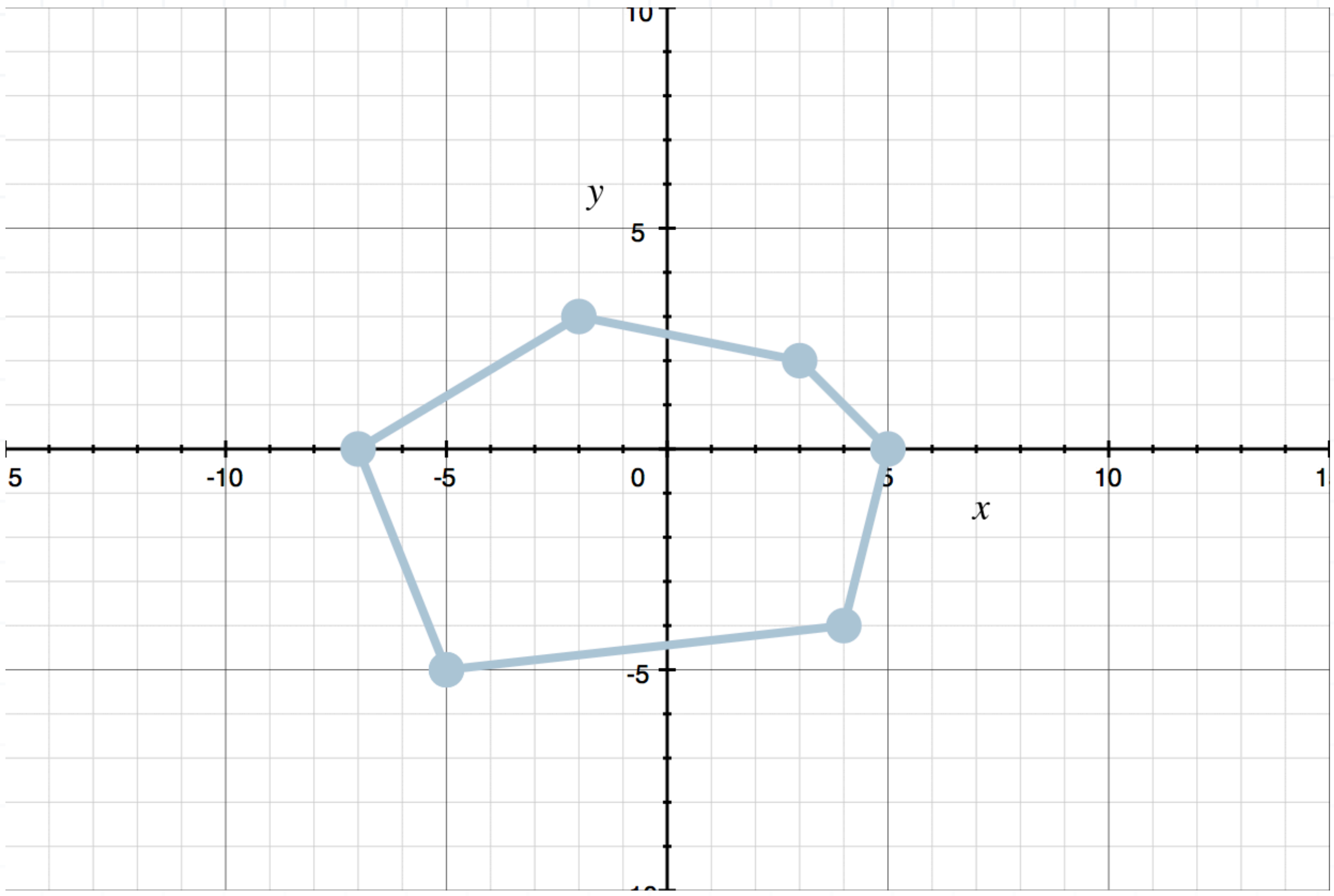
■ 3. Sketch the transformation of the square in the graph after it's transformed by matrix Z.

$$Z=\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}$$



■ 4. Sketch the transformation of the hexagon after it's transformed by matrix Y.

$$Y = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$



5. What happens to the unit vector $\vec{a}=(1,0)$ after the transformation given by matrix K.

$$K=\begin{bmatrix}3 & -5\\ -1 & 0\end{bmatrix}$$

6. What happens to the unit vector $\vec{b}=(0,1)$ after the transformation given by matrix K.

$$K=\begin{bmatrix}3 & -5\\ -1 & 0\end{bmatrix}$$

PREIMAGE, IMAGE, AND THE KERNEL

■ 1. Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 2. Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -1 & -3 & 2 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 3. Find the kernel of the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

$$T(\vec{x}) = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 4. Find the kernel of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$.

$$T(\vec{x}) = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 5. Find the preimage A_1 of the subset B_1 under the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.

$$B_1 = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 6. Find the preimage A_1 of the subset B_1 under the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.

$$B_1 = \left\{ \begin{bmatrix} -4 \\ 4 \\ -12 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 & -6 \\ 1 & -3 & 0 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

LINEAR TRANSFORMATIONS AS MATRIX-VECTOR PRODUCTS

■ 1. Use a matrix-vector product to reflect the square with vertices $(3,-1)$, $(1,-1)$, $(1,1)$, and $(3,1)$ over the y-axis. What are the vertices of the reflected square? Graph the resulting figure.

■ 2. Use a matrix-vector product to transform the triangle with vertices $(-2,-3)$, $(4,2)$, and $(2,-5)$. The transformation T should include a reflection over the x-axis and horizontal stretch by a factor of 5. Graph the resulting figure.

■ 3. Use a matrix-vector product to reflect the triangle with vertices $(3,3)$, $(1,-2)$, and $(-3,3)$ over the line $y=x$. What are the vertices of the reflected triangle? Graph the resulting figure.

■ 4. Use a matrix-vector product to triple the width of the rectangle that has vertices $(-2,-1)$, $(-2,-5)$, $(5,-1)$, and $(5,-5)$, and then compress it vertically by a factor of 2. What are the vertices of the transformed rectangle? Graph the resulting figure.

■ 5. Use a matrix-vector product to reflect the parallelogram with vertices $(-3,1)$, $(0,4)$, $(7,4)$, and $(4,1)$ over the x-axis, and then over the y-axis. What are the vertices of the reflected parallelogram? Graph the resulting figure.

■ 6. Use a matrix-vector product to reflect the triangle with vertices $(2,3)$, $(-5,-4)$, and $(-4,5)$ over the y-axis, and then stretch it horizontally by a factor of 4. What are the vertices of the transformed triangle? Graph the resulting figure.

LINEAR TRANSFORMATIONS AS ROTATIONS

■ 1. Find the rotation of $\vec{x} = (2,-4)$ by an angle of $\theta = 120^\circ$.

■ 2. Find the rotation of $\vec{x} = (1,-5)$ by an angle of $\theta = 60^\circ$.

■ 3. Find the rotation of $\vec{x} = (-7,4)$ by an angle of $\theta = 180^\circ$.

■ 4. Find the rotation of $\vec{x} = (-4,1,3)$ by an angle of $\theta = 90^\circ$ about the x-axis.

■ 5. Find the rotation of $\vec{x} = (-2/\sqrt{2},2,0)$ by an angle of $\theta = 315^\circ$ about the y-axis.

■ 6. Find the rotation of $\vec{x} = (-2,0,3)$ by an angle of $\theta = 150^\circ$ about the z-axis.

ADDING AND SCALING LINEAR TRANSFORMATIONS

■ 1. Find the product of a scalar $c = 5$ and the transformation $T(\vec{x})$.

$$T(\vec{x}) = \begin{bmatrix} 0 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 2. Find the product of a scalar $c = -2$ and the transformation $T(\vec{x})$.

$$T(\vec{x}) = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -5 & 7 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 3. Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 4. Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} 0 & -4 & 1 \\ 1 & -1 & 3 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(\vec{x})=\begin{bmatrix}5 & -3 & 3\\ 2 & 0 & -1\\ 1 & -4 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

■ 5. Find the sum of the transformation $S(\vec{x})$ and the product of a scalar $c=-1/2$ and $T(\vec{x})$.

$$S(\vec{x})=\begin{bmatrix}-5 & 0\\ -2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$T(\vec{x})=\begin{bmatrix}-4 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

■ 6. Find the product of $c=1/3$ with the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x})=\begin{bmatrix}-5 & 4 & -3\\ 0 & 1 & -2\\ 0 & -1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$T(\vec{x})=\begin{bmatrix}-1 & 2 & 0\\ 6 & -4 & 5\\ 9 & 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

PROJECTIONS AS LINEAR TRANSFORMATIONS

1. Find the projection of $\vec{v}$ onto L.

$$L = \left\{ c \begin{bmatrix} 4 \\ 2 \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

2. Find the projection of $\vec{v}$ onto M.

$$M = \left\{ c \begin{bmatrix} -4 \\ 3 \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

3. Find the projection of $\vec{v}$ onto L and the vector complement of $\vec{v}$ orthogonal to L.

$$L = \left\{ c \begin{bmatrix} -3 \\ 1 \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

■ 4. Find the projection of $\vec{v}$ onto L and the vector complement of $\vec{v}$ orthogonal to L.

$$L = \left\{ c \begin{bmatrix} -2 \\ 0 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

■ 5. Find the projection of $\vec{v}$ onto L.

$$L = \left\{ c \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

■ 6. Find the projection of $\vec{v}$ onto L and the vector complement of $\vec{v}$ orthogonal to L.

$$L = \left\{ c \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

COMPOSITIONS OF LINEAR TRANSFORMATIONS

1. If $S: \mathbb{R}^2 \to \mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$, then what is $T(S(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -x_2 + 3x_1 \\ x_1 + 2x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

2. If $S: \mathbb{R}^2 \to \mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$, then what is $S(T(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -2x_1 + x_2 \\ -x_1 - x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -x_1 + 3x_2 \\ -2x_1 + 2x_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

3. If $S: \mathbb{R}^3 \to \mathbb{R}^3$ and $T: \mathbb{R}^3 \to \mathbb{R}^3$, then what is $T(S(\vec{x}))$?

$$S(\overrightarrow{x}) = \begin{bmatrix} -2x_1 + x_2 - x_3 \\ -x_2 + x_3 \\ x_1 + 2x_2 \end{bmatrix}$$

$$T(\overrightarrow{x}) = \begin{bmatrix} -3x_3 \\ x_1 + x_2 - 2x_3 \\ -x_1 - x_2 + x_3 \end{bmatrix}$$

$$\overrightarrow{x} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

■ 4. If $S : \mathbb{R}^2 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^2$, then what is $S(T(\overrightarrow{x}))$?

$$S(\overrightarrow{x}) = \begin{bmatrix} -x_1 \\ x_1 - 3x_2 \\ 2x_2 - 3x_1 \end{bmatrix}$$

$$T(\overrightarrow{x}) = \begin{bmatrix} 2x_1 - x_3 \\ x_2 - x_1 + x_3 \end{bmatrix}$$

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

■ 5. If $S : \mathbb{R}^2 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^2$, then what is $T(S(\overrightarrow{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -x_1 + x_2 \\ 2x_2 - 3x_1 \\ x_1 + 2x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_1 + x_2 - x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

■ 6. If $S: \mathbb{R}^3 \to \mathbb{R}^3$ and $T: \mathbb{R}^3 \to \mathbb{R}^3$, then what are $T(S(\vec{x}))$ and $S(T(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -x_3 + 2x_2 \\ x_1 - x_3 \\ x_1 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -2x_1 + x_2 + 2x_3 \\ 3x_1 \\ x_1 - 2x_2 + x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

INVERSE OF A TRANSFORMATION

■ 1. Given a vector $\vec{v}$ in $\mathbb{R}^3$, what would the identity transformation be?

■ 2. If a transformation T is invertible, what are the three conclusions that we can make about it?

■ 3. If you can prove that a transformation T is both injective and surjective, and if you know that its inverse is unique, then what can you say about the transformation?

■ 4. Is the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ invertible?

$$T(x) = x^2$$

■ 5. Prove that $(T^{-1})^{-1} = T$.

■ 6. Prove that the inverse of a transformation is unique.

INVERTIBILITY FROM THE MATRIX-VECTOR PRODUCT

■ 1. Is the matrix invertible?

$$\begin{bmatrix} 1 & 2 & 0 \\ -3 & 5 & -1 \end{bmatrix}$$

■ 2. Is the matrix invertible?

$$\begin{bmatrix} \pi & -\pi \\ -\pi & \pi \end{bmatrix}$$

■ 3. Is the matrix invertible?

$$\begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix}$$

■ 4. Find the dimensions of the transformation matrix for each transformation, if each transformation were written as a matrix-vector product, $T(\vec{x}) = M\vec{x}$.

$$T : \mathbb{R}^3 \to \mathbb{R}^6$$

$$T : \mathbb{R}^4 \to \mathbb{R}^2$$

$$T : \mathbb{R}^u \to \mathbb{R}^w$$

■ 5. Using the transformations from the previous question, state the dimensions of $\vec{x}$, and then state the dimensions of $T(\vec{x})$.

■ 6. What can we say about the invertibility of the transformation $T : \mathbb{R}^u \to \mathbb{R}^w$ from the last two questions?

INVERSE TRANSFORMATIONS ARE LINEAR

■ 1. Given two $n \times n$ matrices, A and B, if we know that $AB = I$ and $BA = I$, where I is the $n \times n$ identity matrix, then what else do we know about A and B?

■ 2. Find the inverse of the matrix.

$$\begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix}$$

■ 3. Prove that the matrix found in the previous question is actually the inverse of the original matrix.

■ 4. Find the inverse of the matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{bmatrix}$$

■ 5. Prove that the matrix we found in the previous question is actually the inverse of the original matrix.

6. Prove that the inverse of an invertible linear transformation T is also a linear transformation.

MATRIX INVERSES, AND INVERTIBLE AND SINGULAR MATRICES

■ 1. Find the inverse of matrix G.

$$G = \begin{bmatrix} -3 & 8 \\ 0 & -2 \end{bmatrix}$$

■ 2. Find the inverse of matrix N.

$$N = \begin{bmatrix} 11 & -4 \\ 5 & -3 \end{bmatrix}$$

■ 3. What is the inverse of matrix K?

$$K = \begin{bmatrix} 3 & 3 \\ -6 & 0 \end{bmatrix}$$

■ 4. Is the matrix invertible or singular?

$$Z = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$$

■ 5. Is the matrix invertible or singular?

$$Y = \begin{bmatrix} 0 & 6 \\ 2 & -1 \end{bmatrix}$$

■ 6. Is B invertible?

$$B = \begin{bmatrix} -4 & 1 \\ -5 & 0 \end{bmatrix}$$

SOLVING SYSTEMS WITH INVERSE MATRICES

■ 1. Use an inverse matrix to solve the system.

$$-4x + 3y = -14$$

$$7x - 4y = 32$$

■ 2. Use an inverse matrix to solve the system.

$$6x - 11y = 2$$

$$-10x + 7y = -26$$

■ 3. Use an inverse matrix to solve the system.

$$13y - 6x = -81$$

$$7x + 17 = -22y$$

■ 4. Sketch a graph of vectors to visually find the solution to the system.

$$3x = 3$$

$$x - y = -2$$

5. Sketch a graph of vectors to visually find the solution to the system.

$$-y = -4$$

$$2x - y = -2$$

6. Sketch a graph of vectors to visually find the solution to the system.

$$x - y = 0$$

$$x + y = 2$$

DETERMINANTS

■ 1. Use the determinant to say whether the matrix A is invertible.

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 3 \end{bmatrix}$$

■ 2. Use the determinant to say whether the matrix A is invertible.

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

■ 3. Use the determinant to say whether the matrix A is invertible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \\ 4 & -2 & 0 \end{bmatrix}$$

■ 4. Use the determinant to say whether matrix A is invertible.

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

■ 5. Use the Rule of Sarrus to find the determinant.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 0 & -2 & 3 \end{bmatrix}$$

■ 6. Use the Rule of Sarrus to find the determinant.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -2 & -3 \\ 3 & 2 & 1 \end{bmatrix}$$

CRAMER'S RULE FOR SOLVING SYSTEMS

■ 1. Use Cramer's rule to find the expression that would give the value of x. You do not need to solve the system.

$$2x - y = 5$$

$$x + 3y = 15$$

■ 2. Use Cramer's rule to find the expression that would give the value of x. You do not need to solve the system.

$$ax + by = e$$

$$cx + dy = f$$

■ 3. Use Cramer's rule to find the expression that would give the value of y. You do not need to solve the system.

$$3x + 4y = 11$$

$$2x - 3y = -4$$

■ 4. Use Cramer's rule to solve for x.

$$3x + 2y = 1$$

$$6x + 5y = 4$$

5. Use Cramer's rule to solve for y.

$$3x + 2y = 1$$

$$6x + 5y = 4$$

6. Use Cramer's rule to solve for x.

$$3x + 5y = 6$$

$$9x + 10y = 14$$

MODIFYING DETERMINANTS

■ 1. Find the determinant of A if the first row of A gets multiplied by 3.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

■ 2. Find the determinant of A if both rows of A are multiplied by 2.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

■ 3. Find the determinant of C, using only the determinants of A and B.

$$A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 4 \\ -1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 4 \\ 2 & 4 \end{bmatrix}$$

■ 4. Find the determinant of the new matrix if the rows in matrix A are swapped.

$$A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

5. Find the determinant of the new matrix after the second and third rows of matrix A are swapped.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

6. Verify that the row operation $R_2 + 2R_1 \to R_2$ doesn't change the value of $|A|$.

$$A = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$$

UPPER AND LOWER TRIANGULAR MATRICES

■ 1. Find the determinant of the upper-triangular matrix.

$$A = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix}$$

■ 2. Find the determinant of the upper-triangular matrix.

$$A = \begin{bmatrix} -4 & 0 & 1 & 3 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

■ 3. Find the determinant of the lower-triangular matrix.

$$A = \begin{bmatrix} 4 & 0 \\ 5 & 3 \end{bmatrix}$$

■ 4. Find the determinant of the lower-triangular matrix.

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 5 & -3 & 0 \\ 3 & -1 & -1 \end{bmatrix}$$

■ 5. Put A into upper or lower triangular form to find the determinant.

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

■ 6. Put A into upper or lower triangular form to find the determinant.

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 4 \\ 0 & 3 & -4 \end{bmatrix}$$

USING DETERMINANTS TO FIND AREA

■ 1. Find the area of the parallelogram formed by $\vec{v}_1=(1,4)$ and $\vec{v}_2=(-2,1)$, if the two vectors form adjacent edges of the parallelogram.

■ 2. Find the area of a parallelogram formed by $\vec{v}_1=(-3,-3)$ and $\vec{v}_2=(4,-2)$, if the two vectors form adjacent edges of the parallelogram.

■ 3. Find the area of the parallelogram formed by $\vec{v}_1=(4,2)$ and $\vec{v}_2=(1,5)$, if the two vectors form adjacent edges of the parallelogram.

■ 4. The square S is defined by the vertices $(0,3)$, $(0,0)$, $(3,0)$, and $(3,3)$. If the transformation of S by T creates a transformed figure F, find the area of F.

$$T(\vec{x})=\begin{bmatrix}2 & -3\\ 1 & 2\end{bmatrix}\vec{x}$$

■ 5. A rectangle R is defined by the vertices $(-2,2)$, $(2,2)$, $(-2,-3)$, and $(2,-3)$. If the transformation of S by T creates a transformed figure F, find the area of F.

$$T(\vec{x})=\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}\vec{x}$$

■ 6. The rectangle R is defined by the vertices $(2,-6)$, $(2,-1)$, $(8,-1)$, and $(8,-6)$. If the transformation of R by T creates a transformed figure L, find the area of L.

$$T(\vec{x}) = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \vec{x}$$

TRANSPOSES AND THEIR DETERMINANTS

1. Find the transpose A^T.

$$A = \begin{bmatrix} 5 & 6 & 0 & 7 & 5 & -7 \end{bmatrix}$$

2. Find the transpose A^T.

$$A = \begin{bmatrix} 7 & 9 & -6 \\ 0 & -1 & 9 \end{bmatrix}$$

3. Find the transpose A^T.

$$A = \begin{bmatrix} -4 & -7 \\ 5 & 1 \\ 7 & -2 \\ 4 & -2 \end{bmatrix}$$

4. Find the determinant of the transpose of A.

$$A = \begin{bmatrix} 5 & 3 & 6 & -1 \\ 9 & 0 & 1 & -2 \\ 8 & -2 & -4 & 8 \\ 5 & 4 & 9 & 7 \end{bmatrix}$$

■ 5. Find the determinant of the transpose of A.

$$A = \begin{bmatrix} -9 & -3 & -1 \\ -4 & 7 & 3 \\ -4 & 8 & 7 \end{bmatrix}$$

■ 6. Find the determinant of the transpose of A.

$$A = \begin{bmatrix} -8 & 6 & 8 \\ 3 & -9 & -1 \\ 4 & -9 & 9 \end{bmatrix}$$

TRANSPOSES OF PRODUCTS, SUMS, AND INVERSES

■ 1. Find $(AB)^T$.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix}$$

■ 2. Find $(AB)^T$.

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -4 & 1 \\ 0 & -3 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

■ 3. Find $(X+Y)^T$.

$$X = \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix}$$

■ 4. Find $(X+Y)^T$.

$$X=\begin{bmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \\ -2 & 0 & 3 \end{bmatrix}$$

$$Y=\begin{bmatrix} -1 & 2 & -3 \\ 0 & -1 & 2 \\ 4 & -1 & 0 \end{bmatrix}$$

■ 5. Find $(X^T)^{-1}$.

$$X=\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$

■ 6. Find $(A^T)^{-1}$.

$$A=\begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix}$$

NULL AND COLUMN SPACES OF THE TRANSPOSE

■ 1. Find the null and column spaces of the transpose M^T, identify their spaces $\mathbb{R}^i$, and name the dimension of the subspaces.

$$M = \begin{bmatrix} -1 & 0 \\ 2 & 4 \\ -2 & -2 \\ 0 & 4 \end{bmatrix}$$

■ 2. Find the row space and left null space of A, and the dimensions of those spaces.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -4 & 0 \end{bmatrix}$$

■ 3. Find the row space and left null space of B, and the dimensions of those spaces.

$$B = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & -2 & -1 & 4 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

■ 4. Find the row space and left null space of C, and the dimensions of those spaces.

$$C = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

■ 5. Find the row space and left null space of A, and the dimensions of those spaces.

$$A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \\ 0 & -2 \end{bmatrix}$$

■ 6. Find the null and column subspaces of the transpose M^T, identify their spaces $\mathbb{R}^i$, and name the dimension of the subspaces of M^T.

$$M = \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ -1 & -1 \\ 0 & 3 \end{bmatrix}$$

THE PRODUCT OF A MATRIX AND ITS TRANSPOSE

■ 1. Is A^TA invertible?

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$$

■ 2. Is A^TA invertible?

$$A = \begin{bmatrix} -12 & 6 \\ 8 & -4 \end{bmatrix}$$

■ 3. Is A^TA invertible?

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

■ 4. Is A^TA invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

■ 5. Is A^TA invertible?

$$A = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

■ 6. Is A^TA invertible?

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 3 & 3 \end{bmatrix}$$

ORTHOGONAL COMPLEMENTS

1. Find the orthogonal complement of V, $V^{\perp}$.

$$V = \text{Span}\left(\begin{bmatrix}-2\\1\\4\end{bmatrix}, \begin{bmatrix}0\\-3\\2\end{bmatrix}\right)$$

2. Find the orthogonal complement of V, $V^{\perp}$.

$$V = \text{Span}\left(\begin{bmatrix}-1\\2\\-5\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-4\\3\end{bmatrix}\right)$$

3. Rewrite the orthogonal complement of V, $V^{\perp}$, if V is a vector set in $\mathbb{R}^3$.

$$V = \begin{bmatrix}s\\-2s-t\\s+t\end{bmatrix}$$

4. Rewrite the orthogonal complement of W, $W^{\perp}$, if W is a vector set in $\mathbb{R}^4$.

$$W = \begin{bmatrix}-2y-z\\3y+z\\-y\\2y-3z\end{bmatrix}$$

■ 5. Describe the orthogonal component of V, $V^{\perp}$.

$$V = \text{Span}\left(\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\1\\3\end{bmatrix}, \begin{bmatrix}2\\-1\\3\\0\end{bmatrix}\right)$$

■ 6. Describe the orthogonal component of W, $W^{\perp}$.

$$W = \text{Span}\left(\begin{bmatrix}1\\0\\2\\-1\\2\end{bmatrix}, \begin{bmatrix}-3\\2\\4\\1\\-2\end{bmatrix}\right)$$

ORTHOGONAL COMPLEMENTS OF THE FUNDAMENTAL SUBSPACES

■ 1. For the matrix M, find the dimensions of all four fundamental subspaces.

$$M = \begin{bmatrix} -2 & 6 & 0 \\ -1 & 4 & 3 \\ 2 & -5 & 3 \end{bmatrix}$$

■ 2. For the matrix M, find the dimensions of all four fundamental subspaces.

$$M = \begin{bmatrix} -1 & 0 & 2 & -4 \\ -2 & 3 & -5 & 1 \\ 1 & -2 & 4 & 0 \end{bmatrix}$$

■ 3. For the matrix X, find the dimensions of all four fundamental subspaces.

$$X = \begin{bmatrix} 1 & -2 & 4 \\ -3 & 5 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

■ 4. For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} -1 & -3 & 2 & 1 \\ -2 & -5 & 5 & -1 \\ -3 & -7 & 8 & -3 \end{bmatrix}$$

■ 5. For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} 1 & -1 & 3 & 0 & 2 \\ -1 & 4 & -3 & 1 & 0 \\ 2 & -11 & 6 & -3 & -2 \end{bmatrix}$$

■ 6. For the matrix M, find the dimensions of all four fundamental subspaces.

$$M = \begin{bmatrix} -2 & 2 & -4 \\ 1 & -2 & 0 \\ -3 & 5 & -2 \\ 1 & 2 & 8 \end{bmatrix}$$

PROJECTION ONTO THE SUBSPACE

■ 1. If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}-1\\3\\-2\end{bmatrix}, \begin{bmatrix}0\\-2\\3\end{bmatrix}\right)$$

■ 2. If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}-2\\-4\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-3\\-2\end{bmatrix}\right)$$

■ 3. If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 \begin{bmatrix}1\\-3\\1\end{bmatrix} \text{ and } \vec{x}_2 \begin{bmatrix}-1\\0\\-2\end{bmatrix}$$

■ 4. If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 = \frac{1}{2}\begin{bmatrix}1\\-2\\-1\\-1\end{bmatrix} \text{ and } \vec{x}_2 = \frac{1}{2}\begin{bmatrix}-1\\0\\1\\-3\end{bmatrix}$$

■ 5. If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}-1\\0\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\3\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\2\\1\\1\end{bmatrix}\right)$$

■ 6. If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 = \frac{1}{2}\begin{bmatrix}2\\8\\-4\end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$$

LEAST SQUARES SOLUTION

1. Find the least squares solution to the system.

$$x = 2$$

$$x - y = 2$$

$$x + y = 3$$

2. Find the least squares solution to the system.

$$-x + 2y = 6$$

$$3x + 2y = 0$$

$$y - 3x = -2$$

3. Find the least squares solution to the system.

$$y - 2x = 5$$

$$3x + y = -2$$

$$2x - 4y = 5$$

4. Find the least squares solution to the system.

$y - 3x = 5$

$x + y = -3$

$2x - 2y = 3$

■ 5. Find the least squares solution to the system.

$2y - 3x = -4$

$5x + y = -2$

$x + 4y = -1$

■ 6. Find the least squares solution to the system.

$2x - 5y = 4$

$x + 6y = 5$

$4x - 3y = -6$

COORDINATES IN A NEW BASIS

■ 1. The vectors $\overrightarrow{v} = (-2,1)$ and $\overrightarrow{w} = (4,-3)$ form an alternate basis for $\mathbb{R}^2$. Use them to transform $\overrightarrow{x} = 6\mathbf{i} - 2\mathbf{j}$ into the alternate basis.

■ 2. The vectors $\overrightarrow{v} = (1,-5)$ and $\overrightarrow{w} = (2,4)$ form an alternate basis for $\mathbb{R}^2$. Use them, and an inverse matrix, to transform $\overrightarrow{x} = -\mathbf{i}$ into the alternate basis.

■ 3. The vectors $\overrightarrow{v} = (-1,0,4)$, $\vec{s} = (2,-3,1)$, and $\overrightarrow{w} = (1,-1,2)$ form an alternate basis for $\mathbb{R}^3$. Use them to transform $\overrightarrow{x} = -\mathbf{j} + \mathbf{k}$ into the alternate basis.

■ 4. The vectors $\overrightarrow{v} = (1,-3,1)$, $\vec{s} = (-3,-3,2)$, and $\overrightarrow{w} = (5,-3,1)$ form an alternate basis for $\mathbb{R}^3$. Use them, and an inverse matrix to transform $\overrightarrow{x} = 2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ into the alternate basis.

■ 5. The vectors $\overrightarrow{v} = (-2,3)$ and $\overrightarrow{w} = (4,0)$ form an alternate basis for $\mathbb{R}^2$. Use them, and an inverse matrix, to transform $\overrightarrow{x} = 6\mathbf{i} - 3\mathbf{j}$ into the alternate basis.

6. The vectors $\vec{v} = (-1,3,2)$, $\vec{s} = (-2,4,-4)$, and $\vec{w} = (1,-2,0)$ form an alternate basis for $\mathbb{R}^3$. Use them to transform $\vec{x} = -2\mathbf{i} - 4\mathbf{k}$ into the alternate basis.

TRANSFORMATION MATRIX FOR A BASIS

1. Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (2,1)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} 3 & -2 \\ 6 & 0 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \end{bmatrix}\right)$$

2. Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $\vec{x} = (-2,4)$ in the standard basis in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -3 & 1 \\ 4 & 5 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$$

3. Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (-5,2)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix}1\\-3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right)$$

■ 4. Use the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ to transform $\vec{x} = (6, -3)$ in the standard basis in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix}-5 & -4\\2 & -8\end{bmatrix}\vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}0\\4\end{bmatrix}\right)$$

■ 5. Use the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ to transform $[\vec{x}]_B = (-2,4,1)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix}-4 & 1 & 1\\2 & -3 & -1\\0 & 2 & 0\end{bmatrix}\vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$

■ 6. Use the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ to transform $\vec{x} = (-2,3,1)$ in the standard basis in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix}-4 & 2 & 1\\0 & 3 & -5\\1 & -2 & 4\end{bmatrix}\vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\end{bmatrix}\right)$$

ORTHONORMAL BASES

■ 1. Verify that the vector set $V = \{\vec{v}_1, \vec{v}_2\}$ is orthonormal if $\vec{v}_1 = (1,0,0)$ and $\vec{v}_2 = (0,0,-1)$.

■ 2. Determine that the vector set $V = \{\vec{v}_1, \vec{v}_2\}$ is orthonormal.

$$\vec{v}_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

$$\vec{v}_2 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

■ 3. Convert $\vec{x} = (-2,10)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{3}{4} \\ -\frac{\sqrt{7}}{4} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{\sqrt{7}}{4} \\ \frac{3}{4} \end{bmatrix}$$

■ 4. Convert $\vec{x} = (-25,10)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$$

■ 5. Convert $\vec{x} = (-6,3,12)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

■ 6. Convert $\vec{x} = (2,0,-3)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

PROJECTION ONTO AN ORTHONORMAL BASIS

■ 1. Find the projection of $\vec{x}=(-5,0,-2)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\\ 0\end{bmatrix},\begin{bmatrix}0\\ 0\\ -1\end{bmatrix}\right)$$

■ 2. Find the projection of $\vec{x}=(-66,33,11)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ -\frac{2}{\sqrt{6}}\end{bmatrix},\begin{bmatrix}-\frac{3}{\sqrt{11}}\\ \frac{1}{\sqrt{11}}\\ -\frac{1}{\sqrt{11}}\end{bmatrix}\right)$$

■ 3. Find the projection of $\vec{x}=(-6,-3,6)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}-\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix},\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{bmatrix}\right)$$

■ 4. Find the projection of $\vec{x}=(-2,3,5)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}0\\-1\\0\end{bmatrix},\begin{bmatrix}\frac{3}{\sqrt{10}}\\0\\\frac{1}{\sqrt{10}}\end{bmatrix}\right)$$

■ 5. Find the projection of $\vec{x}=(0,-13,4)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{3}{\sqrt{13}}\\\frac{2}{\sqrt{13}}\\0\end{bmatrix},\begin{bmatrix}\frac{2}{\sqrt{17}}\\-\frac{3}{\sqrt{17}}\\\frac{2}{\sqrt{17}}\end{bmatrix}\right)$$

■ 6. Find the projection of $\vec{x}=(-3,10,-10)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{3}{\sqrt{19}}\\-\frac{3}{\sqrt{19}}\\\frac{1}{\sqrt{19}}\end{bmatrix},\begin{bmatrix}0\\\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}\right)$$

GRAM-SCHMIDT PROCESS FOR CHANGE OF BASIS

1. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}\right)$$

2. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}\right)$$

3. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}\right)$$

4. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-3\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\2\end{bmatrix}, \begin{bmatrix}-5\\5\\0\end{bmatrix}\right)$$

■ 5. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}, \begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\right)$$

■ 6. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\0\\-1\\-1\end{bmatrix}\right)$$

EIGENVALUES, EIGENVECTORS, EIGENSPACES

1. Find the eigenvalues of the transformation matrix A.

$$A = \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix}$$

2. For the transformation matrix A, find the eigenvectors associated with each eigenvalue, $\lambda = -2$ and $\lambda = -5$.

$$A = \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix}$$

$$\lambda I_n - A = \begin{bmatrix} \lambda + 2 & -2 \\ 0 & \lambda + 5 \end{bmatrix}$$

3. Find the eigenvalues of the transformation matrix A.

$$A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$$

4. For the transformation matrix A, find the eigenvectors associated with each eigenvalue, $\lambda = -2$ and $\lambda = 4$.

$$A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$$

$$\lambda I_n - A = \begin{bmatrix} \lambda - 3 & 1 \\ 5 & \lambda + 1 \end{bmatrix}$$

■ 5. Find the eigenvectors of the transformation matrix.

$$A = \begin{bmatrix} 5 & 0 \\ -4 & 3 \end{bmatrix}$$

■ 6. Find the eigenvectors of the transformation matrix.

$$A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$$

EIGEN IN THREE DIMENSIONS

■ 1. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

■ 2. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

■ 3. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -3 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

■ 4. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix}$$

5. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

6. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

W W W . K R I S T A K I N G M A T H . C O M