Linear Algebra Workbook

Orthogonality and change of basis

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ORTHOGONAL COMPLEMENTS

1. Find the orthogonal complement of V, $V^{\perp}$.

$$V=\text{Span}\left(\begin{bmatrix}-2\\1\\4\end{bmatrix},\begin{bmatrix}0\\-3\\2\end{bmatrix}\right)$$

2. Find the orthogonal complement of V, $V^{\perp}$.

$$V=\text{Span}\left(\begin{bmatrix}-1\\2\\-5\\3\end{bmatrix},\begin{bmatrix}1\\0\\-4\\3\end{bmatrix}\right)$$

3. Rewrite the orthogonal complement of V, $V^{\perp}$, if V is a vector set in $\mathbb{R}^3$.

$$V=\begin{bmatrix}s\\-2s-t\\s+t\end{bmatrix}$$

4. Rewrite the orthogonal complement of W, $W^{\perp}$, if W is a vector set in $\mathbb{R}^4$.

$$W=\begin{bmatrix}-2y-z\\3y+z\\-y\\2y-3z\end{bmatrix}$$

5. Describe the orthogonal component of V, $V^{\perp}$.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}\right)$$

6. Describe the orthogonal component of W, $W^{\perp}$.

$$W = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 4 \\ 1 \\ -2 \end{bmatrix}\right)$$

ORTHOGONAL COMPLEMENTS OF THE FUNDAMENTAL SUBSPACES

■ 1. For the matrix M, find the dimensions of all four fundamental subspaces.

$$M = \begin{bmatrix} -2 & 6 & 0 \\ -1 & 4 & 3 \\ 2 & -5 & 3 \end{bmatrix}$$

■ 2. For the matrix M, find the dimensions of all four fundamental subspaces.

$$M = \begin{bmatrix} -1 & 0 & 2 & -4 \\ -2 & 3 & -5 & 1 \\ 1 & -2 & 4 & 0 \end{bmatrix}$$

■ 3. For the matrix X, find the dimensions of all four fundamental subspaces.

$$X = \begin{bmatrix} 1 & -2 & 4 \\ -3 & 5 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

■ 4. For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} -1 & -3 & 2 & 1 \\ -2 & -5 & 5 & -1 \\ -3 & -7 & 8 & -3 \end{bmatrix}$$

■ 5. For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} 1 & -1 & 3 & 0 & 2 \\ -1 & 4 & -3 & 1 & 0 \\ 2 & -11 & 6 & -3 & -2 \end{bmatrix}$$

■ 6. For the matrix M, find the dimensions of all four fundamental subspaces.

$$M = \begin{bmatrix} -2 & 2 & -4 \\ 1 & -2 & 0 \\ -3 & 5 & -2 \\ 1 & 2 & 8 \end{bmatrix}$$

PROJECTION ONTO THE SUBSPACE

■ 1. If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}-1\\3\\-2\end{bmatrix}, \begin{bmatrix}0\\-2\\3\end{bmatrix}\right)$$

■ 2. If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}-2\\-4\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-3\\-2\end{bmatrix}\right)$$

■ 3. If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 \begin{bmatrix}1\\-3\\1\end{bmatrix} \text{ and } \vec{x}_2 \begin{bmatrix}-1\\0\\-2\end{bmatrix}$$

■ 4. If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -2 \\ -1 \\ -1 \end{bmatrix} \text{ and } \vec{x}_2 = \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

5. If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}\right)$$

6. If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 = \frac{1}{2}\begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

LEAST SQUARES SOLUTION

■ 1. Find the least squares solution to the system.

$$x = 2$$

$$x - y = 2$$

$$x + y = 3$$

■ 2. Find the least squares solution to the system.

$$-x + 2y = 6$$

$$3x + 2y = 0$$

$$y - 3x = -2$$

■ 3. Find the least squares solution to the system.

$$y - 2x = 5$$

$$3x + y = -2$$

$$2x - 4y = 5$$

■ 4. Find the least squares solution to the system.

$$y - 3x = 5$$

$$x + y = -3$$

$$2x - 2y = 3$$

■ 5. Find the least squares solution to the system.

$$2y - 3x = -4$$

$$5x + y = -2$$

$$x + 4y = -1$$

■ 6. Find the least squares solution to the system.

$$2x - 5y = 4$$

$$x + 6y = 5$$

$$4x - 3y = -6$$

COORDINATES IN A NEW BASIS

■ 1. The vectors $\vec{v}=(-2,1)$ and $\vec{w}=(4,-3)$ form an alternate basis for $\mathbb{R}^2$. Use them to transform $\vec{x}=6\mathbf{i}-2\mathbf{j}$ into the alternate basis.

■ 2. The vectors $\vec{v}=(1,-5)$ and $\vec{w}=(2,4)$ form an alternate basis for $\mathbb{R}^2$. Use them, and an inverse matrix, to transform $\vec{x}=-\mathbf{i}$ into the alternate basis.

■ 3. The vectors $\vec{v}=(-1,0,4)$, $\vec{s}=(2,-3,1)$, and $\vec{w}=(1,-1,2)$ form an alternate basis for $\mathbb{R}^3$. Use them to transform $\vec{x}=-\mathbf{j}+\mathbf{k}$ into the alternate basis.

■ 4. The vectors $\vec{v}=(1,-3,1)$, $\vec{s}=(-3,-3,2)$, and $\vec{w}=(5,-3,1)$ form an alternate basis for $\mathbb{R}^3$. Use them, and an inverse matrix to transform $\vec{x}=2\mathbf{i}+6\mathbf{j}-\mathbf{k}$ into the alternate basis.

■ 5. The vectors $\vec{v}=(-2,3)$ and $\vec{w}=(4,0)$ form an alternate basis for $\mathbb{R}^2$. Use them, and an inverse matrix, to transform $\vec{x}=6\mathbf{i}-3\mathbf{j}$ into the alternate basis.

6. The vectors $\overrightarrow{v}=(-1,3,2)$, $\vec{s}=(-2,4,-4)$, and $\overrightarrow{w}=(1,-2,0)$ form an alternate basis for $\mathbb{R}^3$. Use them to transform $\overrightarrow{x}=-2\mathbf{i}-4\mathbf{k}$ into the alternate basis.

TRANSFORMATION MATRIX FOR A BASIS

1. Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (2,1)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} 3 & -2 \\ 6 & 0 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \end{bmatrix} \right)$$

2. Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $\vec{x} = (-2,4)$ in the standard basis in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -3 & 1 \\ 4 & 5 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right)$$

3. Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (-5,2)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix}1\\-3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right)$$

■ 4. Use the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ to transform $\vec{x} = (6, -3)$ in the standard basis in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix}-5 & -4\\2 & -8\end{bmatrix}\vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}0\\4\end{bmatrix}\right)$$

■ 5. Use the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ to transform $[\vec{x}]_B = (-2,4,1)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix}-4 & 1 & 1\\2 & -3 & -1\\0 & 2 & 0\end{bmatrix}\vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$

■ 6. Use the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ to transform $\vec{x} = (-2,3,1)$ in the standard basis in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix}-4 & 2 & 1\\0 & 3 & -5\\1 & -2 & 4\end{bmatrix}\vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\end{bmatrix}\right)$$

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