Topic: The product of a matrix and its transpose

Question: Is A^TA invertible?

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 4 & 0 \end{bmatrix}$$

Answer choices:

A Yes, because the columns of A are linearly independent

B Yes, but the columns of A aren't linearly independent

C No, because the columns of A aren't linearly independent

D No, but the columns of A are linearly independent

Solution: A

The columns of A are linearly independent, so A^TA is invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & 0 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 4 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1)+0(0)+4(4) & 1(-3)+0(1)+4(0) \\ -3(1)+1(0)+0(4) & -3(-3)+1(1)+0(0) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0+16 & -3+0+0 \\ -3+0+0 & 9+1+0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 17 & -3 \\ -3 & 10 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 1 & -\frac{3}{17} \\ -3 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{17} \\ 0 & \frac{161}{17} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{17} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because we got to the identity matrix, we can say that A^TA is invertible.

Topic: The product of a matrix and its transpose

Question: Is A^TA invertible?

$$A = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$$

Answer choices:

A Yes, because the columns of A are linearly independent

B Yes, but the columns of A aren't linearly independent

C No, because the columns of A aren't linearly independent

D No, but the columns of A are linearly independent

Solution: C

The columns of A aren't linearly independent, so A^TA is not invertible. We can confirm this by finding A^TA, and then verifying that A^TA doesn't simplify to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -6(-6)+4(4) & -6(3)+4(-2) \\ 3(-6)-2(4) & 3(3)-2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 36+16 & -18-8 \\ -18-8 & 9+4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 52 & -26 \\ -26 & 13 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 52 & -26 \\ -26 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ -26 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

Topic: The product of a matrix and its transpose

Question: Is A^TA invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{bmatrix}$$

Answer choices:

A Yes, because the columns of A are linearly independent

B Yes, but the columns of A aren't linearly independent

C No, because the columns of A aren't linearly independent

D No, but the columns of A are linearly independent

Solution: A

The columns of A are linearly independent, so A^TA is invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1)+0(0)+1(1) & 1(0)+0(-2)+1(1) & 1(-2)+0(3)+1(-2) \\ 0(1)-2(0)+1(1) & 0(0)-2(-2)+1(1) & 0(-2)-2(3)+1(-2) \\ -2(1)+3(0)-2(1) & -2(0)+3(-2)-2(1) & -2(-2)+3(3)-2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0+1 & 0+0+1 & -2+0-2 \\ 0-0+1 & 0+4+1 & 0-6-2 \\ -2+0-2 & 0-6-2 & 4+9+4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 17 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 17 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -8 \\ 2 & 1 & -4 \\ -4 & -8 & 17 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -8 \\ 0 & -9 & 12 \\ -4 & -8 & 17 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -8 \\ 0 & -9 & 12 \\ 0 & 12 & -15 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 5 & -8 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 12 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 12 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because we got to the identity matrix, we can say that A^TA is invertible.