**Topic**: Determinants

**Question**: Find the determinant of $L$.

$$L = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{bmatrix}$$

**Answer choices**:

A $|L| = 1$

B $|L| = -1$

C $|L| = 3$

D $|L| = -3$

**Solution**: D

Because $L$ is a $3 \times 3$ matrix, we'll break it down into $2 \times 2$ determinants, remembering to alternate sign using the checkerboard pattern.

$$|L| = \begin{vmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{vmatrix}$$

$$|L| = 1 \begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix} - 0 \begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix} + (-2) \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}$$

Calculate the $2 \times 2$ determinants using the $ad - bc$ rule.

$$|L| = [(-2)(-2) - (3)(1)] - 2[(0)(1) - (-2)(1)]$$

$$|L| = (4 - 3) - 2(0 + 2)$$

$$|L| = 1 - 4$$

$$|L| = -3$$

**Topic**: Determinants

**Question**: Find the determinant of $C$ by working along any row or column.

$$C = \begin{bmatrix} -3 & 1 & 5 & 2 \\ 0 & 0 & -1 & 1 \\ 2 & -2 & 3 & 0 \\ 1 & 4 & 0 & -4 \end{bmatrix}$$

**Answer choices**:

A $|C| = 93$

B $|C| = 76$

C $|C| = 54$

D $|C| = 28$

**Solution**: A

Because there are multiple zeros across the second row, that'll be the easiest row or column to use to find the determinant. So we'll use that row to calculate the $3 \times 3$ determinants, remembering to apply the checkerboard pattern so that we get the signs right.

$$|C| = \begin{vmatrix} -3 & 1 & 5 & 2 \\ 0 & 0 & -1 & 1 \\ 2 & -2 & 3 & 0 \\ 1 & 4 & 0 & -4 \end{vmatrix}$$

$$|C| = -0\begin{vmatrix} 1 & 5 & 2 \\ -2 & 3 & 0 \\ 4 & 0 & -4 \end{vmatrix} + 0\begin{vmatrix} -3 & 5 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & -4 \end{vmatrix} - (-1)\begin{vmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 4 & -4 \end{vmatrix} + 1\begin{vmatrix} -3 & 1 & 5 \\ 2 & -2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$$

Because of the scalars, the first two determinants go to $0$.

$$|C| = \begin{vmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 4 & -4 \end{vmatrix} + \begin{vmatrix} -3 & 1 & 5 \\ 2 & -2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$$

To simplify this first $3 \times 3$ determinant, let's work along the third column, since it includes the $0$.

$$|C| = 2\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} -3 & 1 \\ 1 & 4 \end{vmatrix} + (-4)\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} -3 & 1 & 5 \\ 2 & -2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$$

$$|C| = 2\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} -3 & 1 & 5 \\ 2 & -2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$$

To simplify the second $3 \times 3$ determinant, let's work along the third row, since it includes the $0$.

$$|C| = 2\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ -2 & 3 \end{vmatrix} - 4\begin{vmatrix} -3 & 5 \\ 2 & 3 \end{vmatrix} + 0\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix}$$

$$|C| = 2\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 5 \\ -2 & 3 \end{vmatrix} - 4\begin{vmatrix} -3 & 5 \\ 2 & 3 \end{vmatrix}$$

Now we'll calculate each of the $2 \times 2$ determinants that remain.

$$|C| = 2[(2)(4) - (-2)(1)] - 4[(-3)(-2) - (1)(2)]$$

$$+[(1)(3) - (5)(-2)] - 4[(-3)(3) - (5)(2)]$$

$$|C| = 2(8 + 2) - 4(6 - 2) + (3 + 10) - 4(-9 - 10)$$

$$|C| = 2(10) - 4(4) + (13) - 4(-19)$$

$$|C| = 20 - 16 + 13 + 76$$

$$|C| = 93$$

**Topic**: Determinants

**Question**: Use the determinant to say whether or not $F$ is invertible. Work along any row or column.

$$F = \begin{bmatrix} 1 & 5 & 0 & -1 \\ 3 & -2 & -1 & 2 \\ -1 & 1 & 0 & 3 \\ 1 & 3 & 2 & -2 \end{bmatrix}$$

**Answer choices**:

A $\quad F$ is invertible, and $|F| = 144$

B $\quad F$ is invertible, and $|F| = 126$

C $\quad F$ is invertible, and $|F| = 116$

D $\quad F$ is singular, and $|F| = 0$

**Solution**: B

If the determinant $|F|$ is nonzero, then $F$ is invertible. But if $|F|=0$, then $F$ is singular and won't have a defined inverse. So to say whether or not the matrix is invertible, we'll calculate the determinant.

Because there are multiple zeros across the third column, that'll be the easiest row or column to use to find the determinant. So we'll use that column to calculate the $3\times 3$ determinants, remembering to apply the checkerboard pattern so that we get the signs right.

$$F=\begin{bmatrix}1&5&0&-1\\3&-2&-1&2\\-1&1&0&3\\1&3&2&-2\end{bmatrix}$$

$$|F|=0\begin{vmatrix}3&-2&2\\-1&1&3\\1&3&-2\end{vmatrix}-(-1)\begin{vmatrix}1&5&-1\\-1&1&3\\1&3&-2\end{vmatrix}+0\begin{vmatrix}1&5&-1\\3&-2&2\\1&3&-2\end{vmatrix}-2\begin{vmatrix}1&5&-1\\3&-2&2\\-1&1&3\end{vmatrix}$$

Because of the scalars, the first and third determinants go to $0$.

$$|F|=\begin{vmatrix}1&5&-1\\-1&1&3\\1&3&-2\end{vmatrix}-2\begin{vmatrix}1&5&-1\\3&-2&2\\-1&1&3\end{vmatrix}$$

To simplify this first $3\times 3$ determinant, let's work along the first column, since those entries are all $1$ or $-1$, which is a pretty simple set of entries.

$$|F|=1\begin{vmatrix}1&3\\3&-2\end{vmatrix}-(-1)\begin{vmatrix}5&-1\\3&-2\end{vmatrix}+1\begin{vmatrix}5&-1\\1&3\end{vmatrix}-2\begin{vmatrix}1&5&-1\\3&-2&2\\-1&1&3\end{vmatrix}$$

$$|F| = \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 5 & -1 \\ 3 & -2 & 2 \\ -1 & 1 & 3 \end{vmatrix}$$

To simplify the second $3 \times 3$ determinant, let's work along the first row, since none of those rows or columns are particularly simpler than any other.

$$|F| = \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix}$$

$$-2 \left[ 1 \begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix} - 5 \begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} \right]$$

$$|F| = \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix}$$

$$-2 \begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix} + 10 \begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}$$

Now we'll calculate each of the $2 \times 2$ determinants that remain.

$$|F| = [(1)(-2) - (3)(3)] + [(5)(-2) - (-1)(3)] + [(5)(3) - (-1)(1)]$$

$$-2[(-2)(3) - (2)(1)] + 10[(3)(3) - (2)(-1)] + 2[(3)(1) - (-2)(-1)]$$

$$|F| = (-2 - 9) + (-10 + 3) + (15 + 1) - 2(-6 - 2) + 10(9 + 2) + 2(3 - 2)$$

$$|F| = -11 - 7 + 16 + 16 + 110 + 2$$

$$|F| = 126$$