Invertibility from the matrix-vector product

Remember we said before that any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ can be written as a matrix-vector product,

$T(\vec{x}) = M\vec{x}$, where M is an $m \times n$ matrix

In this form, we can say that the transformation T is surjective (onto) if the column space of M is the entire codomain, $\mathbb{R}^m$.

$C(M) = \mathbb{R}^m$

The column space of M will be the entire codomain $\mathbb{R}^m$ only when the reduced row-echelon form of M has m pivot entries, one pivot entry in every row. But of course, by the definition of a pivot entry, each pivot entry will be in its own column, which means M also has m pivot columns.

That also means that m basis vectors make up the column space of M, $C(M)$, and that the rank of M is m, $\text{Rank}(M) = \text{Dim}(C(M)) = m$.

These are all just different ways of saying that $T: A \to B$ is surjective, which means that every vector $\vec{b}$ in the codomain B is mapped to at least once by some vector $\vec{a}$ in the domain A via the transformation T.

In other words,

- if M becomes the identity matrix in reduced row-echelon form, or
- if M has m pivot entries, or
- if m basis vectors make up the column space of M, $C(M)$, or

- if the rank of M is m,

then you know that the transformation is surjective.

Let's look at a complete example so that we can see how to figure out whether the transformation is surjective, when we've been given the transformation as a matrix-vector product.

Example

The transformation T is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$. Say whether or not T is surjective.

$$T(\overrightarrow{x}) = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix} \overrightarrow{x}$$

We can see that T is given as a matrix-vector product. If we call the matrix M, then

$$M = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix}$$

To determine whether or not T is surjective, first put M into reduced row-echelon form. Find the pivot entry in the first row.

$$\begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & -2 \\ 0 & 7 \end{bmatrix}$$

Find the pivot entry in the second column.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \\ 0 & 7 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now that M is rewritten in reduced row-echelon form,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we can see that it has two pivot entries in two pivot columns. This means there are two columns that make up the basis of the column space of M, $C(M)$, or that the rank of M is 2, Rank$(M)=$ Dim$(C(M))=2$.

Because the codomain is $\mathbb{R}^4$, and since $2\neq 4$, we can say that the transformation T is not surjective; it is not onto.

Only square matrices are invertible

We want to go one step further with our understanding of invertibility, by focusing on the matrix M. Let's say for now that M is any $m\times n$ matrix (it has m rows and n columns).

First, there are two things we know:

1. In order for the transformation to be surjective, the rank of M must be m, Rank$(M)=m$, meaning that we must have a pivot entry in every row.

2. But in order for the transformation to be injective, the rank of M must also be n, Rank$(M)=n$, meaning that we must have a pivot entry in every column.

It's only possible for Rank$(M)=m$ and Rank$(M)=n$ if $m=n$. In other words, a matrix can only be invertible if it's both surjective and injective, but it can only be surjective and injective if $m=n$. So the matrix can only be invertible if it has the same number of rows and columns. And if that's true, then the matrix M has the same number of rows and columns (it's a square matrix), and we can call it an $n \times n$ matrix. So,

Only square matrices can be invertible.

It's important to say that not all square matrices are invertible, but only square matrices have the potential to be invertible. If a square matrix is invertible, then its reduced row-echelon form will be an $n \times n$ matrix where every column is a linearly independent pivot column, which will of course be the $n \times n$ identity matrix I_n.

Because $m = n$, this also implies that the transformation T must map from a domain $\mathbb{R}^n$ to the codomain $\mathbb{R}^n$. In other words, a transformation can only be invertible when it maps $\mathbb{R}^n \to \mathbb{R}^n$.

You could also say it this way. Given a transformation $T(\vec{x}) = M\vec{x}$, if you put the matrix M into reduced row-echelon form and get the identity matrix I_n, then you know that

- the matrix M was a square $n \times n$ matrix, and that

- the transformation T maps $\mathbb{R}^n \to \mathbb{R}^n$, and that

- T is invertible.