Matrix inverses, and invertible and singular matrices

In the last lesson, we looked at how we could find the inverse of a matrix by augmenting it with its associated identity matrix, and then working the matrix into reduced row-echelon form.

In this lesson, we want to talk more about matrix inverses, and when a matrix inverse is defined at all.

Division as multiplication by the reciprocal

In a way, we can think about a matrix inverse as matrix division. To think about matrix division, we want to remember that dividing by some value is the same as multiplying by the reciprocal of that value. For instance, dividing by 4 is the same as multiplying by $1/4$. So if k is a real number, then we know that

$$k \cdot \frac{1}{k} = 1$$

If we call $1/k$ the inverse of k and instead write it as k^{-1}, then we could rewrite this equation as

$$kk^{-1} = 1$$

and read this as "k multiplied by the inverse of k is 1." What we want to know now is whether this is also true for matrices. If I divide matrix K by matrix K, or multiply matrix K by its inverse, do I get back to 1? In other words, we're trying to prove that

$$K \cdot \frac{1}{K} = I \text{ or } KK^{-1} = I$$

where I is the identity matrix, which of course is the matrix equivalent of 1.

Matrix inverses

Of course, as we already know, matrix division is a valid operation, because multiplying a matrix by its inverse will result in the identity matrix.

We also know already how to find a matrix inverse by augmenting the matrix with the identity matrix, and then using row operations to put the matrix into reduced row-echelon form.

But there's another way to find the matrix inverse, and this method will help us identify when the matrix is invertible. This method requires us to use something called the determinant, which we'll learn about in the next section. For now, we'll just say that the determinant of a matrix M,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$\text{Det}(M) = |M| = ad - bc$$

Then the inverse of M is given by

$$M^{-1} = \frac{1}{|M|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Notice that the formula for the inverse matrix is a fraction with a numerator of 1 and the determinant as the denominator, multiplied by another matrix.

The other matrix is called the **adjugate** of M, and the adjugate is the matrix in which the values a and d have been swapped, and the values b and c have been multiplied by -1.

Let's do an example where we use this method with the determinant to find the inverse of a matrix.

Example

Find the inverse of matrix K, then find $K \cdot K^{-1}$ and $K^{-1} \cdot K$ to show that you found the correct inverse.

$$K = \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix}$$

To find the inverse of matrix K, we'll plug into the determinant formula for the inverse of a matrix.

$$K^{-1} = \frac{1}{|K|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$K^{-1} = \frac{1}{\begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix}}\begin{bmatrix} 0 & -4 \\ -3 & -2 \end{bmatrix}$$

$$K^{-1} = \frac{1}{-2(0) - 4(3)}\begin{bmatrix} 0 & -4 \\ -3 & -2 \end{bmatrix}$$

$$K^{-1} = -\frac{1}{12}\begin{bmatrix} 0 & -4 \\ -3 & -2 \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} -\frac{0}{12} & \frac{4}{12} \\ \frac{3}{12} & \frac{2}{12} \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix}$$

This is the inverse of K, but we can prove it to ourselves by multiplying K by its inverse. If we've done our math right, we should get the identity matrix when we multiply them.

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} -2(0) + 4\left(\frac{1}{4}\right) & -2\left(\frac{1}{3}\right) + 4\left(\frac{1}{6}\right) \\ 3(0) + 0\left(\frac{1}{4}\right) & 3\left(\frac{1}{3}\right) + 0\left(\frac{1}{6}\right) \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0+1 & -\frac{2}{3}+\frac{4}{6} \\ 0+0 & 1+0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K \cdot K^{-1} = I_2$$

When we multiplied K by its inverse, we get the identity matrix. We also want to make the point that we can multiply in the other direction, and we still get the identity matrix.

$$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0(-2)+\frac{1}{3}(3) & 0(4)+\frac{1}{3}(0) \\ \frac{1}{4}(-2)+\frac{1}{6}(3) & \frac{1}{4}(4)+\frac{1}{6}(0) \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ -\frac{1}{2}+\frac{1}{2} & 1+0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K^{-1} \cdot K = I_2$$

This example shows how to use the determinant formula to find the inverse matrix, and proves that multiplying by the inverse matrix is commutative. Whether we calculate $K \cdot K^{-1}$ or $K^{-1} \cdot K$, we get back to the identity matrix either way.

Invertible and singular matrices

So what does this determinant formula tell us about when the matrix inverse exists? After all, not every matrix has an inverse. Given the formula for the inverse matrix,

$$K^{-1} = \frac{1}{|K|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we can probably spot right away that, because a fraction is undefined when its denominator is 0, we have to say that $|K| \neq 0$. In other words, if the determinant is not equal to 0, then the inverse matrix is defined, and the matrix K is invertible. But if the determinant is equal to 0, then the inverse matrix is undefined, and the matrix K is not invertible. To be specific, if

$$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the inverse of K is undefined when $ad - bc = 0$, or when $ad = bc$. If we divide both sides of $ad = bc$ by b and d, we get

$$\frac{a}{b} = \frac{c}{d}$$

So if the ratio of a to b (the values in the first row of matrix K) is equal to the ratio of c to d (the values in the second row of matrix K), then you know right away that the matrix K does not have a defined inverse. If the matrix doesn't have an inverse, we call it a **singular matrix**. When the matrix does have an inverse, we say that it's an **invertible matrix**.

Example

Say whether each matrix is invertible or singular.

(a) $M = \begin{bmatrix} 1 & -3 \\ 3 & 5 \end{bmatrix}$

(b) $L = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}$

For each matrix, we'll look at whether or not the ratio of a to b is equal to the ratio of c to d. For matrix M, we get

$$\frac{1}{-3} = -\frac{1}{3} \quad \neq \quad \frac{3}{5}$$

Because these aren't equivalent ratios, matrix M is invertible, which means it has a defined inverse. To confirm this, we'll calculate the determinant.

$$\text{Det}(M) = |M| = 1(5) - (-3)(3) = 5 + 6 = 11$$

Since $|M| \neq 0$, M is invertible.

For matrix L, we get

$$\frac{-6}{2} = -3 \quad = \quad \frac{3}{-1} = -3$$

Because these are equivalent ratios, matrix L is not invertible, which means L does not have a defined inverse, and we can therefore say that it's a singular matrix. To confirm this, we'll calculate the determinant.

$$\text{Det}(L) = |L| = (-6)(-1) - 2(3) = 6 - 6 = 0$$

Since $|L| = 0$, L is not invertible, which means L is a singular matrix.
