Dot products

We've already learned about the dot product as a method we can use to multiply matrices. The dot product of two two-dimensional vectors is

$$\overrightarrow{u} \cdot \overrightarrow{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1v_1 + u_2v_2$$

Notice how this dot product formula is really just summing the products of the first components, second components, etc. In other words, we just multiply u_1 by v_1, then add the sum of u_2 and v_2, etc.

And of course, we can expand the dot product concept to matrix multiplication with matrices of any size. (We're not using u here to indicate the unit vector; we're using u as a regular vector name, just like v.)

But the dot product isn't just a matrix multiplication tool. It actually has a definition in and of itself.

The dot product in two dimensions

Remember that we talked earlier about the length of a vector, and we said that the length of a vector in two dimensions is

$$||\overrightarrow{u}|| = \sqrt{u_1^2 + u_2^2}$$

If we dot (take the dot product) a vector with itself, we get

$$\vec{u} \cdot \vec{u} = [u_1 \quad u_2] \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1u_1 + u_2u_2 = u_1^2 + u_2^2$$

The dot product $\vec{u} \cdot \vec{u}$ is $u_1^2 + u_2^2$, which we can see is also the value under the root in the formula for the length of a vector. So if we square both sides of the length formula, we get

$$(||\vec{u}||)^2 = \left(\sqrt{u_1^2 + u_2^2}\right)^2$$

$$||\vec{u}||^2 = u_1^2 + u_2^2$$

And then if we substitute $\vec{u} \cdot \vec{u}$ for $u_1^2 + u_2^2$, we get

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$$

In words, this formula tells us:

> *"The square of the length of a vector is equal to the vector dotted with itself."*

As we continue working with vectors, this will be a super helpful formula in particular, since it directly relates the length of a vector to its dot product.

Example

Use the dot product to find the length the vector $\vec{v} = (3,4)$.

We know we'll get the square of the length if we dot the vector with itself. Use the formula $||\vec{u}||^2 = \vec{u} \cdot \vec{u}$.

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$$

$$||\vec{u}||^2 = u_1u_1 + u_2u_2$$

Plugging $\vec{v}$ into this formula gives

$$||\vec{v}||^2 = (3)(3) + (4)(4)$$

$$||\vec{v}||^2 = 9 + 16$$

$$||\vec{v}||^2 = 25$$

Take the square root of both sides. We can ignore the negative value of the root, since we're looking for the length of the vector, and length will always be positive.

$$\sqrt{||\vec{v}||^2} = \sqrt{25}$$

$$||\vec{v}|| = 5$$

The length of $\vec{v} = (3,4)$ is 5.

The dot product in n dimensions

We looked at these dot product formulas for two-dimensional vectors, but these formulas are equally valid for three-dimensional vectors, and even n-dimensional vectors. The dot product of two n-dimensional vectors is

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

And if we dot an n-dimensional vector with itself, we get

$$\vec{u} \cdot \vec{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1u_1 + u_2u_2 + \dots + u_nu_n$$

The length of a vector in n-dimensions is

$$||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

Just like in two dimensions, the value of a vector dotted with itself is what we see under this square root, so the formula that relates the length of an n-dimensional vector to the dot product of an n-dimensional vector is still the same as the formula in two dimensions:

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$$

Let's do an example with a higher-dimensional vector.

Example

Use the dot product to find the length the vector $\vec{w} = (2, -3, 4, 0)$.

We know we'll get the square of the length if we dot the vector with itself. Use the formula $||\vec{u}||^2 = \vec{u} \cdot \vec{u}$.

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$$

$$||\vec{u}||^2 = u_1u_1 + u_2u_2 + u_3u_3 + u_4u_4$$

Plugging $\vec{w}$ into this formula gives

$$||\vec{w}||^2 = (2)(2) + (-3)(-3) + (4)(4) + (0)(0)$$

$$||\vec{w}||^2 = 4 + 9 + 16 + 0$$

$$||\vec{w}||^2 = 29$$

Take the square root of both sides. We can ignore the negative value of the root, since we're looking for the length of the vector, and length will always be positive.

$$\sqrt{||\vec{w}||^2} = \sqrt{29}$$

$$||\vec{w}|| = \sqrt{29}$$

The length of $\vec{w} = (2, -3, 4, 0)$ is $\sqrt{29}$.

Properties of dot products

Dot products in any dimension are commutative, distributive, and associative, meaning that the following properties all hold:

Commutative

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

Distributive

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

$$(\vec{u} - \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} - \vec{v} \cdot \vec{w}$$

Associative

$$(c\vec{u}) \cdot \vec{v} = c(\vec{v} \cdot \vec{u})$$

Let's do a quick example with some of these properties.

Example

Simplify the expression if $\vec{u} = (3, -2)$, $\vec{v} = (-1,4)$, and $\vec{w} = (0,5)$.

$$\vec{w} \cdot (3\vec{u} + 2\vec{v})$$

First, we need to apply the scalars to the vectors to find $3\vec{u}$ and $2\vec{v}$.

$$3\vec{u} = 3(3, -2) = (9, -6)$$

$$2\vec{v} = 2(-1,4) = (-2,8)$$

Then the sum $3\vec{u} + 2\vec{v}$ is

$$3\vec{u} + 2\vec{v} = \begin{bmatrix} 9 \\ -6 \end{bmatrix} + \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$3\vec{u}+2\vec{v}=\begin{bmatrix}9-2\\-6+8\end{bmatrix}$$

$$3\vec{u}+2\vec{v}=\begin{bmatrix}7\\2\end{bmatrix}$$

Calculate the dot product of $\vec{w}$ and $3\vec{u}+2\vec{v}$.

$$\vec{w}\cdot(3\vec{u}+2\vec{v})=\begin{bmatrix}0 & 5\end{bmatrix}\begin{bmatrix}7\\2\end{bmatrix}$$

$$\vec{w}\cdot(3\vec{u}+2\vec{v})=0(7)+5(2)$$

$$\vec{w}\cdot(3\vec{u}+2\vec{v})=0+10$$

$$\vec{w}\cdot(3\vec{u}+2\vec{v})=10$$