

# Orthonormal bases

We've talked about changing bases from the standard basis to an alternate basis, and vice versa. Now we want to talk about a specific kind of basis, called an **orthonormal basis**, in which every vector in the basis is both 1 unit in length and orthogonal to each of the other basis vectors.

In other words,  $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  is the orthonormal basis for  $V$  if each vector  $\vec{v}_i$  in the set  $V$  has length 1, such that  $\vec{v}_i \cdot \vec{v}_i = 1$  or  $||\vec{v}_i||^2 = 1$ . And if each vector in the set  $\vec{v}_i$  is orthogonal to every other vector in the set  $\vec{v}_j$ , then  $\vec{v}_i \cdot \vec{v}_j = 0$  for  $i \neq j$ .

Realize too that if a set of vectors is orthonormal, that means that all the vectors in the set are also linearly independent.

## Example

Verify that the vector set  $V = \{\vec{v}_1, \vec{v}_2\}$  is an orthonormal set, if  $\vec{v}_1 = (0,0,1)$  and  $\vec{v}_2 = (0,1,0)$ .

For the set to be orthonormal, each vector needs to have length 1. So we'll find the length of each vector.

$$||\vec{v}_1||^2 = \vec{v}_1 \cdot \vec{v}_1 = 0(0) + 0(0) + 1(1) = 1$$

$$||\vec{v}_2||^2 = \vec{v}_2 \cdot \vec{v}_2 = 0(0) + 1(1) + 0(0) = 1$$



Both vectors have length 1, so now we'll just confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = 0(0) + 0(1) + 1(0) = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1,  $\vec{v}_1$  and  $\vec{v}_2$  form an orthonormal set, so  $V$  is orthonormal.

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## Converting into an orthonormal basis

Notice in this example that we used two of the three standard basis vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  for  $\mathbb{R}^3$ . In fact, it should make intuitive sense that the standard basis vectors will form an orthonormal set in any  $\mathbb{R}^n$ .

Considering the fact that the standard basis vectors are extremely easy to use as a basis, it should make sense then that orthonormal bases in general make for good bases.

For instance, to convert a vector from the standard basis to an alternate basis  $B$ , normally we would solve a matrix equation like

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ v_1 & v_2 & v_3 \\ \vdots & \vdots & \vdots \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ where } \vec{x} = (x_1, x_2, x_3)$$

Which means we would create an augmented matrix, put it in rref, and from that rref matrix pull out the components of  $[\vec{x}]_B$ .



But if we're converting to an orthonormal basis specifically, that means the column vectors  $v_1, v_2, v_3$  form an orthonormal set of vectors. An **orthogonal matrix** is a square matrix whose columns form an orthonormal set of vectors. If a matrix is rectangular, but its columns still form an orthonormal set of vectors, then we call it an **orthonormal matrix**.

When a matrix is orthogonal, we know that its transpose is the same as its inverse. So given an orthogonal matrix  $A$ ,

$$A^T = A^{-1}$$

Orthogonal matrices are always square (an orthonormal matrix can be rectangular, but if we call a matrix orthogonal, we specifically mean that it's a square matrix), so its inverse can be defined, assuming that the square matrix is invertible.

Using an orthonormal basis simplifies many of the operations and formulas that we've learned. Here, if the column vectors  $v_1, v_2, v_3$  in

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ v_1 & v_2 & v_3 \\ \vdots & \vdots & \vdots \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

form an orthonormal set, then the matrix problem simplifies to a dot product problem. Specifically, the equation above becomes

$$[\vec{x}]_B = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \\ \vec{v}_3 \cdot \vec{x} \end{bmatrix}$$



In other words, instead of putting the augmented matrix into reduced row-echelon form, we just need to take dot products of the vectors that define the orthonormal basis  $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  and the vector  $\vec{x}$ .

Especially as the dimension gets larger, for instance if we're in  $\mathbb{R}^{100}$  instead of just  $\mathbb{R}^3$ , the matrix problem becomes unmanageable without a computer. But with an orthonormal set, the matrix problem becomes a dot product problem, which is a much easier calculation.

### Example

Convert  $\vec{x} = (5, 6, -1)$  from the standard basis to the alternate basis  $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ .

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}$$

First, let's confirm that the set is orthonormal. Confirm that the length of each vector is 1.

$$\|\vec{v}_1\|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + 0 + \frac{1}{2} = 1$$

$$\|\vec{v}_2\|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$$



$$||\vec{v}_3||^2 = \left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$$

Confirm that each vector is orthogonal to the others.

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}} \left(\frac{1}{2}\right) + 0 \left(\frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}} \left(\frac{1}{2}\right) = \frac{1}{2\sqrt{2}} + 0 - \frac{1}{2\sqrt{2}} = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}} \left(\frac{1}{2}\right) + 0 \left(-\frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}} \left(\frac{1}{2}\right) = \frac{1}{2\sqrt{2}} + 0 - \frac{1}{2\sqrt{2}} = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = \frac{1}{2} \left(\frac{1}{2}\right) + \frac{\sqrt{2}}{2} \left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{2} \left(\frac{1}{2}\right) = \frac{1}{4} - \frac{2}{4} + \frac{1}{4} = 0$$

Each of the vectors  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  has a length 1, and is orthogonal to the other vectors in the set, so the set is orthonormal.

Because the set is orthonormal, the vector  $\vec{x} = (5, 6, -1)$  can be converted to the alternate basis  $B$  with dot products. In other words, instead of solving

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$$

which would require us to put the augmented matrix into reduced row-echelon form, we can simply take dot products to get the value of  $[\vec{x}]_B$ .



$$[\vec{x}]_B = \begin{bmatrix} \frac{1}{\sqrt{2}}(5) + 0(6) - \frac{1}{\sqrt{2}}(-1) \\ \frac{1}{2}(5) + \frac{\sqrt{2}}{2}(6) + \frac{1}{2}(-1) \\ \frac{1}{2}(5) - \frac{\sqrt{2}}{2}(6) + \frac{1}{2}(-1) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{5}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} \\ \frac{5}{2} + 3\sqrt{2} - \frac{1}{2} \\ \frac{5}{2} - 3\sqrt{2} - \frac{1}{2} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{6}{\sqrt{2}} \\ 2 + 3\sqrt{2} \\ 2 - 3\sqrt{2} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{6\sqrt{2}}{2} \\ 2 + 3\sqrt{2} \\ 2 - 3\sqrt{2} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 3\sqrt{2} \\ 2 + 3\sqrt{2} \\ 2 - 3\sqrt{2} \end{bmatrix}$$

This result tells us that  $\vec{x} = (5, 6, -1)$  can be expressed in the alternate basis  $B$  as



$$[\vec{x}]_B = \begin{bmatrix} 3\sqrt{2} \\ 2 + 3\sqrt{2} \\ 2 - 3\sqrt{2} \end{bmatrix}$$

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