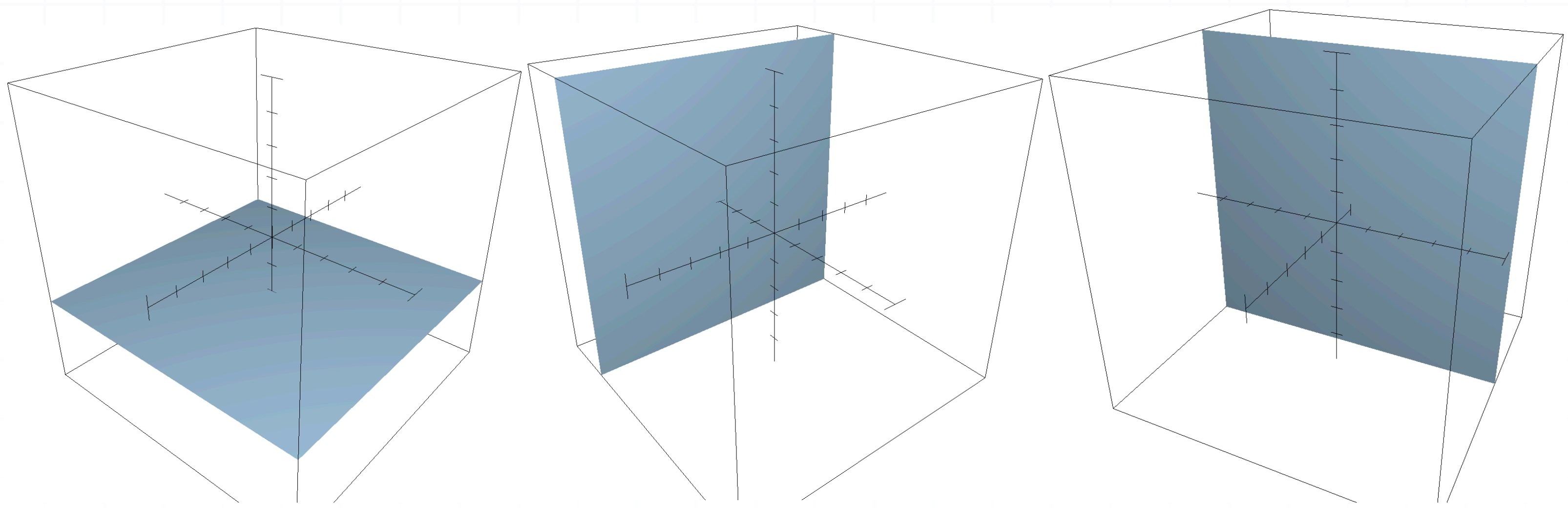
Equation of a plane, and normal vectors

A **plane** is a perfectly flat surface that goes on forever in every direction in three-dimensional space.

Because it's impossible to show a plane going on forever, when we sketch a plane in space, we only sketch a portion of it. These are all examples of planes in space:



Mathematically, we define a **plane** as the set of all vectors that are perpendicular (orthogonal) to one given **normal vector**, which is the vector that's perpendicular (orthogonal) to the plane.

So if you imagine the infinite number of vectors that all lie in the same plane, a normal vector $\vec{n}$ will be perpendicular (orthogonal) to every vector in the plane.

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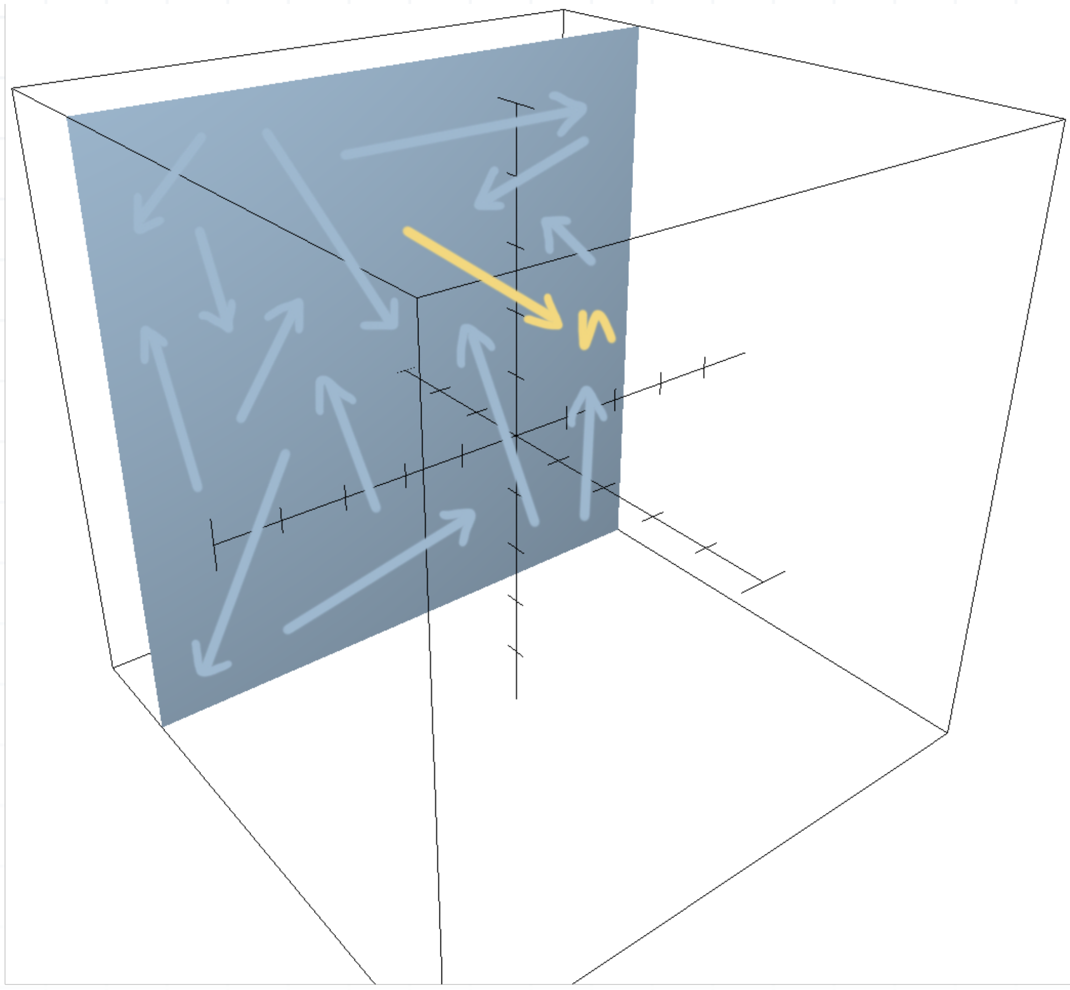

Equation of a plane from the dot product

The standard equation of a plane is given by

$$Ax + By + Cz = D$$

where the normal vector is $\overrightarrow{n} = (A, B, C)$. But you'll also see the equation of a plane written as

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

The reason this second equation represents a plane is because it's the dot product of the normal vector and a vector in the plane.

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Here's why. Remember that two vectors are perpendicular (or orthogonal) when their dot product is 0, which means that we can get the equation of a plane by setting the dot product of the normal vector and a vector in the plane equal to 0.

In other words, let's say that we define two points on the plane (x_0, y_0, z_0) and (x, y, z) with the associated position vectors $\vec{x_0}$ and $\vec{x}$.

$$\vec{x_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Because (x_0, y_0, z_0) and (x, y, z) are both points on the plane, the vector that connects those two points must lie in the plane as well. That vector $(\vec{x} - \vec{x_0})$ is given by

$$(\vec{x} - \vec{x_0}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$$

If the normal vector to the plane is given by $\vec{n} = (a, b, c)$, then we know that the vector $(\vec{x} - \vec{x_0})$ and the normal vector $\vec{n}$ are orthogonal, which means their dot product is 0.

$$\vec{n} \cdot (\vec{x} - \vec{x_0}) = 0$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

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This is how we get the equation of the plane. Then, given a point on the plane and a normal vector to the plane, the plane's equation will simplify to the standard form $Ax + By + Cz = D$.

For example, given the plane defined by the point $(2,5,-1)$ and the normal vector $\vec{n} = (-1,-2,1)$, the plane's equation would be

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$a(x - 2) + b(y - 5) + c(z - (-1)) = 0$$

$$-1(x - 2) - 2(y - 5) + 1(z - (-1)) = 0$$

$$-x + 2 - 2y + 10 + z + 1 = 0$$

$$-x - 2y + z + 13 = 0$$

$$-x - 2y + z = -13$$

Notice how this plane equation matches the standard equation of the plane, $Ax + By + Cz = D$, with $A = -1$, $B = -2$, $C = 1$, and $D = -13$.

Let's do an example where we build the equation of a plane from its normal vector and a point that lies in the plane.

Example

Find the equation of a plane with normal vector $\vec{n} = (-3,2,5)$ and that passes through $(1,0,-2)$.

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Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$-3(x-1)+2(y-0)+5(z-(-2))=0$$

Now we'll simplify and get the equation of the plane into standard form.

$$-3x+3+2y+5z+10=0$$

$$-3x+2y+5z+13=0$$

$$-3x+2y+5z=-13$$

This is the equation of the plane in standard form, with $A=-3$, $B=2$, $C=5$, and $D=-13$.

Identifying the normal vector

We've looked at how to build the equation of a plane when we start with a point on the plane and a normal vector to the plane. But we can also work backwards. When we start with the equation of the plane in standard form, we can identify the normal vector from that equation.

Given a plane $Ax+By+Cz=D$, the normal vector to that plane is

$$\overrightarrow{n}=(A,B,C)$$

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Because the value of D doesn't change the orientation of the plane, it only shifts it, changing the value of D won't change the normal vector. So the normal vector

$$\vec{n} = (A, B, C)$$

would be normal to all of these planes:

$$Ax + By + Cz = 0$$

$$Ax + By + Cz = \pi$$

$$Ax + By + Cz = -7$$

$$Ax + By + Cz = e$$

This is pretty straightforward, but let's do a quick example just to make sure we know how to grab the normal vector.

Example

Find the normal vector to the plane.

$$-3x + 2y + 5z = -13$$

This is the same plane we found in the last example, so we already know the components of the normal vector. But if we didn't know the normal, we could still simply pick it out of the equation of the plane.

Pulling out the coefficients on x, y, and z, the normal vector is

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$$\vec{n} = (-3,2,5)$$

Notice how the plane equation could be multiplied through by -1, and therefore be rewritten as

$$-1(-3x + 2y + 5z) = -1(-13)$$

$$3x - 2y - 5z = 13$$

This is still the same plane, so another normal vector to the plane could be

$$\vec{n} = (3, -2, -5)$$

Furthermore, we could multiply both sides of the equation by any other scalar, and we'd get another normal vector. All of these are normal vectors to the same plane, some of which are pointing away from the plane on one side, and others of which are pointing away from the plane on the other side.

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