Preimage, image, and the kernel

In the last lesson we looked at a quadrilateral Q, and we said that Q was a subset of $\mathbb{R}^2$. We used the transformation T to transform Q, and said that $T(Q)$ was the image of Q under T. In other words, applying the transformation T to the preimage Q, we were able to find the image $T(Q)$.

We'll talk more about preimages and images in this lesson, but let's take a quick moment just to say that every transformation is a subspace.

Transformations as subspaces

Now we want to talk about $T(Q)$ as a subspace. Remember that in order for a set to be a subspace, it needs to be closed under addition and closed under scalar multiplication.

As it turns out, by the definition of transformations, a transformation will always be closed under addition and closed under scalar multiplication, which means that the transformation will always be a subspace. So sticking with the example where we transformed Q by the transformation T, we can say that $T(Q)$ is a subspace.

Finding the preimage from the image

Normally, when we talk about the transformation of a set, like $T : A \to B$, we say that the transformation T transforms vectors from the domain A into the codomain B.

But we can also talk about subsets of A and B. For instance, let's say we have a subset A_1 that's inside A. We write that as $A_1 \subseteq A$, which means that A_1 is contained within A as a subset of A. The transformation T will map A_1 to a subset of B, and we can write this transformation of A_1 as $T(A_1)$, and call it the image of A_1 under T.

But sometimes we want to work backwards, starting with a subset of B, we'll call it B_1 where $B_1 \subseteq B$, and trying to find all the points in A that map to B_1. The collection of all the points in A that map to the subset B_1 is the preimage of B_1. Since it's kind of like we're doing a reverse transformation (trying to find all the vectors in A that will map to the subset B_1 under the transformation T), we write the preimage of B_1 under T as $T^{-1}(B_1)$.

In other words,

- if we start with a subset of the domain, under the transformation T it'll map to the image of the subset in the codomain, but

- if we start with a subset of the codomain, under the inverse transformation T^{-1} it'll map to the preimage of the subset in the domain.

Let's actually look at how we'd find the preimage of a set B_1 that's in the codomain, if the set B_1 is made up of two vectors.

Example

Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We're trying to find the preimage of B_1 under T, which we'll call $T^{-1}(B_1)$.

$$T^{-1}(B_1) = \{ \vec{x} \in \mathbb{R}^2 \ | \ T(\vec{x}) \in B_1 \}$$

$$T^{-1}(B_1) = \left\{ \vec{x} \in \mathbb{R}^2 \ \middle| \ \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

In other words, we're just trying to find all the vectors $\vec{x}$ in $\mathbb{R}^2$ that satisfy either of these matrix equations. So let's rewrite both augmented matrices in reduced row-echelon form. We get

$$\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & | & 0 \\ 2 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & | & 0 \\ -3 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ -3 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & | & 2 \\ 2 & 0 & | & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & | & 3 \\ -3 & 1 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{3}{2} \\ -3 & 1 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{3}{2} \\ 0 & 1 & | & \frac{13}{2} \end{bmatrix}$$

From the first augmented matrix, we get $x_1 = 0$ and $x_2 = 0$. And from the second augmented matrix we get $x_1 = 3/2$ and $x_2 = 13/2$. Therefore,

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the subset B_1 under T

$\begin{bmatrix} \frac{3}{2} \\ \frac{13}{2} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in the subset B_1 under T

Keep in mind that the vector $\vec{x}$ in

$$\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is the null space of the transformation matrix. The vector (or set of vectors) $\vec{x}$ that makes this matrix equation true is called the kernel of the transformation T, Ker(T).

In other words, the **kernel** of a transformation T is all of the vectors that result in the zero vector under the transformation T:

$$\text{Ker}(T) = \left\{ \vec{x} \in \mathbb{R}^2 \ \middle|\ T(\vec{x}) = \vec{O} \right\}$$