Topic: Transposes of products, sums, and inverses

Question: Find $(AB)^T$.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & -6 & 1 \\ 0 & -8 & 5 \\ 1 & 1 & -2 \end{bmatrix}$$

Answer choices:

A $\quad (AB)^T = \begin{bmatrix} 2 & 9 & 16 \\ -16 & -35 & 10 \\ 10 & 15 & -20 \end{bmatrix}$

B $\quad (AB)^T = \begin{bmatrix} 10 & 15 & -20 \\ -16 & -35 & 10 \\ 2 & 9 & 16 \end{bmatrix}$

C $\quad (AB)^T = \begin{bmatrix} 2 & -16 & 10 \\ 9 & -35 & 15 \\ 16 & 10 & -20 \end{bmatrix}$

D $\quad (AB)^T = \begin{bmatrix} 10 & -16 & 2 \\ 15 & -35 & 9 \\ -20 & 10 & 16 \end{bmatrix}$

Solution: A

There are two ways we could go about finding $(AB)^T$. Given the rule for products of transposes,

$$(XY)^T = Y^T X^T$$

we could either calculate the left side, finding the product AB and then taking its transpose, or we could calculate the right side, finding A^T and B^T individually, and then taking their product.

Let's use the second method, where we start by taking the transposes individually. We'll swap rows and columns in A and B to get A^T and B^T.

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & -3 \\ -2 & 1 & 4 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 4 & 0 & 1 \\ -6 & -8 & 1 \\ 1 & 5 & -2 \end{bmatrix}$$

Now we'll find the product of these transposes.

$$B^T A^T = \begin{bmatrix} 4 & 0 & 1 \\ -6 & -8 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & -3 \\ -2 & 1 & 4 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 4(1) + 0(1) + 1(-2) & 4(2) + 0(3) + 1(1) & 4(3) + 0(-3) + 1(4) \\ -6(1) - 8(1) + 1(-2) & -6(2) - 8(3) + 1(1) & -6(3) - 8(-3) + 1(4) \\ 1(1) + 5(1) - 2(-2) & 1(2) + 5(3) - 2(1) & 1(3) + 5(-3) - 2(4) \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 4+0-2 & 8+0+1 & 12+0+4 \\ -6-8-2 & -12-24+1 & -18+24+4 \\ 1+5+4 & 2+15-2 & 3-15-8 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 2 & 9 & 16 \\ -16 & -35 & 10 \\ 10 & 15 & -20 \end{bmatrix}$$

So we can say

$$(AB)^T = B^TA^T = \begin{bmatrix} 2 & 9 & 16 \\ -16 & -35 & 10 \\ 10 & 15 & -20 \end{bmatrix}$$

Topic: Transposes of products, sums, and inverses

Question: Find $(A+B)^T$.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & -6 & 1 \\ 0 & -8 & 5 \\ 1 & 1 & -2 \end{bmatrix}$$

Answer choices:

A $\quad (A+B)^T = \begin{bmatrix} -1 & -5 & 5 \\ 6 & -5 & 2 \\ 2 & -2 & 4 \end{bmatrix}$

B $\quad (A+B)^T = \begin{bmatrix} 5 & -5 & -1 \\ 2 & -5 & 6 \\ 4 & -2 & 2 \end{bmatrix}$

C $\quad (A+B)^T = \begin{bmatrix} -1 & 6 & 2 \\ -5 & -5 & -2 \\ 5 & 2 & 4 \end{bmatrix}$

D $\quad (A+B)^T = \begin{bmatrix} 5 & 2 & 4 \\ -5 & -5 & -2 \\ -1 & 6 & 2 \end{bmatrix}$

Solution: D

There are two ways we could go about finding $(A+B)^T$. Given the rule for sums of transposes,

$$(X+Y)^T = X^T + Y^T$$

we could either calculate the left side, finding the sum $A+B$ and then taking its transpose, or we could calculate the right side, finding A^T and B^T individually, and then taking their sum.

Let's use the first method, where we start by finding the sum of the matrices.

$$A+B = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -6 & 1 \\ 0 & -8 & 5 \\ 1 & 1 & -2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1+4 & 1-6 & -2+1 \\ 2+0 & 3-8 & 1+5 \\ 3+1 & -3+1 & 4-2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 5 & -5 & -1 \\ 2 & -5 & 6 \\ 4 & -2 & 2 \end{bmatrix}$$

Now we'll find the transpose of the sum by swapping the rows and columns of $A+B$.

$$(A+B)^T = \begin{bmatrix} 5 & 2 & 4 \\ -5 & -5 & -2 \\ -1 & 6 & 2 \end{bmatrix}$$

So we can say

$$(A+B)^T = \begin{bmatrix} 5 & 2 & 4 \\ -5 & -5 & -2 \\ -1 & 6 & 2 \end{bmatrix}$$

Topic: Transposes of products, sums, and inverses

Question: Find $(A^{-1})^T$.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{bmatrix}$$

Answer choices:

A $\quad (A^{-1})^T = \begin{bmatrix} -\frac{3}{8} & \frac{1}{8} & \frac{3}{8} \\ -\frac{1}{20} & -\frac{1}{4} & -\frac{3}{20} \\ -\frac{7}{40} & \frac{1}{8} & -\frac{1}{40} \end{bmatrix}$

B $\quad (A^{-1})^T = \begin{bmatrix} -\frac{3}{8} & -\frac{1}{20} & -\frac{7}{40} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \\ \frac{3}{8} & -\frac{3}{20} & -\frac{1}{40} \end{bmatrix}$

C $\quad (A^{-1})^T = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{1}{20} & \frac{1}{4} & \frac{3}{20} \\ \frac{7}{40} & -\frac{1}{8} & \frac{1}{40} \end{bmatrix}$

D $\quad (A^{-1})^T = \begin{bmatrix} \frac{3}{8} & \frac{1}{20} & \frac{7}{40} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$

Solution: C

There are two ways we could go about finding $(A^{-1})^T$. Given the rule for the inverse of a transpose,

$$(X^T)^{-1} = (X^{-1})^T$$

we could either calculate the left side, finding the transpose A^T and then taking its inverse, or we could calculate the right side, finding the inverse A^{-1} and then taking its transpose.

Let's use the second method, where we start by finding the inverse A^{-1}. We'll augment A with I_3, and then put the left side of the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & -2 & | & 1 & 0 & 0 \\ 2 & 3 & 1 & | & 0 & 1 & 0 \\ 3 & -3 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

We already have the pivot entry in the first row, so let's work on zeroing out the rest of the first column.

$$\begin{bmatrix} 1 & 1 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 3 & -3 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & -6 & 10 & | & -3 & 0 & 1 \end{bmatrix}$$

We already have the pivot in the second column, so let's work on zeroing out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -7 & | & 3 & -1 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & -6 & 10 & | & -3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -7 & | & 3 & -1 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & 0 & 40 & | & -15 & 6 & 1 \end{bmatrix}$$

Find the pivot in the third column, then zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & -7 & | & 3 & -1 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{8} & \frac{1}{20} & \frac{7}{40} \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{8} & \frac{1}{20} & \frac{7}{40} \\ 0 & 1 & 0 & | & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \\ 0 & 0 & 1 & | & -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$$

Now that the left side of the augmented matrix is the identity matrix, the right side is the inverse A^{-1}.

$$A^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{20} & \frac{7}{40} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$$

To find $(A^{-1})^T$, we'll take the transpose of this inverse.

$$(A^{-1})^T = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{1}{20} & \frac{1}{4} & \frac{3}{20} \\ \frac{7}{40} & -\frac{1}{8} & \frac{1}{40} \end{bmatrix}$$