Eigen in three dimensions

The process for finding the eigenvalues for a 3×3 matrix is the same as the process we used to do so for the 2×2 matrix.

But calculating the 3×3 determinant and factoring the third-degree characteristic polynomial will be more complex than finding the 2×2 determinant or factoring the second-degree characteristic polynomial.

Let's walk through an example so that we can see the full process.

Example

Find the eigenvalues and eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

We need to find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \right|$$

$$\left|\begin{bmatrix} \lambda-(-1) & 0-1 & 0-0 \\ 0-1 & \lambda-2 & 0-1 \\ 0-0 & 0-3 & \lambda-(-1) \end{bmatrix}\right|$$

$$\left|\begin{bmatrix} \lambda+1 & -1 & 0 \\ -1 & \lambda-2 & -1 \\ 0 & -3 & \lambda+1 \end{bmatrix}\right|$$

Then the determinant of this resulting matrix is

$$(\lambda+1)\begin{vmatrix} \lambda-2 & -1 \\ -3 & \lambda+1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & \lambda+1 \end{vmatrix} + 0\begin{vmatrix} -1 & \lambda-2 \\ 0 & -3 \end{vmatrix}$$

$$(\lambda+1)[(\lambda-2)(\lambda+1)-(-1)(-3)] - (-1)[(-1)(\lambda+1)-(-1)(0)]$$

$$+0[(-1)(-3)-(\lambda-2)(0)]$$

$$(\lambda+1)[(\lambda-2)(\lambda+1)-3] - (\lambda+1)$$

$$(\lambda+1)(\lambda^2+\lambda-2\lambda-2-3) - (\lambda+1)$$

$$(\lambda+1)(\lambda^2-\lambda-5) - (\lambda+1)$$

Remember that we're trying to satisfy $|\lambda I_n - A| = 0$, so we can set this characteristic polynomial equal to 0 to get the characteristic equation:

$$(\lambda+1)(\lambda^2-\lambda-5) - (\lambda+1) = 0$$

To solve for λ, we'll factor.

$$(\lambda+1)[(\lambda^2-\lambda-5)-1] = 0$$

$$(\lambda+1)(\lambda^2-\lambda-5-1) = 0$$

$$(\lambda+1)(\lambda^2-\lambda-6)=0$$

$$(\lambda+1)(\lambda-3)(\lambda+2)=0$$

$\lambda=-2$ or $\lambda=-1$ or $\lambda=3$

So assuming non-zero eigenvectors, we're saying that $A\vec{v}=\lambda\vec{v}$ can be solved for $\lambda=-2$, $\lambda=-1$, and $\lambda=3$.

With these three eigenvalues, we'll have three eigenspaces, given by $E_\lambda=N(\lambda I_n-A)$. Given

$$E_\lambda=N\left(\begin{bmatrix}\lambda+1 & -1 & 0\\ -1 & \lambda-2 & -1\\ 0 & -3 & \lambda+1\end{bmatrix}\right)$$

we get

$$E_{-2}=N\left(\begin{bmatrix}-2+1 & -1 & 0\\ -1 & -2-2 & -1\\ 0 & -3 & -2+1\end{bmatrix}\right)$$

$$E_{-2}=N\left(\begin{bmatrix}-1 & -1 & 0\\ -1 & -4 & -1\\ 0 & -3 & -1\end{bmatrix}\right)$$

and

$$E_{-1}=N\left(\begin{bmatrix}-1+1 & -1 & 0\\ -1 & -1-2 & -1\\ 0 & -3 & -1+1\end{bmatrix}\right)$$

$$E_{-1} = N\left(\begin{bmatrix} 0 & -1 & 0 \\ -1 & -3 & -1 \\ 0 & -3 & 0 \end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3+1 & -1 & 0 \\ -1 & 3-2 & -1 \\ 0 & -3 & 3+1 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -3 & 4 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -3 & -1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 & | & 0 \\ -1 & -4 & -1 & | & 0 \\ 0 & -3 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ -1 & -4 & -1 & | & 0 \\ 0 & -3 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & -3 & -1 & | & 0 \\ 0 & -3 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & -3 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & \frac{1}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} & | & 0 \\ 0 & 1 & \frac{1}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 - \frac{1}{3}v_3 = 0$$

$$v_2 + \frac{1}{3}v_3 = 0$$

or

$$v_1 = \frac{1}{3}v_3$$

$$v_2 = -\frac{1}{3}v_3$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_{-1} will satisfy

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -3 & -1 \\ 0 & -3 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 & | & 0 \\ -1 & -3 & -1 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -3 & -1 & | & 0 \\ 0 & -1 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & | & 0 \\ 0 & -1 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 + v_3 = 0$$

$$v_2 = 0$$

or

$$v_1 = -v_3$$

$$v_2 = 0$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-1} is defined by

$$E_{-1} = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_3 will satisfy

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -3 & 4 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 0 & | & 0 \\ -1 & 1 & -1 & | & 0 \\ 0 & -3 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & -1 & | & 0 \\ 4 & -1 & 0 & | & 0 \\ 0 & -3 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 4 & -1 & 0 & | & 0 \\ 0 & -3 & 4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 3 & -4 & | & 0 \\ 0 & -3 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 3 & -4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & -\frac{4}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & | & 0 \\ 0 & 1 & -\frac{4}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 - \frac{1}{3}v_3 = 0$$

$$v_2 - \frac{4}{3}v_3 = 0$$

or

$$v_1 = \frac{1}{3}v_3$$

$$v_2 = \frac{4}{3}v_3$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}$$

Which means that E_3 is defined by

$$E_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} \right)$$

Let's put these results together. For the eigenvalues $\lambda = -2$, $\lambda = -1$, and $\lambda = 3$, respectively, we got

$$E_{-2} = \text{Span}\left(\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right), \ E_{-1} = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right), \text{ and } E_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} \right)$$

Each of these spans represents a line in $\mathbb{R}^3$. So for any vector $\vec{v}$ along any of these lines, when you apply the transformation T to the vector $\vec{v}$, $T(\vec{v})$ will be a vector along the same line, it might just be scaled up or scaled down.

Specifically,

- since $\lambda = -2$ in the eigenspace E_{-2}, any vector $\vec{v}$ in E_{-2}, under the transformation T, will be scaled by -2, meaning that $T(\vec{v}) = \lambda\vec{v} = -2\vec{v}$,

- since $\lambda = -1$ in the eigenspace E_{-1}, any vector $\vec{v}$ in E_{-1}, under the transformation T, will be scaled by -1, meaning that $T(\vec{v}) = \lambda\vec{v} = -1\vec{v} = -\vec{v}$, and

- since $\lambda = 3$ in the eigenspace E_3, any vector $\vec{v}$ in E_3, under the transformation T, will be scaled by 3, meaning that $T(\vec{v}) = \lambda\vec{v} = 3\vec{v}$.