Topic: Orthogonal complements of the fundamental subspaces

Question: For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} -1 & 3 & 5 & -1 \\ 2 & 0 & 2 & -4 \\ -3 & -5 & 9 & 0 \end{bmatrix}$$

Answer choices:

A Dim$(C(A)) = 3$, Dim$(N(A)) = 0$, Dim$(C(A^T)) = 3$, Dim$(N(A^T)) = 1$

B Dim$(C(A)) = 1$, Dim$(N(A)) = 3$, Dim$(C(A^T)) = 1$, Dim$(N(A^T)) = 2$

C Dim$(C(A)) = 3$, Dim$(N(A)) = 1$, Dim$(C(A^T)) = 3$, Dim$(N(A^T)) = 0$

D Dim$(C(A)) = 1$, Dim$(N(A)) = 3$, Dim$(C(A^T)) = 3$, Dim$(N(A^T)) = 2$

Solution: C

Put A into reduced row-echelon form.

$$\begin{bmatrix} -1 & 3 & 5 & -1 \\ 2 & 0 & 2 & -4 \\ -3 & -5 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 & 1 \\ 2 & 0 & 2 & -4 \\ -3 & -5 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 6 & 12 & -6 \\ -3 & -5 & 9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 6 & 12 & -6 \\ 0 & -14 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -14 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 22 & -11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 22 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

In reduced row-echelon form, we can see that there are three pivots, which means the rank of A is $r = 3$.

The matrix A is a 3×4 matrix, which means there are $m = 3$ rows and $n = 4$ columns. Therefore, the dimensions of the four fundamental subspaces of A are:

Column space, $C(A)$	$r = 3$
Null space, $N(A)$	$n - r = 4 - 3 = 1$
Row space, $C(A^T)$	$r = 3$
Left null space, $N(A^T)$	$m - r = 3 - 3 = 0$

Topic: Orthogonal complements of the fundamental subspaces

Question: For the matrix A, which of these dimensions of the four fundamental subspaces is incorrect?

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ 1 & -1 & 2 \end{bmatrix}$$

Answer choices:

A $\quad$ Dim$(C(A)) = 2$

B $\quad$ Dim$(N(A)) = 0$

C $\quad$ Dim$(C(A^T)) = 2$

D $\quad$ Dim$(N(A^T)) = 1$

Solution: B

Put A into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

In reduced row-echelon form, we can see that there are two pivots, which means the rank of A is $r = 2$.

The matrix A is a 3×3 matrix, which means there are $m = 3$ rows and $n = 3$ columns. Therefore, the dimensions of the four fundamental subspaces of A are:

Column space, $C(A)$ $\qquad r = 2$

Null space, $N(A)$ $\qquad n - r = 3 - 2 = 1$

Row space, $C(A^T)$ $\qquad r = 2$

Left null space, $N(A^T)$ $\qquad m - r = 3 - 2 = 1$

Topic: Orthogonal complements of the fundamental subspaces

Question: For the matrix A, which of these dimensions of the four fundamental subspaces is incorrect?

$$A = \begin{bmatrix} 2 & -3 & 6 & -5 & -6 \\ 4 & -5 & 12 & -11 & -14 \\ 2 & -2 & 6 & -6 & -8 \end{bmatrix}$$

Answer choices:

A Dim$(C(A)) = 2$

B Dim$(N(A)) = 3$

C Dim$(C(A^T)) = 2$

D Dim$(N(A^T)) = 2$

Solution: D

Put A into reduced row-echelon form.

$$\begin{bmatrix} 2 & -3 & 6 & -5 & -6 \\ 4 & -5 & 12 & -11 & -14 \\ 2 & -2 & 6 & -6 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & 3 & -\frac{5}{2} & -3 \\ 4 & -5 & 12 & -11 & -14 \\ 2 & -2 & 6 & -6 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & 3 & -\frac{5}{2} & -3 \\ 0 & 1 & 0 & -1 & -2 \\ 2 & -2 & 6 & -6 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & 3 & -\frac{5}{2} & -3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & 3 & -\frac{5}{2} & -3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -4 & -6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In reduced row-echelon form, we can see that there are two pivots, which means the rank of A is $r = 2$.

The matrix A is a 3×5 matrix, which means there are $m = 3$ rows and $n = 5$ columns. Therefore, the dimensions of the four fundamental subspaces of A are:

Column space, $C(A)$	$r = 2$
Null space, $N(A)$	$n - r = 5 - 2 = 3$
Row space, $C(A^T)$	$r = 2$
Left null space, $N(A^T)$	$m - r = 3 - 2 = 1$