Cramer's rule for solving systems

Cramer's rule is a simple little rule that lets us use determinants to solve a system of linear equations. The rule says that you can solve for any variable in the system by calculating

$$\frac{D_v}{D}$$

where D_v is the determinant of the coefficient matrix with the answer column values in the variable column you're trying to solve, and where D is the determinant of the coefficient matrix.

Which means that, if we want to find the value of x, we need to find D_x/D, and if we want to find the value of y, we need to find D_y/D.

For example, given the linear system of two equations in two unknowns,

$$a_1x + b_1y = d_1$$

$$a_2x + b_2y = d_2$$

we can say that

$$x = \frac{D_x}{D}, \ y = \frac{D_y}{D}, \text{ with } D \neq 0$$

where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \ D_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}, \ D_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$$

All that sounds tricky, but let's look at an example to break it down.

Example

Solve for x in the system.

$$2x - 3y = 5$$

$$3x + 12y = -8$$

Because we're looking for the value of x, we want to find D_x/D. We need to start with the coefficient matrix for the system.

$$\begin{bmatrix}2 & -3\\3 & 12\end{bmatrix}$$

The answer column matrix is built from the constants from the right side of the system,

$$\begin{bmatrix}5\\-8\end{bmatrix}$$

D_x is the determinant of the coefficient matrix with the answer column matrix substituted into the x-column, so

$$D_x = \begin{vmatrix}5 & -3\\-8 & 12\end{vmatrix}$$

D is the determinant of the coefficient matrix, so

$$D = \begin{vmatrix} 2 & -3 \\ 3 & 12 \end{vmatrix}$$

Then, putting these values together, Cramer's rule tells us that the value of x in the system is

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 5 & -3 \\ -8 & 12 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & 12 \end{vmatrix}}$$

$$x = \frac{(5)(12) - (-3)(-8)}{(2)(12) - (3)(-3)}$$

$$x = \frac{36}{33} = \frac{12}{11}$$

Let's do another example where we use Cramer's rule to solve for y.

Example

Use Cramer's rule to solve for the value of y that satisfies the system.

$$3x - 2y = 7$$

$$5x - 8y = 21$$

The coefficient matrix is

$$\begin{bmatrix} 3 & -2 \\ 5 & -8 \end{bmatrix}$$

The answer column matrix is

$$\begin{bmatrix} 7 \\ 21 \end{bmatrix}$$

Then D_y is what we get when we plug the answer column matrix into the second column of the coefficient matrix (because the second column holds the coefficients on y), and then take the determinant of the result.

$$D_y = \begin{vmatrix} 3 & 7 \\ 5 & 21 \end{vmatrix}$$

The determinant of the coefficient matrix is

$$D = \begin{vmatrix} 3 & -2 \\ 5 & -8 \end{vmatrix}$$

Then, putting these values together, Cramer's rule tells us that the value of y in the system is

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & 7 \\ 5 & 21 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 5 & -8 \end{vmatrix}}$$

$$y = \frac{(3)(21) - (7)(5)}{(3)(-8) - (-2)(5)}$$

$$y = \frac{28}{-14} = -2$$

Or given a linear system of three equations in three unknowns,

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

we can say that

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D},\ \text{with } D \neq 0$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Let's look at an example of a linear system with three equations.

Example

Use Cramer's rule to solve the system.

$$-7x + 6z = 99$$

$$-4x - 8y - 6z = 40$$

$$8x - y - 9z = -121$$

The coefficient matrix is

$$\begin{bmatrix} -7 & 0 & 6 \\ -4 & -8 & -6 \\ 8 & -1 & -9 \end{bmatrix}$$

The answer column matrix is

$$\begin{bmatrix} 99 \\ 40 \\ -121 \end{bmatrix}$$

Then D_x, D_y, and D_z are what we get when we plug the answer column matrix into the first, second, and third columns, respectively, of the coefficient matrix, and then take the determinant of the result.

$$D_x = \begin{vmatrix} 99 & 0 & 6 \\ 40 & -8 & -6 \\ -121 & -1 & -9 \end{vmatrix}$$

$$D_x = 99 \begin{vmatrix} -8 & -6 \\ -1 & -9 \end{vmatrix} - 0 \begin{vmatrix} 40 & -6 \\ -121 & -9 \end{vmatrix} + 6 \begin{vmatrix} 40 & -8 \\ -121 & -1 \end{vmatrix}$$

$$D_x = 99\left[(-8)(-9) - (-6)(-1)\right] + 6\left[(40)(-1) - (-8)(-121)\right]$$

$$D_x = 99(72 - 6) + 6(-40 - 968)$$

$$D_x = 99(66) + 6(-1{,}008)$$

$$D_x = 6{,}534 - 6{,}048$$

$$D_x = 486$$

and

$$D_y = \begin{vmatrix} -7 & 99 & 6 \\ -4 & 40 & -6 \\ 8 & -121 & -9 \end{vmatrix}$$

$$D_y = -7 \begin{vmatrix} 40 & -6 \\ -121 & -9 \end{vmatrix} - 99 \begin{vmatrix} -4 & -6 \\ 8 & -9 \end{vmatrix} + 6 \begin{vmatrix} -4 & 40 \\ 8 & -121 \end{vmatrix}$$

$$D_y = -7\left[(40)(-9) - (-6)(-121)\right] - 99\left[(-4)(-9) - (-6)(8)\right]$$

$$+6\left[(-4)(-121) - (40)(8)\right]$$

$$D_y = -7(-360 - 726) - 99(36 + 48) + 6(484 - 320)$$

$$D_y = -7(-1{,}086) - 99(84) + 6(164)$$

$$D_y = 7{,}602 - 8{,}316 + 984$$

$$D_y = 270$$

and

$$D_z = \begin{vmatrix} -7 & 0 & 99 \\ -4 & -8 & 40 \\ 8 & -1 & -121 \end{vmatrix}$$

$$D_z = -7 \begin{vmatrix} -8 & 40 \\ -1 & -121 \end{vmatrix} - 0 \begin{vmatrix} -4 & 40 \\ 8 & -121 \end{vmatrix} + 99 \begin{vmatrix} -4 & -8 \\ 8 & -1 \end{vmatrix}$$

$$D_z = -7\left[(-8)(-121) - (40)(-1)\right] + 99\left[(-4)(-1) - (-8)(8)\right]$$

$$D_z = -7(968 + 40) + 99(4 + 64)$$

$$D_z = -7(1{,}008) + 99(68)$$

$$D_z = -7{,}056 + 6{,}732$$

$$D_z = -324$$

The determinant of the coefficient matrix is

$$D = \begin{vmatrix} -7 & 0 & 6 \\ -4 & -8 & -6 \\ 8 & -1 & -9 \end{vmatrix}$$

$$D = -7 \begin{vmatrix} -8 & -6 \\ -1 & -9 \end{vmatrix} - 0 \begin{vmatrix} -4 & -6 \\ 8 & -9 \end{vmatrix} + 6 \begin{vmatrix} -4 & -8 \\ 8 & -1 \end{vmatrix}$$

$$D = -7\left[(-8)(-9) - (-6)(-1)\right] + 6\left[(-4)(-1) - (-8)(8)\right]$$

$$D = -7(72 - 6) + 6(4 + 64)$$

$$D = -7(66) + 6(68)$$

$$D = -462 + 408$$

$$D = -54$$

Then, putting these values together, Cramer's rule tells us that the values of x, y, and z in the system are

$$x = \frac{D_x}{D} = \frac{486}{-54} = -9$$

$$y = \frac{D_y}{D} = \frac{270}{-54} = -5$$

$$z = \frac{D_z}{D} = \frac{-324}{-54} = 6$$

The solution to the system is $(x, y, z) = (-9, -5, 6)$.
