$$S_\theta = \begin{pmatrix} S_{11}(\theta) & S_{12}(\theta) \\ S_{21}(\theta) & S_{22}(\theta) \end{pmatrix}$$

$$S_\theta = \frac{1}{2}\left(I + x(\theta)\sigma_x + y(\theta)\sigma_y + z(\theta)\sigma_z\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1+z(\theta) & x(\theta)-iy(\theta) \\ x+iy(\theta) & 1-z(\theta) \end{bmatrix}$$

$$S_{11}(\theta) = \frac{1}{2}(1+z(\theta))$$

$$S_{12}(\theta) = \frac{1}{2}(x(\theta)-iy(\theta))$$

$$S_{21}(\theta) = \frac{1}{2}(x(\theta)+iy(\theta))$$

$$S_{22}(\theta) = \frac{1}{2}(1-z(\theta))$$

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$. エルミート行列

$$E_\theta[T] = \mathrm{Tr}\, S_\theta T = \mathrm{Tr}\begin{pmatrix} S_{11}(\theta)t_{11} + S_{12}(\theta)t_{21} & S_{11}(\theta)t_{12} + S_{12}(\theta)t_{22} \\ S_{21}(\theta)t_{11} + S_{22}(\theta)t_{21} & S_{21}(\theta)t_{12} + S_{22}(\theta)t_{22} \end{pmatrix}$$

$$= S_{11}(\theta)t_{11} + S_{12}(\theta)t_{21} + S_{21}(\theta)t_{12} + S_{22}(\theta)t_{22} \quad - \theta$$

$$\frac{d}{d\theta}\mathrm{Tr}\, S_\theta T = S_{11}'(\theta)t_{11} + \cdots + S_{22}'(\theta)t_{22}$$

$$= \mathrm{Tr}\left(\frac{d}{d\theta}S_\theta\right)T$$

$$\frac{d}{d\theta}S_\theta \underset{定義}{=} \begin{pmatrix} S_{11}'(\theta) & S_{12}'(\theta) \\ S_{21}'(\theta) & S_{22}'(\theta) \end{pmatrix}$$

$$(A+B)^* = A^* + B^*$$

$$(AB)^* = B^* A^*$$

$$A^{**} = A$$

$$(\log x)' = \frac{1}{x}$$

$$(\log S_\theta)' = \frac{1}{S_\theta}(S_\theta)'$$

$$A^* = \hat{A}$$

P.15

$$\frac{dS_\theta}{d\theta} = \frac{1}{2}\left[S_\theta L_\theta + L_\theta^* S_\theta\right]$$

をみたす L_θ のうち Hermite 行列
であるものが唯一存在する

$\mathcal{L}_h$: エルミート行列全体の集合

$\boxed{実線型空间}$

和 $\mathcal{L}_h \ni A, B$ $\quad \underline{A+B} \in \mathcal{L}_h$

・スカラー積 $\mathcal{L}_h \ni A,\ \lambda \in \mathbb{R}$ $\quad \lambda \in \mathcal{L}_h$

$\mathcal{L}_h \ni A \Leftrightarrow A^* = A$

$\mathcal{L}_h \ni B \Leftrightarrow B^* = B$

$(A+B)^* = A^* + B^*$
$= A + B$

$\therefore A+B \in \mathcal{L}_h$

$(\lambda A)^* = \bar{\lambda} A^*$
$= \lambda A$

$$\begin{array}{cccc} \mathcal{T}: & \mathcal{L}_h & \longrightarrow & \mathcal{L}_h \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & A & \longmapsto & \mathcal{T}A \end{array}$$

$\mathcal{T}$ 実線型写像 行列表示可能

$\mathcal{T}(A+B) = \mathcal{T}A + \mathcal{T}B$

・$\mathcal{T}(\lambda A) = \lambda \mathcal{T}(A)$, $\lambda \in \mathbb{R}$

$$\mathcal{T}: L \longrightarrow \underbrace{\frac{1}{2}\left[S_\theta L + L S_\theta\right]}_{= \mathcal{T}L}$$

$$\mathcal{T}(L_1 + L_2) = \frac{1}{2}\left[S_\theta(L_1+L_2) + (L_1+L_2)S_\theta\right]$$
$$= \frac{1}{2}\left[S_\theta L_1 + L_1 S_\theta\right] + \frac{1}{2}\left[S_\theta L_2 + L_2 S_\theta\right] = \mathcal{T}L_1 + \mathcal{T}L_2$$

$$\mathcal{T}(\lambda L) = \frac{1}{2}\left[S_\theta \lambda L + \lambda L S_\theta\right] = \lambda \frac{1}{2}\left[S_\theta L + L S_\theta\right] = \lambda \mathcal{T}L$$

$S_\theta > 0$ 固有値 : $\lambda_1, \lambda_2, \cdots, \lambda_n$ ($\lambda_1 > 0,\ \lambda_2 > 0,\ \ldots,\ \lambda_n > 0$)

固有ベクトル : $\psi_1\ \psi_2\ \cdots\ \psi_n$

$$\mathrm{Ker}\, \mathcal{T} = \{ A \in \mathcal{L}_n \ ; \ \mathcal{T}A = 0 \}$$

$$\mathrm{Ker}\, \mathcal{T} = \{0\} \iff \underset{\underset{\mathcal{S}_n}{\uparrow}}{{}^{\forall}A \neq 0} \quad \mathcal{T}A \neq 0$$

↑ このとき $\mathcal{T}$ は 全単射 となる

$\mathcal{T}$ が全単射なら

$$\mathcal{T}: \underset{\underset{\mathcal{L}_n}{\uparrow}}{L} \longrightarrow \frac{1}{2}[S_\theta \underset{\underset{\mathcal{I}_n}{\uparrow}}{L} + LS_\theta]$$

$\frac{dS_\theta}{d\theta} = \mathcal{T}(L) = \frac{1}{2}[S_\theta L + LS_\theta]$ をみたす

L がただ1つ存在する.

$K \in \mathrm{Ker}\, \mathcal{T}$ とすると, $\mathcal{T}(K) = 0$ なので

$$\mathcal{T}(K) = \frac{1}{2}(S_\theta K + KS_\theta) = 0$$

$S_\theta \psi_i = \lambda_i \psi_i$

$$0 = \langle \psi_i | (S_\theta K + KS_\theta)\psi_j \rangle$$

ただし ψ_i は S_θ の固有ベクトル (λ_i に対応)

$$= \langle \psi_i | S_\theta K \psi_j \rangle + \langle \psi_i | KS_\theta \psi_j \rangle$$

$$= \langle S_\theta \psi_i | K \psi_j \rangle + \langle \psi_i | K \underline{S_\theta \psi_j} \rangle$$

$$= \langle \lambda_i \psi_i | K \psi_j \rangle + \langle \psi_i | K \lambda_j \psi_j \rangle$$

$$= \bar{\lambda}_i \langle \psi_i | K \psi_j \rangle + \lambda_j \langle \psi_i | K \psi_j \rangle$$

$$= \lambda_i \underline{\langle \psi_i | K \psi_j \rangle} + \lambda_j \underline{\langle \psi_i | K \psi_j \rangle} = \underset{\underset{0}{\vee}}{(\lambda_i + \lambda_j)} \langle \psi_i | K \psi_j \rangle$$

$${}^{\forall}\psi_i, {}^{\forall}\psi_j \quad \langle \psi_i | K \psi_j \rangle = 0 \implies \boxed{K = 0}$$

正規直交基底

$|e_1\rangle, \cdots, |e_n\rangle$

$|\psi\rangle = x_1|e_1\rangle + \quad + x_n|e_n\rangle$

$x_1, \cdots, x_n \in \mathbb{C}$

2次元の例

$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|\psi\rangle = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

正規 $\langle e_i | e_i \rangle = 1 \quad i = 1, \ , n$

直交 $\langle e_i | e_j \rangle = 0 \quad i \neq j$

$$U = \begin{pmatrix} \boxed{|e_1\rangle} & \cdots & \boxed{|e_n\rangle} \end{pmatrix}$$

$$U^* = \begin{pmatrix} \langle e_1| \\ \vdots \\ \langle e_n| \end{pmatrix}$$

$|e_n\rangle$

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{1n} \\ u_{21} & & \vdots \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}$$

$|e_1\rangle$

$\langle e_1|$

$$U^* = \begin{pmatrix} \bar{u}_{11} & \bar{u}_{21} \cdots & \bar{u}_{n1} \\ \vdots & & \\ \bar{u}_{1n} & \cdots & \bar{u}_{nn} \end{pmatrix}$$

$\langle e_n|$

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$U^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

$$U^*U = \begin{pmatrix} \langle e_1| \\ \vdots \\ \langle e_n| \end{pmatrix} \begin{pmatrix} |e_1\rangle \cdots |e_n\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle e_1|e_1\rangle & \langle e_1|e_2\rangle \cdots & \langle e_1|e_n\rangle \\ \vdots & & \\ \langle e_n|e_1\rangle & \cdots\cdots & \langle e_n|e_n\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ & & 1 & 0 \\ 0 & 0 & \cdots\ 0 & 1 \end{pmatrix} = I$$

$U^* = U^{-1}$

このとき

$UU^* = I$

$$\left[\ x \xrightarrow{U} y \xrightarrow{U^*} x \qquad y \xrightarrow{U^*} x \xrightarrow{U} y\ \right]$$

$$UU^* = \begin{pmatrix} |e_1\rangle \cdots |e_n\rangle \end{pmatrix} \begin{pmatrix} \langle e_1| \\ \vdots \\ \langle e_n| \end{pmatrix}$$

$$= |e_1\rangle\langle e_1| + \cdots + |e_n\rangle\langle e_n| = I$$

$$\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$$

$$AB = \begin{pmatrix} \sum_i a_{1i} b_{i1} & & & \\ & \sum_i a_{2i} b_{i2} & & \\ & & \ddots & \\ & & & \sum_i a_{ni} b_{in} \end{pmatrix}$$

$$\therefore \mathrm{Tr}\,AB = \sum_j \sum_i a_{ji} b_{ij}$$

$$BA = \begin{pmatrix} \sum_i b_{1i} a_{i1} & & \\ & \ddots & \\ & & \sum b_{ni} a_{in} \end{pmatrix}$$

$$\mathrm{Tr}\,BA = \sum_j \sum_i b_{ji} a_{ij} = \mathrm{Tr}\,AB$$

$$\mathrm{Tr}\,A = \mathrm{Tr}\,A\left(|e_1\rangle\langle e_1| + \cdots |e_n\rangle\langle e_n|\right)$$

$|e_1\rangle, \ldots, |e_n\rangle$: 正規直交基底

$$= \mathrm{Tr}\,\boxed{A\left(|e_1\rangle \cdots |e_n\rangle\right)} \begin{pmatrix} \langle e_1| \\ \vdots \\ \langle e_n| \end{pmatrix}$$

$$= \mathrm{Tr} \begin{pmatrix} \langle e_1| \\ \vdots \\ \langle e_n| \end{pmatrix} A \left(|e_1\rangle \cdots |e_n\rangle\right)$$

$$= \mathrm{Tr} \begin{pmatrix} \langle e_1| \\ \vdots \\ \langle e_n| \end{pmatrix} \left(A|e_1\rangle \cdots A|e_n\rangle\right)$$

$$= \mathrm{Tr} \begin{pmatrix} \langle e_1|A|e_1\rangle & & \\ & \ddots & \\ & & \langle e_n|A|e_n\rangle \end{pmatrix}$$

$$= \sum_{i=1}^{n} \langle e_i|A|e_i\rangle = \sum_{i=1}^{n} \langle e_i|Ae_i\rangle$$

$$(A, B)_{S_0} = \mathrm{Tr}\,S_0 BA^*$$

$$(A, A)_{S_0} \geq 0$$

$$(A, A)_{S_0} = \mathrm{Tr}\,\boxed{S_0 A}\boxed{A^*}$$

$$= \mathrm{Tr}\,A^* S_0 A$$

$$= \sum_{i=1}^{n} \langle e_i | A^* S_0 A\, e_i \rangle$$

$$= \sum_{i=1}^{n} \langle A e_i | S_0 A e_i \rangle$$

ここで S_0 は 正定値

$\langle \psi | S_0 | \psi \rangle > 0$

$\psi = 0$ のときは

$\langle \psi | S_0 | \psi \rangle = 0$

$(A, A)_{S_0} \geq 0$

とくに

$(A, A)_{S_0} = 0$ のとき

$\langle Ae_i | S_0 A e_i \rangle = 0 \quad \forall_i$

$\Leftrightarrow \quad A|e_i\rangle = 0 \quad \forall_i \quad \cdots ①$

$\forall \psi \quad \psi = x_1|e_1\rangle + \cdots + x_n|e_n\rangle$ だったので

①が成立すると

$A|\psi\rangle = x_1 A|e_1\rangle + \cdots + x_n A|e_n\rangle$

$= 0$

すなわち $A = 0$.

$(\lambda A, B)_{S_0} = \operatorname{Tr} S_0 B \underbrace{(\lambda A)^*}_{= \bar{\lambda} A^*}$

$\lambda A = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$

$(\lambda A)^* = \begin{pmatrix} \bar{\lambda}\bar{a} & \bar{\lambda}\bar{c} \\ \bar{\lambda}\bar{b} & \bar{\lambda}\bar{d} \end{pmatrix} = \bar{\lambda} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \bar{\lambda} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^*$

$= \bar{\lambda} A^*$

$\therefore (\lambda A, B)_{S_0} = \operatorname{Tr} S_0 B \bar{\lambda} A^*$

$= \bar{\lambda} \operatorname{Tr} S_0 B A^*$

$= \bar{\lambda} (A, B)_{S_0}$.

$S_0^* = S_0$

$\operatorname{Tr} A^* = \overline{\operatorname{Tr} A}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \operatorname{Tr} A = a + d$

$A^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \quad \operatorname{Tr} A^* = \bar{a} + \bar{d}$

$= \overline{a + d}$

$= \overline{\operatorname{Tr} A}$

$(AB)^* = B^* A^*$

$\overline{(A, B)_{S_0}} = \overline{\operatorname{Tr} S_0 B A^*}$

$(B, A)_{S_0} = \operatorname{Tr} S_0 A B^*$

$= \overline{\operatorname{Tr} (\boxed{S_0 A} B^*)^*}$

$= \overline{\operatorname{Tr} B^{**} (S_0 A)^*} = \overline{\operatorname{Tr} \boxed{B A^*} S_0}$

$= \overline{\operatorname{Tr} S_0 B A^*} = \overline{(A, B)_{S_0}}$

$$J(L_\theta) = (L_\theta, L_\theta)_{S_\theta}$$

$$\downarrow$$

$$J_\theta = \sum_{x\in\Omega} P_\theta(x)\left(\frac{d}{d\theta}\log P_\theta(x)\right)^2$$ 古典版

$$S_\theta = \begin{pmatrix} P_\theta(1) & & \\ & \ddots & \\ & & P_\theta(n) \end{pmatrix} \quad \Omega = \{1, \cdots, n\}$$

$$L_\theta = \frac{d}{d\theta}\log S_\theta$$

$$= \frac{d}{d\theta}\begin{pmatrix} \log P_\theta(1) & & \\ & \ddots & \\ & & \log P_\theta(n) \end{pmatrix}$$

S_θ は対角行列

$$= \begin{pmatrix} \frac{d}{d\theta}\log P_\theta(1) & & \\ & \ddots & \\ & & \frac{d}{d\theta}\log P_\theta(n) \end{pmatrix}$$

$$\left[= \begin{pmatrix} \frac{1}{P_\theta(1)}\frac{d}{d\theta}P_\theta(1) & & \\ & \ddots & \\ & & \frac{1}{P_\theta(n)}\frac{d}{d\theta}P_\theta(n) \end{pmatrix} \right]$$

この時

$$(L_\theta, L_\theta)_{S_\theta}$$

$$= \mathrm{Tr}\, S_\theta L_\theta^2$$

$$= \mathrm{Tr}\begin{pmatrix} P_\theta(1) & & \\ & \ddots & \\ & & P_\theta(n) \end{pmatrix}\begin{pmatrix} \frac{d}{d\theta}\log P_\theta(1) & & \\ & \ddots & \\ & & \frac{d}{d\theta}\log P_\theta(n) \end{pmatrix}^2$$

$$= \mathrm{Tr}\begin{pmatrix} P_\theta(1)\left(\frac{d}{d\theta}\log P_\theta(1)\right)^2 & & \\ & \ddots & \\ & & P_\theta(n)\left(\frac{d}{d\theta}\log P_\theta(n)\right)^2 \end{pmatrix}$$

$$= \sum_{x\in\Omega} P_\theta(x)\left(\frac{d}{d\theta}\log_\theta P_\theta(x)\right)^2$$

$$\frac{d}{d\theta}S_\theta = \begin{pmatrix} \frac{d}{d\theta}P_\theta(1) & & \\ & \ddots & \\ & & \frac{d}{d\theta}P_\theta(n) \end{pmatrix}$$

$$\frac{1}{2}(S_\theta L_\theta + L_\theta^* S_\theta)$$

$$= \frac{1}{2}\left\{\begin{pmatrix} P_\theta(1) & & \\ & \ddots & \\ & & P_\theta(n) \end{pmatrix}\begin{pmatrix} \frac{d}{d\theta}\log P_\theta(1) & & \\ & \ddots & \\ & & \frac{d}{d\theta}\log P_\theta(n) \end{pmatrix} + \begin{pmatrix} \frac{d}{d\theta}\log P_\theta(1) & & \\ & \ddots & \\ & & \frac{d}{d\theta}\log P_\theta(n) \end{pmatrix}\begin{pmatrix} P_\theta(1) & & \\ & \ddots & \\ & & P_\theta(n) \end{pmatrix}\right\}$$

$$= \begin{pmatrix} P_\theta(1)\frac{d}{d\theta}\log P_\theta(1) & & \\ & \ddots & \\ & & P_\theta(n)\frac{d}{d\theta}\log P_\theta(n) \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{P_\theta(1)}\frac{1}{\cancel{P_\theta(1)}}\frac{d}{d\theta}P_\theta(1) & & \\ & \ddots & \\ & & \cancel{P_\theta(n)}\frac{1}{\cancel{P_\theta(n)}}\frac{d}{d\theta}P_\theta(n) \end{pmatrix}$$

$$= \frac{d}{d\theta}S_\theta$$