$$\Omega = \{\omega_1, \cdots, \omega_n\}$$

$$P : P_\theta(\omega_1), \cdots, P_\theta(\omega_n) \qquad \sum P_\theta(\omega_i) = 1$$

$$\hat{\theta} : \hat{\theta}(\omega_1), \cdots, \hat{\theta}(\omega_n)$$

This is a estimation that is described as a r.v.

$$E_\theta[\hat{\theta}] = \sum_{i=1}^{n} \hat{\theta}(\omega_i) P_\theta(\omega_i) = \theta$$

$$V_\theta[\hat{\theta}] = \sum_{i=1}^{n} \left(\hat{\theta}(\omega_i) - \theta\right)^2 P_\theta(\omega_i)$$

$$\left(= \sum_i \hat{\theta}(\omega_i)^2 P_\theta(\omega_i) - 2\theta^2 + \theta^2 \right.$$

$$\left. = \sum_i \hat{\theta}(\omega_i)^2 P_\theta(\omega_i) - \theta^2 \right)$$

$$\hat{\theta}(\omega_i) = \theta + k \frac{l_\theta(\omega_i)}{J_\theta}$$

$$J_\theta = \sum_{i=1}^{n} P_\theta(\omega_i) \left(\frac{d}{d\theta} \log P_\theta(\omega_i) \right)^2$$

$P_\theta(\omega_i)$

$$V_\theta[\hat{\theta}] \cdot J_\theta \geqq \sum_{i=1}^{n} \left(\hat{\theta}(\omega_i) - \theta\right) \left(\frac{d}{d\theta} \log P_\theta(\omega_i) \right)$$

$$= \sum_{i=1}^{n} \left(\hat{\theta}(\omega_i) - \theta\right) \frac{d}{d\theta} P_\theta(\omega_i)$$

$$= \sum_{i=1}^{n} \hat{\theta}(\omega_i) \frac{d}{d\theta} P_\theta(\omega_i) - \theta \underbrace{\sum_{i=1}^{n} \frac{d}{d\theta} P_\theta(\omega_i)}_{=0}$$

$$= 1.$$

$$V_\theta[\hat{\theta}] \geqq \frac{1}{J_\theta}$$

$$\therefore k = \frac{1}{J_\theta}.$$

$$\hat{\theta}(\omega_i) - \theta = k\, l_\theta(\omega_i)$$

$$V_\theta[\hat{\theta}] = k^2 \sum_{i=1}^{n} l_\theta(\omega_i)^2 P_\theta(\omega_i) = k^2 J_\theta = \frac{1}{J_\theta}$$

$$T_{\hat\theta}^{M} = \sum \hat\theta(\omega_i) M(\omega_i)$$

$$E[\hat\theta, M] = \mathrm{Tr}\, S_\theta \sum \hat\theta(\omega_i) M(\omega_i)$$

$$= \mathrm{Tr}\, S_\theta T$$

M : simple Meas.

$$V_\theta[M, \hat\theta] = \mathrm{Tr}\, S_\theta (T - \theta I)^2$$

$$= \mathrm{Tr}\, S_\theta T^2 - E[\hat\theta, M]^2$$

$\Omega = \{\omega_1, \cdots, \omega_n\}$

P_θ : $P_\theta(\omega_1) \cdots P_\theta(\omega_n)$

$\hat\theta$: $\hat\theta(\omega_1), \cdots \hat\theta(\omega_n)$

$E_\theta[\hat\theta] = \sum \hat\theta(\omega_i) P_\theta(\omega_i) = \theta$

$V_\theta[\hat\theta] = \sum (\hat\theta(\omega_i) - \theta)^2 P_\theta(\omega_i)$

$\uparrow$

$\hat\theta$ を optimize!

$\Omega = \{\omega_1, \cdots\cdots, \omega_n\}$

S_θ : 状態

$M = \{M(\omega_1), \cdots, M(\omega_n)\}$: 測定

$P_\theta^M(\omega_i) = \mathrm{Tr}\, S_\theta M(\omega_i)$: 確率

$\hat\theta(\omega_i)$: 推定値.

$$E_\theta[M, \hat\theta] = \sum \hat\theta(\omega_i) P_\theta^M(\omega_i) = \theta$$

$$V_\theta[M, \hat\theta] = \sum (\hat\theta(\omega_i) - \theta)^2 P_\theta^M(\omega_i)$$

$$S_\theta = \begin{pmatrix} P_\theta(\omega_1) & & \\ & \ddots & \\ & & P_\theta(\omega_n) \end{pmatrix}, \quad M(\omega_i) = \mathrm{diag}\,[0, \cdots 0, 1, 0 \cdots 0] : \text{fix.}$$

$$P_\theta^M(\omega_i) = P_\theta(\omega_i)$$

$$T_{\hat\theta}^M = \begin{pmatrix} \hat\theta(\omega_1) & & \\ & \ddots & \\ & & \hat\theta(\omega_n) \end{pmatrix} \leftarrow \hat\theta \text{ を optimize!}$$

$$E_\theta[\hat\theta, M] = \sum \hat\theta(\omega_i) P_\theta(\omega_i) = \theta$$

$$V_\theta[\hat\theta, M] = \sum (\hat\theta(\omega_i) - \theta)^2 P_\theta(\omega_i).$$

by being aware

$$\frac{dS_\theta}{d\theta} = S_\theta \circ L_\theta = \frac{1}{2}(S_\theta L_\theta + L_\theta S_\theta)$$

$$P_\theta^M(x) = \mathrm{Tr}\, S_\theta M(x) \quad . \quad x \in \Omega$$

$$\frac{d}{d\theta} P_\theta^M(x) = \mathrm{Tr} \frac{d}{d\theta} S_\theta M(x).$$

$$\frac{d P_\theta^M(x)}{d\theta} = \mathrm{Tr} \frac{d}{d\theta} S_\theta M(x)$$

$$= \mathrm{Tr}\, S_\theta \circ L_\theta M(x)$$

$$= \frac{1}{2}\left(\mathrm{Tr}\, S_\theta L_\theta M(x) + \mathrm{Tr}\, L_\theta S_\theta M(x)\right)$$

$$= \frac{1}{2}\left(\mathrm{Tr}\, L_\theta M(x) S_\theta + \mathrm{Tr}\, M(x) L_\theta S_\theta\right)$$

$$= \mathrm{Tr}\left(L_\theta \circ M(x)\, S_\theta\right) = \langle L_\theta, M(x) \rangle_{S_\theta}$$

$$T_M = \sum_{x \in \Omega} \hat{\theta}(x) M(x)$$

$$\langle T_M, L_\theta \rangle_{S_\theta} = \sum \hat{\theta}(x) \langle M(x), L_\theta \rangle_{S_\theta}$$

$$= \sum \hat{\theta}(x) \frac{d P_\theta^M(x)}{d\theta}$$

$$E_\theta[\hat{\theta}, M] = \sum \hat{\theta}(x) P_\theta^M(x)$$

$$\frac{d}{d\theta} E_\theta[\hat{\theta}, M] = \sum \hat{\theta}(x) \frac{d}{d\theta} P_\theta^M(x)$$

$$= \sum \hat{\theta}(x) \langle L_\theta, M(x) \rangle_{S_\theta}$$

$$= \langle L_\theta, T_M \rangle_{S_\theta}$$

$$V_\theta[\hat{\theta}, M] = \sum_x (\hat{\theta}(x) - \theta)^2 P_\theta^M(x)$$

$$= \sum_x \hat{\theta}(x)^2 P_\theta^M(x) - \theta^2$$

$$T_M = \sum_{x\in\Omega} \hat\theta(x)\, M(x)$$

$$\sum_y \left[\hat\theta(y) - \sum_x \hat\theta(x) M(x)\right] M(y) \left[\hat\theta(y) - \sum_x \hat\theta(x) M(x)\right] \geq 0$$

$$\sum_y \hat\theta(y)^2 M(y) - \sum_y \hat\theta(y) M(y) \sum_x \hat\theta(x) M(x) - \sum_y \left(\sum_x \hat\theta(x) M(x)\right) \hat\theta(y) M(y)$$

$$+ \left(\sum_x \hat\theta(x) M(x)\right)^2 \geq 0$$

$$\sum_y \hat\theta(y)^2 M(y) \geq \left(\sum_y \hat\theta(y) M(y)\right)^2$$

$$\mathrm{Tr}\sqrt{S} \sum_y \hat\theta(y)^2 M(y) \sqrt{S} \geq \mathrm{Tr}\sqrt{S} \left(\qquad \right)^2 \sqrt{S}$$

$$\mathrm{Tr} \sum_y \hat\theta(y)^2 M(y)\, S \geq \mathrm{Tr}\left(\qquad \right)^2 S.$$

$$E_\theta\{M\} = \int \hat\theta \, \mu_\theta(d\hat\theta) \quad , \quad \mu_\theta(d\hat\theta) = \mathrm{Tr}\, S_\theta M(d\hat\theta)$$

$$E_\theta\{M\} = \sum_{x\in\Omega} \hat\theta(x)\, \mu_\theta(x) \qquad \mu_\theta(x) = \mathrm{Tr}\, S_\theta M(x)$$

$$\hat\theta : \Omega \longrightarrow \underset{\cup}{\Theta} \subset \mathbb{R}$$
$$B$$

$$\begin{cases} \mu_\theta(B) = \mu_\theta^\Omega(\hat\theta^{-1}(B)) \\ M(B) = M^\Omega(\hat\theta^{-1}(B)) \end{cases}$$

$$\frac{d}{d\theta} E_\theta\{M\} = \frac{d}{d\theta} \int \hat\theta \, \mu_\theta(d\hat\theta)$$
$$= \int \hat\theta \, \frac{d\mu_\theta}{d\theta}(d\hat\theta)$$
$$= 1$$

The condition of *locally unbiased*

$$\begin{cases} \int \hat\theta \mu_\theta(d\hat\theta) = \theta \leftarrow \text{Is this not used?} \\ \int \hat\theta^2 \mu_\theta(d\hat\theta) < \infty \\ \boxed{\int \hat\theta \, \frac{d\mu_\theta}{d\theta}(d\hat\theta) = 1} \end{cases}$$

$$\underline{\langle L_\theta, X_M \rangle_\theta = 1}$$

Definition of inner product

X : vec. sp / $\mathbb{R}$

$\langle \cdot , \cdot \rangle : X \times X \longrightarrow \mathbb{R}$

i) $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0 \iff x = 0$

ii) $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$

iii) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

iv) $\langle x, y \rangle = \langle y, x \rangle$

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v) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

vi) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

When X : vec. sp on $\mathbb{R}$

$\langle x, y \rangle = \overline{\langle y, x \rangle}$

This means $\langle \alpha x, y \rangle = \bar{\alpha} \langle x, y \rangle$

$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$

$\because$ $x = 0 \rightarrow$ ok

$x \neq 0$ $\quad x' = \dfrac{x}{\sqrt{\langle x, x \rangle}}$

$0 \leq \langle \langle x', y \rangle x' - y , \langle x', y \rangle x' - y \rangle$

$= |\langle x', y \rangle|^2 \langle x', x' \rangle - |\langle x', y \rangle|^2 - |\langle x', y \rangle|^2 + \langle y, y \rangle$

$\langle x', x' \rangle = \dfrac{\langle x, x \rangle}{\langle x, x \rangle} = 1$

$= -\dfrac{|\langle x, y \rangle|^2}{\langle x, x \rangle} + \langle y, y \rangle \quad \therefore |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$

equality : $\langle x', y \rangle x' = y$

$\dfrac{\langle x, y \rangle}{\langle x, x \rangle} x = y$

$t = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \; S = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \quad \langle t, s \rangle_\theta = \sum_{i=1}^{n} t_i s_i P_\theta(\omega_i)$

$P_\theta(x) > 0 \quad \forall x \in \Omega = \{\omega_1, \ldots, \omega_n\}$

内积 $\langle A, B \rangle_{S_\theta} = \frac{1}{2} \mathrm{Tr} S_\theta (AB + BA) = \mathrm{Tr} A \circ B S_\theta = \mathrm{Tr} B \circ S_\theta A$

$l_\theta(\omega_i)$ $\quad \dfrac{d}{d\theta} S_\theta = S_\theta \circ L_\theta = \frac{1}{2}(S_\theta L_\theta + L_\theta S_\theta)$

$$\langle A, B\rangle_{S_\theta} = \mathrm{Tr}\, S_\theta (A\circ B)$$

with $A\circ B = \frac{1}{2}(AB+BA)$

$$\langle A, A\rangle_{S_\theta} = \mathrm{Tr}\, S_\theta A^2$$

$$= \mathrm{Tr}\, A S_\theta A$$

by a ONB $\{|e_i\rangle\}$

$$= \sum_i \langle e_i | A S_\theta A | e_i\rangle$$

$$= \sum_i \langle A e_i | S_\theta | A e_i\rangle$$

$$\geq 0$$

$Ae_i = 0$ $\forall i$ のとき等号成立

$$\langle \lambda A, B\rangle_{S_\theta} = \lambda \langle A, B\rangle$$

$$\langle A_1 + A_2, B\rangle_{S_\theta} = \langle A_1, B\rangle_{S_\theta} + \langle A_2, B\rangle_{S_\theta}$$

$$V_\theta[\hat\theta, M] = \sum_{x\in\Omega} (\hat\theta(x) - \theta)^2 P_\theta^M(x)$$

$$= \sum_{x\in\Omega} (\hat\theta(x) - \theta)^2\, \mathrm{Tr}\, S_\theta M(x)$$

$$= \mathrm{Tr}\, S_\theta \sum_x \underbrace{(\hat\theta(x)-\theta)^2}_{= f(x)} M(x)$$

$$\geq \mathrm{Tr}\, S_\theta \Big(\sum_x \underbrace{(\hat\theta(x)-\theta)}_{f(x)} M(x)\Big)^2$$

$$= \mathrm{Tr}\, S_\theta \Big(\underbrace{\sum_x \hat\theta(x) M(x)}_{= T_M} - \theta I\Big)^2$$

$$= \langle T_M - \theta, T_M - \theta\rangle_{S_\theta}$$

$$\mathrm{Tr}\, S_\theta \sum_x f(x)^2 M(x) \geq \mathrm{Tr}\, S_\theta \Big(\sum f(x) M(x)\Big)^2$$

$$V_\theta[\hat\theta, M] \geq \langle T_M - \theta, T_M - \theta\rangle_{S_\theta}$$

$$\langle T_M - \theta, T_M - \theta\rangle_{S_\theta} \langle L_\theta, L_\theta\rangle_{S_\theta} \geq \langle T_M - \theta, L_\theta\rangle_{S_\theta}^2$$

$$= \langle T_M, L_\theta\rangle_{S_\theta}^2$$

$$\langle I, L_\theta \rangle_{S_\theta} = \mathrm{Tr}\, S_\theta L_\theta = \mathrm{Tr}\, S_\theta \circ L_\theta$$

$$= \mathrm{Tr} \frac{d}{d\theta} S_\theta = \frac{d}{d\theta} \mathrm{Tr}\, S_\theta = 0$$

$$\langle T_M, L_\theta \rangle_{S_\theta} = \left\langle \sum_x \hat\theta(x) M(x), L_\theta \right\rangle_{S_\theta}$$

$$= \sum_x \hat\theta(x) \langle M(x), L_\theta \rangle_{S_\theta}$$

$$\frac{d}{d\theta} E_\theta[\hat\theta, M] = \frac{d}{d\theta} \sum_x \hat\theta(x)\, \mathrm{Tr}\, S_\theta M(x)$$

$$= \sum_x \hat\theta(x)\, \mathrm{Tr} \frac{d}{d\theta} S_\theta M(x)$$

$$= \sum_x \hat\theta(x)\, \mathrm{Tr}\, S_\theta \circ L_\theta M(x)$$

$$= \sum_x \hat\theta(x)\, \mathrm{Tr}\left(L_\theta \circ M(x)\right) S_\theta$$

$$= \sum_x \hat\theta(x) \langle L_\theta, M(x) \rangle_{S_\theta} = \langle T_M, L_\theta \rangle_{S_\theta}$$

$$\Omega := \{\omega_1, \cdots, \omega_n\}$$

$$P_\theta : \Omega \to \mathbb{R} \quad \sum_x P_\theta(x) = 1,\ P_\theta(x) > 0$$

$$\hat\theta : \Omega \to \mathbb{R}$$

$$E_\theta[\hat\theta] = \sum_{i=1}^{n} \hat\theta(\omega_i) P_\theta(\omega_i) = \theta$$

$$V_\theta[\hat\theta] = \sum_{i=1}^{n} (\hat\theta(\omega_i) - \theta)^2 P_\theta(\omega_i)$$

↑ min にする $\hat\theta$ をみつける

$$J_\theta = \langle l_\theta, l_\theta \rangle_\theta = \sum_{i=1}^{n} l_\theta(\omega_i)^2 P_\theta(\omega_i)$$

- $l_\theta(x) = \frac{d}{d\theta} \log P_\theta(x) = \frac{1}{P_\theta(x)} \frac{d}{d\theta} P_\theta(x)$

$$l_\theta := \begin{pmatrix} l_\theta(\omega_1) \\ \vdots \\ l_\theta(\omega_n) \end{pmatrix}, \quad \hat\theta := \begin{pmatrix} \hat\theta(\omega_1) \\ \vdots \\ \hat\theta(\omega_n) \end{pmatrix} \qquad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\langle a, b \rangle_\theta := \sum_{i=1}^{n} a_i b_i P_\theta(\omega_i)$$

$$1 := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad V_\theta[\hat\theta] = \langle \hat\theta - \theta 1, \hat\theta - \theta 1 \rangle_\theta$$

$$\langle \hat\theta, l_\theta \rangle_\theta = \sum_i \hat\theta(\omega_i) l_\theta(\omega_i) P_\theta(\omega_i)$$

$$= \sum_i \hat\theta(\omega_i) \frac{d}{d\theta} P_\theta(\omega_i) = \frac{d}{d\theta} \sum_i \hat\theta(\omega_i) P_\theta(\omega_i) = 1$$

$$V_\theta[\hat\theta] \langle l_\theta, l_\theta \rangle_\theta \geq |\langle \hat\theta, l_\theta \rangle_\theta|^2 = 1$$

$\Omega = \{w_1, \cdots, w_n\}$

S_θ 状态

$M := \{M(w_1), \cdots, M(w_n)\}$ 测定 $\quad M(w_i) \geq 0$

$$\sum_{x \in \Omega} M(x) = I$$

$P_\theta^M(x) = \mathrm{Tr}\, S_\theta M(x)$

$E_\theta[\hat{\theta}, M] = \sum_x \hat{\theta}(x) P_\theta^M(x) = \theta$

$V_\theta[\hat{\theta}, M] = \sum_x (\hat{\theta}(x) - \theta)^2 P_\theta(x)$

↑ min にする.

$$\frac{dS_\theta}{d\theta} = S_\theta \circ L_\theta = \frac{1}{2}(S_\theta L_\theta + L_\theta S_\theta) \quad \Bigg| \quad \frac{d}{d\theta} P_\theta(x) = P_\theta(x)\, l_\theta(x)$$

$$\hat{\Theta}_M = \sum_x \hat{\theta}(x) M(x), \quad I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$V_\theta[\hat{\theta}, M] \geq \langle \hat{\Theta}_M - \theta I, \hat{\Theta}_M - \theta I \rangle_{S_\theta}$$

$$\langle A, B \rangle_{S_\theta} = \mathrm{Tr}\, S_\theta (A \circ B), \quad \langle L_\theta, L_\theta \rangle_{S_\theta} =: J(L_\theta)$$

$$\langle \hat{\Theta}_M, L_\theta \rangle_{S_\theta} = \frac{d}{d\theta} \sum_x \hat{\theta}(x) P_\theta^M(x) = 1$$

$$V_\theta[\hat{\theta}, M] \cdot J(L_\theta) \geq |\langle \hat{\Theta}_M, L_\theta \rangle_{S_\theta}|^2 = 1.$$

$\Omega := \{w_1, \cdots, w_n\}$

$P_\theta : \Omega \to \mathbb{R} \quad \sum_x P_\theta(x) = 1, \; P_\theta(x) > 0$

$\hat{\theta} : \Omega \to \mathbb{R}$

$E_\theta[\hat{\theta}] = \sum_{i=1}^n \hat{\theta}(w_i) P_\theta(w_i) = \theta$

$V_\theta[\hat{\theta}] = \sum_{i=1}^n (\hat{\theta}(w_i) - \theta)^2 P_\theta(w_i)$

↑ min にする $\hat{\theta}$ をみつける

$$J_\theta = \langle l_\theta, l_\theta \rangle_\theta = \sum_{i=1}^n l_\theta(w_i)^2 P_\theta(w_i)$$

- $l_\theta(x) = \frac{d}{d\theta} \log P_\theta(x) = \frac{1}{P_\theta(x)} \frac{d}{d\theta} P_\theta(x)$

$$\frac{d}{d\theta} P_\theta(x) = P_\theta(x)\, l_\theta(x)$$

$$l_\theta := \begin{pmatrix} l_\theta(w_1) \\ \vdots \\ l_\theta(w_n) \end{pmatrix}, \quad \hat{\theta} := \begin{pmatrix} \hat{\theta}(w_1) \\ \vdots \\ \hat{\theta}(w_n) \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \; b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\langle a, b \rangle_\theta := \sum_{i=1}^n a_i b_i P_\theta(w_i)$$

$$1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad V_\theta[\hat{\theta}] = \langle \hat{\theta} - \theta 1, \hat{\theta} - \theta 1 \rangle_\theta$$

$$\langle \hat{\theta}, l_\theta \rangle_\theta = \sum_i \hat{\theta}(w_i) l_\theta(w_i) P_\theta(w_i) = \sum_i \hat{\theta}(w_i) \frac{d}{d\theta} P_\theta(w_i) = \frac{d}{d\theta} \sum_i \hat{\theta}(w_i) P_\theta(w_i) = 1$$

$$V_\theta[\hat{\theta}] \cdot \langle l_\theta, l_\theta \rangle_\theta \geq |\langle \hat{\theta}, l_\theta \rangle_\theta|^2 = 1$$

$$\langle \hat{\theta} - \theta \cdot 1,\ l_\theta \rangle_\theta = \langle \hat{\theta}, l_\theta \rangle_\theta - \theta \langle 1, l_\theta \rangle_\theta$$

$$\langle 1, l_\theta \rangle_\theta = \sum_\theta l_\theta(x)\, p_\theta(x) = \sum_\theta \frac{d}{d\theta} p_\theta(x)$$

$$= \frac{d}{d\theta} \sum_\theta p_\theta(x) = \frac{d}{d\theta} 1 = 0$$

$$\langle \hat{\Theta} - \theta I,\ L_\theta \rangle_{S_\theta} = \langle \hat{\Theta}, L_\theta \rangle_{S_\theta} - \theta \langle I, L_\theta \rangle_{S_\theta}$$

$$\langle I, L_\theta \rangle_{S_\theta} = \mathrm{Tr}\, L_\theta S_\theta = \mathrm{Tr}\, S_\theta \circ L_\theta$$

$$= \mathrm{Tr} \frac{dS_\theta}{d\theta} = \frac{d}{d\theta} \mathrm{Tr}\, S_\theta = \frac{d}{d\theta} \cdot 1 = 0$$