$\mathcal{M}$: finite dimensional v. N. alg.

ω : faithful state on $\mathcal{M}$

The Wedderburn theorem :

$$\exists\, \varphi : \mathcal{M} \longrightarrow \mathcal{M}_f = M(m_1, \mathbb{C}) \oplus \cdots \oplus M(m_n, \mathbb{C})$$

$\longrightarrow$ Let us consider GNS representation for $(\mathcal{M}, \omega)$

$\pi_\omega : \mathcal{M} \longrightarrow \mathcal{L}(\mathfrak{H}_\omega)$: we have the Hilbert space $\mathfrak{H}_\omega = \mathcal{M}$ with the inner product $\langle X, Y \rangle_\omega = \omega(X^* Y)$, and $\pi_\omega(X)Y = XY$ for $X \in \mathcal{M}$, $Y \in \mathfrak{H}_\omega$.

Then $\Omega_\omega := \pi_\omega(I) \in \mathfrak{H}_\omega$ satisfies

$$\boxed{\omega(X) = \langle \Omega_\omega, \pi_\omega(X)\Omega_\omega \rangle_\omega, \quad X \in \mathcal{M}, \quad \Omega_\omega \in \mathfrak{H}_\omega}$$

- We define a faithful tracial state on $\mathcal{M}_f$:

$$\tau(A) = \frac{1}{m} \operatorname{Tr} A, \quad m = m_1 + \cdots + m_n$$

$\longrightarrow$ Let us consider GNS representation for $(\mathcal{M}_f, \tau)$

$\pi_\tau : \mathcal{M}_f \to \mathcal{L}(\mathfrak{H}_\tau)$: we have the Hilbert space $\mathfrak{H}_\tau = \mathcal{M}_f$ with the inner product $\langle A, B \rangle_\tau = \tau(A^* B)$, and $\pi_\tau(A)B = AB$ for $A \in \mathcal{M}_f$, $B \in \mathfrak{H}_\tau$.

- $$\begin{tikzcd} \Omega_\omega \in \mathcal{M} \arrow[r, "\omega"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C} \\ \mathcal{M}_f \arrow[ur, "\tilde{\omega}"'] & \end{tikzcd}$$

$\tilde{\omega} = \omega \circ \varphi^{-1}$ gives a state on $\mathcal{M}_f$.

$\longrightarrow$ $\tilde{\omega}(A) = \omega(\varphi^{-1}(A)) = \langle \Omega_\omega, \pi_\omega(\varphi^{-1}(A))\Omega_\omega \rangle_\omega$

- $\exists\, H \in \underbrace{\mathcal{M}_f}_{= \mathfrak{H}_\tau}$ s.t.

$$\boxed{\begin{aligned} & A \in \mathcal{M}_f,\ H \in \mathfrak{H}_\tau \\ & \tilde{\omega}(A) = \tau(AH) = \tau(H^{1/2} A H^{1/2}) \\ & \quad = \langle H^{1/2}, \pi_\tau(A) H^{1/2} \rangle_\tau \end{aligned}}$$

$U : \mathfrak{H}_\tau \longrightarrow \mathfrak{H}_\omega$

$$\left\{ \begin{aligned} & \pi_\tau(A)H^{1/2} \longrightarrow \pi_\omega(X)\Omega_\omega \\ & \qquad\qquad A = \varphi(X) \\ & \qquad\qquad B = \varphi(Y) \\ & \pi_\tau(B)H^{1/2} \longrightarrow \pi_\omega(Y)\Omega_\omega \end{aligned} \right.$$

$$\langle \pi_\tau(B)H^{1/2}, \pi_\tau(A)H^{1/2} \rangle_\tau = \langle \pi_\omega(Y)\Omega_\omega, \pi_\omega(X)\Omega_\omega \rangle_\omega$$