内積 (実ベクトル空间)

V：実ベクトル空间

$V \ni \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad x_i \in \mathbb{R}$

$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{R}$

$\qquad (x, y) \longmapsto \langle x, y \rangle$

例 $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$

以下の性質をみたしていれば $\langle \cdot , \cdot \rangle$ を内積と呼ぶことにする.

i) $\langle x, x \rangle \geqq 0$, $\langle x, x \rangle = 0 \iff x = 0$

ii) $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$. $\alpha \in \mathbb{R}$

iii) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

iv) $\langle x, y \rangle = \langle y, x \rangle$

v) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ $\quad$ vi) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

$$\boxed{\langle x, y \rangle^2 \leqq \langle x, x \rangle \langle y, y \rangle}$$

$x \parallel y$

∵ $x = 0$ のとき $\rightarrow$ ok

$x \neq 0$ のとき $\quad x' = \dfrac{x}{\sqrt{\langle x, x \rangle}}$

$0 \leq \langle \langle x', y \rangle x' - y , \langle x', y \rangle x' - y \rangle$

$= \langle \langle x', y \rangle x', \langle x', y \rangle x' \rangle - \langle \langle x', y \rangle x', y \rangle - \langle y, \langle x', y \rangle x' \rangle + \langle y, y \rangle$

$= \langle x', y \rangle^2 \langle x', x' \rangle - \langle x', y \rangle \langle x', y \rangle - \langle x', y \rangle \langle y, x' \rangle + \langle y, y \rangle$

$\langle x', x' \rangle = \left\langle \dfrac{x}{\sqrt{\langle x, x \rangle}}, \dfrac{x}{\sqrt{\langle x, x \rangle}} \right\rangle = \dfrac{1}{\langle x, x \rangle} \langle x, x \rangle = 1$

$= \langle x', y \rangle^2 - \langle x', y \rangle^2 - \langle x', y \rangle^2 + \langle y, y \rangle = - \langle x', y \rangle^2 + \langle y, y \rangle$

$\langle x', y \rangle^2 \leqq \langle y, y \rangle$

$\langle x', y \rangle^2 = \left\langle \dfrac{x}{\sqrt{\langle x, x \rangle}}, y \right\rangle^2 = \dfrac{1}{\langle x, x \rangle} \langle x, y \rangle^2$

$\therefore \dfrac{\langle x, y \rangle^2}{\langle x, x \rangle} \leqq \langle y, y \rangle$ 」

等号：

$\langle x', y \rangle x' - y = 0$

$\left\langle \dfrac{x}{\langle x, x \rangle}, y \right\rangle \dfrac{x}{\langle x, x \rangle} = y$

$\dfrac{\langle x, y \rangle}{\langle x, x \rangle^2} x = y$

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Ω = {ω_1, ……, ω_n} 事象

古典

$P_\theta : \Omega \to \mathbb{R}$ ←固定 , $P_\theta(\omega_i) > 0$, $\sum_{i=1}^{n} P_\theta(\omega_i) = 1$

$\hat{\theta} : \Omega \to \mathbb{R}$: 推定

不偏性

$$E_\theta[\hat{\theta}] = \sum_{i=1}^{n} \hat{\theta}(\omega_i) P_\theta(\omega_i) = \theta$$

$$\frac{d}{d\theta} E_\theta[\hat{\theta}] = \sum_{i=1}^{n} \hat{\theta}(\omega_i) \frac{d}{d\theta} P_\theta(\omega_i) = 1$$

$$V_\theta[\hat{\theta}] = \sum_{i=1}^{n} (\hat{\theta}(\omega_i) - \theta)^2 P_\theta(\omega_i)$$ ← minimize

$$l_\theta(\omega_i) = \frac{d}{d\theta} \log P_\theta(\omega_i) = \frac{1}{P_\theta(\omega_i)} \frac{d}{d\theta} P_\theta(\omega_i)$$

$$\left(\frac{d}{d\theta} P_\theta(\omega_i) = P_\theta(\omega_i)\, l_\theta(\omega_i) \right)$$

量子

S_θ ←固定 : 状態 , $M = \{M(\omega_1), \cdots, M(\omega_n)\}$ ← optimized : 測定

$M(\omega_i) > 0$, $\sum_{i=1}^{n} M(\omega_i) = I$

$$P_\theta^M(\omega_i) = \mathrm{Tr}\, S_\theta M(\omega_i)$$

(注) $S_\theta = \begin{pmatrix} P_\theta(\omega_1) & & \\ & \ddots & \\ & & P_\theta(\omega_n) \end{pmatrix}$ ←固定

$M_0(\omega_i) := \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$.

$P_\theta^{M_0}(\omega_i) = P_\theta(\omega_i)$: 古典と同じ

$\hat{\theta} : \Omega \to \mathbb{R}$ & M : 量子推定

不偏性

$$E_\theta[\hat{\theta}, M] = \sum_{i=1}^{n} \hat{\theta}(\omega_i) P_\theta^M(\omega_i) = \theta$$

$$\frac{d}{d\theta} E_\theta[\hat{\theta}, M] = \sum_{i=1}^{n} \hat{\theta}(\omega_i) \frac{d}{d\theta} P_\theta^M(\omega_i) = 1$$

$$= \sum_{i=1}^{n} \hat{\theta}(\omega_i) \frac{d}{d\theta} \mathrm{Tr}\, S_\theta M(\omega_i)$$

$$= \sum_{i=1}^{n} \hat{\theta}(\omega_i) \mathrm{Tr} \frac{d}{d\theta} S_\theta M(\omega_i)$$

$$V_\theta[\hat{\theta}, M] = \sum_{i=1}^{n} (\hat{\theta}(\omega_i) - \theta)^2 P_\theta^M(\omega_i)$$ ← minimize

$$\frac{d}{d\theta} S_\theta = S_\theta \circ L_\theta = \frac{1}{2}(S_\theta L_\theta + L_\theta S_\theta)$$

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$\Omega = \{\omega_1, \cdots\cdots, \omega_n\}$: 事象

古典

$$\ell_\theta = \begin{pmatrix} \ell_\theta(\omega_1) \\ \vdots \\ \ell_\theta(\omega_n) \end{pmatrix}, \quad t = \begin{pmatrix} \hat{\theta}(\omega_1) \\ \vdots \\ \hat{\theta}(\omega_n) \end{pmatrix}$$

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \to \langle a, b \rangle_\theta := \sum_{i=1}^{n} a_i b_i P_\theta(\omega_i) \quad : \text{内積}$$

$$1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$V_\theta[\hat{\theta}] = \langle t - \theta \cdot 1, t - \theta \cdot 1 \rangle_\theta = \left\langle \begin{pmatrix} \hat{\theta}(\omega_1) - \theta \\ \vdots \\ \hat{\theta}(\omega_n) - \theta \end{pmatrix}, \begin{pmatrix} \hat{\theta}(\omega_1) - \theta \\ \vdots \\ \hat{\theta}(\omega_n) - \theta \end{pmatrix} \right\rangle_\theta$$

$$= \sum_{i=1}^{n} (\hat{\theta}(\omega_i) - \theta)^2 P_\theta(\omega_i)$$

$$J(\ell_\theta) = \langle \ell_\theta, \ell_\theta \rangle_\theta = \sum_{i=1}^{n} \ell_\theta^2(\omega_i) P_\theta(\omega_i)$$

量子

$$L_\theta, \quad T_M = \sum_{i=1}^{n} \hat{\theta}(\omega_i) M(\omega_i)$$

$$T_{M_0} = \begin{pmatrix} \hat{\theta}(\omega_1) & & \\ & \ddots & \\ & & \hat{\theta}(\omega_n) \end{pmatrix}$$

古典対応

$A, B \in \mathcal{L}_h$ (エルミート行列全体) $\quad \langle A, B \rangle_{S_\theta} := \mathrm{Tr}\, S_\theta (A \circ B)$: 内積

但し $A \circ B = \frac{1}{2}(AB + BA)$

$$\langle A, B \rangle_{S_\theta} = \frac{1}{2} \mathrm{Tr}\, S_\theta (AB + BA) = \frac{1}{2}(\mathrm{Tr}\, S_\theta A B + \mathrm{Tr}\, S_\theta B A)$$

$$= \frac{1}{2}(\mathrm{Tr}\, B S_\theta A + \mathrm{Tr}\, S_\theta B A) = \mathrm{Tr}\, \underbrace{\frac{1}{2}(B S_\theta + S_\theta B)}_{S_\theta \circ B} A$$

$$= \mathrm{Tr}(S_\theta \circ B) A$$

$$I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$V_\theta[\hat{\theta}, M] \geqq \langle T_M - \theta I, T_M - \theta I \rangle_{S_\theta}$$

$$J(L_\theta) := \langle L_\theta, L_\theta \rangle_{S_\theta}$$

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$\Omega = \{\omega_1, \cdots\cdots, \omega_n\}$: 事象

古典

$$V_\theta[\hat\theta]\, J(l_\theta) = \langle t-\theta 1, t-\theta 1\rangle_\theta \langle l_\theta, l_\theta\rangle_\theta$$

$$\geq \langle t-\theta\cdot 1, l_\theta\rangle_\theta^2$$

$$= \left(\langle t, l_\theta\rangle_\theta - \theta\langle 1, l_\theta\rangle_\theta\right)^2 = 1$$

$$\langle 1, l_\theta\rangle_\theta = \sum_{i=1}^n 1\cdot l_\theta(\omega_i) P_\theta(\omega_i) = \sum_{i=1}^n \frac{1}{\cancel{P_\theta(\omega_i)}}\left(\frac{d}{d\theta}P_\theta(\omega_i)\right)\cancel{P_\theta(\omega_i)}$$

$$= \sum_{i=1}^n \frac{d}{d\theta}P_\theta(\omega_i) = \frac{d}{d\theta}\boxed{\sum_{i=1}^n P_\theta(\omega_i)} = 0$$

$$\langle t, l_\theta\rangle_\theta = \sum_{i=1}^n \hat\theta(\omega_i)\, l_\theta(\omega_i) P_\theta(\omega_i) = \sum_{i=1}^n \hat\theta(\omega_i)\frac{d}{d\theta}P_\theta(\omega_i)$$

$$= 1$$

$$\therefore\quad V_\theta[\hat\theta] \geq \frac{1}{J(l_\theta)}$$

量子

$$V_\theta[\hat\theta, M]\, J(L_\theta) \geq \langle T_M - \theta I, T_M - \theta I\rangle_{S_\theta} \langle L_\theta, L_\theta\rangle_{S_\theta}$$

$$\geq \langle T_M - \theta I, L_\theta\rangle_{S_\theta}^2$$

$$= \left(\langle T_M, L_\theta\rangle_{S_\theta} - \theta\langle I, L_\theta\rangle_{S_\theta}\right)^2$$

$$\langle I, L_\theta\rangle_{S_\theta} = \mathrm{Tr}\, S_\theta \frac{1}{2}(I\cdot L_\theta + L_\theta I) = \mathrm{Tr}\, S_\theta L_\theta = \frac{1}{2}(\mathrm{Tr} S_\theta L_\theta + \mathrm{Tr} S_\theta L_\theta)$$

$$= \frac{1}{2}(\mathrm{Tr} S_\theta L_\theta + \mathrm{Tr} L_\theta S_\theta) = \mathrm{Tr}\frac{1}{2}(S_\theta L_\theta + L_\theta S_\theta) = \mathrm{Tr} S_\theta \circ L_\theta$$

$$= \mathrm{Tr}\frac{d}{d\theta}S_\theta = \frac{d}{d\theta}\mathrm{Tr} S_\theta = 0$$

状態のトレースは必ず1

$$\langle T_M, L_\theta\rangle_{S_\theta} = \left\langle \sum_{i=1}^n \hat\theta(\omega_i) M(\omega_i), L_\theta\right\rangle_{S_\theta} = \sum_{i=1}^n \hat\theta(\omega_i)\underline{\langle M(\omega_i), L_\theta\rangle_{S_\theta}}$$

$$= \sum_{i=1}^n \hat\theta(\omega_i)\underline{\mathrm{Tr}(S_\theta\circ L_\theta) M(\omega_i)} = \sum_{i=1}^n \hat\theta(\omega_i)\mathrm{Tr}\frac{dS_\theta}{d\theta} M(\omega_i)$$

$$= \sum_{i=1}^n \hat\theta(\omega_i)\frac{d}{d\theta}\mathrm{Tr}\, S_\theta M(\omega_i) = 1$$

補足

$$T_M = \sum_{x\in\Omega} \hat{\theta}(x) M(x) \ , \Omega = \{\omega_1, \cdots, \omega_n\}$$

$$T_M - \theta I = \sum_{x\in\Omega} (\hat{\theta}(x) - \theta) M(x)$$

$M(y) \geqq 0$ より，以下が成立する（★をつかった）

$$\sum_y \Big[\underbrace{(\hat{\theta}(y)-\theta) - \sum_x (\hat{\theta}(x)-\theta) M(x)}_{エルミート}\Big] M(y) \Big[\underbrace{(\hat{\theta}(y)-\theta) - \sum_x (\hat{\theta}(x)-\theta) M(x)}_{エルミート}\Big] \geqq 0$$

★ $M \geqq 0$. A：エルミート

$\Rightarrow AMA \geqq 0$

∵ $M \geqq 0 \iff \forall \psi \ \langle \psi | M \psi \rangle \geqq 0$ —①

$\langle \psi | AMA | \psi \rangle = \langle A\psi | M | A\psi \rangle \geqq 0$ ①より

よって $\forall \psi$, $\langle \psi | AMA | \psi \rangle \geqq 0$

∴ $AMA \geqq 0$

左辺を展開して

$$\sum_y (\hat{\theta}(y)-\theta) M(y) (\hat{\theta}(y)-\theta) - \Big(\sum_y (\hat{\theta}(y)-\theta) M(y)\Big)\Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big) - \sum_y \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big) M(y) (\hat{\theta}(y)-\theta) + \sum_y \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big) M(y) \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big) \geqq 0$$

$S_\theta = \sqrt{S_\theta}\sqrt{S_\theta}$

$$\therefore \sum_x (\hat{\theta}(x)-\theta)^2 M(x) - \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big)^2 - \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big)^2 + \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big)^2 \underbrace{\sum_y M(y)}_{I} \geqq 0$$

$$\therefore \sum_x (\hat{\theta}(x)-\theta)^2 M(x) \geqq \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big)^2$$

$\sqrt{S_\theta} \geqq 0$ と ★を使った

$$\therefore \sqrt{S_\theta} \sum_x (\hat{\theta}(x)-\theta)^2 M(x) \sqrt{S_\theta} \geqq \sqrt{S_\theta} \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big)^2 \sqrt{S_\theta}$$

$$\therefore \mathrm{Tr} \sqrt{S_\theta} \sum_x (\hat{\theta}(x)-\theta)^2 M(x) \sqrt{S_\theta} \geqq \mathrm{Tr} \sqrt{S_\theta} \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big)^2 \sqrt{S_\theta}$$

$$\to \mathrm{Tr} \sum_x (\hat{\theta}(x)-\theta)^2 M(x) S_\theta \geqq \mathrm{Tr} \Big(\sum_x (\hat{\theta}(x)-\theta) M(x)\Big)^2 S_\theta$$

$$\sum_x (\hat{\theta}(x)-\theta)^2 \underbrace{\mathrm{Tr} M(x) S_\theta}_{= P_\theta^M(x)} \geqq \mathrm{Tr} (T_M - \theta I)^2 S_\theta$$

$$\therefore V_\theta[\hat{\theta}, M] \geqq \langle T_M - \theta I, T_M - \theta I \rangle_{S_\theta}$$