von Neumann alg.

Assume $\mathfrak{A} \subset \mathcal{L}(\mathcal{H})$ where $\dim \mathcal{H} < \infty$.

Then any linear functional $\omega : \mathfrak{A} \to \mathbb{C}$

can be represented as

$$\omega(A) = \operatorname{Tr} \rho A$$

$\because$

$\mathcal{L}(\mathcal{H})$ is an inner product space with

the inner product $\langle A, B \rangle_2 = \operatorname{Tr} A^* B$.

Then $\mathfrak{A}$ is also an inner product space

with the inner product $\langle \cdot , \cdot \rangle_2$.

So $\exists\, \rho$ s.t. $\omega(A) = \langle \rho, A \rangle_2$.

Since $\omega(A^*A) \geq 0$, $\rho \geq 0$.

Since $\omega(I) = 1$, $\operatorname{Tr} \rho = 1$

These mean that $\rho$ is a d.o.

$\therefore\ \omega(A) = \operatorname{Tr} \rho^* A = \operatorname{Tr} \rho A$ //