1

A：エルミート行列

$A = \lambda_1 |e_1\rangle\langle e_1| + \cdots + \lambda_n |e_n\rangle\langle e_n|$

λ_i：固有値　$|e_i\rangle$：固有ベクトル

$\|e_i\| = \sqrt{\langle e_i|e_i\rangle} = 1$　（スナワチ $\langle e_i|e_i\rangle = 1$）①

$\langle e_i|e_j\rangle = 0 \quad i \neq j$ ②

$$\left.\begin{array}{l} A \geqq 0 \iff \lambda_1 \geqq 0, \cdots, \lambda_n \geqq 0 \\ \mathrm{Tr}A = 1 \iff \lambda_1 + \cdots + \lambda_n = 1 \end{array}\right\} \iff A:\text{状態}$$

測定　$M = \{M_i\}$　$M_i \geqq 0$，$\sum_i M_i = I$

単純測定　$M = \{M_i\}$．$M_i^2 = M_i$，$M_i M_j = 0 \quad i \neq j$

A：正射影行列 $\iff \lambda_i = 0$ または 1．

$\iff A^2 = A$，A：エルミート行列．

$\uparrow$

$A^2 = (\lambda_1|e_1\rangle\langle e_1| + \cdots + \lambda_n|e_n\rangle\langle e_n|)(\lambda_1|e_1\rangle\langle e_1| + \cdots + \lambda_n|e_n\rangle\langle e_n|)$

$= \lambda_1^2 |e_1\rangle\underbrace{\langle e_1|\cdot|e_1\rangle}_{\langle e_1|e_1\rangle=1}\langle e_1| + \lambda_1\lambda_2|e_1\rangle\underbrace{\langle e_1|\cdot|e_2\rangle}_{\langle e_1|e_2\rangle=0}\langle e_2| + \cdots$

$+ \lambda_2\lambda_1|e_2\rangle\underbrace{\langle e_2|\cdot|e_1\rangle}_{\langle e_2|e_1\rangle=0}\langle e_1| + \lambda_2^2|e_2\rangle\underbrace{\langle e_2|\cdot|e_2\rangle}_{\langle e_2|e_2\rangle=1}\langle e_2| + \cdots$

$= \lambda_1^2|e_1\rangle\langle e_1| + \cdots + \lambda_n^2|e_n\rangle\langle e_n|$

$A = \lambda_1|e_1\rangle\langle e_1| + \cdots + \lambda_n|e_n\rangle\langle e_n|$

$A^2 = A \iff \lambda_1^2 = \lambda_1, \cdots + \lambda_n^2 = \lambda_n$

$\iff \lambda_1^2 - \lambda_1 = 0$　　$\iff \lambda_n^2 - \lambda_n = 0$

$\lambda_1(\lambda_1 - 1) = 0$　　$\lambda_n(\lambda_n - 1) = 0$

$\lambda_1 = 0$ or 1　　$\lambda_n = 0$ or 1

2

A：2×2 のエルミート行列．$\left(A = \lambda_1|e_1\rangle\langle e_1| + \lambda_2|e_2\rangle\langle e_2|\right)$

$A \geq 0,\ \mathrm{Tr}A = 1 \iff A = \frac{1}{2}\left(I + x\sigma_x + y\sigma_y + z\sigma_z\right)$

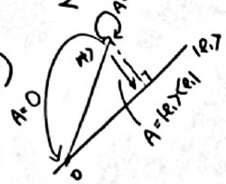
（A：状態）

$x^2+y^2+z^2 \leq 1,\ x, y, z \in \mathbb{R}$

・A の固有値は $\dfrac{1 \pm \sqrt{x^2+y^2+z^2}}{2}$

・$x^2+y^2+z^2 = 1$ のとき A の固有値は $\dfrac{1\pm 1}{2} = 1$ or 0

すなわち A は正射影行列．—㊍



・A：正射影行列．

$\lambda_1 = \lambda_2 = 1,\quad A = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| = \begin{pmatrix}1&0\\0&1\end{pmatrix} \longrightarrow \mathrm{Tr}A = 2$ ①

$\lambda_1 = 1,\ \lambda_2 = 0 \quad A = |e_1\rangle\langle e_1| \longrightarrow \mathrm{Tr}A = 1 \longrightarrow$ ㊍

$\lambda_1 = 0,\ \lambda_2 = 0 \quad A = \begin{pmatrix}0&0\\0&0\end{pmatrix} \longrightarrow \mathrm{Tr}A = 0$ ②

まとめると．2×2 の正射影行列は以下の形しかとらない

・$A = \begin{pmatrix}1&0\\0&1\end{pmatrix},\quad A = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

・$A = \frac{1}{2}\left(I + x\sigma_x + y\sigma_y + z\sigma_z\right),\quad x^2+y^2+z^2 = 1$

$x, y, z \in \mathbb{R}$

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② の証明：$|e_1\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$，$\langle e_1|e_1\rangle = 1$ より．$(\bar{\alpha}\ \bar{\beta})\begin{pmatrix}\alpha\\\beta\end{pmatrix} = |\alpha|^2+|\beta|^2 = 1$

$$A = |e_1\rangle\langle e_1| = \begin{pmatrix}\alpha\\\beta\end{pmatrix}(\bar{\alpha}\ \bar{\beta}) = \begin{pmatrix}\alpha\bar{\alpha} & \alpha\bar{\beta}\\ \bar{\alpha}\beta & \beta\bar{\beta}\end{pmatrix} = \begin{pmatrix}|\alpha|^2 & \alpha\bar{\beta}\\ \bar{\alpha}\beta & |\beta|^2\end{pmatrix}$$

$\therefore\ \mathrm{Tr}A = |\alpha|^2+|\beta|^2 = 1.$

① の証明．$|e_1\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix},\ |e_2\rangle = \begin{pmatrix}\gamma\\\delta\end{pmatrix}$ ……> 計算がメンドクサクなる

$|\psi\rangle = \alpha|e_1\rangle + \beta|e_2\rangle.\quad \alpha, \beta \in \mathbb{C}$

$$A|\psi\rangle = \left(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|\right)\left(\alpha|e_1\rangle + \beta|e_2\rangle\right) = \alpha|e_1\rangle\underbrace{\langle e_1|e_1\rangle}_{1} + \beta|e_1\rangle\underbrace{\langle e_1|e_2\rangle}_{0}$$
$$+\ \alpha|e_2\rangle\underbrace{\langle e_2|e_1\rangle}_{0} + \beta|e_2\rangle\underbrace{\langle e_2|e_2\rangle}_{1}$$
$$= \alpha|e_1\rangle + \beta|e_2\rangle = |\psi\rangle$$

よって $A = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ である．

3

A : 2×2 のエルミート行列. $\left(A = \lambda_1|e_1\rangle\langle e_1| + \lambda_2|e_2\rangle\langle e_2|\right)$

単純測定 : $M = \{M_1, M_2\}$　$M_i^2 = M_i$, $M_i \geq 0$, $M_i M_j = 0$ $\quad i \neq j$

$M_1 + M_2 = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}$ —①

$M_1 = I$ のとき ①より. $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \leftarrow O$

$M_1 = O$ のとき ①より $M_2 = I$

$M = \{I, O\}, \{O, I\}$: 自明な単純測定 (trivial)

$\begin{cases} p_1 = 0 \\ p_2 = 1 \end{cases}$

$\begin{cases} p_1 = \mathrm{Tr} S M_1 = \mathrm{Tr} S I = \mathrm{Tr} S = 1 \\ p_2 = 0 \end{cases}$

$M = \{M_1, M_2\}$, $\mathrm{Tr} M_1 = \mathrm{Tr} M_2 = 1$: 自明でない単純測定 (non trivial)

$M_j = \frac{1}{2}\{I + \alpha_j \sigma_x + \beta_j \sigma_y + \gamma_j \sigma_z\}$, $j = 1, 2$ $\quad (\alpha_j^2 + \beta_j^2 + \gamma_j^2 = 1)$

$$M_j = \frac{1}{2}\begin{bmatrix} 1 + \gamma_j & \alpha_j - i\beta_j \\ \alpha_j + i\beta_j & 1 - \gamma_j \end{bmatrix}$$

$$M_1 + M_2 = \frac{1}{2}\begin{bmatrix} 2 + \gamma_1 + \gamma_2 & \alpha_1 + \alpha_2 - i(\beta_1 + \beta_2) \\ \alpha_1 + \alpha_2 + i(\beta_1 + \beta_2) & 2 - \gamma_1 - \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $\cancel{2} + \gamma_1 + \gamma_2 = \cancel{2}$ $\quad \alpha_1 + \alpha_2 - i(\beta_1 + \beta_2) = 0$

$(\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2) = 0$ $\quad 2 - \gamma_1 - \gamma_2 = 2$

$\therefore$ $\gamma_2 = -\gamma_1$, $\alpha_2 = -\alpha_1$, $\beta_2 = -\beta_1$

Stern-Gerlach 測定と呼ばれる.

$$\boxed{M_1 = \frac{1}{2}\begin{bmatrix} 1 + \gamma_1 & \alpha_1 - i\beta_1 \\ \alpha_1 + i\beta_1 & 1 - \gamma_1 \end{bmatrix} , \quad M_2 = \frac{1}{2}\begin{bmatrix} 1 - \gamma_1 & -\alpha_1 + i\beta_1 \\ -\alpha_1 - i\beta_1 & 1 + \gamma_1 \end{bmatrix}}$$

自明でない単純測定はすべてこの形で書ける.

テキストP12 下から5行目

$$M_1 M_2 = \frac{1}{4}\begin{bmatrix} 1 - \gamma_1^2 - \alpha_1^2 - \beta_1^2 & \cancel{-\alpha_1} + \cancel{i\beta_1} - \cancel{\gamma_1\alpha_1} + \cancel{i\beta_1\gamma_1} + \cancel{\alpha_1} + \cancel{\alpha_1\gamma_1} - \cancel{i\beta_1} - \cancel{i\beta_1\gamma_1} \\ \cancel{\alpha_1} - \cancel{\alpha_1\gamma_1} + \cancel{i\beta_1} - \cancel{i\beta_1\gamma_1} - \cancel{\alpha_1} - \cancel{i\beta_1} + \cancel{\alpha_1\gamma_1} - \cancel{i\beta_1\gamma_1} & -\alpha_1^2 - \beta_1^2 + 1 - \gamma_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$