Quantum Mechanics.

$\mathcal{H}$: Hilbert space

ψ : state

A : observable

$\omega_\psi(A) = \langle \psi | A \psi \rangle$: expectation of observable A for state ψ.

$\Downarrow$

$\mathfrak{A}_S := \underbrace{\text{l.i.h.}}_{\text{linear hull}}(\{M_1 M_2 \cdots M_n \,;\, n \in \mathbb{N},\ M_j : \text{observable},\ j=1,\cdots,n\})$

$*$- algebra.

$\uparrow$ It is important to consider the totality of observables.

Algebraic Quantum Mechanics

- We start with C^*-alg (or more generally $*$-alg.) without considering representation (Hilbert) space.

Axiom 1

For each Quantum System, there exists a unital C^*-alg. $\mathfrak{A}$.
where observales are described by self-adj. elem. of $\mathfrak{A}$
and the state of the system are described by a state on $\mathfrak{A}$.

Axiom 2

- For state $\omega : \mathfrak{A} \to \mathbb{C}$, expectation value of measuring result for observable A.
is given by $\omega(A)$

$\mathfrak{A}$: C^*-alg $\subset \mathfrak{B}(\mathcal{H}_{\mathfrak{A}})$

ω : $\mathfrak{A} \longrightarrow \mathbb{C}$: state

A : observable

$$\left(A = A^* , \quad A \in \mathfrak{B}(\mathcal{H}_{\mathfrak{A}}) \right)$$

$$A = \int \lambda \, E_A(d\lambda)$$

Borel set of $\mathbb{R}$

$E_A(J) \in \mathfrak{A}$, $J \in \mathcal{B}^1$

- Axiom II'. (ii) $\omega(E_A(J))$ gives probability that "measuring result" $\in J$.
  (i). ω is weak-continuous

- Axiom II' $\Rightarrow$ Axiom II

$\because$ Expectation is given by

$$\int_J \lambda \, \omega(E_A(d\lambda)) \underset{\text{Axiom II}'(i)}{=} \omega\left(\int_J \lambda E_A(d\lambda) \right)$$

$$= \omega(E_A(J))$$