Quantum Mechanics

$\mathcal{H}$: Hilbert space

ψ : state

A : observable

$\omega_\psi(A) = \langle \psi | A \psi \rangle$: expectation of observable A for state ψ.

$\Downarrow$

$\mathfrak{A}_S := \underbrace{\text{l.i.h.}}_{\text{linear hull}}(\{ M_1 M_2 \cdots M_n \ ; \ n \in \mathbb{N} \ , \ M_j : \text{observable} \ , \ j=1,\cdots,n \})$

$*$ - algebra.

It is important to consider the totality of observables.

Algebraic Quantum Mechanics

· We start with C^*-alg without considering representation (Hilbert) space.

or more generally $*$-alg.

Axiom 1

For each Quantum System, there exists a unital C^*-alg. $\mathfrak{A}$.
where observales are described by self-adj. elem. of $\mathfrak{A}$
and the state of the system are described by a state on $\mathfrak{A}$.

Axiom 2

· For state $\omega : \mathfrak{A} \to \mathbb{C}$, expectation value of measuring result for observable A.
is given by $\omega(A)$

Axiom 3

$\mathfrak{A}$: C^*- alg of observables

$\{\alpha_t\}_{t \in \mathbb{R}}$: one-parameter autom group.

Time evolution of $A \in \mathfrak{A}$ is given by $\alpha_t(A)$.

$\mathfrak{A}$: C^*-alg $\subset \mathfrak{B}(\mathcal{H}_{\mathfrak{A}})$

ω : $\mathfrak{A} \longrightarrow \mathbb{C}$: state

A : observable

$$\left(A = A^* , \quad A \in \mathfrak{B}(\mathcal{H}_{\mathfrak{A}}) \right)$$

$$A = \int \lambda \, E_A(d\lambda)$$

$E_A(J) \in \mathfrak{A}$, $J \in \mathcal{B}^1$ ← Borel set of $\mathbb{R}$

- Axiom II$'$. (ii) $\omega(E_A(J))$ gives probability that "measuring result" $\in J$.
  (i). ω is weak-continuous

- Axiom II$'$ $\Rightarrow$ Axiom II

∵ Expectation is given by

$$\int_J \lambda \, \omega(E_A(d\lambda)) = \omega\left(\int_J \lambda E_A(d\lambda) \right)$$

Axiom II$'$(i) $= \omega(E_A(J))$

H is a Hamiltonean of the system. (A : s.a. on $\mathcal{H}$)

$$\boxed{\alpha_t^H(A) = e^{itH/\hbar} A e^{-itH/\hbar}}$$

↑ This gives the observable at t.

. We may take an unbounded operator as A.

$\Updownarrow$ $\quad$ $\mathfrak{A}$ is a C^*-alg. $\subset \mathfrak{B}(\mathcal{H})$

. We can consider α_t^H as a map $\mathfrak{A} \longrightarrow \mathfrak{A}$.

assuming $\alpha_t^H(A) \in \mathfrak{A}$ for $\forall t \in \mathbb{R}$ and $\forall A \in \mathfrak{A}$.

Prop.

1) α_t^H is Linear.

2) $\alpha_t^H(AB) = \alpha_t^H(A)\alpha_t^H(B)$

3) $\alpha_t^H(A)^* = \alpha_t^H(A^*)$

means α_t^H is $*$-hom.

4) $\alpha_t^H(I) = I$

5) $\alpha_{t+s}^H = \alpha_t^H \alpha_s^H$

6) $\alpha_0^H = I_{\mathfrak{A}}$

7) $\lim_{s\to 0} \| \alpha_{t+s}^H(A)\psi - \alpha_t^H(A)\psi \| = 0$, $\forall A \in \mathfrak{A}$, $\forall \psi \in \mathcal{H}$.

$$\begin{cases} \alpha_\cdot^H : \mathbb{R} \to \mathrm{Aut}(\mathfrak{A}) \\ \quad t \mapsto \alpha_t^H \\ \alpha_t^H : \mathfrak{A} \longrightarrow \mathfrak{A} \end{cases}$$

From (5) and (6), $I_{\mathfrak{A}} = \alpha_0^H = \alpha_t^H \alpha_{-t}^H$,

which means α_t^H is bij. (i.e. $(\alpha_t^H)^{-1} = \alpha_{-t}^H$)

∴ α_t^H forms a one-parameter group of $*$-automorphism.

Def. $\mathfrak{A}$: unital C^*-alg. When a family of $*$-autom. $\{\alpha_t\}_{t\in\mathbb{R}}$ satisfies $\alpha_t \alpha_s = \alpha_{t+s}$, $t, s \in \mathbb{R}$, $\{\alpha_t\}_{t\in\mathbb{R}}$ is called <u>one-parameter automorphism group</u>. α_t^H gives an example of one-param. autom. group