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$E_\theta[T] = \mathrm{Tr}\, S_\theta T$ — ①

状態 S_θ の物理量 T を測定した時の測定値の期待値

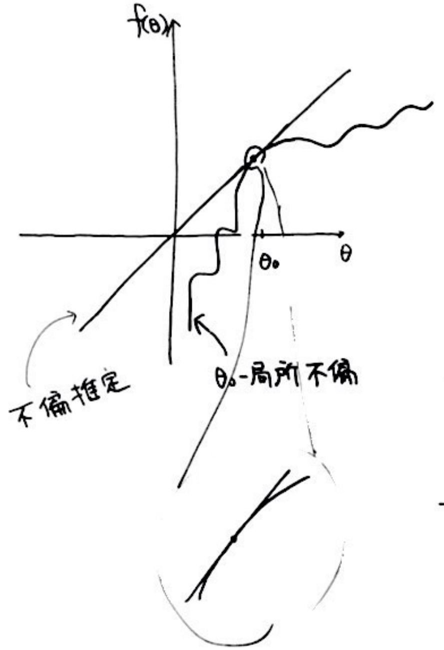
$E_\theta[T] = \theta \iff T$ は不偏推定量

① より $E_\theta[T]=\theta \iff \mathrm{Tr}\, S_\theta T = \theta$ — ②

$$\Big\downarrow \text{微分}$$

$\frac{d}{d\theta} \mathrm{Tr}\, S_\theta T = 1$ — ③

② と ③ が θ_0 で成立するとき
T は θ_0-局所不偏



$f(\theta) = \mathrm{Tr}\, S_\theta T$

$\theta \longmapsto f(\theta)$

T が不偏推定量

$\iff f(\theta) = \theta$

$\mathrm{Tr}\, S_{\theta_0} T = \theta_0$ 　 $f(\theta_0) = \theta_0$

$\frac{d}{d\theta} \mathrm{Tr}\, S_\theta T \Big|_{\theta=\theta_0} = 1$ 　 $\frac{d}{d\theta} f(\theta_0) = 1$

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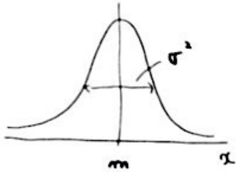
局所不偏だが不偏ではない例（古典）

X ． 平均 $\frac{1}{2a}$ 分散 θ^2 の正規分布



$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m)^2/2\sigma^2}$$

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

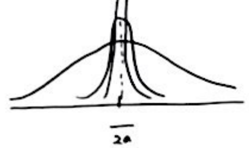
$a \le x \le b$ となる確率 $\int_a^b p(x)\,dx$

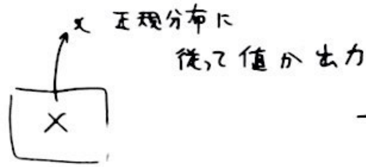
注 $p(x)$ は x の出現する確率ではない

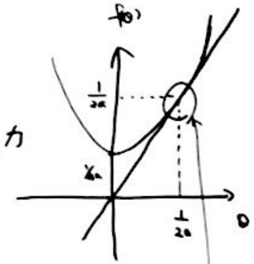
$p(x)$ はかくりつ密度をあらわす．

$m = \frac{1}{2a}$ 固定

$\sigma = \theta \leftarrow$ 推定







この辺りで見ると $E_\theta[\delta(x)] \fallingdotseq \theta$

$\underset{\sim\sim\sim}{\delta(x) = ax^2}$　　局所不偏推定量　$\theta_0 = \frac{1}{2a}$

$m = E_\theta[X]$　　$\theta^2 = E_\theta[(X-m)^2] = E_\theta(X^2) - m^2$

$$E_\theta[\delta(X)] = E_\theta[aX^2] = a\,E_\theta[X^2] = a(\theta^2 + m^2)$$

$$= a\left(\theta^2 + \left(\frac{1}{2a}\right)^2\right) = a\theta^2 + \frac{1}{4a} = f(\theta)$$

$E_\theta[\delta(X)] \ne \theta$ ∴ δ は不偏推定ではない．

$\theta = \frac{1}{2a}$ のとき $E_\theta[\delta(X)] = a\left(\frac{1}{2a}\right)^2 + \frac{1}{4a} = \frac{1}{2a} = \theta$

$$\frac{d}{d\theta}E_\theta[\delta(X)] = 2a\theta \quad \therefore \frac{d}{d\theta}E_\theta[\delta(X)]\Big|_{\theta=\frac{1}{2a}} = 2a\frac{1}{2a} = 1$$

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Ω：事象系

$\Omega = \{\omega_1, \cdots, \omega_n\}$

Ω 上の確率： $P(\omega_1) = p_1(\theta), \cdots, P(\omega_n) = p_n(\theta)$, $\theta \in \Theta \subset \mathbb{R}$

確率変数 $X(\omega_1) = t_1, \cdots, X(\omega_n) = t_n$ （$t_1, \cdots, t_n$ は互いに異なる）

$P(t_i) = p_i(\theta)$

期待値 $= t_1 p_1(\theta) + \cdots + t_n p_n(\theta)$

状態 $S_\theta = \begin{pmatrix} p_1(\theta) & & & \\ & p_2(\theta) & & \\ & & \ddots & \\ & & & p_n(\theta) \end{pmatrix}$

測定 $T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$

T の固有ベクトル $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, |e_n\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

固有値 $t_1, t_2, \cdots, t_n$

$\left(T|e_i\rangle = \begin{pmatrix} t_1 & & & \\ & t_i & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} i = t_i \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} i = t_i |e_i\rangle \right)$

$M_i = |e_i\rangle\langle e_i|$ $M = \{M_i\}$: POVM

$= \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} (0 \cdots 0 1 0 \cdots 0) = i \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$

$\underline{P(t_i) = \mathrm{Tr}\, S_\theta M_i} = \mathrm{Tr} \begin{pmatrix} p_1(\theta) & & \\ & \ddots & \\ & & p_n(\theta) \end{pmatrix} \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} i$

$= p_i(\theta)$

$E_\theta[T] = \underline{\mathrm{Tr}\, S_\theta T} = \mathrm{Tr} \begin{pmatrix} p_1(\theta) t_1 & & & \\ & p_2(\theta) t_2 & & \\ & & \ddots & \\ & & & p_n(\theta) t_n \end{pmatrix} = \underline{p_1(\theta) t_1 + \cdots + p_n(\theta) t_n}$

$V_\theta[T] = \mathrm{Tr}\, S_\theta (T - \theta I)^2 = \mathrm{Tr}\, S_\theta \begin{pmatrix} (t_1 - \theta)^2 & & \\ & \ddots & \\ & & (t_n - \theta)^2 \end{pmatrix} = p_1(\theta)(t_1 - \theta)^2 + \cdots + p_n(\theta)(t_n - \theta)^2$

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$$V_\theta[T] = \mathrm{Tr}\, S_\theta (T - \theta I)^2$$

$$= \sum_{i=1}^{n} p_i(\theta)(t_i - \theta)^2$$

$$L_\theta = \begin{pmatrix} l_1(\theta) & & \\ & \ddots & \\ & & l_n(\theta) \end{pmatrix} = \begin{pmatrix} \frac{1}{p_1(\theta)}\frac{d}{d\theta}p_1(\theta) & & \\ & \ddots & \\ & & \frac{1}{p_n(\theta)}\frac{d}{d\theta}p_n(\theta) \end{pmatrix}$$

対数微分

$$l_i(\theta) = \frac{d}{d\theta}\log p_i(\theta)$$

$$(L_\theta, L_\theta)_{S_\theta} = \mathrm{Tr}\, S_\theta L_\theta L_\theta^*$$

$$= \mathrm{Tr}\begin{pmatrix} p_1(\theta) & & \\ & \ddots & \\ & & p_n(\theta) \end{pmatrix}\begin{pmatrix} l_1(\theta) & & \\ & \ddots & \\ & & l_n(\theta) \end{pmatrix}\begin{pmatrix} l_1(\theta) & & \\ & \ddots & \\ & & l_n(\theta) \end{pmatrix}$$

$$= \sum_{i=1}^{n} p_i(\theta)\, l_i^2(\theta)$$

$$V_\theta[T]\ (L_\theta, L_\theta)_{S_\theta} = \left[\sum_{i=1}^{n} p_i(\theta)(t_i-\theta)^2 \sum_{i=1}^{n} p_i(\theta)\, l_i(\theta)^2\right]$$

$$\sum_{i=1}^{n} p_i(\theta)\, l_i(\theta)^2 = J_\theta$$

$$\geq \left(\sum_{i=1}^{n} p_i(\theta)(t_i-\theta)\, l_i(\theta)\right)^2$$

Shwartz の不等式

$$V_\theta[T] \geq \frac{1}{(L_\theta, L_\theta)_{S_\theta}} = \frac{1}{\sum_{i=1}^{n} p_i(\theta)\, l_i(\theta)^2}$$

$$= \left(\sum_{i=1}^{n} \cancel{p_i(\theta)}(t_i-\theta)\frac{1}{\cancel{p_i(\theta)}}\frac{d}{d\theta}p_i(\theta)\right)^2$$

$$= \left(\sum_{i=1}^{n} \frac{d}{d\theta}p_i(\theta)\ (t_i-\theta)\right)^2$$

$$= \left(\underbrace{\sum_{i=1}^{n} \frac{d}{d\theta}p_i(\theta)\, t_i}_{=1} - \theta \underbrace{\sum_{i=1}^{n}\frac{d}{d\theta}p_i(\theta)}_{=0}\right)^2 = 1$$

$$\mathrm{Tr}\, S_\theta T = \sum_{i=1}^{n} t_i\, p_i(\theta) = \theta$$

$$\frac{d}{d\theta}\mathrm{Tr}\, S_\theta T = \sum_{i=1}^{n} t_i \frac{d}{d\theta}p_i(\theta) = 1$$

$$\sum p_i(\theta) = 1$$

$$\sum \frac{d}{d\theta}p_i(\theta) = 0$$