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n 次元複素 線型空间 $\mathcal{H}$ に話を限定

$\mathcal{H} \ni \psi, \varphi$

(ψ, φ) : 内積 $\left(\langle \psi | \varphi \rangle \right)$

$(\alpha\psi, \beta\varphi) = \bar{\alpha}\beta(\psi, \varphi)$ $\left(\langle \alpha\psi | \beta\varphi \rangle = \bar{\alpha}\beta \langle \psi | \varphi \rangle \right)$

物理の習慣

$(\alpha\psi, \beta\varphi) = \alpha\bar{\beta}(\psi, \varphi)$

数学の習慣

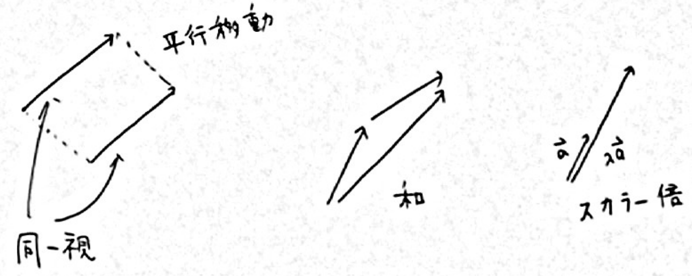
○ 内積に関する $\mathcal{H}$ の正規直交基底を1つ固定すれば $\mathcal{H}$ は $\mathbb{C}^n$ と同一視でき, $\mathcal{H}$ 上の線型作用素は n 次正方行列とみなせる.

$\mathbb{C}^n$ + 距離

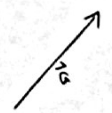
Euclid 空间

線型空间 = ベクトル空间 V

↑ ベクトルの集合



成分表示



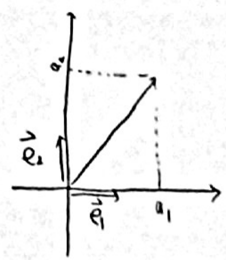
$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$

$\vec{a} = (a_1, a_2)$

↑

今後

$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$

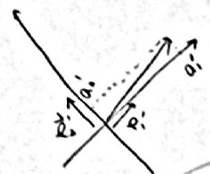


$\vec{e}_1, \vec{e}_2$: 基本ベクトル (基底)

V と $\mathbb{R}^2$ は同視できた.

$\vec{a} = a_1'\vec{e}_1' + a_2'\vec{e}_2'$

$\vec{a} = \begin{pmatrix} a_1' \\ a_2' \end{pmatrix}$



基底のとり方がかわると V と $\mathbb{R}^2$ の対応関係がかわる

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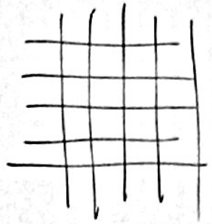
例題5

基底 $\vec{e}_1$ と $\vec{e}_2$ を使った表現

$\vec{c} = \begin{pmatrix} 8 \\ 9 \end{pmatrix} = 8\vec{e}_1 + 9\vec{e}_2$



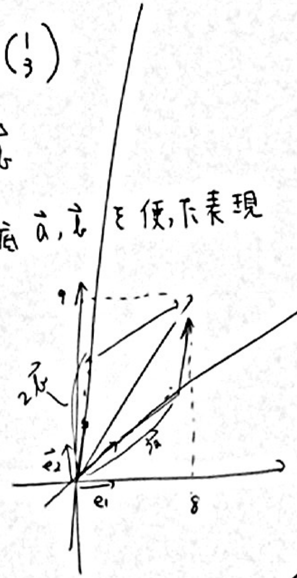
$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

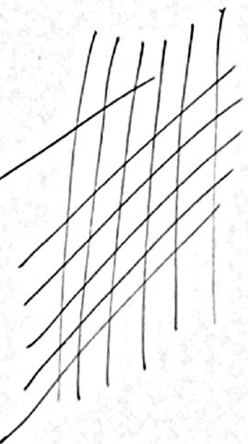
$\vec{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\vec{c} = 3\vec{a} + 2\vec{b}$

$\updownarrow$

$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$: 基底 $\vec{a}, \vec{b}$ を使った表現





- $\vec{a}, \vec{b}$ を $\vec{e}_1, \vec{e}_2$ をつかってあらわす

$$\begin{cases} \vec{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\vec{e}_1 + \vec{e}_2 \\ \vec{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \vec{e}_1 + 3\vec{e}_2 \end{cases}$$

$$(\vec{a} \quad \vec{b}) = (2\vec{e}_1 + \vec{e}_2 \quad \vec{e}_1 + 3\vec{e}_2)$$
$$= (\vec{e}_1 \quad \vec{e}_2)\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$(\vec{a} \quad \vec{b}) = (\vec{e}_1 \quad \vec{e}_2)\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$\updownarrow$ $\qquad$ $\updownarrow$

$$\begin{pmatrix} \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 3 \end{matrix}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

- 事実

$$\underbrace{(\vec{b}_1 \quad \vec{b}_2)}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (a_{11}\vec{b}_1 + a_{21}\vec{b}_2 \quad a_{12}\vec{b}_1 + a_{22}\vec{b}_2)$$

$$\left((b_1 \quad b_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (b_1 a_{11} + b_2 a_{21} \quad b_1 a_{12} + b_2 a_{22}) \right)$$

との類似が成立する

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$$(\vec{b}_1\ \vec{b}_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \Bigg(\underbrace{\boxed{\begin{matrix} b_{11} \\ b_{21} \end{matrix}}}_{\vec{b}_1}\ \underbrace{\boxed{\begin{matrix} b_{12} \\ b_{22} \end{matrix}}}_{\vec{b}_2}\Bigg)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\vec{b}_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix},\ \vec{b}_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{\begin{matrix} b_{11}a_{11} + b_{12}a_{21} \\ b_{21}a_{11} + b_{22}a_{21} \end{matrix}} & \begin{matrix} b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{12} + b_{22}a_{22} \end{matrix} \end{pmatrix}$$

$$\begin{matrix} b_{11}a_{11} + b_{12}a_{21} \\ b_{21}a_{11} + b_{22}a_{21} \end{matrix} = a_{11}\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} + a_{21}\begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = a_{11}\vec{b}_1 + a_{21}\vec{b}_2$$

$$\begin{matrix} b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{12} + b_{22}a_{22} \end{matrix} = a_{12}\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} + a_{22}\begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = a_{12}\vec{b}_1 + a_{22}\vec{b}_2$$

$$= \left(a_{11}\vec{b}_1 + a_{21}\vec{b}_2 \quad a_{12}\vec{b}_1 + a_{22}\vec{b}_2\right)$$

よって

$$(\vec{b}_1\ \vec{b}_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \left(a_{11}\vec{b}_1 + a_{21}\vec{b}_2 \quad a_{12}\vec{b}_1 + a_{22}\vec{b}_2\right)$$

ベクトル $\vec{c}$ を
基底 $\vec{a}, \vec{b}$ で表示 $\longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\vec{c} = x_1\vec{a} + x_2\vec{b} = (\vec{a}\ \vec{b})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

ベクトル $\vec{c}$ を
基底 $\vec{e}_1, \vec{e}_2$ で表示 $\longleftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$= y_1\vec{e}_1 + y_2\vec{e}_2 = (\vec{e}_1\ \vec{e}_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$(a\ b)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1 + bx_2$$

先程のギロンにより $\boxed{(\vec{a}\ \vec{b}) = (\vec{e}_1\ \vec{e}_2)\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}}$

よって $\vec{c} = (\vec{a}\ \vec{b})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\vec{e}_1\ \vec{e}_2)\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= (\vec{e}_1\ \vec{e}_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$(\vec{e}_1\ \vec{e}_2)$ は逆行列 $(\vec{e}_1\ \vec{e}_2)^{-1}$ をもつので

$$\cancel{(\vec{e}_1\ \vec{e}_2)^{-1}(\vec{e}_1\ \vec{e}_2)}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \cancel{(\vec{e}_1\ \vec{e}_2)^{-1}(\vec{e}_1\ \vec{e}_2)}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$(\vec{e}_1\ \vec{e}_2)^{-1}(\vec{e}_1\ \vec{e}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}$$

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$f : V \longrightarrow V$

V : ベクトル空间

基底 $\vec{e}_1, \vec{e}_2$

基底 $\vec{a}, \vec{b}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ | $P = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ | $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ | $\begin{pmatrix} \ \\ \ \end{pmatrix}$ | $\begin{pmatrix} \ \\ \ \end{pmatrix}$ $\mathbb{R}^2$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}$

$$\begin{tikzcd}
 & V \arrow[r, "f"] & V \\
\vec{e}_1, \vec{e}_2 & \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \\
\vec{a}, \vec{b} & \mathbb{R}^2 \arrow[u, "P"] \arrow[r, "A'"] & \mathbb{R}^2 \arrow[u, "P"']
\end{tikzcd}$$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$P^{-1}AP \leftarrow A'$