Prop ρ : inj.

$\mathcal{B}_{fin}(\mathcal{H}) = \{ \text{finite rank opr on } \mathcal{H} \} \subset \mathfrak{A}$

Then $\mathcal{H}_\rho = \mathcal{L}_2(\mathcal{H})$

Notes $H_\rho \subset \mathcal{L}_2(\mathcal{H})$ $\left(H_\rho := \overline{\ell(\mathfrak{A}) T_\rho} \right)$ ← Completion w.r.t. HS-norm

(∵) It suffices to show

$$\{ \ell(A) T_\rho \; ; \; A \in \mathcal{B}_{fin}(\mathcal{H}) \}^\perp = 0 .$$

$$S \in \{ \ell(A) T_\rho \; ; \; A \in \mathcal{B}_{fin}(\mathcal{H}) \}^\perp$$

$$\Updownarrow$$

$$\langle S, \ell(A) T_\rho \rangle_2 = 0$$

$$\Updownarrow$$

$$\langle S, A\rho^{1/2} \rangle_2 = 0$$

$$\| $$

$$\mathrm{Tr}\, S^* A \rho^{1/2} = \mathrm{Tr}\, \rho^{1/2} S^* A = \mathrm{Tr}\, (S\rho^{1/2})^* A$$

$$= \langle S\rho^{1/2}, A \rangle_2$$

Since $\mathcal{B}_{fin}(\mathcal{H})$ is dense in $\mathcal{L}_2(\mathcal{H})$,

$\langle S\rho^{1/2}, A \rangle_2 = 0 \quad \forall A \in \mathcal{B}_{fin}(\mathcal{H})$

$$\Updownarrow$$

$$S\rho^{1/2} = 0 \iff (S\rho^{1/2})^* = 0$$

$$\iff \rho^{1/2} S^* = 0 \iff \rho S^* = 0$$

$$\underset{\rho:\text{inj}}{\iff} S^* = 0 \iff S = 0$$

⌟