Modular operator and Holevo's commutation operator

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1 Introduction

This paper reviews results about modular operators [2] and commutation operators [1] in the simplest case. We mainly deal with the von Neumann algebra $\mathfrak{N} = M_n(\mathbb{C})$, which is the ensemble of $n \times n$ complex matrices and can be considered as the algebra $\mathfrak{B}(\mathcal{H})$ of bounded linear operators on the Hilbert space $\mathcal{H} = \mathbb{C}^n$ with the inner product $(x, y) = \bar{x}^T y$. Let ρ be a non-degenerated density operator and ω be the corresponding normal state given by $\omega(A) = \mathrm{Tr}\rho A, A \in \mathfrak{N}$. We regard $\mathfrak{N}$ as a Hilbert space with the inner product

$$\langle A, B \rangle = \frac{1}{2}\omega(BA^* + A^*B), \tag{1}$$

and denote it by $\mathfrak{H}$. In the quantum theory we consider additional bilinear form on $\mathfrak{H}$

$$[A, B] = i\omega(A^*B - BA^*). \tag{2}$$

and obtain fundamental inequalities

$$\langle X, X \rangle \geq \pm\frac{i}{2}[X, X],$$

which yield the uncertainty relation of the most general form [1]. We define a commutation operator $\mathfrak{D}$ so that it satisfies

$$[A, X] = \langle A, \mathfrak{D}X \rangle. \tag{3}$$

The operator $\mathfrak{D}$, firstly introduced by Holevo [1], plays an important role in the non-commutative statistical theory. From Eq. (2) it holds that

$$1 \pm \frac{i}{2}\mathfrak{D} \geq 0. \tag{4}$$

On the other hand we can also regard $\mathfrak{N}$ as a Hilbert-Schmidt space with another inner product

$$\langle A, B\rangle_2 = \mathrm{Tr}(A^* B),$$

by virtue of its finite-dimensionality and denote it by $\mathfrak{H}_2$. Let us consider a $*$-representation on $\mathfrak{H}_2$

$$\ell : \mathfrak{N} \to \mathfrak{B}(\mathfrak{H}_2), \tag{5}$$

where $\mathfrak{B}(\mathfrak{H}_2)$ is the ensemble of bounded operators on $\mathfrak{H}_2$ and $\ell(A)B = AB$. Then the state ω can be written by the inner product $\langle \cdot, \cdot\rangle_2$ as

$$\omega(A) = \mathrm{Tr}(\rho^{1/2} A \rho^{1/2}) = \langle \rho^{1/2}, \ell(A)\rho^{1/2}\rangle_2.$$

Remark: In the present case we have $\mathfrak{H}_2 = \mathfrak{H}$, but when we consider an infinite dimensional Hilbert space $\mathcal{H}$ the equality does not hold, i.e. $\mathfrak{H}_2 \subset \mathfrak{B}(\mathcal{H}) \subset \mathfrak{H}$. This may make it difficult to extend the discussion in Sec. 4 to the infinite dimensional case.

As stated in Sec. 3, we can define the modular operator Δ for the von Neumann algebra $\mathfrak{M} = \ell(\mathfrak{N})$ and its cyclic separating vector $\rho^{1/2} \in \mathfrak{H}_2$. In this paper we derive a simple relation between such derived modular operator and the commutation operator:

$$\Delta = \left(1 + \frac{i}{2}\mathfrak{D}\right)\left(1 - \frac{i}{2}\mathfrak{D}\right)^{-1},$$

which is originally shown in [1].

2 Representation on $\mathcal{H} \otimes \mathcal{H}$

We identify the Hilbert-Schmidt space $\mathfrak{H}_2 (= \mathfrak{N} = M_n(\mathbb{C}))$ on $\mathcal{H} = \mathbb{C}^n$ with $\mathcal{H} \otimes \mathcal{H}$ by a unitary operator satisfying

$$v(e_j e_k^T) = e_j \otimes e_k,$$

where $e_j = (\delta_{j1}, \delta_{j2}, ..., \delta_{jn})^T$ with the Kronecker delta δ_{jl}. Here, for $\psi = \sum_j \lambda_j e_j$ and $\phi = \sum_k \mu_k e_k$ we have

$$\begin{aligned} v(\psi\phi^*) &= \sum_{j,k} \lambda_j \bar{\mu}_k v(e_j e_k^T) \\ &= \sum_{j,k} \lambda_j \bar{\mu}_k e_j \otimes e_k = \psi \otimes \overline{\phi}. \end{aligned} \tag{6}$$

In the $*$-representation (5), $\ell(A)$ is given by $\tilde{\ell}(A) = A \otimes I_n$ on $\mathcal{H} \otimes \mathcal{H}$. In fact

$$\begin{aligned} v(\ell(A) e_j e_k^T) &= v(A e_j e_k^T) = A e_j \otimes e_k \\ &= (A \otimes I_n) e_k \otimes e_j = \tilde{\ell}(A) v(e_j e_k^T), \end{aligned} \tag{7}$$

and hence $\ell(A) = v^*\tilde{\ell}(A)v$.
On the other hand,

$$r(A) : \mathfrak{N} \in X \to XA \ni \mathfrak{N}$$

is represented by $\tilde{r}(A) = I_n \otimes A^T$ on $\mathcal{H} \otimes \mathcal{H}$. In fact

$$\begin{aligned} v(r(A)e_j e_k^T) &= v(e_j e_k^T A) = v(e_j(A^* e_k)^*) = e_j \otimes \overline{A^* e_k} \\ &= e_j \otimes A^T e_k = (I_n \otimes A^T)e_j \otimes e_k = \tilde{r}(A)v(e_j e_k^T), \end{aligned} \tag{8}$$

and hence $r(A) = v^*\tilde{r}(A)v$. Remark that $\tilde{\ell}(\mathfrak{N}) = \mathfrak{N} \otimes I_n$ and $\tilde{r}(\mathfrak{N}) = I_n \otimes \mathfrak{N}$ are von Neumann algebras in

$$\mathfrak{B}(\mathcal{H} \otimes \mathcal{H}) = \mathfrak{B}(\mathcal{H}) \otimes \mathfrak{B}(\mathcal{H}) = \mathfrak{N} \otimes \mathfrak{N},$$

and we have

$$\tilde{\ell}(\mathfrak{N})' = \tilde{r}(\mathfrak{N}), \quad \tilde{r}(\mathfrak{N})' = \tilde{\ell}(\mathfrak{N}). \tag{9}$$

3 Modular Operator

We give a proof of the main results of Tomita-Takesaki theory in the case of $\mathfrak{M} = \ell(\mathfrak{N}) \subset \mathfrak{B}(\mathfrak{H}_2)$ with $\mathfrak{N} = M_n(\mathbb{C})$, where all difficulties in the theory vanish. From Eq. (9), we have

$$\begin{aligned} \mathfrak{M}' &= v^*\tilde{\ell}(\mathfrak{N})'v = v^*\tilde{r}(\mathfrak{N})v = r(\mathfrak{N}) \\ \mathfrak{M}'' &= v^*\tilde{r}(\mathfrak{N})'v = v^*\tilde{\ell}(\mathfrak{N})v = \ell(\mathfrak{N}) = \mathfrak{M}. \end{aligned} \tag{10}$$

We introduce a cyclic separating vector $\rho^{1/2}$, satisfying

$$\mathfrak{H}_2 = \mathfrak{M}\rho^{1/2} = \mathfrak{M}'\rho^{1/2},$$

and consider anti-linear operators on $\mathfrak{H}_2$

$$S : \ell(A)\rho^{1/2} \to \ell(A)^*\rho^{1/2}, \tag{11}$$

$$F : r(A)\rho^{1/2} \to r(A)^*\rho^{1/2}. \tag{12}$$

Here

$$\ell(A)^* = v^*(A \otimes I_n)^*v = v^*\tilde{\ell}(A^*)v = \ell(A^*),$$

and

$$r(A)^* = v^*(I_n \otimes A^T)^*v = v^*\tilde{r}(A^*)v = r(A^*).$$

The linear operator $\Delta = FS$ on $\mathfrak{H}_2$ is known as a modular operator. For the operator S, we have

$$S(X) = \rho^{-1/2}X^*\rho^{1/2}.$$

In fact, putting $X = \ell(A)\rho^{1/2} = A\rho^{1/2}$,$Y = \ell(A)^*\rho^{1/2} = \ell(A^*)\rho^{1/2} = A^*\rho^{1/2}$,

$$Y = (X\rho^{-1/2})^*\rho^{1/2} = \rho^{-1/2}X^*\rho^{1/2}.$$

On the other hand, for the operator F, we have

$$F(X) = \rho^{1/2} X^* \rho^{-1/2}.$$

In fact, putting $X = r(A)\rho^{1/2} = \rho^{1/2}A$,$Y = r(A)^*\rho^{1/2} = r(A^*)\rho^{1/2} = \rho^{1/2}A^*$,

$$Y = \rho^{1/2}(\rho^{-1/2}X)^* = \rho^{1/2}X^*\rho^{-1/2}.$$

Thus

$$\Delta(X) = FS(X) = F(\rho^{-1/2}X^*\rho^{1/2}) = \rho^{1/2}(\rho^{-1/2}X^*\rho^{1/2})^*\rho^{-1/2} = \rho X \rho^{-1},$$

that is,

$$\Delta = v^*(\rho \otimes (\rho^{-1})^T)v. \tag{13}$$

It follows that $\Delta^* = v^*(\rho\otimes(\rho^{-1})^T)^*v = \Delta$. Since $\Delta^{-1/2} = v^*(\rho^{-1/2}\otimes(\rho^{1/2})^T)v$,

$$\Delta^{-1/2}(X) = \rho^{-1/2}X\rho^{1/2}$$

and hence

$$S(X) = \Delta^{-1/2}(X^*) = \Delta^{-1/2}J(X),$$

where J is an anti-linear operator defined by $J(X) = X^*$. In a similar way we have

$$F(X) = \Delta^{1/2}J(X).$$

Since

$$\begin{aligned}\Delta^{-it}\ell(A)\Delta^{it}(X) &= \Delta^{-it}(A\rho^{it}X\rho^{-it}) = \rho^{-it}A\rho^{it}X\rho^{-it}\rho^{it} \\ &= \rho^{-it}A\rho^{it}X = \ell(\rho^{-it}A\rho^{it})(X),\end{aligned} \tag{14}$$

we have

$$\Delta^{-it}\ell(A)\Delta^{it} = \ell(\rho^{-it}A\rho^{it}).$$

On the other hand, we have

$$J\ell(A)J(X) = J(AX^*) = (AX^*)^* = XA^* = r(A^*)(X)$$

Thus we obtain the main results of Tomita-Takesaki theory in our case:

$$\Delta^{-it}\mathfrak{M}\Delta^{it} = \mathfrak{M}, \tag{15}$$

$$J\mathfrak{M}J = \mathfrak{M}'. \tag{16}$$

Remark that the above discussion can be easily extend to the case where $\mathfrak{N} = M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_n}(\mathbb{C})$. The proof of Tomita-Takesaki theory for a finite dimensional von Neumann algebra is given in the Appendix.

4 Relation between Holevo's commutation operator and modular operator

Let us see how the commutation operator defined by (3) is described on $\mathcal{H}\otimes\mathcal{H}$. Since it holds for $A, X, Y = \mathfrak{D}X \in \mathfrak{H}(= \mathfrak{N} = \mathfrak{H}_2)$ that

$$\begin{aligned}
[A, X] &= i\omega(A^*X - XA^*) = i\mathrm{Tr}\rho(A^*X - XA^*) \\
&= \mathrm{Tr}A^* i(X\rho - \rho X) = \langle A, i(X\rho - \rho X)\rangle_2 \\
\langle A, Y\rangle &= \omega((YA^* + A^*Y)/2) = \mathrm{Tr}\rho((YA^* + A^*Y)/2) \\
&= \mathrm{Tr}A^*(\rho Y + Y\rho)/2 = \langle A, \rho Y + Y\rho\rangle_2,
\end{aligned} \tag{17}$$

we have

$$(\rho Y + Y\rho)/2 = i(X\rho - \rho X),$$

which can be represented on $\mathcal{H}\otimes\mathcal{H}$ as

$$(\rho\otimes I_n + I_n\otimes\rho^T)v(Y) = 2i(I_n\otimes\rho^T - \rho\otimes I_n)v(X).$$

Thus

$$v(\mathfrak{D}X) = v(Y) = 2i(\rho\otimes I_n + I_n\otimes\rho^T)^{-1}(I_n\otimes\rho^T - \rho\otimes I_n)v(X),$$

that is,

$$\mathfrak{D} = v^*[2i(\rho\otimes I_n + I_n\otimes\rho^T)^{-1}(I_n\otimes\rho^T - \rho\otimes I_n)]v.$$

Moreover

$$1 + \frac{i}{2}\mathfrak{D} = v^*[2(\rho\otimes I_n + I_n\otimes\rho)^{-1}\rho\otimes I_n]v, \tag{18}$$

$$1 - \frac{i}{2}\mathfrak{D} = v^*[2(\rho\otimes I_n + I_n\otimes\rho)^{-1}I_n\otimes\rho^T]v, \tag{19}$$

$$1 + \frac{1}{4}\mathfrak{D}^2 = v^*[4(\rho\otimes I_n + I_n\otimes\rho)^{-2}\rho\otimes\rho^T]v. \tag{20}$$

From our assumption stated in Sec. 1, $\rho^{1/2}$ is a non-degenerated operator and hence we can use it as a cyclic separating vector in Sec. 3. Thus, from Eqs. (18), (19) and (13) we conclude

$$\Delta = \left(1 + \frac{i}{2}\mathfrak{D}\right)\left(1 - \frac{i}{2}\mathfrak{D}\right)^{-1},$$

and

$$\frac{i}{2}\mathfrak{D} = (\Delta - 1)(\Delta + 1)^{-1}.$$

5 Appendix

In this section we prove the main results of Tomita-Takesaki theorem for a finite dimensional von Neumann algebras $\tilde{\mathfrak{N}}$ on a Hilbert space $\mathcal{K}$. We assume there exists a cyclic separating vector $\xi \in \mathcal{K}$; $\mathcal{K} = \tilde{\mathfrak{N}}\xi = \tilde{\mathfrak{N}}'\xi$. Then we define operators $\tilde{S}, \tilde{F}, \tilde{\Delta}$ and $\tilde{J}$ on $\mathcal{K}$ as

$$\begin{aligned} &\tilde{S} : X\xi \to X^*\xi, X \in \tilde{\mathfrak{N}} \\ &\tilde{F} : Y\xi \to Y^*\xi, Y \in \tilde{\mathfrak{N}}' \\ &\tilde{\Delta} = \tilde{F}\tilde{S}, \\ &\tilde{J} = \tilde{\Delta}^{1/2}\tilde{S}. \end{aligned} \tag{21}$$

The Wedderburn theorem states that a finite dimensional C^* algebra is $*$-isomorphic to a direct sum of simple matrix algebras. That is, there exists $*$-isomorphic function φ for von Neumann algebra $\tilde{\mathfrak{N}}$ such that

$$\varphi : \tilde{\mathfrak{N}} \simeq \mathfrak{N} := M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_n}(\mathbb{C}).$$

Let us consider the faithful state on $\tilde{\mathfrak{N}}$ as

$$\omega_\xi(\tilde{A}) = (\xi, \tilde{A}\xi)_{\mathcal{K}}, \quad \tilde{A} \in \tilde{\mathfrak{N}},$$

where $(\cdot,\cdot)_{\mathcal{K}}$ is an inner product of the Hilbert space $\mathcal{K}$. Using this state we can define the state on $\mathfrak{N}$

$$\omega(A) = \omega_\xi(\varphi^{-1}(A)), A \in \mathfrak{N},$$

which is normal by virtue of finite-dimensionality and is faithful because ξ is separating, i.e. there exists a non-degenerated density operator ρ such that $\omega(A) = \mathrm{Tr}\rho A$. Applying the discussion in Sec. 3 to the von Neumann algebra $\mathfrak{M} = \ell(\mathfrak{N})$ and the cyclic separating vector $\rho^{1/2}$, we get the operators, S, F, J and Δ. In particular the operators S and F are given by Eqs. (11) and (12). Here we have

$$\langle \ell(A)\rho^{1/2}, \ell(B)\rho^{1/2}\rangle_2 = (\varphi^{-1}(A)\xi, \varphi^{-1}(B)\xi)_{\mathcal{K}},$$

which means

$$U : \mathfrak{H}_2 \ni \ell(A)\rho^{1/2} \to \varphi^{-1}(A)\xi \in \mathcal{K}, \quad A \in \mathfrak{N}$$

gives a unitary operator from $\mathfrak{H}_2$ to $\mathcal{K}$. Using this unitary operator we obtain the following relations

$$\begin{aligned} \tilde{S} &= USU^*, \\ \tilde{F} &= UFU^*, \end{aligned} \tag{22}$$

and hence

$$\tilde{\Delta} = U\Delta U^*, \tag{23}$$

$$\tilde{J} = UJU^*. \tag{24}$$

Moreover

$$\tilde{\mathfrak{N}} = U\mathfrak{M}U^*, \tag{25}$$

since $\mathfrak{M} = \ell(\mathfrak{N})$ and $\varphi^{-1}(A) = U\ell(A)U^*$ for $A \in \mathfrak{N}$. From Eqs. (15),(16),(23),(24) and (25), we conclude the main result of Tomita-Takesaki theory,

$$\begin{aligned} \tilde{\Delta}^{-it}\tilde{\mathfrak{N}}\tilde{\Delta}^{it} &= \tilde{\mathfrak{N}} \\ \tilde{J}\tilde{\mathfrak{N}}\tilde{J} &= \tilde{\mathfrak{N}}'. \end{aligned} \tag{26}$$

References

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[2] Roberto Longo. A simple proof of the existence of modular automorphisms in approximately finite-dimensional von Neumann algebras. *Pacific Journal of Mathematics*, 75(1):199–205, mar 1978.