```
Algorithm GreedyJob($d, J, n$)
// $J$ is a set of jobs that can be completed by their deadlines.
{
    $J := \{1\}$;
    for $i := 2$ to $n$ do
    {
        if (all jobs in $J \cup \{i\}$ can be completed
            by their deadlines) then $J := J \cup \{i\}$;
    }
}
```

Algorithm 4.5 High-level description of job sequencing algorithm

... , $J[k]$ are changed after the insertion. Hence, it is necessary to verify only that these jobs (and also job i) do not violate their deadlines following the insertion. The algorithm that results from this discussion is function JS (Algorithm 4.6). The algorithm assumes that the jobs are already sorted such that $p_1 \geq p_2 \geq \cdots \geq p_n$. Further it assumes that $n \geq 1$ and the deadline $d[i]$ of job i is at least 1. Note that no job with $d[i] < 1$ can ever be finished by its deadline. Theorem 4.5 proves that JS is a correct implementation of the greedy strategy.

Theorem 4.5 Function JS is a correct implementation of the greedy-based method described above.

Proof: Since $d[i] \geq 1$, the job with the largest p_i will always be in the greedy solution. As the jobs are in nonincreasing order of the p_i's, line 8 in Algorithm 4.6 includes the job with largest p_i. The **for** loop of line 10 considers the remaining jobs in the order required by the greedy method described earlier. At all times, the set of jobs already included in the solution is maintained in J. If $J[i]$, $1 \leq i \leq k$, is the set already included, then J is such that $d[J[i]] \leq d[J[i+1]]$, $1 \leq i < k$. This allows for easy application of the feasibility test of Theorem 4.3. When job i is being considered, the **while** loop of line 15 determines where in J this job has to be inserted. The use of a fictitious job 0 (line 7) allows easy insertion into position 1. Let w be such that $d[J[w]] \leq d[i]$ and $d[J[q]] > d[i]$, $w < q \leq k$. If job i is included into J, then jobs $J[q]$, $w < q \leq k$, have to be moved one position up in J (line 19). From Theorem 4.3, it follows that such a move retains feasibility of J iff $d[J[q]] \neq q$, $w < q \leq k$. This condition is verified in line 15. In addition, i can be inserted at position $w+1$ iff $d[i] > w$. This is verified in line 16 (note $r = w$ on exit from the **while** loop if $d[J[q]] \neq q$, $w < q \leq k$). The correctness of JS follows from these observations. □

```
1   **Algorithm** JS($d, j, n$)
2   // $d[i] \geq 1$, $1 \leq i \leq n$ are the deadlines, $n \geq 1$. The jobs
3   // are ordered such that $p[1] \geq p[2] \geq \cdots \geq p[n]$. $J[i]$
4   // is the $i$th job in the optimal solution, $1 \leq i \leq k$.
5   // Also, at termination $d[J[i]] \leq d[J[i+1]]$, $1 \leq i < k$.
6   {
7       $d[0] := J[0] := 0$; // Initialize.
8       $J[1] := 1$; // Include job 1.
9       $k := 1$;
10      **for** $i := 2$ **to** $n$ **do**
11      {
12          // Consider jobs in nonincreasing order of $p[i]$. Find
13          // position for $i$ and check feasibility of insertion.
14          $r := k$;
15          **while** $((d[J[r]] > d[i])$ **and** $(d[J[r]] \neq r))$ do $r := r - 1$;
16          **if** $((d[J[r]] \leq d[i])$ **and** $(d[i] > r))$ **then**
17          {
18              // Insert $i$ into $J[\,]$.
19              **for** $q := k$ **to** $(r + 1)$ **step** $-1$ **do** $J[q + 1] := J[q]$;
20              $J[r + 1] := i$; $k := k + 1$;
21          }
22      }
23      **return** $k$;
24  }
```

Algorithm 4.6 Greedy algorithm for sequencing unit time jobs with deadlines and profits

For JS there are two possible parameters in terms of which its complexity can be measured. We can use n, the number of jobs, and s, the number of jobs included in the solution J. The **while** loop of line 15 in Algorithm 4.6 is iterated at most k times. Each iteration takes $\Theta(1)$ time. If the conditional of line 16 is true, then lines 19 and 20 are executed. These lines require $\Theta(k - r)$ time to insert job i. Hence, the total time for each iteration of the **for** loop of line 10 is $\Theta(k)$. This loop is iterated $n - 1$ times. If s is the final value of k, that is, s is the number of jobs in the final solution, then the total time needed by algorithm JS is $\Theta(sn)$. Since $s \leq n$, the worst-case time, as a function of n alone is $\Theta(n^2)$. If we consider the job set $p_i = d_i = n - i + 1$, $1 \leq i \leq n$, then algorithm JS takes $\Theta(n^2)$ time to determine J. Hence, the worst-case computing time for JS is $\Theta(n^2)$. In addition to the space needed for d, JS needs $\Theta(s)$ amount of space for J.

Note that the profit values are not needed by JS. It is sufficient to know that $p_i \geq p_{i+1}$, $1 \leq i < n$.

The computing time of JS can be reduced from $O(n^2)$ to nearly $O(n)$ by using the disjoint set union and find algorithms (see Section 2.5) and a different method to determine the feasibility of a partial solution. If J is a feasible subset of jobs, then we can determine the processing times for each of the jobs using the rule: if job i hasn't been assigned a processing time, then assign it to the slot $[\alpha - 1, \alpha]$, where α is the largest integer r such that $1 \leq r \leq d_i$ and the slot $[\alpha - 1, \alpha]$ is free. This rule simply delays the processing of job i as much as possible. Consequently, when J is being built up job by job, jobs already in J do not have to be moved from their assigned slots to accommodate the new job. If for the new job being considered there is no α as defined above, then it cannot be included in J. The proof of the validity of this statement is left as an exercise.

Example 4.3 Let $n = 5, (p_1, \ldots, p_5) = (20, 15, 10, 5, 1)$ and $(d_1, \ldots, d_5) = (2, 2, 1, 3, 3)$. Using the above feasibility rule, we have

J	assigned slots	job considered	action	profit
$\emptyset$	none	1	assign to [1, 2]	0
{1}	[1, 2]	2	assign to [0, 1]	20
{1, 2}	[0, 1], [1, 2]	3	cannot fit; reject	35
{1, 2}	[0, 1], [1, 2]	4	assign to [2, 3]	35
{1, 2, 4}	[0, 1], [1, 2], [2, 3]	5	reject	40

The optimal solution is $J = \{1, 2, 4\}$ with a profit of 40. □

Since there are only n jobs and each job takes one unit of time, it is necessary only to consider the time slots $[i-1, i]$, $1 \leq i \leq b$, such that $b = \min\,\{n, \max\,\{d_i\}\}$. One way to implement the above scheduling rule is to partition the time slots $[i-1, i]$, $1 \leq i \leq b$, into sets. We use i to represent the time slots $[i-1, i]$. For any slot i, let n_i be the largest integer such that $n_i \leq i$ and slot n_i is free. To avoid end conditions, we introduce a fictitious slot $[-1, 0]$ which is always free. Two slots i and j are in the same set iff $n_i = n_j$. Clearly, if i and j, $i < j$, are in the same set, then $i, i+1, i+2, \ldots, j$ are in the same set. Associated with each set k of slots is a value $f(k)$. Then $f(k) = n_i$ for all slots i in set k. Using the set representation of Section 2.5, each set is represented as a tree. The root node identifies the set. The function f is defined only for root nodes. Initially, all slots are free and we have $b+1$ sets corresponding to the $b+1$ slots $[i-1, i]$, $0 \leq i \leq b$. At this time $f(i) = i$, $0 \leq i \leq b$. We use $p(i)$ to link slot i into its set tree. With the conventions for the union and find algorithms of Section 2.5, $p(i) = -1$, $0 \leq i \leq b$, initially. If a job with deadline d is to be scheduled, then we need to find the root of the tree containing the slot $\min\{n, d\}$. If this root is j,