Matrix Chain Order Problem

- Multiplying non-square matrices:
  - A is n x m, B is m x p — must be equal
  - AB is n x p whose (i,j) entry is $\sum a_{ik}\, b_{kj}$
- Computing AB takes nmp scalar multiplications and $n(m-1)p$ scalar additions (using basic algorithm).
- Suppose we have a sequence of matrices to multiply.
- What is the best order?

Why Order Matters

- Suppose we have 4 matrices:
  - A, 30 x 1
  - B, 1 x 40
  - C, 40 x 10
  - D, 10 x 25
- $((AB)(CD))$: requires 41,200 mults.
- $(A((BC)D))$: requires 1400 mults.

Matrix Chain Order Problem

- Given matrices $A_1, A_2, \ldots, A_n$, where A_i is d_{i-1} x d_i:

  [1] What is minimum number of scalar mults required to compute $A_1 \cdot A_2 \cdot \ldots \cdot A_n$?

  [2] What order of matrix multiplications achieves this minimum?

A Possible Solution

- Try all possibilities and choose the best one.
- Let P(n)= # of alternative parenthesizations of a seq of n matrices.
- Then $P(n) = \sum_{k=1}^{n-1} P(k)P(n-k)$ if $n \geq 2$; $P(n)=1$ if $n=1$;
- Sol. $\Omega(4^n/n^{3/2})$ (*Hint:* **Catalan number**)
- Drawback is there are too many of them (exponential in the number of matrices to be multiplied)
- Need to be more clever - try dynamic programming!

Step 1: Develop a Recursive Solution

- Define $M(i,j)$ to be the minimum number of mults. needed to compute $A_i \cdot A_{i+1} \cdot \ldots \cdot A_j$
- Goal: Find $M(1,n)$.
- Basis: $M(i,i) = 0$.
- Recursion: How to define $M(i,j)$ recursively?

Defining $M(i,j)$ Recursively

- Consider all possible ways to split A_i through A_j into two pieces.
- Compare the costs of all these splits:
  - best case cost for computing the product of the two pieces
  - plus the cost of multiplying the two products
- Take the best one
- $M(i,j) = \min_k(M(i,k) + M(k+1,j) + d_{i-1}d_k d_j)$

Defining $M(i,j)$ Recursively

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{P_1} \cdot \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{P_2}$$

- minimum cost to compute P_1 is $M(i,k)$
- minimum cost to compute P_2 is $M(k+1,j)$
- cost to compute $P_1 \cdot P_2$ is $d_{i-1}d_k d_j$
- $M(i,j) = M(i,k) + M(k+1,j) + d_{i-1}d_k d_j$ for a k
- k run from i to $j-1$
- $M(i,j) = \min_k(M(i,k) + M(k+1,j) + d_{i-1}d_k d_j)$

Step 2: Find Dependencies Among Subproblems

M:

	1	2	3	4	5
1	0				
2	n/a	0			
3	n/a	n/a	0		
4	n/a	n/a	n/a	0	
5	n/a	n/a	n/a	n/a	0

GOAL!

computing the pink square requires the purple ones: to the left and below.

Defining the Dependencies

- Computing $M(i,j)$ uses
  - everything in same row to the left:
    $M(i,i)$, $M(i,i+1)$, …, $M(i,j-1)$
  - and everything in same column below:
    $M(i,j)$, $M(i+1,j)$,…,$M(j,j)$

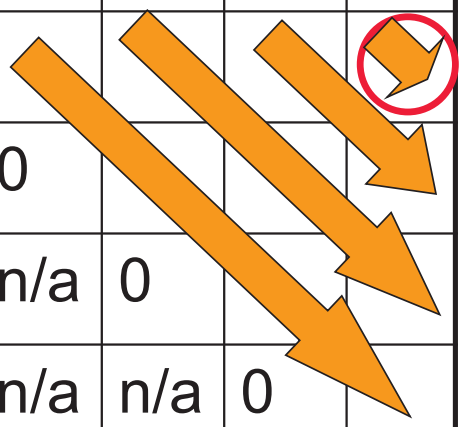
Step 3: Identify Order for Solving Subproblems

- Recall the dependencies between subproblems just found
- Solve the subproblems (i.e., fill in the table entries) this way:
  - go along the diagonal
  - start just above the main diagonal
  - end in the upper right corner (goal)

Order for Solving Subproblems

M:

	1	2	3	4	5
1	0				
2	n/a	0			
3	n/a	n/a	0		
4	n/a	n/a	n/a	0	
5	n/a	n/a	n/a	n/a	0



Pseudocode

```
for i := 1 to n do M[i,i] := 0
for d := 1 to n-1 do  // diagonals
   for i := 1 to n-d to // rows w/ an entry on d-th diagonal
      j := i + d          // column corresponding to row i on d-th diagonal
      M[i,j] := infinity
      for k := i to j-1 to
        M[i,j] := min(M[i,j], M[i,k]+M[k+1,j]+d_{i-1}d_k d_j)
      endfor
   endfor
endfor
```

pay attention here to remember actual sequence of mults.

running time $O(n^3)$

Example

M:

	1	2	3	4
1	0	1200	700	1400
2	n/a	0	400	650
3	n/a	n/a	0	10,000
4	n/a	n/a	n/a	0

1: A is 30x1
2: B is 1x40
3: C is 40x10
4: D is 10x25

Example

```
for i := 1 to n do M[i,i] := 0
for d := 1 to n-1 do  // diagonals
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      endfor
   endfor
endfor
```

M:

	1	2	3	4
1				
2	n/a			
3	n/a	n/a		
4	n/a	n/a	n/a	

1: A is 30x1
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Example

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```

M:

	1	2	3	4
1	0			
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      endfor
   endfor
endfor
```

d=1 i= 1, 2, 3
j= 2, 3, 4
k=1, 2, 3
M[1,2]
M[2,3]
M[3,4]

M:

	1	2	3	4
1	0	1200		
2	n/a	0		
3	n/a	n/a	0	
4	n/a	n/a	n/a	0

1: A is 30x1
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Example

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```

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k=1, 2, 3
M[1,2]
M[2,3]
M[3,4]

M:

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k=1, 2, 3
M[1,2]
M[2,3]
M[3,4]

M:

	1	2	3	4
1	0	1200		
2	n/a	0	400	
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4	n/a	n/a	n/a	0

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k=1, 2, 3
M[1,2]
M[2,3]
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M:

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      endfor
   endfor
endfor
```

d=2 i= 1, 2
j= 3, 4
k=1,2, 2,3
M[1,3]
M[2,4]

A(BC)

M:

	1	2	3	4
1	0	1200	700	
2	n/a	0	400	
3	n/a	n/a	0	10,000
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      endfor
   endfor
endfor
```

d=2 i= 1, 2
j= 3, 4
k=1,2, 2,3
M[1,3]
M[2,4]

(AB)C

M:

	1	2	3	4
1	0	1200	700	
2	n/a	0	400	
3	n/a	n/a	0	10,000
4	n/a	n/a	n/a	0

13200

1: A is 30x1
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Example

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k=1,2, 2,3
M[1,3]
M[2,4]

A(BC)

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      endfor
   endfor
endfor
```

d=2 i= 1, 2
j= 3, 4
k=1,2, 2,3
M[1,3]
M[2,4]

(BC)D

M:

	1	2	3	4
1	0	1200	700	
2	n/a	0	400	650
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Example

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      endfor
   endfor
endfor
```

d=3 i= 1
j= 4
k=1, 2, 3
M[1,4]

A((BC)D)

M:

	1	2	3	4
1	0	1200	700	1400
2	n/a	0	400	650
3	n/a	n/a	0	10,000
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1: A is 30x1
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      endfor
   endfor
endfor
```

d=3 i= 1
j= 4
k=1, 2, 3
M[1,4]

A((BC)D)

M:

	1	2	3	4
1	0	1200	700	∞
2	n/a	0	400	650
3	n/a	n/a	0	10,000
4	n/a	n/a	n/a	0

1400

1: A is 30x1
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Example

```
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for d := 1 to n-1 do  // diagonals
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      endfor
   endfor
endfor
```

d=3 i= 1
j= 4
k=1, 2, 3
M[1,4]

(AB)(CD)

M:

	1	2	3	4
1	0	1200	700	1400
2	n/a	0	400	650
3	n/a	n/a	0	10,000
4	n/a	n/a	n/a	0

41200

1: A is 30x1
2: B is 1x40
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Example

```
for i := 1 to n do M[i,i] := 0
for d := 1 to n-1 do  // diagonals
   for i := 1 to n-d to // rows w/ an entry on d-th diagonal
      j := i + d          // column corresponding to row i on d-th diagonal
      M[i,j] := infinity
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         M[i,j] := min(M[i,j], M[i,k]+M[k+1,j]+d_{i-1}d_k d_j)
      endfor
   endfor
endfor
```

d=3 i= 1
j= 4
k=1, 2, 3
M[1,4]

(A(BC))D

M:

	1	2	3	4
1	0	1200	700	1400
2	n/a	0	400	650
3	n/a	n/a	0	10,000
4	n/a	n/a	n/a	0

8200

1: A is 30x1
2: B is 1x40
3: C is 40x10
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Example

```
for i := 1 to n do M[i,i] := 0
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   for i := 1 to n-d to // rows w/ an entry on d-th diagonal
      j := i + d          // column corresponding to row i on d-th diagonal
      M[i,j] := infinity
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      endfor
   endfor
endfor
```

M:

	1	2	3	4
1	0	1200	700	1400
2	n/a	0	400	650
3	n/a	n/a	0	10,000
4	n/a	n/a	n/a	0

1: A is 30x1
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A((BC)D)

Keeping Track of the Order

- It's fine to know the cost of the cheapest order, but what *is* that cheapest order?
- Keep another array S and update it when computing the minimum cost in the inner loop
- After M and S have been filled in, then call a recursive algorithm on S to print out the actual order

Modified Pseudocode

```
for i := 1 to n do M[i,i] := 0
for d := 1 to n-1 do  // diagonals
   for i := 1 to n-d to // rows w/ an entry on d-th diagonal
      j := i + d          // column corresponding to row i on d-th diagonal
      M[i,j] := infinity
      for k := 1 to j -1 to
         M[i,j] := min(M[i,j], M[i,k]+M[k+1,j]+d_{i-1}d_k d_j)
      if previous line changed value of M[i,j] then S[i,j] := k
      endfor
   endfor
endfor
```

keep track of cheapest split point found so far: between A_k and A_{k+1})

Example

M:

	1	2	3	4
1	0	1200	700	1400
2	n/a	0	400	650
3	n/a	n/a	0	10,000
4	n/a	n/a	n/a	0

1: A is 30x1
2: B is 1x40
3: C is 40x10
4: D is 10x25

S:

	1	2	3	4
1	n/a	1	1	1
2	n/a	n/a	2	3
3	n/a	n/a	n/a	3
4	n/a	n/a	n/a	n/a

Using S to Print Best Ordering

Call Print(S,1,n) to get the entire ordering.

Print(S,i,j):

```
if i = j then output "A" i
else k := S[i,j]
   output "("
            Print(S,i,k)
            Print(S,k+1,j)
          ")"
```

M:

S:

	1	2	3	4
1	0	1200 1	700 1	1400 1
2	n/a	0	400 2	650 3
3	n/a	n/a	0	10,000 3
4	n/a	n/a	n/a	0

A((BC)D)

$(A_1((A_2A_3)A_4))$