Active Learning and Covering Problems with Precedence

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Abstract

In the Bayesian Active Learning a hidden hypothesis is required to be uncovered. To do so, the learner is allowed to perform tests, each of which reveals partial information about the hidden hypothesis. Upon receiving this information, the learner adaptively selects the next test to be performed. The goal is to uncover the hidden hypothesis while performing as few tests as possible in the worst or average case.

In the covering problems, we are given a set of items and a collection of subsets that cover these items. The objective is to select a sequence of subsets that covers all items, which minimizing the worst or average covering cost.

For both types of problems, a natural constraint may arise that some tests can only be performed only after certain other tests (or some subsets can only be selected after selecting certain other subsets). We model such constraints using directed acyclic graphs (DAGs) that impose precedence on the tests or subsets. This paper explores the connection of active learning and covering problems under such constraints.

We show that given any bicriteria $(O(1)\,, \alpha)$-approximation ratio for the Precedence Constrained Set Cover, we can obtain an $O(\alpha \cdot \log n)$-approximation ratio for the Worst Case Active Learning with precedence constraints, where n is the number of hypothesis. Similarly, we prove that given any $O(\beta)$-approximation ratio for the Precedence Constrained Min-Sum Set Cover, we can obtain an $O(\beta \cdot \log n)$-approximation ratio for the Average Case Active Learning with Precedence Constraints. Finally, we provide several approximation algorithms for the Set Cover and Min-Sum Set Cover problems with various types of precedence constraints.

Keywords: Bayesian active learning, Set cover, Precedence constraints, Approximation Algorithms, Decision Trees

Ogolne uwagi:

- **wszelkie uwagi pisze jako komenda ... dzieki czemu na koniec latwo sie pozbyc; nie bede raczej uwag dorzucal w mailach, aby nie zniknelo; wszystko co ponizej oczywiscie do dyskusji, a gdy bedzie zgoda i bedzie zaimplementowane, to bede usuwal artefakty.**
- **przejrzawszy sporo papierow z poprzedniego COLT, mam obserwacje, ze dobrym/typowym ukladem papieru jest: intro; our contribution; related work; outline(opcjonalnie); preliminaries; wyniki; appendix.**
- **intro zwykle nie jest zbyt rozlekle oraz prawie zawsze pozbawione lania wody. Czesto od razu definicja problemu, aby wprowadzic pojecia, aby moc szybko formalnie podac wyniki (our contribution)**
- **“front” artykulu, czyli wszystko do preliminaries to typowo 4-5 stron,**
- **front we wszystkich miejcach zawiera zwykle odnosniki do literatury a sekcja “related work” jest czesto tytulowana “other related work” lub cos w tym rodzaju**

- **czytelnik powiniem poza dowodami rozumiec baze przeczytawszy front (czyli rozumiec wyniki, widziec co papier robi) a jesli chce sie dowiedziec jak/dlaczego (dowody) to idzie dalej. Czesto recenzent jest leniwy i nie zajrzy dalej niz front... niestety.**
- **W zwiazku z powyzszym sekcja “our contribution” (potencjalnie w tytule dodamy “and techniques” jak Michal sugeruje) powinna sie pochwalic takze jakimis ciekawszymi trickami lub technikami uzytymi pozniej w dowodach.**

1. Introduction

pomysl na intro: zdefiniowac dwa glowne problemy: learning + set cover; zapowiedziec, ze sa powiazane ze soba i celem papieru jest przestudiowanie tych zaleznosci plus usyskanie konkretnych wynikow; pytanie/do sprawdzenia: czy ktores wyniki przypadkiem poprawiaja lub sa tozsame z najlepszymi znanymi bez precedensow; ewentualne inna "marketingowe" uwagi.

Consider a set $\mathcal{H}$ of n *hypotheses*, a set $\mathcal{T}$ of m *tests* and an unknown *target hypothesis* $h^* \in \mathcal{H}$ that needs to be discovered through an adaptive learning process. Each test $t \in \mathcal{T}$ is a partition of $\mathcal{H}$, that is, t consists of subsets of $\mathcal{H}$ such that $x \cap y = \emptyset$ for any $x, y \in t$ and $\bigcup t = \mathcal{H}$. As a result of executing a test $t \in \mathcal{T}$, questioner recives a *reply* that reveals $x \in t$ such that $h^* \in x$. That is, the questioner learns which subset of $\mathcal{H}$ that belongs to t contains the target. Each subsequent test is selected by questioner by taking into acocunt replies from all test to date. Without formally stating an optimization criterion we refer to the above as the *Adaptive Learing Process* (AL). (Another widely used name in the literature is the decision tree construction). The goal for the questioner is to output h^*.

Consider an arbitrary partial order $(\mathcal{T}, \preceq)$ that introduces a precedence relation between tests. This leads us to the two adaptive learning problems in which order to perform a test t, all its predecesors had to be performed previously. Hence we have the *Precedence Constrained Worst Case Adaptive Learing* (PCWCAL) in which the goal is to compute the AL that respects the precedence constraints and outputs the target h^* by performing the minimum number of tests in the worst case. Similarly, in the *Precedence Constrained Average Case Adaptive Learing* (PCACAL) the optimization criterion changes to minimizing the number of queries done on average. Whenever the criterion is not important or we want to make a claim that applies to both we use the symbol PCAL to refer to a *Precedence Constrained Adaptive Learning* instance.

In this work we study connections between adaptive lerning with precedences and the covering problems defined as follows. We are given a set $\mathcal{U}$ of n items, a collection $\mathcal{S}$ of m subsets of $\mathcal{U}$, such that $\bigcup \mathcal{S} = \mathcal{U}$, an arbitrary partial order $(\mathcal{S}, \preceq)$ on these subsets and an integer k. We say that a subfamily $\mathcal{C} \subseteq \mathcal{S}$ *covers* at least k items from $\mathcal{U}$ if $|\bigcup \mathcal{C}| \geq k$. We ask for a $\mathcal{C} \subseteq \mathcal{S}$ that covers at least k items from $\mathcal{U}$ and for each $x \in \mathcal{C}$ and each $y \in \mathcal{S}$ such that $y \preceq x$ it holds $y \in \mathcal{C}$. In the *Precedence Constrained Set Cover* (PCSC) the goal is to minimize $|\mathcal{C}|$. A permutation $C_1, \ldots, C_l)$ of the elements in $\mathcal{C}$ is *consistent* with the partial order $(\mathcal{S}, \preceq)$ if for any C_i and C_j such that $C_i \preceq C_j$ it holds $i < j$. The *coverage time* of a $x \in \bigcup \mathcal{C}$ is the minimum index i such that $x \in C_i$. In the *Precedence Constrained Min-Sum Set Cover* (PCMSSC) the goal is to find a sequence $(C_1, \ldots, C_l)$ that covers at least k items in $\mathcal{U}$ and minimizes the total coverage time of all items in $C_1 \cup \cdots \cup C_k$.

czy powyzsze jest dobrze zdefiniowane? tzn. jest pewna subtelnosc w tym, ze do coverage time wliczone sa wszystkie elementy, ktore pokrywa sekwencja. Chodzi o to, ze moze ich byc duzo wiecej niz k, wiec niejako placimy kare za pokrywanie za duzo. Nie do konca to intuicyjne, wiec chcialem przedyskutowac.

1.1. Our results and techniques

Pomysl na rozdzial:

- **zajawka, że wprpwadzimy nowe inne problemy (raz - jak pomocnicze; dwa - jako dopelnienie obrazu roznych rzeczy z litearatury)**

- **zdefiniowac pozostałe problemy z "diagramu" zaleznosci miedzy nimi**
- **diagram**
- **najwazniejsze twierdzenia**
- **tabelka na podsumowanie**

Our main goal is to derive several complexity results regarding the problems introduced above. Hence we start with stating the main results of this work.

Tutaj poszłyby główne twierdzenia dot. PCAL, PCMSSC oraz PCSC, czyli tych głównych z intro.

Ten przykład jest super, ale chyba musimy znaleźć sposób na jego kompresję, tzn. nie stać nas na zapłacenie całej strony; około 1/3 strony byłaby ok (może tabelka obok drzewa decyzyjnego (moze zamiana kolumn vs wierszy aby ją powęzić, a może mocno zminimalizować przestrzeń między kolumnami, a partial order jako łuki pomiędzy etykietami wierszy tabelki?); też do przemyślenia gdzie ten przykład umieścić (raczej nie w "contribution", ale albo w intro, albo w preliminaries chyba

na razie zakomentowałem rysunek dot zbiorów

Our main results are obtained through several algorithmic "reductions" by which we mean that we use approximation algorithms for selected problems to obtain approximations for others. In order to show a full picture of our method we introduce three remaining problems that paly an important role in our approach.

Definition 1 (Precedence constrained test cover (PCTC)) *Given a set $\mathcal{H}$ of n hypotheses, a set $\mathcal{T}$ of m tests, and a partial order $\{\mathcal{T}, \preceq\}$ encoding precedence constraints for $\mathcal{T}$, find a precedence-closed subfamily of tests that distinguishes all pairs of hypotheses.*

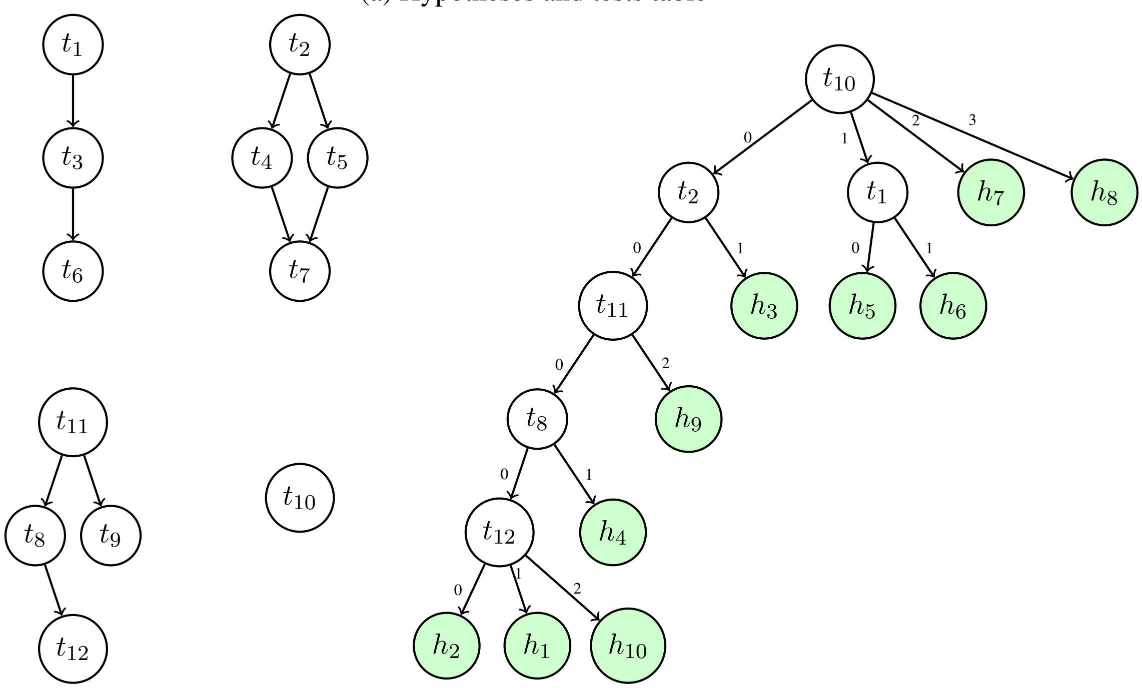
czyli taka inna nazwa na test cover zaaplikowany bezposrednio do hipotez? co dokładnie znaczy "distinguishing a pair"?

precedence/problem	**PCSC**	**PCMSSC**	**PCWCAL**	**PCACAL**
none	$O(\log n)$	4	$O(\log n)$	$O(\log n)$
inforest	$O(\log n)$*	4	$O(\log n)$*	$O(\log n)$*
outforest	$O(\log^2 n)$**	$O(\log n)$**	$O(\log^2 n)$*	$O(\log^2 n)$*
general	$O(\sqrt{n}\log n)$*	$O(\sqrt{n})$	$O(\sqrt{n}\log n)$*	$O(\sqrt{n}\log n)$*

Table 1: Approximation algorithms for various covering and active learning problems under different precedence constraints. (* denotes new results, ** denotes previously unmentioned corollaries of known results)

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}
h_1	0	0	0	0	0	0	2	0	0	0	0	1
h_2	0	0	0	0	0	1	0	0	0	0	0	0
h_3	0	1	0	1	0	0	0	0	0	0	0	0
h_4	0	0	0	0	1	0	0	1	0	0	0	0
h_5	0	0	0	0	0	0	1	0	0	1	1	0
h_6	1	0	0	2	0	0	0	0	0	1	0	0
h_7	0	0	0	0	0	2	0	0	1	2	0	0
h_8	0	0	1	0	0	0	0	0	0	3	0	0
h_9	0	0	0	0	0	0	0	1	1	0	2	0
h_{10}	0	0	0	1	0	0	0	0	0	0	0	2

(a) Hypotheses and tests table



(b) Precedence DAG

(c) Valid decision tree

Figure 1: Example of a PCAL instance with 10 hypotheses and 12 tests. (a) Hypotheses-tests table. (b) Precedence DAG with four components. (c) A valid decision tree solution respecting precedence constraints.

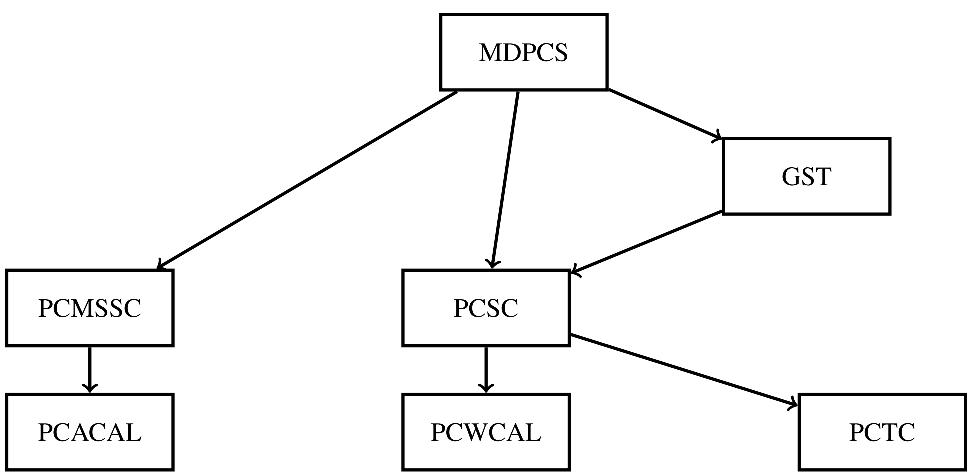


Figure 2: Relationships between covering and active learning problems, $\Pi_1 \to \Pi_2$ denotes that an approximation algorithm for problem Π_1 implies an approximation algorithm for problem Π_2.

2. Preliminaries

Propozycje do notacji:

- **odnosnie czesciowych porzadkow, to moze pozostac przy akademickim** $(\mathcal{T}, \preceq)$
- **drzewo decyzyjne to** D**, to niech** D_v **oznacza poddrzewo ukorzenione w tescie** v**; wowczas** $V(D_v)$ **donosiloby sie do wierzcholkow/testow ponizej** v**, wlacznie z** v
- **moze cos w rodzaju leaves(**D'**) oraz internal(**D'**) do oznaczanie lisci i wierzch. wewnetrznych drzewa**
- **do oznaczania sekwencji zapytan do wierzcholka** v **w** D **moznaby uzywac standardowego** $D_{r\to v}$
- **proponuje, aby wszelkie parametry byly komendami latex (**$cov()$**, nazwy problemow (to juz wprowadzilem).**

In all adaptive learning problems, the goal is to design a *strategy* $\mathcal{S}$ understood as an adaptive algorithm, that based on the previous responses provides the questioner with the next test to be performed. We represent this strategy as a rooted tree D called a *decision tree*, where each internal node represents a test to be performed and each leaf represents a hypothesis. Formally, a decision tree is defined recursively as follows. If $r = (R_1, \ldots, R_l) \in \mathcal{T}$ is the first test performed by $\mathcal{S}$, then r is the root of D. For each reply R_i, $i \in \{1, \ldots, l\}$, take inductively defined decision tree D_i derived from the queries done by $\mathcal{S}$ when the reply to r is R_i. Then, D is obtained by making the roots of $D_1, \ldots, D_l$ the children of r. To complete the inductive definition, whenever $\mathcal{S}$ finalizes the search by declaring that $h \in \mathcal{H}$ is the target, the corresponding decistion subtree is the vertex h. Hence, each edge outgoing from the root r of D is associated with a unique response to a test r, and the same holds for all internal It should be remarked that it is possible for a test to appear multiple times in the decision tree. Hence, with a slight abuse of notation we use tests as vertices of D but whenever a given test provides many vertices in D we make clear which one we refer to.

Let $\mathrm{tests}(D, h)$**, denote the sequence of tests performed when the hidden hypothesis is** h **and the learner follows the strategy represented by decision tree** D**.** $\leftarrow$ **to chyba nie jest uzywane (check todo).** The cost of identifying hypothesis h using a decision tree D is denoted by $\texttt{COST}(D, h)$ and defined as the number of tests performed by D when the target is h. We consider two cost measures for decision trees: the worst-case cost $\texttt{COST}_W(D) = \max_{h\in\mathcal{H}} \texttt{COST}(D, h)$ and the average-case cost $\texttt{COST}_A(D) = \sum_{h\in\mathcal{H}} \texttt{COST}(D, h)$ (up to a multiplicative factor of $1/|\mathcal{H}|$). The criteria we consider for any PCAL instance I are denoted by

$$\texttt{OPT}_A(I) = \min\{\texttt{COST}_A(D) \mid D \text{ is a decision tree}\},$$

$$\texttt{OPT}_W(I) = \min\{\texttt{COST}_W(D) \mid D \text{ is a decision tree}\}.$$

Skoro mamy już te problemy PCWCAL oraz PCACAL zdefiniowane, to tutaj aby nie powtarzac w nowej notacji drze decyzyjnych zamienie to na uwage/obserwacje w ramach jednego paragrafu do optymalizacji czego to sie sprowadza odnosnie drzew dec. Hence the PCACAL aims at finding a decision tree D such that $\texttt{COST}_A(D) = \texttt{OPT}_A(I)$ and in case of PCWCAL the goal is to find D such that $\texttt{COST}_W(D) = \texttt{OPT}_W(I)$.

For any subtree D' of D, $\text{tests}(D',)$ denotes the subset of $\mathcal{T}$ that appear in D'. $\leftarrow$ to chyba nie jest uzywane (check todo) We write $\text{inner}(D)$ we denote the set of nodes in D that are not leaves. Hence, each usage of a test in D corresponds to a unique element of $\text{inner}(D)$. **W sumie sie wole upewnic co mamy na mysli piszac "not distinguished"? Ja to czytam jako zbior hipotez bedacych potencjalnymi targetami?** Let $\mathcal{H}_t$ denote the set of hypotheses that are not yet distinguished by the tests selected before test $t \in \text{inner}(D)$ in the decision tree. **Trzebaby zrobic jaki fix na tą notację, może $\mathcal{H}(D_v)$ jako zbiór hipotez gdy dochodzimy do podrzewa D_v. Odnoszenie się do wierzchołka drzewa zamiast hipotezy ma tą zaletę, że czasem mamy warianty gdzie jedna hipoteza jest w wielu miejscach w drzewie, a wówczas ta notacja się psuje.** Then we immediately obtain the following simple observation:

Observation 1 *Let D be any decision tree for $\mathcal{I} = (\mathcal{H}, \mathcal{T}, \mathcal{F})$. Then we have that:*

$$\mathtt{COST}_A(D) = \sum_{t \in \text{inner}(D)} |\mathcal{H}_t|$$

unifikacja I vs $\mathcal{I}$

We will also make use of the following folklore lemmas which are due to the fact that any decision tree for I can be restricted to a decision tree for each subproblem $(\mathcal{H}_i, \mathcal{T}, \mathcal{F})$ without increasing the cost:

Lemma 2 *Let $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$ be any* PCAL *instance. Let $\mathcal{H}' \subseteq \mathcal{H}$. Then $\mathtt{OPT}((\mathcal{H}', \mathcal{T}, \mathcal{F})) \leq \mathtt{OPT}(I)$.*

Lemma 3 *Let $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$ be an instance of* PCACAL. *Let $\mathcal{H}_1, \ldots, \mathcal{H}_t \subseteq \mathcal{H}$ such that $\bigcup_{i=1}^{t} \mathcal{H}_i \subseteq \mathcal{H}$ and for any $i \neq j$, $\mathcal{H}_i \cap \mathcal{H}_j = \emptyset$. Then we have that:*

$$\mathtt{OPT}_A(I) \geq \sum_{i=1}^{t} \mathtt{OPT}_A((\mathcal{H}_i, \mathcal{T}, \mathcal{F}))$$

Definition 4 (Group Steiner Tree (GST)) *Given an undirected graph $G = (V, E)$ with edge costs, a root vertex $r \in V$, and groups $g_1, \ldots, g_k \subseteq V$, find a minimum-cost tree T rooted at r that contains at least one vertex from each group g_i.*

For a subfamily $\mathcal{A} \subseteq \mathcal{F}$, we define the coverage as:

$$\text{cov}(\mathcal{A}) \equiv \bigcup_{A \in \mathcal{A}} A$$

For a subset X of the universe, the coverage on X is:

$$\text{cov}(\mathcal{A}, X) = \text{cov}(\mathcal{A}) \cap X$$

The density Δ of a nonempty subfamily $\mathcal{A}$ on subset X is:

$$\Delta(\mathcal{A}, X) \equiv \frac{|\text{cov}(\mathcal{A}, X)|}{|\mathcal{A}|}$$

For convenience, we define $\Delta(\emptyset, X) < 0$.

Definition 5 (Max-Density Precedence-Closed Subfamily (MDPCS)) *Given a family of m sets $\mathcal{G}$, a precedence relation $\prec$, and a set of n items to be covered $R \subseteq \mathrm{cov}(\mathcal{G})$, the MDPCS problem asks to find a precedence-closed subfamily $\mathcal{A} \subseteq \mathcal{G}$ that maximizes $\Delta(\mathcal{A}, R)$.*

For $S \in \mathcal{G}$, let $P[S]$ denote the minimal precedence-closed subfamily of $\mathcal{G}$ containing S (i.e., the ancestors of S including S itself).

3. A warm up: Binary search with precedences

In this section, we consider the special case of the PCWCAL and PCACAL where the instance $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$ is an instance of the binary search problem with precedence constraints. In this setup we are given a linearly ordered set of n elements $\mathcal{H} = \{h_1, h_2, \ldots, h_n\}$ with $h_1 \prec h_2 \prec \ldots \prec h_n$ and a set of tests $t_{i,j} \in \mathcal{T}$ corresponding to performing a comparison operation informing the learner whether the target element is less than or equal to h_j or greater than h_j. This problem is NP-hard (see Section **??**) however it is possible to derive an $O(\log n)$-approximation algorithm for both the PCWCAL and PCACAL.

The algorithm for a worst case is simple. We use the equivalence between binary searching in an ordered set and the edge ranking coloring of a path. An edge ranking of a path is a coloring of its edges such that any path between two edges of the same color contains an edge of a lower color **?**. Intuitevely, the color corresponds to the level of the decision tree where the test corresponding to the edge is performed. Let the input path be P. For any test $t \in \mathcal{T}$ define its *depth* as $d(t) = \max_{\tau \in \mathcal{T}, P_{\tau,t} \in F}\{d(\tau, t)\} + 1$. Let the height of $\mathcal{F}$ be defined as $h(\mathcal{F}) = \max_{t \in \mathcal{T}}\{d(t)\}$. The algorithm starts by partitioning $\mathcal{T}$ into $h(\mathcal{F})$ sets $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{h(\mathcal{F})}$ such that for any $t \in \mathcal{T}_i$, $d(t) = i$. Then, the algorithm builds a decision tree processing layer one by one. Let $\mathcal{T}_i$ be such layer. We concatenate the (possibly disjoint) edges in T_i into a path P_i. Then, we compute an optimal edge ranking coloring of P_i using $\log n$ colors starting from color $(i-1) \cdot \log n$. Finally, we build a decision tree for the tests in $\mathcal{T}_i$ according to the edge ranking coloring. Observe that the resulting coloring is a valid edge ranking of P and that each edge has a color greater than all its predecessors in $\mathcal{F}$. Thus, the precedence constraints are respected. The final decision tree D is built recursively picking the root of the decision tree to be edge in P with the smallest color. it is easy to see that $\texttt{COST}_W(D, P) \leq h(\mathcal{F}) \cdot \log n$. Since the optimal decision tree has depth at least $h(\mathcal{F})$, the algorithm is an $O(\log n)$-approximation.

To obtain an algorithm for the average case, we use a different approach. A precedence constrained $1/2$-cut in a path is a subset of edges whose removal splits the path into subpaths of size at most $n/2$, such that the subset is precedence-closed. We wish to minimize the size of this set. We have the following observation: **troche nie widze tego lematu: co jesli precedensy to dwa lancuchy biegnace od srodka sciezki do jej konca/poczatku? Być może źle czytam, gdyz nie bylo zdefiniowane jak rozumiemy zbior krawedzi bedacy precedence closed?**

Lemma 6 *Let S^* be the optimal precedence constrained $1/2$-cut in a path $P = \langle v_1, \ldots, v_n \rangle$. Then either $S^* = F[\{(v_{\lfloor n/2 \rfloor} v_{\lfloor n/2 \rfloor + 1})\}]$ or there exist edges e_1, e_2 such that $S^* = F[\{e_1, e_2\}]$.*

Proof Observe that if S^* contains $(v_{\lfloor n/2 \rfloor} v_{\lfloor n/2 \rfloor + 1})$ then it is a $1/2$ cut. In such case we are done, since minimal precedence closed subset of edges containing $(v_{\lfloor n/2 \rfloor} v_{\lfloor n/2 \rfloor + 1})$ is $F[\{(v_{\lfloor n/2 \rfloor} v_{\lfloor n/2 \rfloor + 1})\}]$. Otherwise, let e_1 be the edge in S^* closest to v_1 and e_2 be the edge in S^* closest to v_n. Since e_1 and e_2 suffice to be a $1/2$-cut, the minimal precedence closed subset of edges with this property is $F[\{e_1, e_2\}]$. ■

Therefore one can easily compute such a cut in polynomial time by checking all possible candidates. We also have the following lemma relating the size of the optimal cut with the cost of the optimal decision tree:

Lemma 7 *Let P be the path induced by the instance $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$ of the binary search problem with precedence constraints. Additionally, let $S^* \subseteq \mathcal{T}$ be the optimal precedence constrained $1/2$-cut in P. Then we have that $n \cdot |S^*|/2 \leq \texttt{OPT}_A(I)$.*

Proof Let D^* be the optimal decision tree for I. Let S_D be the set of all tests which appeared at least once in D^* before the number of hypotheses consistent with previous responses dropped below $n/2$. Clearly, S_D is a precedence-closed $1/2$-cut in P. Additionally, each test in S_D contributes at least $n/2$ to the average cost of D^*. Thus, we have that $\texttt{COST}_A(D^*) \geq n \cdot |S_D|/2 \geq n \cdot |S^*|/2$. ■

The two lemmas above suggest a natural approach to finding a good strategy. Find the optimal precedence constrained $1/2$-cut S^* in P and use the tests in S^* to build any precedence-respecting decision tree D that splits the hypotheses into subproblems of size at most $n/2$. Then, recursively build decision trees for each subproblem and attach them to the leaves of D. It is easy to see that the resulting decision tree respects precedence constraints. We have the following theorem:

Theorem 8 *The above algorithm is an $O(\log n)$-approximation for the* PCACAL *in the case of binary search with precedence constraints.*

Proof Let D be the decision tree built by the algorithm. We prove by induction on n that $\frac{\texttt{COST}_A(D)}{\texttt{OPT}_A(I)} \leq 2\log n$. The base case for $n = 1$ is trivial since there is only one hypothesis and no tests are needed. For the inductive step, observe that the cost of D can be upper bounded as follows:

$$\begin{aligned}
\frac{\texttt{COST}_A(D, P)}{\texttt{OPT}_A(I)} &\leq \frac{|S^*| \cdot n + \sum_{P' \in P - S^*} \texttt{COST}_A(D_{P'}, P')}{\texttt{OPT}_A(I)} \\
&\leq 2 + \frac{\sum_{P' \in P - S^*} 2 \cdot \log\left(\frac{n}{2}\right) \cdot \texttt{OPT}(I_{P'})}{\sum_{P' \in P - S^*} \texttt{OPT}(I_{P'})} \\
&= 2 + 2\log\left(\frac{n}{2}\right) = 2\log n.
\end{aligned}$$

where the first inequality follows from the definition of average cost, the second inequality follows from the Lemma 7, inductive hypothesis, the fact that each subproblem has size at most $n/2$ and Lemma 3. The claim follows. ■

4. Active Learning via Covering Problems

To build our algorithm we will make use of the following definition:

Definition 9 (Sepcover) *Let D be any decision tree for $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$. We define a sequence of tests P_D called* sepcover *as follows. Initially, P_D is empty and $\mathcal{H}' = \mathcal{H}$. While $|\mathcal{H}'| > |\mathcal{H}|/2$, we append to P_D the test $r(D_{\mathcal{H}'})$ and update $\mathcal{H}'$ to be the set of hypotheses corresponding to the child of $D_{\mathcal{H}'}$ that contains the most hypotheses. If $\texttt{COST}_W(D) = \texttt{OPT}_W(I)$, then we denote $P^*(I) = P_D$ (ties broken arbitrarily).*

It should be remarked that P_D is well-defined, as each test in P_D can have at most one child associated with more than half of the hypotheses in $\mathcal{H}'$. Since P_D is a subpath of D, we also have the following simple observation. Figure 3 illustrates the definition of sepcover.

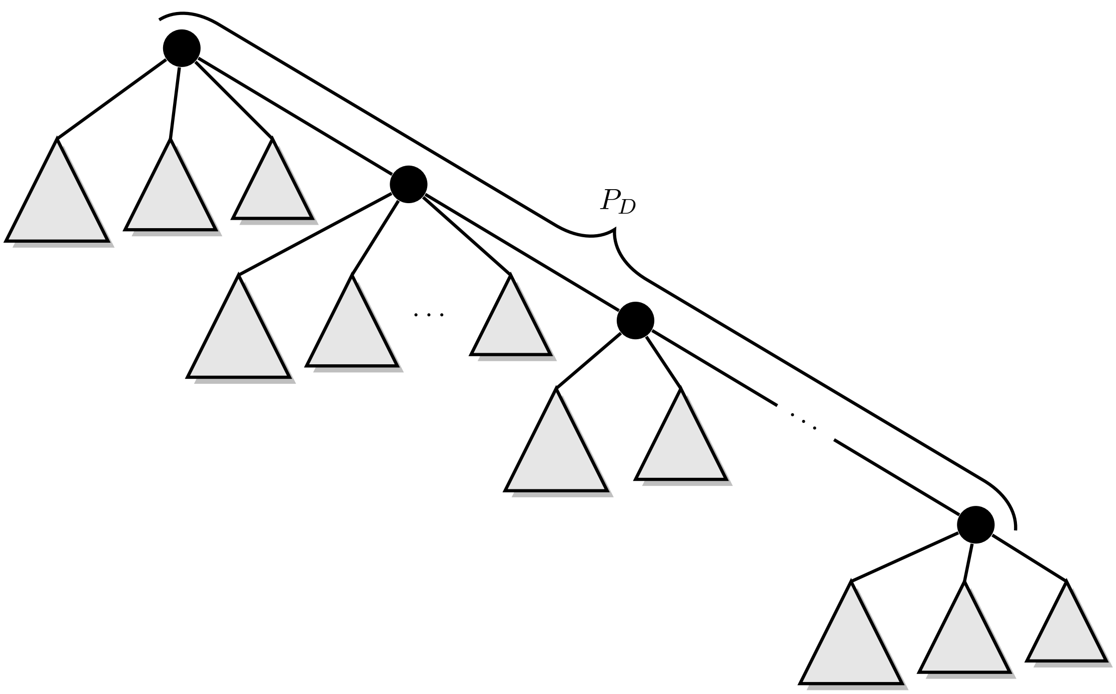


Figure 3: Sepcover sequence in a decision tree

We will say that a test in $P^*(I)$ *sepcovers* an element $u \in \mathcal{H}$ in C if after applying the test t in C, u belongs to a response $\mathcal{H}'$ of size at most $|\mathcal{H}|/2$. **tutaj jest cos dla mnie niejasne: nie wiem czy powyzsze definiuje P^*, gdyz jest to uzywane w lemacie ponizej i lemat nie mowi czy zachodzi dla kazdego P^* czy sa jakies warunki? W powyzszym takze z frazy “applyting the test t in C – to by sugerowalo, ze C zawiera jeden test (bo ”the“). Innymi slowy, nie wiem czym jest P^*.**

4.1. Worst Case

Observation 2 *Let I be any instance of . Then $|P^*(I)| \leq \texttt{OPT}_W(I)$.*

We will use $|P^*(I)|$ as a lower bound on $\texttt{OPT}_W(I)$ in the analysis of the approximation algorithm for . We have the following lemma:

Lemma 10 *Let $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$ be any instance. Let S^* be the optimal solution for the PCSC on instance $(\mathcal{H}, \mathcal{T}, \mathcal{F}, f)$ where $f = 1/4$ and a test t covers $h \in \mathcal{H}$ if $\{U_t(u)\} \leq \frac{3}{4} \cdot \{\mathcal{U}\}$. Then, $|S^*| \leq |P^*(I)|$.*

Proof We show that $P^*(I)$ is a feasible solution for the PCSC instance $(\mathcal{H}, \mathcal{T}, \mathcal{F}, K)$. Assume towards a contradiction that this is not the case, i. e. less than $|\mathcal{H}|/4$ are covered by $P^*(I)$. Therefore there exists $t \in P^*(I)$ and a response $\mathcal{H}'$ of size $|\mathcal{H}'| \leq |\mathcal{H}|/2$ such that hypotheses in $\mathcal{H}'$ are not covered by $P^*(I)$, otherwise the claim holds trivially, since all hypotheses are covered. Let $\mathcal{H}' \subseteq U_{t,j}$ (since $\mathcal{H}'$ is a response to a test t, such $U_{t,j}$ always exists). By assumption, we have that $|U_{t,j} - \mathcal{H}'| < |\mathcal{H}|/4$. Therefore, we have that $|U_{t,j}| = |\mathcal{H}'| + |U_{t,j} - \mathcal{H}'| < 3/4 \cdot |\mathcal{H}|$ which by definition means that h is covered by $P^*(I)$, a contradiction. ■

Theorem 11 *If there is an (γ, α)-bicriteria approximation algorithm for PCSC then there is an $O\left(\frac{\alpha}{\log\left(\frac{2\gamma}{2\gamma-1}\right)} \cdot \log n\right)$-approximation algorithm for . In particular when $\gamma = O(1)$, the approximation is $O(\alpha \cdot \log n)$.*

Proof The algorithm 1 is recursive and works as follows: Given an instance $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$, if $|\mathcal{H}| = 1$ we return the trivial decision tree with a single leaf corresponding to the only hypothesis in $\mathcal{H}$. Otherwise, we run the (γ, α)-approximation algorithm for PCSC on instance $(\mathcal{H}, \mathcal{T}, \mathcal{F}, f)$ with $f = 1/4$, where a test t covers element $u \in \mathcal{H}$ if for $u \in U_{t,j}$, $|U_{t,j}| \leq \frac{3}{4} \cdot |\mathcal{H}|$. Let S be the returned set of tests. We build a decision tree D_S on tests from S closed under $\mathcal{F}$. For each $\mathcal{H}' \in \mathcal{H} - S$, we recursively call WORSTDECISIONTREE on instance $(\mathcal{H}', \mathcal{T} - S, \mathcal{F} - S)$ and attach the returned decision tree to the leaf of D_S corresponding to $\mathcal{H}'$. Finally, we return the constructed decision tree D. The following observation follows by Lemmas 2 and 10.:

Algorithm 1: The $O(\alpha \cdot \log n)$-approximation algorithm for the PCWCAL

```
procedure WORSTDECISIONTREE(H, T, F)
    if |H| = 1 then
        return the trivial decision tree with a single leaf corresponding to the only hypothesis in H.
    end
    foreach t ∈ T do
        Set t to cover u ∈ H if for u ∈ U_{t,j}, |U_{t,j}| ≤ 3/4 · |H|.
    end
    S ← Run the (γ, α)-approximation algorithm for PCSC on instance (H, T, F, f) with f = 1/4.
    D ← D_S ← any decision tree built on tests from S respecting the precedence constraints F.
    foreach H' ∈ H − S do
        D' ← WORSTDECISIONTREE(H', T − S, F − S).
        Attach D' to the leaf of D corresponding to H'.
    end
    return D.
```

Observation 3 *Let D_S be the decision tree built on tests from S respecting the precedence constraints $\mathcal{F}$. Then, $\textsc{cost}_W(D_S) \leq \alpha \cdot |P^*(I)|$.*

We are now ready to prove the theorem.

Lemma 12 *Let D be the decision tree returned by* WORSTDECISIONTREE *on input $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$. Then,* $\texttt{COST}_W(D, I) \leq \frac{\alpha}{\log\left(\frac{4\gamma}{4\gamma-1}\right)} \cdot \log n \cdot \texttt{OPT}_W(I)$.

Proof We prove the lemma by induction on n. The base case when $n = 1$ is trivial since the the cost of the decision tree is 0. Assume by induction that for every $I' = (\mathcal{H}', \mathcal{T}, \mathcal{F})$ such that $\mathcal{H}' \in \mathcal{H} - S$ and $n' = |\mathcal{H}'|$ we have $\texttt{COST}_W(D', I') \leq \frac{\alpha}{\log\left(\frac{4\gamma}{4\gamma-1}\right)} \cdot \log n' \cdot \texttt{OPT}_W(I')$, where D' is the decision tree returned by WORSTDECISIONTREE on input I'. We have that:

$$
\begin{aligned}
\texttt{COST}_W(D, I) \leq& \texttt{COST}_W(D_S, I) + \max_{\mathcal{H}' \in \mathcal{H} - S} \texttt{COST}_W\left(D', I'\right) \\
\leq& \alpha \cdot |S^*| + \max_{\mathcal{H}' \in \mathcal{H} - S} \frac{\alpha}{\log\left(\frac{4\gamma}{4\gamma-1}\right)} \cdot \log n' \cdot \texttt{OPT}_W\left(I'\right) \\
\leq& \alpha \cdot |P^*(I)| + \frac{\alpha}{\log\left(\frac{4\gamma}{4\gamma-1}\right)} \cdot \log\left(\frac{(4\gamma - 1) \cdot n}{4\gamma}\right) \cdot \texttt{OPT}_W(I) \\
=& \alpha \cdot \texttt{OPT}_W(I) + \frac{\alpha}{\log\left(\frac{4\gamma}{4\gamma-1}\right)} \cdot \log n \cdot \texttt{OPT}_W(I) - \alpha \cdot \texttt{OPT}_W(I) \\
=& \frac{\alpha}{\log\left(\frac{4\gamma}{4\gamma-1}\right)} \cdot \log n \cdot \texttt{OPT}_W(I)
\end{aligned}
$$

where the second inequality follows by the induction hypothesis, the third inequality follows by Lemma 2 and the fact that $|\mathcal{H}'| \leq \frac{(4\gamma-1)}{4\gamma} \cdot n$ for every $\mathcal{H}' \in \mathcal{H} - S$ and the equalities follow by rearranging terms. This concludes the proof of the lemma. ■

■

4.2. Average Case

We follow a similar idea, however we use the connection to PCMSSC instead of PCSC. In order to lower bound the cost of the optimal decision tree, we will need the following notion: Let S be any sequence of tests in a decision tree D. Then let:

$$\texttt{COST}_A(S, I) = \sum_{t \in S} |\mathcal{H}_t| \,.$$

w powyższym nie widać jakie znaczenie ma wybór D.

We have the following observations:

Observation 4 *Let D be any decision tree for $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$ and let S be any sequence of tests in D. Then,*

$$\texttt{COST}_A(D, I) = \texttt{COST}_A(S, I) + \sum_{D' \in D - S} \texttt{COST}_A\left(D', I\right) .$$

As an immediate corollary for $D = D^*(I)$ and $S = P^*(I)$ we have:

Observation 5 *Let I be any instance of PCACAL. Then* $\texttt{COST}_A(P^*(I), I) \leq \texttt{OPT}_A(I)$.

This allows to use $\texttt{COST}_A(P^*(I), I)$ as a lower bound on $\texttt{OPT}_A(I)$ in the analysis of the approximation algorithm for PCACAL. We have the following lemma, analogous to Lemma 10.

Lemma 13 *Let $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$ be any PCACAL instance. Let S^* be the optimal solution for the PCMSSC on instance $(\mathcal{H}, \mathcal{T}, \mathcal{F}, f)$ with $f = 1/4$, where a test t covers element $u \in \mathcal{H}$ if for $u \in U_{t,j}$, $|U_{t,j}| \leq \frac{3}{4} \cdot |\mathcal{H}|$. Then,* $\texttt{COST}_A(S^*) \leq \texttt{COST}_A(P^*(I))$.

Proof Assume towards a contradiction that $\texttt{COST}_A(S^*) > \texttt{COST}_A(P^*(I))$. We will show that in such case there exists a cover $\sigma \subseteq P^*(I)$ such that $c_A(\sigma) < \texttt{COST}_A(P^*(I))$, a contradiction with the optimality of S^*. Let σ be the smallest subsequence of $P^*(I)$ which covers at least $|\mathcal{H}|/4$ elements. Such a subsequence exists, by repeating the argument used in the proof of Lemma 10. Let $h \in \mathcal{H}$. There are two cases to consider:

- σ covers h. Consider the first test t that sepcovered h in $P^*(I)$. By definition, tests previous to t in σ cover at most $|\mathcal{H}|/4$ elements. Since at the moment of sepcovering, h belonged to a response of size at most $|\mathcal{H}|/2$, we know that t also covers h in σ. This means that the contribution of h to $c_A(\sigma)$ is at most its contribution to $\texttt{COST}_A(P^*(I))$.
- σ does not cover h. In such case h is sepcovered by some test t in $P^*(I)$ but not in σ. therefore, the contribution of h to $c_A(\sigma)$ is $|\sigma|$ and its contribution to $\texttt{COST}_A(P^*(I))$ is at least $|\sigma|$.

Thus, we have that $c_A(\sigma) < \texttt{COST}_A(P^*(I))$, a contradiction. ■

Theorem 14 *If there is a β- approximation algorithm for PCMSSC then there is an $O(\beta \cdot \log n)$-approximation algorithm for PCACAL.*

Proof The idea behind Algorithm 2 is the same as for the worst case version of the problem except the fact that we use a solution to PCMSSC instead of PCSC.

Algorithm 2: The $O(\beta \cdot \log n)$-approximation algorithm for the PCACAL

```
procedure AVERAGEDECISIONTREE(H, T, F)
    if |H| = 1 then
        return the trivial decision tree with a single leaf corresponding to the only hypothesis in H.
    end
    foreach t ∈ T do
        Set t to cover u ∈ H if for u ∈ U_{t,j}, |U_{t,j}| ≤ 3/4 · |H|.
    end
    S ← Run the β-approximation algorithm for PCMSSC on instance (H, T, F, f) with f = 1/4.
    D ← D_S ← decision tree which consists of sequence of tests S.
    foreach H' ∈ H − S do
        D' ← AVERAGEDECISIONTREE(H', T − S, F − S).
        Attach D' to the leaf of D corresponding to H'.
    end
    return D.
```

Observation 6 *Let D_S be the decision tree consisting of the sequence of tests S. Then, $\mathit{COST}_A(D_S, I) \leq \beta \cdot \mathit{COST}_A(P^*(I), I)$.*

We are now ready to prove the theorem:

Lemma 15 *Let D be the decision tree returned by* AVERAGEDECISIONTREE *on input $I = (\mathcal{H}, \mathcal{T}, \mathcal{F})$. Then, $\mathit{COST}_A(D, I) \leq \beta \cdot \log_{4/3} n \cdot \mathit{OPT}_A(I)$.*

Proof We prove the lemma by induction on n. The base case when $n = 1$ is trivial since the the cost of the decision tree is 0. Assume by induction that for every $I' = (\mathcal{H}', \mathcal{T}, \mathcal{F})$ such that $\mathcal{H}' \in \mathcal{H} - S$ and $n' = |\mathcal{H}'|$ we have $\mathtt{COST}_A(D', I') \leq \beta \cdot \log_{4/3} n' \cdot \mathtt{OPT}_A(I')$, where D' is the decision tree returned by AVERAGEDECISIONTREE on input I'. We have that:

$$
\begin{aligned}
\mathtt{COST}_A(D, I) \leq & \mathtt{COST}_A(D_S, I) + \sum_{\mathcal{H}' \in \mathcal{H} - S} \mathtt{COST}_A\big(D', I'\big) \\
\leq & \beta \cdot \mathtt{COST}_A(P^*(I), I) + \sum_{\mathcal{H}' \in \mathcal{H} - S} \beta \cdot \log_{4/3} n' \cdot \mathtt{OPT}_A\big(I'\big) \\
\leq & \beta \cdot \mathtt{COST}_A(P^*(I), I) + \sum_{\mathcal{H}' \in \mathcal{H} - S} \beta \cdot \log_{4/3}\left(\frac{3}{4} \cdot n\right) \cdot \mathtt{OPT}_A(I) \\
= & \beta \cdot \mathtt{COST}_A(P^*(I), I) + \beta \cdot \Big(\log_{4/3} n - 1\Big) \cdot \mathtt{OPT}_A(I) \\
\leq & \beta \cdot \log_{4/3} n \cdot \mathtt{OPT}_A(I)
\end{aligned}
$$

where the second inequality follows by the induction hypothesis, the third inequality follows by Lemma 2 and the fact that $|\mathcal{H}'| \leq \frac{3}{4} \cdot n$ for every $\mathcal{H}' \in \mathcal{H} - S$ and the equalities follow by rearranging terms. This concludes the proof of the lemma. ∎

∎

5. Set covering with constraints

5.1. Max-Density Precedence-Closed Subfamily (MDPCS)

The key to solce PCSC and PCMSSC is to solve the MDPCS problem. An approximation algorithm for MDPCS can be used as an essential subroutine in our algorithms for PCSC and PCMSSC. By **?**, the following greedy algorithm achieves an $O(\sqrt{m})$-approximation for MDPCS:

Algorithm 3: The greedy algorithm for MDPCS

procedure MDPCS-GREEDY($\mathcal{G}, \prec, R$)
  $\mathcal{A} \leftarrow \mathcal{G}$
  foreach $S \in \mathcal{G}$ **do**
    if $\Delta(P[S], R) > \Delta(\mathcal{A}, R)$ **then**
      $\mathcal{A} \leftarrow P[S]$
    end
  end
  return $\mathcal{A}$

Let $\delta = \max_{S \in \mathcal{G}} \Delta(P[S], R)$. When $\delta \geq 1$, then the approximation factor of the greedy can also be bounded by $O(\sqrt{n})$. We show that if we enforce a certain condition on the input called *ϵ-shallow ancestry*, then for $\epsilon < 1$ the greedy algorithm achieves an $O(n^\epsilon)$-approximation .

For each $S \in \mathcal{G}$, let $p(S) = |P[S]|$ and $c(S, R) = |\mathrm{cov}(P[S], R)|$.

Theorem 16 *Suppose there exists a constant $C > 0$ and $\epsilon \in (0, 1)$ such that for all $S \in \mathcal{G}$, $p(S) \leq C \cdot c(S)^\epsilon$ (ϵ-shallow ancestry). Then MDPCS-Greedy provides an $O(n^\epsilon)$-approximation.*

Proof Let $\mathcal{A}^*$ be an optimal solution consisting of sets $S_1, \ldots, S_k$. There are two cases:

1. If $\delta \geq n^{1-\epsilon}$, then, we observe that $\Delta(\mathcal{A}, R) = \delta \geq n^{1-\epsilon}$. Since we can cover at most n elements with at least one set, we have $\Delta(\mathcal{A}^*, R) \leq n$. Therefore:
$$\frac{\Delta(\mathcal{A}^*, R)}{\Delta(\mathcal{A}, R)} \leq \frac{n}{n^{1-\epsilon}} = n^\epsilon$$

2. Else, if $\delta \leq n^{1-\epsilon}$, we proceed as follows: By definition of density, for any $S \in \mathcal{G}$, $c(S) \leq \delta \cdot p(S)$. Combining this with the ϵ-shallow ancestry condition, we have that for all $S \in \mathcal{G}$, $c(S) \leq \delta \cdot C \cdot c(S)^\epsilon$. Rearraning this inequality, we get that $c(S) \leq (\delta \cdot C)^{\frac{1}{1-\epsilon}}$. We have that:
$$|\mathrm{cov}(\mathcal{A}^*)| = \left| \bigcup_{j=1}^{k} \mathrm{cov}(S_j, R) \right| \leq \sum_{j=1}^{k} c(S_j) \leq k \cdot (\delta \cdot C)^{\frac{1}{1-\epsilon}}$$
Therefore:
$$\Delta(\mathcal{A}^*, R) = \frac{|\mathrm{cov}(\mathcal{A}^*, R)|}{k} \leq (\delta \cdot C)^{\frac{1}{1-\epsilon}}$$
By the greedy choice, $\Delta(\mathcal{A}, R) \geq \delta$ and by assumption $\delta \leq n^{1-\epsilon}$. Thus:
$$\frac{\Delta(\mathcal{A}^*, R)}{\Delta(\mathcal{A}, R)} \leq \frac{(\delta \cdot C)^{\frac{1}{1-\epsilon}}}{\delta} = C^{\frac{1}{1-\epsilon}} \cdot \delta^{\frac{\epsilon}{1-\epsilon}} \leq C^{\frac{1}{1-\epsilon}} \cdot (n^{1-\epsilon})^{\frac{\epsilon}{1-\epsilon}} = C^{\frac{1}{1-\epsilon}} \cdot n^\epsilon$$

Since C is constant, the theorem follows. ∎

5.2. Precedence constrained set cover

Tu poniżej miałem jakąś próbe pisania tego, ale sie pokomplikowało więc ten pseuudokod jest niekompletny. To jest do zmiany wszystko. Zostawiam na razie te sekcje tobie Darku.

We show the following:

Theorem 17 *If there exists an γ approximation algorithm for the MDPCS problem, then there exists an $(H_K + 1) \cdot \gamma$ - approximate algorithm for the PCSC problem.*

Proof ■

Algorithm 4: The γ-greedy algorithm for PCSC

procedure PCSC$(\mathcal{U}, \mathcal{S}, \mathcal{F}, K)$
- $\mathcal{C} \leftarrow \emptyset$
- **while** $|\text{cov}(\mathcal{C}, \mathcal{U})| < K$ **do**
  - $\mathcal{A} \leftarrow$ Run the γ-approx. algorithm for MDPCS on $(\mathcal{U} - \mathcal{C}, \mathcal{S} - \mathcal{C}, \mathcal{F} - \mathcal{C}, m)$
  - **if** $|\text{cov}(\mathcal{C} \cup \mathcal{A}, \mathcal{U})| \geq K$ **then**
    - Find the minimum budget $B \in [|\mathcal{A}|]$, such that the γ-approx. algorithm for MDPCS on $(\mathcal{U} - \mathcal{C}, \mathcal{S} - \mathcal{C}, \mathcal{F} - \mathcal{C}, B)$ returns a set $\mathcal{B}$ with $|\text{cov}(\mathcal{B}, \mathcal{U} - \mathcal{C})| \geq \frac{K - \text{cov}(\mathcal{C}, \mathcal{U})}{\alpha}$
    - **while** $|\text{cov}(\mathcal{C}, \mathcal{U})| < K$ **do**
      - $\mathcal{B} \leftarrow$ Run the γ-approx. algorithm for MDPCS on $(\mathcal{U} - \mathcal{C}, \mathcal{S} - \mathcal{C}, \mathcal{F} - \mathcal{C}, B)$
      - $\mathcal{C} \leftarrow \mathcal{C} \cup \mathcal{B}$
    - **end**
    - **return** $\mathcal{C}$
  - **end**
  - **foreach** $u \in \text{cov}(\mathcal{A}, \mathcal{U} - \mathcal{C})$ **do**
    - $c(u) \leftarrow \Delta(\mathcal{A}, \mathcal{U} - \mathcal{C})$
  - **end**
  - $\mathcal{C} \leftarrow \mathcal{C} \cup \mathcal{A}$
- **end**

Theorem 18 *If the precedence constraints form an outforest, then there exists an bicriteria $(4, O(\log n))$-approximation algorithm for PCSC which be converted to an $O(\log^2 n)$ approximation algorithm.*

5.3. Precedence constrained min sum set cover

By **?**:

Theorem 19 *If there exists an γ approximation algorithm for the MDPCS problem, then there exists an $4 \cdot \gamma$ - approximate algorithm for the PCMSSC problem.*

Theorem 20 *If the precedence constraints form an outforest, then there exists an $(O(\log n))$-approximation algorithm for PCMSSC.*

6. Hardness

6.1. Binary search with precedence constrains

Theorem 21 *Worst-Case Binary Search With Precedence Caseonstrains is NP-hard.*

Theorem 22 *Average-Case Binary Search With Precedence Caseonstrains is NP-hard.*

6.2. Outforest precendce constrains

Theorem 23 *PCWCAL with outforest precedence constraints is NP-hard to approximate within a factor of* $O(\log^2 n)$ *unless P = NP.*

6.3. General precedence constraints

Theorem 24 *PCWCAL cannot be approximated within a factor of* $o(m^{1/6})$ *nor* $o(n^{1/12})$ *condition to Planted Dense Subgraph Conjecture.*

Theorem 25 *PCACAL cannot be approximated within a factor of* $o(m^{1/6})$ *nor* $o(n^{1/12})$ *condition to Planted Dense Subgraph Conjecture.*

7. Conclusions and Future Work

Appendix A. My Proof of Theorem 1

This is a boring technical proof.

Appendix B. My Proof of Theorem 2

This is a complete version of a proof sketched in the main text.