Searching in Graphs

Michał Szyfelbein

Faculty of Electronics, Telecommunications and Informatics
Gdańsk University of Technology, Poland

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Binary Search

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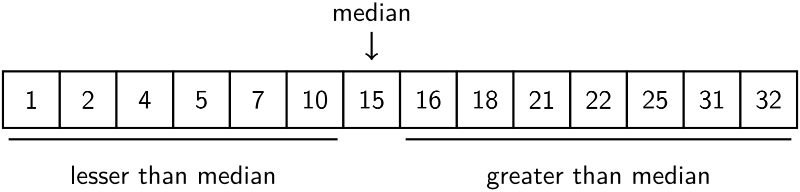

Figure: Example of a sorted array containing 14 elements.

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Easy to generalize to graphs:

A **query** to a vertex v returns information whether v is the target x, and if not, which connected component of $G - v$ contains x.

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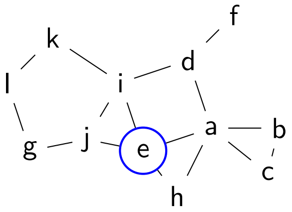

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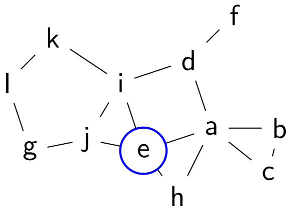

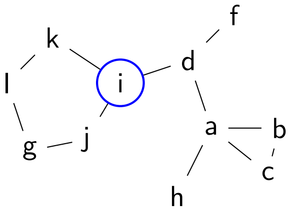

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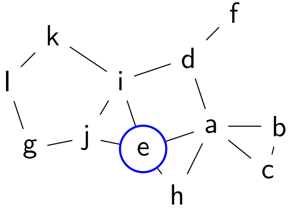

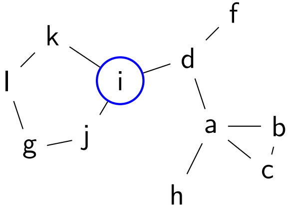

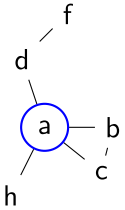

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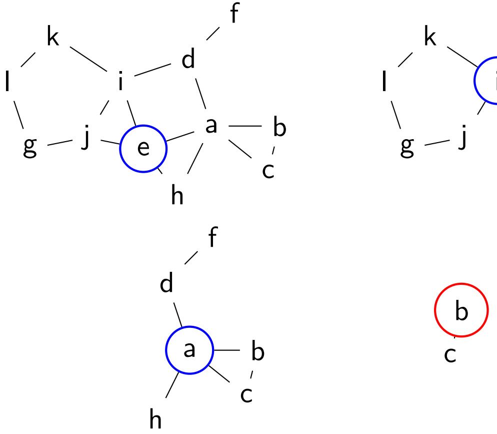

Strategy of searching

We wish to find a strategy of searching.

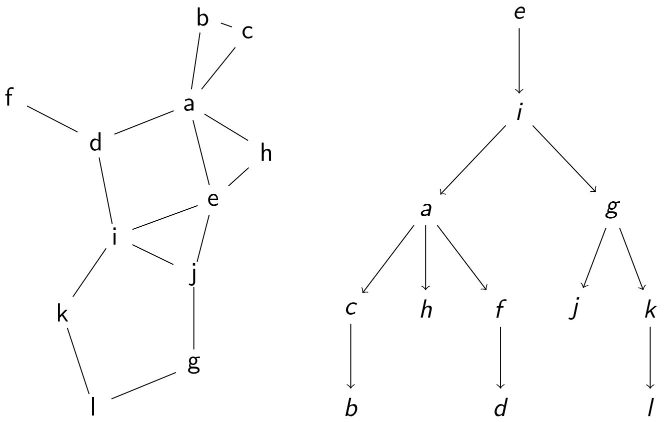

Figure: Sample input graph a decision tree for it.

Additional parameters

We introduce additional information about the input:

- To each vertex v we assign an arbitrary **cost** $c(v)$, which denotes a cost of performing a query to v.

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- To each vertex v we assign an arbitrary **cost** $c(v)$, which denotes a cost of performing a query to v.
- Additionally, to each vertex v we also assign an arbitrary **weight** $c(v)$, which denotes the importance of v.
- For any $S \subseteq V(G)$, we denote $c(S) = \sum_{v \in S} c(v)$ and $w(S) = \sum_{v \in S} w(v)$.

Our setup

What is the best strategy of searching in a graph?

Graph Search Problem (GSP)

Input: Graph G, a query cost function $c\colon V(G) \to \mathbb{N}$ and a weight function $w\colon V(G) \to \mathbb{N}$.

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Graph Search Problem (GSP)

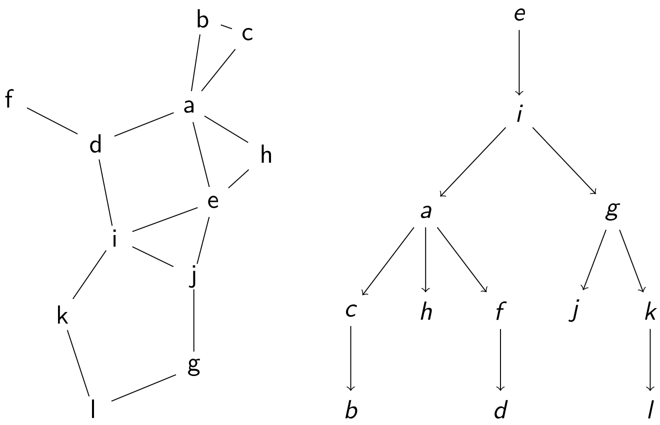
Input: Graph G, a query cost function $c\colon V(G) \to \mathbb{N}$ and a weight function $w\colon V(G) \to \mathbb{N}$.

Output: A decision tree D minimizing the weighted average search cost:

$$c_G(D) = \sum_{x \in V(G)} w(x) \cdot c(Q_G(D, x))$$

where $Q_G(D, x)$ denotes the set of queries performed along the unique path in D from the root $r(D)$ to x.

Decision Tree



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Why do we care?

Useful in:

- Automated bug detection in computer code,
- Hierarchical clustering of data.

Related to:

- Scheduling of parallel database join operations,
- Parallel Cholesky factorization of matrices,
- Parallel assembly of multi–part products from their components.

Why do we care?

Why our setup?:

1. **Average case** – it is natural to assume that the search strategies we design are intended to be used repeatedly.
2. **Weight function** – some vertices may serve as targets more frequently than the others.
3. **Query costs** – performing a query may require significant resources, such as time or money.
4. **Have not yet been investigated.**

Many names

- Binary Search [Knu73; OP06; Der+17],
- Tree Search Problem [Jac+10; Cic+16],
- Binary Identification Problem [Cic+12],
- Ranking Colorings [LY98; Der06],
- Ordered Colorings [KMS95],
- Elimination Trees [Pot88],
- Hub Labeling [Ang18],
- Tree–Depth [NO06],
- Partition Trees [Høg24],
- Hierarchical Clustering [Das16; Coh+19; CC17],
- Search Trees on Trees [BK22; Ber+22],
- LIFO–Search [GHT12].

How to tackle the problem

Bad news: The Graph Search Problem is **NP–hard** even when restricted to bounded degree trees and bounded diameter trees.

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We want to find an algorithm providing a good **approximation** for the problem:

- $(4 + \epsilon)$–approximation for **trees**.
- $O\left(\sqrt{\log n}\right)$–approximation for **general graphs**.

Weighted α–Separator Problem

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Input: Graph G, a cost function $c\colon V \to \mathbb{N}$, a weight function $w\colon V \to \mathbb{N}$ and a real number α.

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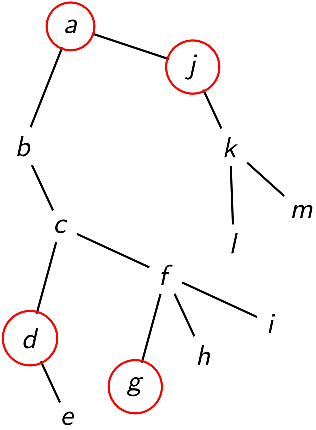
Input: Graph G, a cost function $c\colon V \to \mathbb{N}$, a weight function $w\colon V \to \mathbb{N}$ and a real number α.

Output: A set $S \subseteq V(G)$ called **separator** such that for every $H \in G - S$, $w(H) \leq w(G)/\alpha$ and $c(S)$ is minimized.

Example of a separator



	$w(v)$	$c(v)$
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Figure: Sample input tree T and a weighted 3-separator (the circled vertices) of cost 8.

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Theorem

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2. $w(H) \leq \frac{(1+\delta)\cdot w(T)}{\alpha}$ *for every* $H \in T - S'$.

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1. $c(S') \leq c(S)$.
2. $w(H) \leq \frac{(1+\delta)\cdot w(T)}{\alpha}$ *for every* $H \in T - S'$.
3. *The algorithm runs in* $O\left(n^3/\delta^2\right)$ *time.*

Notation

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- $\mathcal{L}_k^*$ – the k–th **level** of $\mathtt{OPT}(G)$: the subset of $\mathcal{R}_{D^*}(G)$ consisting of all maximal elements H of $\mathcal{R}_{D^*}(G)$ with $w(H) \leq k$.

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- $S_k^* = V(G) - \mathcal{L}_k^*$ – vertices belonging to the separator at the level $\mathcal{L}_k^*$. S_k^* forms a weighted $w(G)/k$–separator of G.

Example of the connection

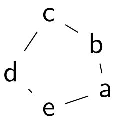
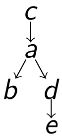


Figure: G.



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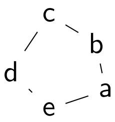

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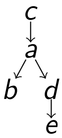

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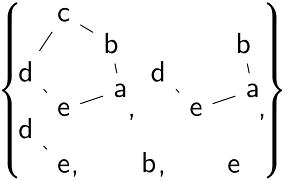

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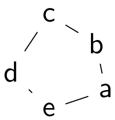

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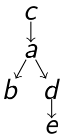

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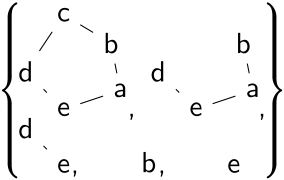

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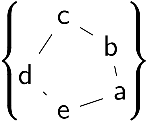
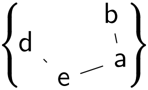


Figure: $\mathcal{L}_5^*, S_5^* = \{\}$.



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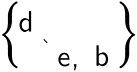
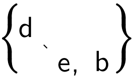


Figure: $\mathcal{L}_3^*, S_3^* = \{a, c\}$.



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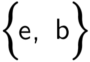
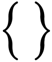


Figure: $\mathcal{L}_1^*, S_1^* = \{a, c, d\}$.



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Basic lemmas

Lemma

Let $G_{D,v}$ be the candidate subgraph of G in which v is queried when using D. Then, $c_G(D) = \sum_{v \in V(G)} w(G_{D,v}) \cdot c(v)$.

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Lemma

$OPT(G) = \sum_{k=0}^{w(G)-1} c(S_k^*)$.

Proof.

Consider any vertex v. For every $0 \le k < w(G_{D^*,v})$, $v \notin \bigcup_{H \in \mathcal{L}_k^*} H$, so $v \in S_k^*$ and the contribution of v to the cost is $w(G_{D^*,v}) \cdot c(v)$:

$$\sum_{k=0}^{w(G)-1} c(S_k^*) = \sum_{v \in V(G)} \sum_{k=0}^{w(G_{D^*,v})-1} c(v) = OPT(G).$$

□

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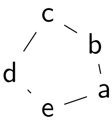

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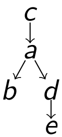

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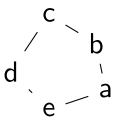

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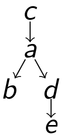

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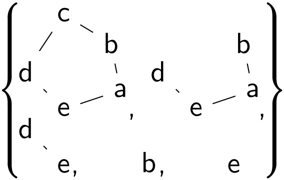

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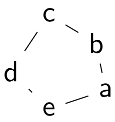

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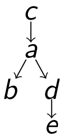

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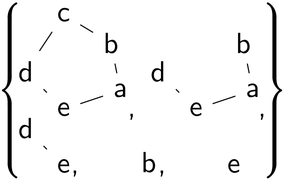

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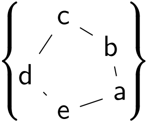

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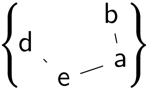

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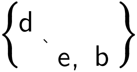

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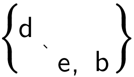

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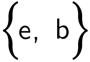

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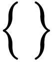

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We will iteratively use the **FPTAS** for the Weighted α–separator problem to create an $(4+\epsilon)$–approximation algorithm for the Tree Search Problem:

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Theorem

For any $\epsilon > 0$ there exists an $(4+\epsilon)$–approximation algorithm for the Tree Search Problem running in $O\left(n^4/\epsilon^2\right)$ time.

The algorithm

proc $\texttt{DecisionTree}(T, c, w, \epsilon)$:

1. $S_T \leftarrow \texttt{SeparatorFPTAS}\left(T, c, w, \alpha = 2, \delta = \frac{\epsilon}{4+\epsilon}\right)$.
2. $D_T \leftarrow$ arbitrary partial decision tree for T, built from vertices of S_T.
3. For each $H \in T - S_T$:
   - 3.1 $D_H \leftarrow \texttt{DecisionTree}(H, c, w, \epsilon)$.
   - 3.2 Hang D_H in D_T below the last query to $v \in N_T(H)$.
4. Return D_T.

Structure of the solution

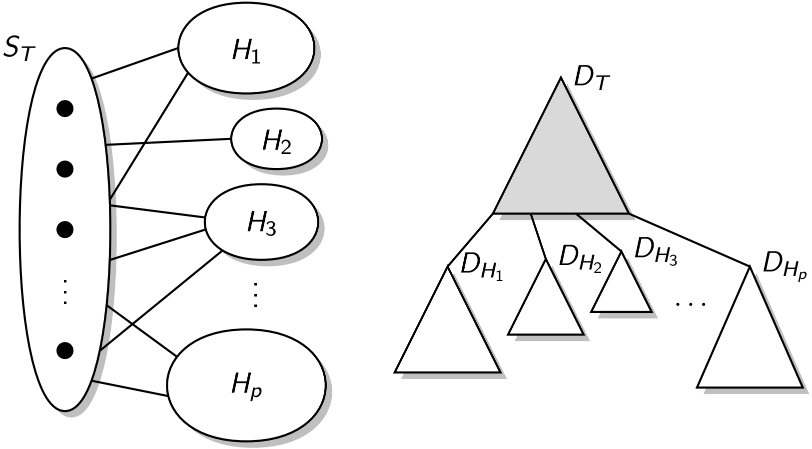

Figure: The separator S_T produced by the algorithm and the structure of the decision tree built using S_T.

How does it work

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b	1	4
c	3	6
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l	2	3
m	3	4

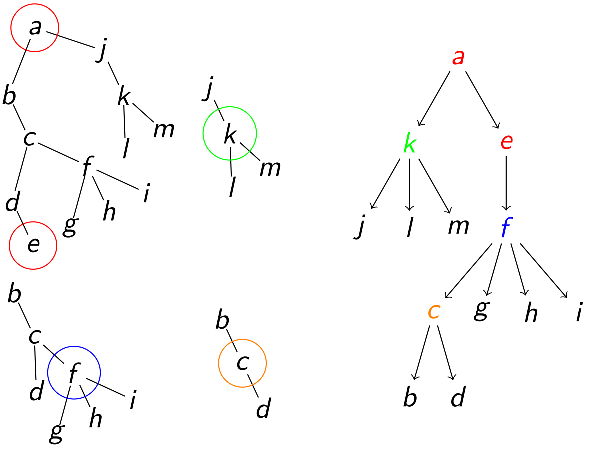

Figure: Example solution of cost 259 built by `DecisionTree`.

Sketch of the proof

- $\mathcal{T}$ – subtree of T for which the procedure was called.
- Step 1. Bound the cost of **single** recurrence call:

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$$w(\mathcal{T}) \cdot c(S_{\mathcal{T}}) \leq \frac{2}{1-\delta} \cdot \sum_{k=\frac{1+\delta}{2}\cdot w(\mathcal{T})+1}^{w(\mathcal{T})} c\left(S^*_{\lfloor k/2 \rfloor} \cap \mathcal{T}\right).$$

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- Step 2. Bound the cost of the **whole** solution:

$$\begin{aligned} c_T(D) &\leq \frac{2}{1-\delta} \cdot \sum_{\mathcal{T}} \sum_{k=\frac{1+\delta}{2}\cdot w(\mathcal{T})+1}^{w(\mathcal{T})} c\left(S^*_{\lfloor k/2 \rfloor} \cap \mathcal{T}\right) \\ &\leq \frac{2}{1-\delta} \cdot \sum_{k=0}^{w(T)} c\left(S^*_{\lfloor k/2 \rfloor}\right) \leq (4+\epsilon) \cdot \mathtt{OPT}(T). \end{aligned}$$

Min–Ratio Vertex Cut Problem

Min–Ratio Vertex Cut Problem

Input: Graph $G = (V(G), E(G))$, the cost function $c\colon V \to \mathbb{N}$ and the weight function $w\colon V \to \mathbb{N}$.

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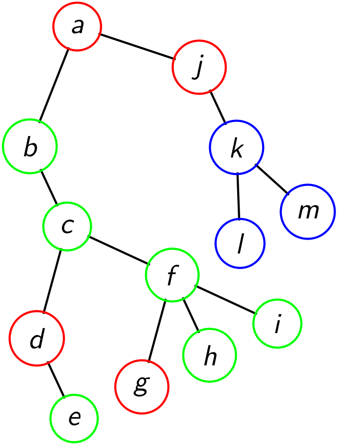
Input: Graph $G = (V(G), E(G))$, the cost function $c\colon V \to \mathbb{N}$ and the weight function $w\colon V \to \mathbb{N}$.

Output: A partition (A, S, B) of $V(G)$ called **vertex–cut**, such that there are no $u \in A$ and $v \in B$ for which $uv \in E(G)$, minimizing the ratio:

$$\alpha_{c,w}(A, S, B) = \frac{c(S)}{w(A \cup S) \cdot w(B \cup S)}.$$

Example of a vertex cut



	$w(v)$	$c(v)$
a	2	3
b	1	4
c	3	6
d	2	2
e	4	1
f	0	3
g	1	1
h	4	3
i	2	3
j	5	2
k	1	2
l	2	3
m	3	4

Figure: A tree and a vertex cut of ratio
$\alpha_{c,w}(A, S, B) = \frac{c(S)}{w(A \cup S) \cdot w(B \cup S)} = \frac{8}{(6+10)\cdot(14+10)} = \frac{8}{384} = \frac{1}{48}$.

How to find the cut

Min–Ratio Vertex Cut Problem is also **NP–hard**. However, we use the following result of [FHL05]:

Theorem

Given a graph $G = (V(G), E(G))$, the cost function $c : V \to \mathbb{N}$ and the weight function $w : V \to \mathbb{N}$, there exists a polynomial–time algorithm, which computes a partition (A, S, B), such that:

$$\alpha_{c,w}(A, S, B) = O\left(\sqrt{\log n}\right) \cdot \alpha_{c,w}(G) .$$

Searching in Graphs

We will iteratively use the f_n–approximation algorithm for the Min–Ratio Vertex Cut Problem to create an $O(f_n)$–approximation algorithm for the Graph Search Problem:

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We will iteratively use the f_n–approximation algorithm for the Min–Ratio Vertex Cut Problem to create an $O(f_n)$–approximation algorithm for the Graph Search Problem:

Theorem

Let f_n be the approximation ratio of any polynomial time algorithm for the Min–Ratio Vertex Cut Problem. Then, there exists an $O(f_n)$–approximation algorithm for the Graph Search Problem, running in polynomial time.

The algorithm

proc `DecisionTree`(T, c, w, ϵ):

1. $A_G, S_G, B_G \leftarrow$ `AlgorithmMinCut`(G, c, w).
2. $D_G \leftarrow$ arbitrary partial decision tree for G, built from vertices of S_G.
3. For each $H \in G - S_G$:
   3.1 $D_H \leftarrow$ `DecisionTree`(H, c, w).
   3.2 Hang D_H in D_G below the last query to $v \in N_G(H)$.
4. Return D_G.

Sketch of the proof

- $\mathcal{G}$ – any subgraph of G, for which the procedure was called.
- Step 1. Bound the cost of a **single** recurrence call:

$$w(\mathcal{G}) \cdot c(S_{\mathcal{G}}) \leq 11 \cdot f_n \cdot \sum_{k=w(A_{\mathcal{G}})+1}^{w(\mathcal{G})} c\left(S^*_{\lfloor k/2 \rfloor} \cap \mathcal{G}\right).$$

Sketch of the proof

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- Step 2. Bound the cost of the **whole** solution:

$$\begin{aligned} c_G(D) &\leq 11 \cdot f_n \cdot \sum_{\mathcal{G}} \sum_{k=w(A_{\mathcal{G}})+1}^{w(\mathcal{G})} c\left(S^*_{\lfloor k/2 \rfloor} \cap \mathcal{G}\right) \\ &\leq 11 \cdot f_n \cdot \sum_{k=0}^{w(G)} c\left(S^*_{\lfloor k/2 \rfloor}\right) \leq 22 \cdot f_n \cdot \mathtt{OPT}(G). \end{aligned}$$

Thank you for your attention!

Questions?

Bibliography I

[Ang18] Haris Angelidakis. "Shortest path queries, graph partitioning and covering problems in worst and beyond worst case settings". In: *ArXiv* abs/1807.09389 (2018). URL: https://api.semanticscholar.org/CorpusID:51718679.

[BK22] Benjamin Berendsohn and László Kozma. "Splay trees on trees". In: Jan. 2022, pp. 1875–1900. ISBN: 978-1-61197-707-3. DOI: 10.1137/1.9781611977073.75.

[Ber+22] Benjamin Berendsohn et al. *Fast approximation of search trees on trees with centroid trees*. Sept. 2022. DOI: 10.48550/arXiv.2209.08024.

[CC17] Moses Charikar and Vaggos Chatziafratis. "Approximate hierarchical clustering via sparsest cut and spreading metrics". In: *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*. SODA '17. Barcelona, Spain: Society for Industrial and Applied Mathematics, 2017, pp. 841–854.

[Cic+12] Ferdinando Cicalese et al. "The binary identification problem for weighted trees". In: *Theoretical Computer Science* 459 (2012), pp. 100–112. ISSN: 0304-3975. DOI: https://doi.org/10.1016/j.tcs.2012.06.023.

[Cic+16] Ferdinando Cicalese et al. "On the tree search problem with non-uniform costs". In: *Theoretical Computer Science* 647 (2016), pp. 22–32. ISSN: 0304-3975. DOI: https://doi.org/10.1016/j.tcs.2016.07.019.

[Coh+19] Vincent Cohen-addad et al. "Hierarchical Clustering: Objective Functions and Algorithms". In: *J. ACM* 66.4 (June 2019). ISSN: 0004-5411. DOI: 10.1145/3321386. URL: https://doi.org/10.1145/3321386.

Bibliography II

[Das16] Sanjoy Dasgupta. “A cost function for similarity-based hierarchical clustering”. In: *Proceedings of the Forty-Eighth Annual ACM Symposium on Theory of Computing*. STOC ’16. Cambridge, MA, USA: Association for Computing Machinery, 2016, pp. 118–127. ISBN: 9781450341325. DOI: 10.1145/2897518.2897527. URL: https://doi.org/10.1145/2897518.2897527.

[Der06] Dariusz Dereniowski. “Edge ranking of weighted trees”. In: *Discrete Applied Mathematics* 154.8 (2006), pp. 1198–1209. ISSN: 0166-218X. DOI: https://doi.org/10.1016/j.dam.2005.11.005.

[Der+17] Dariusz Dereniowski et al. “Approximation Strategies for Generalized Binary Search in Weighted Trees”. In: *44th International Colloquium on Automata, Languages, and Programming (ICALP 2017)*. Ed. by Ioannis Chatzigiannakis et al. Vol. 80. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2017, 84:1–84:14. ISBN: 978-3-95977-041-5. DOI: 10.4230/LIPIcs.ICALP.2017.84.

[FHL05] Uriel Feige, MohammadTaghi Hajiaghayi, and James R. Lee. “Improved approximation algorithms for minimum-weight vertex separators”. In: *Proceedings of the Thirty-Seventh Annual ACM Symposium on Theory of Computing*. STOC ’05. Baltimore, MD, USA: Association for Computing Machinery, 2005, pp. 563–572. ISBN: 1581139608. DOI: 10.1145/1060590.1060674. URL: https://doi.org/10.1145/1060590.1060674.

[GHT12] Archontia C. Giannopoulou, Paul Hunter, and Dimitrios M. Thilikos. “LIFO-search: A min–max theorem and a searching game for cycle-rank and tree-depth”. In: *Discrete Applied Mathematics* 160.15 (2012), pp. 2089–2097. ISSN: 0166-218X. DOI: https://doi.org/10.1016/j.dam.2012.03.015. URL: https://www.sciencedirect.com/science/article/pii/S0166218X12001199.

Bibliography III

[Høg24] Svein Høgemo. “Tight Approximation Bounds on a Simple Algorithm for Minimum Average Search Time in Trees”. In: *ArXiv* abs/2402.05560 (2024). URL: https://api.semanticscholar.org/CorpusID:267547530.

[Jac+10] Tobias Jacobs et al. “On the Complexity of Searching in Trees: Average-Case Minimization”. In: *Automata, Languages and Programming*. Ed. by Samson Abramsky et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 527–539. ISBN: 978-3-642-14165-2.

[KMS95] Meir Katchalski, William McCuaig, and Suzanne Seager. “Ordered colourings”. In: *Discrete Mathematics* 142.1 (1995), pp. 141–154. ISSN: 0012-365X. DOI: https://doi.org/10.1016/0012-365X(93)E0216-Q. URL: https://www.sciencedirect.com/science/article/pii/0012365X93E0216Q.

[Knu73] Donald Knuth. *The Art Of Computer Programming, vol. 3: Sorting And Searching*. Addison-Wesley, 1973, pp. 391–392.

[LY98] Tak Wah Lam and Fung Ling Yue. “Edge ranking of graphs is hard”. In: *Discrete Applied Mathematics* 85.1 (1998), pp. 71–86. ISSN: 0166-218X. DOI: https://doi.org/10.1016/S0166-218X(98)00029-8.

[NO06] Jaroslav Nešetřil and Patrice Ossona de Mendez. “Tree-depth, subgraph coloring and homomorphism bounds”. In: *European Journal of Combinatorics* 27.6 (2006), pp. 1022–1041. ISSN: 0195-6698. DOI: https://doi.org/10.1016/j.ejc.2005.01.010. URL: https://www.sciencedirect.com/science/article/pii/S0195669805000570.

[OP06] Krzysztof Onak and Pawel Parys. “Generalization of Binary Search: Searching in Trees and Forest-Like Partial Orders”. In: *2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS’06)*. 2006, pp. 379–388. DOI: 10.1109/FOCS.2006.32.

Bibliography IV

[Pot88] Alex Pothen. *The Complexity of Optimal Elimination Trees*. Technical Report CS-88-16. Penn State University CS-88-16. Pennsylvania State University, Department of Computer Science, Apr. 1988.