Searching in Graphs

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Binary Search

Binary Search - a classical strategy used to efficiently locate a hidden target element t in a linearly ordered set S using $O(\log n)$ comparison operations.

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median

$\downarrow$

1	2	4	5	7	10	15	16	18	21	22	25	31	32

lesser than median — greater than median

Figure: Example of a sorted array containing 14 elements.

Searching in Graphs

Easy to generalize to trees: A **query** to a vertex v returns information whether v is the target, and if not, which connected component of $G - v$ contains t. The question is:

What is the best strategy of searching in a graph?

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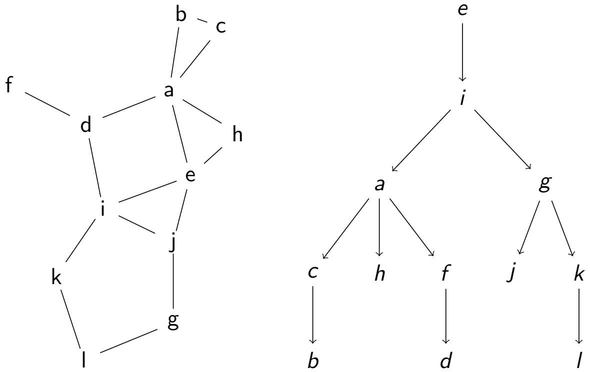

Figure: Sample input graph a decision tree for it.

Our setup

We consider the following variant of this problem:

Graph Search Problem (GSP)

Input: Tree G, a query cost function $c : V(G) \to \mathbb{N}$ and a weight function $w : V(G) \to \mathbb{N}$.

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We consider the following variant of this problem:

Graph Search Problem (GSP)

Input: Tree G, a query cost function $c : V(G) \to \mathbb{N}$ and a weight function $w : V(G) \to \mathbb{N}$.

Output: A decision tree D minimizing the weighted average search cost:

$$c_G(D) = \sum_{x \in V(G)} w(x) \cdot \sum_{q \in Q_G(D,x)} c(q)\,.$$

where $Q_G(D, x)$ denotes the sequence of queries performed along the unique path in D from the root $r(D)$ to x.

Why do we care?

Useful in:

1. Scheduling of parallel database join operations,
2. Automated bug detection in computer code,
3. Parallel Cholesky factorization of matrices,
4. Hierarchical clustering of data,
5. Parallel assembly of multi-part products from their components.

Why do we care?

Our setup:

1. Average case - it is natural to assume that the search strategies we design are intended to be used repeatedly.
2. Weight function - some vertices may serve as targets more frequently than the others.
3. Query costs - performing a query may require significant resources, such as time or money.
4. Have not yet been investigated.

Many names

- Binary Search [OP06; Der+17; DMS19; EKS16; DW22; DW24; DŁU25; DGW24; DŁU21; DGP23],
- Tree Search Problem [Jac+10; Cic+14; Cic+16],
- Binary Identification Problem [Cic+12],
- Ranking Colorings [Knu73; Der06; Der08; DK06; DN06; LY98],
- Ordered Colorings [KMS95],
- Elimination Trees [Pot88],
- Hub Labeling [Ang18],
- Tree-Depth [NO06; BDO23],
- Partition Trees [Høg24],
- Hierarchical Clustering [Das16; Coh+19; CC17],
- Search Trees on Trees [BK22; Ber+22],
- LIFO-Search [GHT12].

How to tackle the problem

Bad news: The Graph Search Problem is **NP-hard** even when restricted to bounded degree trees and bounded diameter diamter trees.

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We want to find an algorithm providing a good **approximation** for the problem:

- $(4 + \epsilon)$-approximation for trees.
- $O\left(\sqrt{\log n}\right)$-approximation for general graphs.

Weighted α-Separator Problem

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Input: Graph G, a cost function $c : V \to \mathbb{N}$, a weight function $w : V \to \mathbb{N}$ and a real number α.

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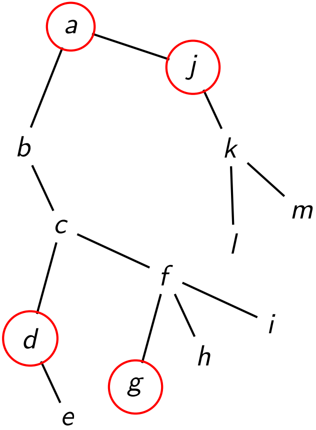
Input: Graph G, a cost function $c : V \to \mathbb{N}$, a weight function $w : V \to \mathbb{N}$ and a real number α.

Output: A set $S \subseteq V(G)$ called **separator** such that for every $H \in G - S$: $w(H) \leq w(G)/\alpha$ and $c(S)$ is minimized.

Example of a separator



	$w(v)$	$c(v)$
a	2	3
b	1	4
c	3	6
d	2	2
e	4	1
f	0	3
g	1	1
h	4	3
i	2	3
j	5	2
k	1	2
l	2	3
m	3	4

Figure: Sample input tree T and a weighted 3-separator (the circled vertices) of cost 8.

How to find the separator

Bad news again: The Weighted α-separator Problem is **NP-hard** as well, but there exists a biciriteria **FPTAS** for trees:

How to find the separator

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Theorem

Let S be an optimal weighted α-separator for (T, c, w, α). For any $\delta > 0$ there exists an algorithm `SeparatorFPTAS`*, which returns a separator S', such that:*

1. $c(S') \leq c(S)$.
2. $w(H) \leq \frac{(1+\delta)\cdot w(T)}{\alpha}$ *for every* $H \in T - S'$.
3. *The algorithm runs in* $O\left(n^3/\delta^2\right)$ *time.*

Min-Ratio Vertex Cut Problem

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Input: Graph $G = (V(G), E(G))$, the cost function $c : V \to \mathbb{N}$ and the weight function $w : V \to \mathbb{N}$.

Output: A partition (A, S, B) of $V(G)$ called *vertex-cut*, such that there are no $u \in A$ and $v \in B$ for which $uv \in E(G)$, minimizing the ratio:

$$\alpha_{c,w}(A, S, B) = \frac{c(S)}{w(A \cup S) \cdot w(B \cup S)}.$$

How to find the cut

Min-Ratio Vertex Cut Problem is also **NP-hard**. However, we invoke the following result of [FHL05]:

Theorem

Given a graph $G = (V(G), E(G))$, *the cost function* $c : V \to \mathbb{N}$ *and the weight function* $w : V \to \mathbb{N}$, *there exists a polynomial-time algorithm, which computes a partition* (A, S, B), *such that:*

$$\alpha_{c,w}(A, S, B) = O\left(\sqrt{\log n}\right) \cdot \alpha_{c,w}(G).$$

Notation

- $\mathcal{R}_D(G) = \{V(G_{D,v}) \,|\, v \in V(G)\}$ - the family of all candidate subsets of D in G.
- D^* - optimal decision tree.
- $\mathcal{L}_k^*$ - the subfamily of $\mathcal{R}_{D^*}(G)$ consisting of all maximal elements H of $\mathcal{R}_{D^*}(G)$ with $w(H) \leq k$. We call such a set the k-th *level* of $\mathtt{OPT}(G)$.
- $S_k^* = V(G) - \mathcal{L}_k^*$ - vertices belonging to the separator at the level $\mathcal{L}_k^*$. S_k^* forms a Weighted $w(G)\,/k$-separator of G.
- For any $H_1, H_2 \in \mathcal{R}_D(G)$, we have $H_1 \cup H_2 \neq \emptyset$ if and only if $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$, so $\mathcal{R}_D(G)$ is laminar. Thus, for any $k_1 \neq k_2$, we have $\mathcal{L}_{k_1}^* \cap \mathcal{L}_{k_2}^* = \emptyset$.

Example of the connection

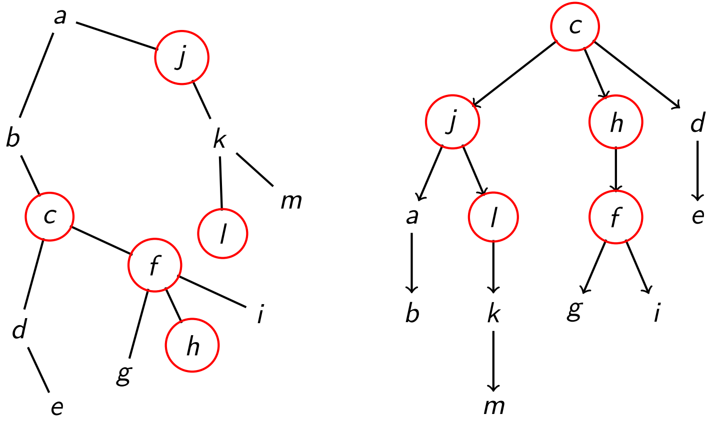

Figure: A 13/2-separator induced by the subtree of the decision tree consisting of circled vertices.

Basic lemmas

Lemma

Let $G_{D,v}$ be the candidate subgraph of G in which v is queried when using D. Then, $c_G(D) = \sum_{v \in V(G)} w(G_{D,v}) \cdot c(v)$.

Basic lemmas

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Lemma

$\mathit{OPT}(G) = \sum_{k=0}^{w(G)-1} c(S_k^*)$.

Proof.

Consider any vertex v. For every $0 \leq k < w(G_{D^*,v})$, $v \notin \bigcup_{H \in \mathcal{L}_k^*} H$, so $v \in S_k^*$ and the contribution of v to the cost is $w(G_{D^*,v}) \cdot c(v)$:

$$\sum_{k=0}^{w(G)-1} c(S_k^*) = \sum_{v \in V(G)} \sum_{k=0}^{w(G_{D^*,v})-1} c(v) = \mathit{OPT}(G).$$

□

Basic lemmas

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$$2 \cdot \mathit{OPT}(G) = 2 \cdot \sum_{k=0}^{w(T)-1} c(S_k^*) \geq \sum_{k=0}^{w(T)} c\left(S_{\lfloor k/2 \rfloor}^*\right).$$

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Lemma

Let $\mathcal{G}$ be any subgraph of G and $0 \leq \beta \leq 1$. Then:

$$\beta \cdot w(\mathcal{G}) \cdot c\left(S^*_{\lfloor w(\mathcal{G})/2 \rfloor} \cap \mathcal{G}\right) \leq \sum_{k=(1-\beta)w(\mathcal{G})+1}^{w(\mathcal{G})} c\left(S^*_{\lfloor k/2 \rfloor} \cap \mathcal{G}\right).$$

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Theorem

For any $\epsilon > 0$ there exists an $(4+\epsilon)$-approximation algorithm for the Tree Search Problem running in $O\left(n^4/\epsilon^2\right)$ time.

The algorithm

proc $\texttt{DecisionTree}(T, c, w, \epsilon)$:

1. $S_T \leftarrow \texttt{SeparatorFPTAS}\left(T, c, w, \alpha = 2, \delta = \frac{\epsilon}{4+\epsilon}\right)$.
2. $D_T \leftarrow$ arbitrary partial decision tree for T, built from vertices of S_T.
3. For each $H \in T - S_T$:
   3.1 $D_H \leftarrow \texttt{DecisionTree}(H, c, w, \epsilon)$.
   3.2 Hang D_H in D_T below the last query to $v \in N_T(H)$.
4. Return D_T.

Structure of the solution

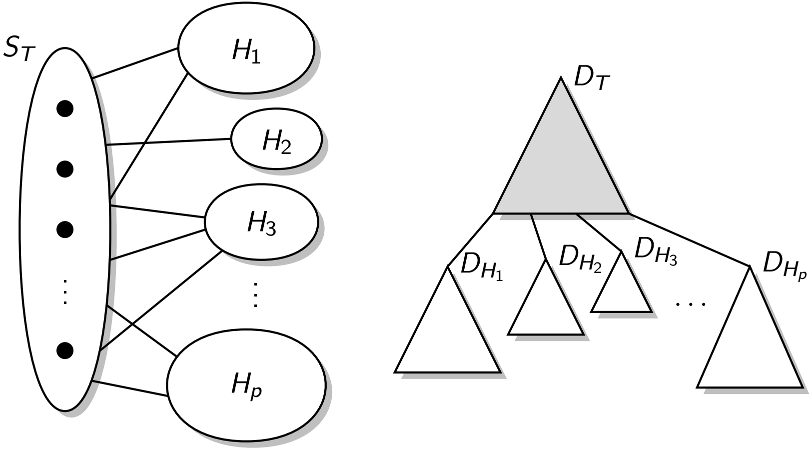

Figure: The separator S_T produced by the algorithm and the structure of the decision tree built using S_T.

Bounding the cost of a single recurrence call

- $\mathcal{T}$ - subtree of T for which the procedure was called.

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$$w(\mathcal{T}) \cdot c(S_{\mathcal{T}})$$

$$\leq w(\mathcal{T}) \cdot c(S^*_{\mathcal{T}}) \leq \frac{2}{1-\delta} \cdot \sum_{k=\frac{1+\delta}{2} \cdot w(\mathcal{T})+1}^{w(\mathcal{T})} c\left(S^*_{\lfloor k/2 \rfloor} \cap \mathcal{T}\right).$$

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\begin{aligned}
c_T(D) &\leq \sum_{\mathcal{T}} w(\mathcal{T}) \cdot c(S_{\mathcal{T}}) \\
&\leq \frac{2}{1-\delta} \cdot \sum_{\mathcal{T}} \sum_{k=\frac{1+\delta}{2}\cdot w(\mathcal{T})+1}^{w(\mathcal{T})} c\left(S^*_{\lfloor k/2 \rfloor} \cap \mathcal{T}\right) \\
&\leq \frac{2}{1-\delta} \cdot \sum_{k=0}^{w(T)} c\left(S^*_{\lfloor k/2 \rfloor}\right) \leq \frac{4}{1-\delta} \cdot \mathtt{OPT}(T) \\
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- $1/\delta = (4+\epsilon)/\epsilon = 4/\epsilon + 1$ so the running time is $O\left(n^4/\epsilon^2\right)$.

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Theorem

Let f_n be the approximation ratio of any polynomial time algorithm for the Min-Ratio Vertex Cut Problem. Then, there exists an $O(f_n)$-approximation algorithm for the GSP, running in polynomial time.

The algorithm

proc `DecisionTree`(T, c, w, ϵ):

1. $A_G, S_G, B_G \leftarrow$ `AlgorithmMinCut`(G, c, w).
2. $D_G \leftarrow$ arbitrary partial decision tree for G, built from vertices of S_G.
3. For each $H \in G - S_G$:
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Key technical lemma

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Lemma

Let $\mathcal{H} = \mathcal{G} - S^*_{\mathcal{G}}$ and let $\lambda = 6 + 2\sqrt{5}$ be the unique, positive solution of the equation $\frac{1}{4} - \frac{1}{2\sqrt{\lambda}} = \frac{1}{\lambda}$. Then, we can partition $\mathcal{H}$ into two sets, $\mathcal{A}$ and $\mathcal{B}$ such that for $A = \bigcup_{H \in \mathcal{A}} V(H)$ and $B = \bigcup_{H \in \mathcal{B}} V(H)$, we have:

$$w\left(A \cup S^*_{\mathcal{G}}\right) \cdot w\left(B \cup S^*_{\mathcal{G}}\right) \geq w(\mathcal{G})^2 / \lambda.$$

Key technical lemma

Proof.

There are two cases:

1. $w\left(S^*_{\mathcal{G}}\right) \geq w(\mathcal{G})/\sqrt{\lambda}$. Take arbitrary partition $\mathcal{A}, \mathcal{B}$ of $\mathcal{H}$. We have: $w\left(A \cup S^*_{\mathcal{G}}\right) \cdot w\left(B \cup S^*_{\mathcal{G}}\right) \geq w\left(S^*_{\mathcal{G}}\right)^2 \geq w(\mathcal{G})^2/\lambda$.

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2. $w\left(S^*_{\mathcal{G}}\right) \leq w(\mathcal{G})/\sqrt{\lambda}$. For any $\mathcal{A}, \mathcal{B}$, $\frac{w(A \cup B)}{w(\mathcal{G})} \geq 1 - \frac{1}{\sqrt{\lambda}}$. Pick $\mathcal{A}, \mathcal{B}$, so that $w(A) \geq w(B) \geq \left(\frac{1}{2} - \frac{1}{\sqrt{\lambda}}\right) \cdot w(\mathcal{G})$ (always possible as $\frac{1}{2} - \frac{1}{\sqrt{\lambda}} > 0$ and for each H, $w(H) \leq w(\mathcal{G})/2$):

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1. $w\left(S_{\mathcal{G}}^{*}\right) \geq w(\mathcal{G}) / \sqrt{\lambda}$. Take arbitrary partition $\mathcal{A}, \mathcal{B}$ of $\mathcal{H}$. We have: $w\left(A \cup S_{\mathcal{G}}^{*}\right) \cdot w\left(B \cup S_{\mathcal{G}}^{*}\right) \geq w\left(S_{\mathcal{G}}^{*}\right)^{2} \geq w(\mathcal{G})^{2} / \lambda$.
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$$\begin{aligned} w\left(A \cup S_{\mathcal{G}}^{*}\right) \cdot w\left(B \cup S_{\mathcal{G}}^{*}\right) &\geq w(A) \cdot w(B) \\ &\geq \left(\left(1 - 1/\sqrt{\lambda}\right) \cdot w(\mathcal{G}) - w(B)\right) \cdot w(B) \\ &\geq w(\mathcal{G})^{2}/2 \cdot \left(1/2 - 1/\sqrt{\lambda}\right) = w(\mathcal{G})^{2}/\lambda. \end{aligned}$$

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- $(A, S^*_{\mathcal{G}}, B)$ in the above lemma is a vertex cut of $\mathcal{G}$. We have:

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$$\begin{aligned}\alpha_{c,w}(\mathcal{G}) &\leq \alpha_{c,w}\left(A, S^*_{\mathcal{G}}, B\right) \\ &= \frac{c\left(S^*_{\mathcal{G}}\right)}{w\left(A \cup S^*_{\mathcal{G}}\right) \cdot w\left(B \cup S^*_{\mathcal{G}}\right)} \leq \frac{\lambda \cdot c\left(S^*_{\mathcal{G}}\right)}{w(\mathcal{G})^2}.\end{aligned}$$

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- Let $(A_{\mathcal{G}}, S_{\mathcal{G}}, B_{\mathcal{G}}) = \texttt{AlgorithmMinCut}(\mathcal{G}, c, w)$, assume without loss of generality that $w(A_{\mathcal{G}}) \geq w(B_{\mathcal{G}})$. We have:

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$$\alpha_{c,w}(A_{\mathcal{G}}, S_{\mathcal{G}}, B_{\mathcal{G}}) = \frac{c(S_{\mathcal{G}})}{w(A_{\mathcal{G}} \cup S_{\mathcal{G}}) \cdot w(B_{\mathcal{G}} \cup S_{\mathcal{G}})} \leq f_n \cdot \frac{\lambda \cdot c\left(S^*_{\mathcal{G}}\right)}{w(\mathcal{G})^2}.$$

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- $(1 - \beta) \cdot w(\mathcal{G}) = w(A_{\mathcal{G}})$. We have:

$$
\begin{aligned}
w(\mathcal{G}) \cdot c(S_{\mathcal{G}}) &\le \lambda \cdot f_n \cdot \frac{w(A_{\mathcal{G}} \cup S_{\mathcal{G}}) \cdot w(B_{\mathcal{G}} \cup S_{\mathcal{G}})}{w(\mathcal{G})} \cdot c\left(S^*_{\mathcal{G}}\right) \\
&\le \lambda \cdot f_n \cdot w(B_{\mathcal{G}} \cup S_{\mathcal{G}}) \cdot c\left(S^*_{\mathcal{G}}\right) \\
&\le \lambda \cdot f_n \cdot \sum_{k=w(A_{\mathcal{G}})+1}^{w(\mathcal{G})} c\left(S^*_{\lfloor k/2 \rfloor} \cap \mathcal{G}\right).
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c_G(D) &\leq \sum_{\mathcal{G}} w(\mathcal{G}) \cdot c(S_{\mathcal{G}}) \\
&\leq \lambda \cdot f_n \cdot \sum_{\mathcal{G}} \sum_{k=w(A_{\mathcal{G}})+1}^{w(\mathcal{G})} c\left(S^*_{\lfloor k/2 \rfloor} \cap \mathcal{G}\right) \\
&\leq \lambda \cdot f_n \cdot \sum_{k=0}^{w(G)} c\left(S^*_{\lfloor k/2 \rfloor}\right) \\
&\leq 2 \cdot \lambda \cdot f_n \cdot \mathtt{OPT}(G) = \left(12 + 4\sqrt{5}\right) \cdot f_n \cdot \mathtt{OPT}(G)\,.
\end{aligned}
$$

Thank you for your attention!

Questions?

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