GDAŃSK UNIVERSITY
OF TECHNOLOGY

Experimental analysis of binary search models in graphs

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GDAŃSK UNIVERSITY OF TECHNOLOGY | Basic information

The aim: Experimental analysis of selected generalized binary search problems. The aim of the analysis is to verify the hypotheses regarding the effectiveness of search algorithms.

Schedule:

- 2024.01 – 2024.06: Researching and selection of the query model.
- 2024.06 – 2025.05: Development and formal analysis of the proposed algorithms.
- 2025.06 – 2025.10: Selection of models and algorithms for comparison, implementation, thesis writing.
- 2025.11 – 2025.12: execution of experiments, analysis and interpretation of the results, finalization of the thesis.

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State of the work

- Implementation, about 60% complete:
  - Language: **python**,
  - Libraries: **networkx**,
  - Environment: **PyCharm**,
  - Versioning: **git** + **github**, https://github.com/MSzyfel/Binary-Search.
- Thesis, about 80% complete:
  - Language: **LaTeX**,
  - Environment: **Visual Studio Code**,
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- What is left:
  - Experiments,
  - Advanced data generation,
  - Optimization,
  - Bug fixing,
  - Data visualization.

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Binary Search

median

↓

1	2	4	5	7	10	15	16	18	21	22	25	31	32

lesser than median

greater than median

Figure: Example of a sorted array containing 14 elements.

Generalized binary search

Definition

A **searcher** is required to find a hidden **target** vertex x in a graph G. To do so, they iteratively perform **queries** to an **oracle**, each about a chosen vertex v. After each such call, the oracle responds whether the target was found and if not, the searcher receives as a reply the connected component of $G - v$ containing the target.

A further generalization is to associate with each vertex a **cost** function $c : V(G) \to \mathbb{R}_{\geq 0}$ representing the time required to query a given vertex.

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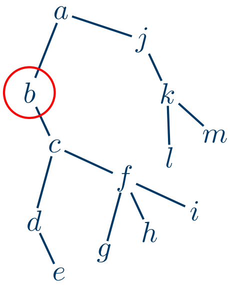

Figure: Query to b

Figure: Query to f

Figure: Query to d

Figure: Query to c

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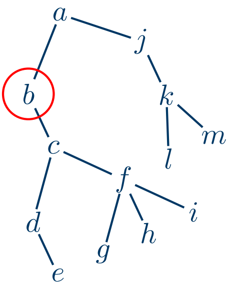

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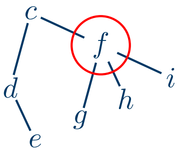

Figure: Query to f

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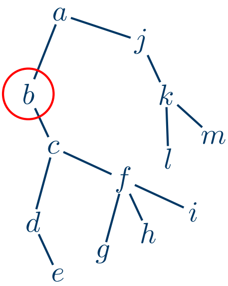

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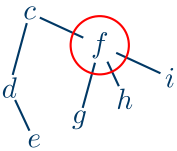

Figure: Query to f

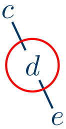

Figure: Query to d

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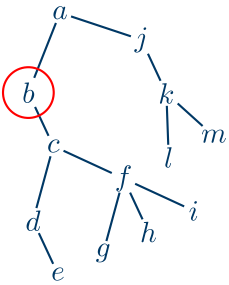

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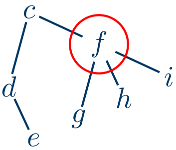

Figure: Query to f

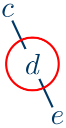

Figure: Query to d

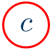

Figure: Query to c

Classes of graphs considered

There are three main classes of graphs to be considered:

- **Paths** - equivalent to searching in a sorted array.
- **Trees** - The most extensively studied model. **Our choice**.
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Definition

A decision tree:

- $D = (V(D), E(D))$, $V(D) = V(T)$ are vertices and $E(D)$ are edges of D.
- $Q_D(T, x)$ - sequence of queries performed in order to find x.
- Cost of D in (T, c):

$$\texttt{COST}_D(T, c) = \max_{x \in V(T)} \left\{ \sum_{q \in Q_D(T,x)} c(q) \right\}.$$

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Example of decision tree

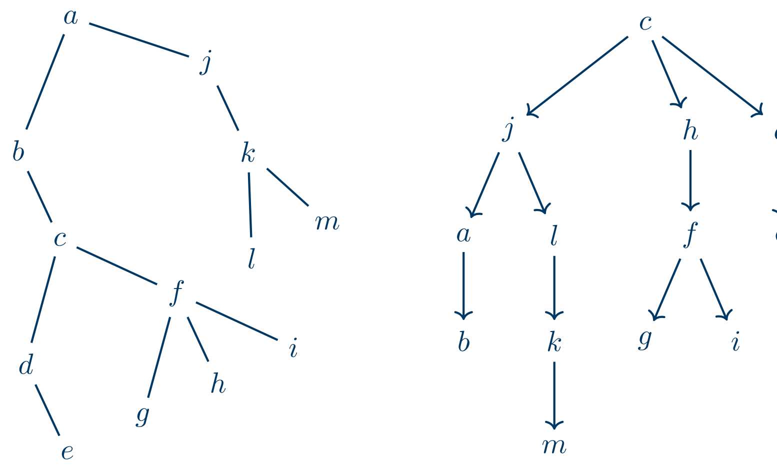

Figure: Sample input tree and a decision tree for it.

Problem statement

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Given a tree T and weight function c, the **Tree Search Problem** consists of finding a decision tree D, such that $\texttt{COST}_D(T, c) = \texttt{OPT}(T, c)$.

Unluckily, the Tree Search Problem is **strongly NP-Hard** even when restricted to binary trees and spiders of diameter at most 6. However, one can find **approximate** solutions.

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Claims

Theorem

Let $c(v) = 1$ for every $v \in V(T)$. There exists an exact algorithm called `RankingBasedDT` *for the Tree Search Problem running in linear time, such that the resulting decision tree uses at most $\lfloor \log n \rfloor + 1$ queries.*

Theorem

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Implementation results I

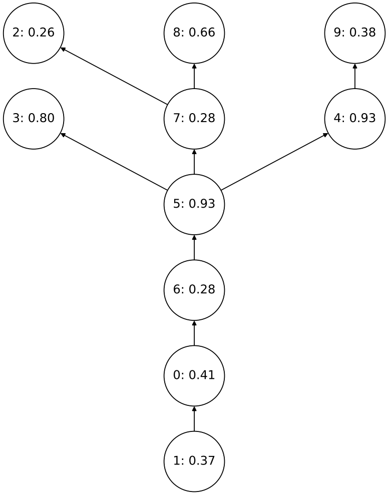

Figure: Input tree of size 10.

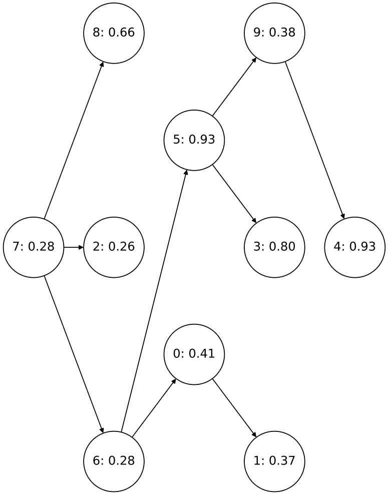

Figure: Decision tree of cost 2.8.

Implementation results II

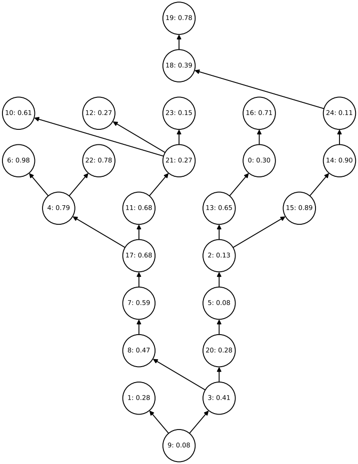

Figure: Input tree of size 25.

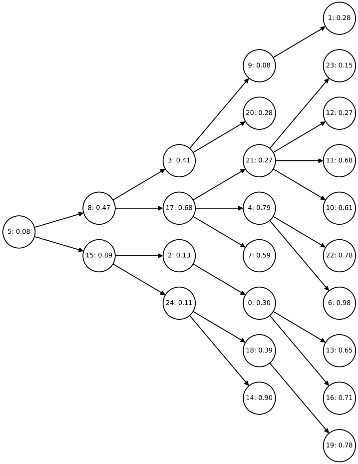

Figure: Decision tree of cost 3.

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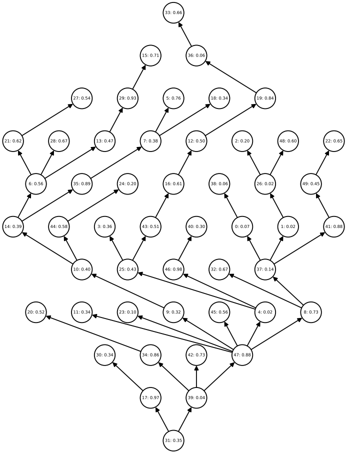

Figure: Input tree of size 50.

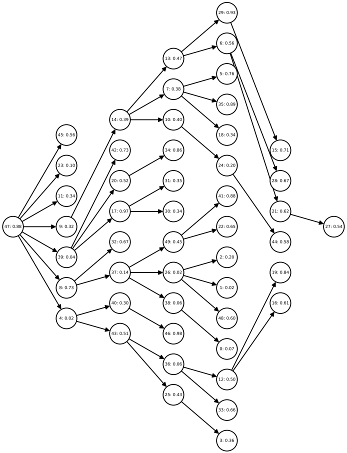

Figure: Decision tree of cost 3.78.

Implementation results IV

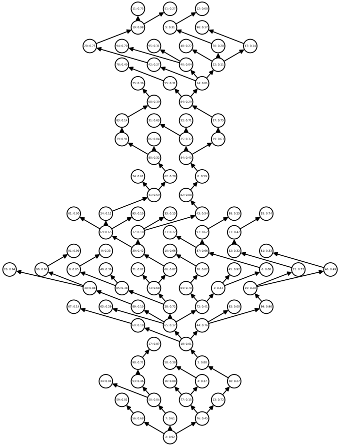

Figure: Input tree of size 100.

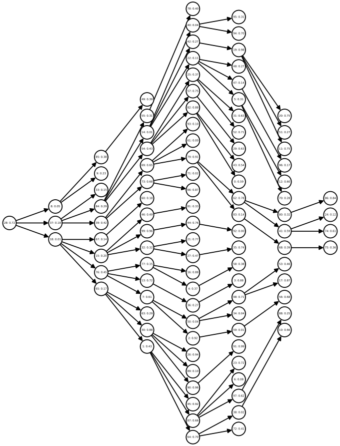

Figure: Decision tree of cost 4.85.

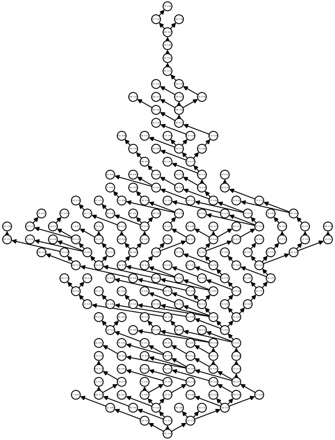

Figure: Input tree of size 200.

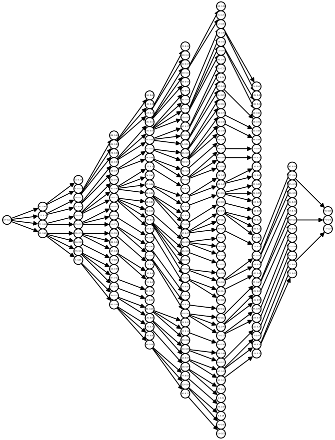

Figure: Decision tree of cost 4.66

proc `ApproxDT` (T):

1. Set a global parameter $k = 2^{\lfloor \sqrt{\log n} \rfloor + 2}$.
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3. Else, find a set $\mathcal{X} \subseteq V(T)$, such that $|\mathcal{X}| \leq k$ and for every $H \in T - \mathcal{X}$, $n(H) \leq n(T) / 2^{\sqrt{\log n}}$.
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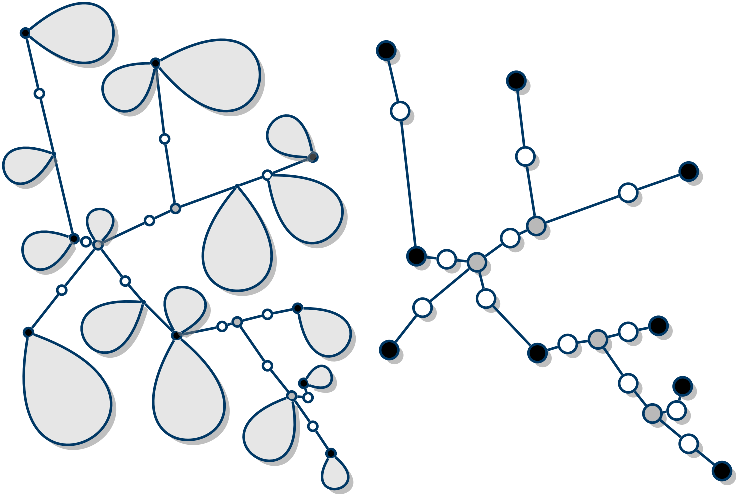
Auxiliarry tree structure



Solution structure

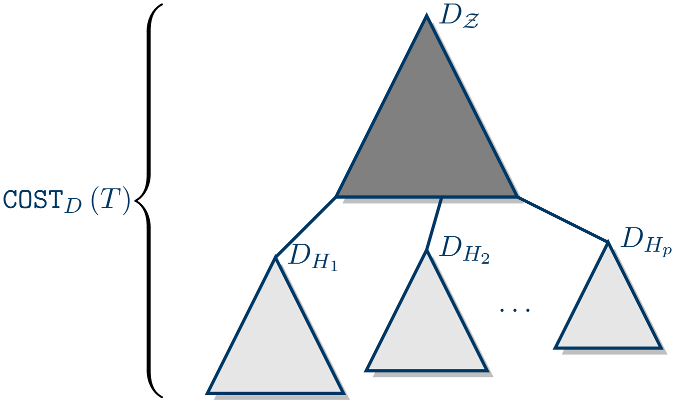


Figure: Structure of the decision tree D

Approximation guarantee

Since $\texttt{COST}_{D_{\mathcal{Z}}} \leq 2 \cdot \texttt{OPT}(T)$, we have:

$$\begin{aligned}
\texttt{COST}_D(T) &\leq \texttt{COST}_{D_{\mathcal{Z}}} + \max_{H \in T - \mathcal{Z}} \{\texttt{COST}_{D_H}(H)\} \\
&\leq 2 \cdot \log_{2^{\sqrt{\log n}}}(n) \cdot \texttt{OPT}(T) \\
&= \frac{2 \log n}{\sqrt{\log n}} \cdot \texttt{OPT}(T) \\
&= 2\sqrt{\log n} \cdot \texttt{OPT}(T).
\end{aligned}$$

Rounding trick

- Let $p \in \mathbb{N}$,
- Let $k = a/pn$, for some $a \in \mathbb{N}$.
- We define new cost function c', called **aligned cost function**:

$$c'(v) = \begin{cases} \lceil c(c) \rceil_k, & \text{if } c(v) > pk, \textit{ heavy vertex}, \\ \lceil c(c) \rceil_{\frac{1}{pn}}, & \text{otherwise, } \textit{light vertex}. \end{cases}$$

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Lemma

$$\mathtt{OPT}(T, c') \leq \left(1 + \frac{2}{p}\right) \cdot \mathtt{OPT}(T, c).$$

Lemma

There exists a decision tree D for (T, c'), such that:

1. $\mathtt{COST}_D(T, c') \leq \left(1 + \frac{3}{p}\right) \cdot \mathtt{OPT}(T, c')$
2. *Starting point of each heavy query is aligned to a multiple of c,*
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We call a decision tree fulfiling the requirements 2. and 3. of the above lemma an **aligned decision tree**.

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Lemmas

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Aligned cost

For any vertex $v \in V(T)$ and query $q \in Q_D(T, v)$ the **contribution** $\kappa_{T,c,k}(q, v)$ of u is defined as:

$$\kappa_{T,c}(q, v) = \begin{cases} 0, & \text{if its a light down response,} \\ c(q), & \text{otherwise.} \end{cases}$$

Then, the **aligned cost** of D is defined as:

$$\mathtt{COST}'_D(T, c', k) = \max_{v \in V(T)} \left\{ \sum_{q \in Q_D(T,v)} \kappa_{T,c',k}(q, v) \right\}.$$

Let $\mathtt{OPT}'(T, c', k)$ denote the optimal aligned cost.

Aligned cost

For any vertex $v \in V(T)$ and query $q \in Q_D(T, v)$ the **contribution** $\kappa_{T,c,k}(q, v)$ of u is defined as:

$$\kappa_{T,c}(q, v) = \begin{cases} 0, & \text{if its a light down response,} \\ c(q), & \text{otherwise.} \end{cases}$$

Then, the **aligned cost** of D is defined as:

$$\texttt{COST}'_D(T, c', k) = \max_{v \in V(T)} \left\{ \sum_{q \in Q_D(T,v)} \kappa_{T,c',k}(q, v) \right\}.$$

Let $\texttt{OPT}'(T, c', k)$ denote the optimal aligned cost.

Heavy module contraction

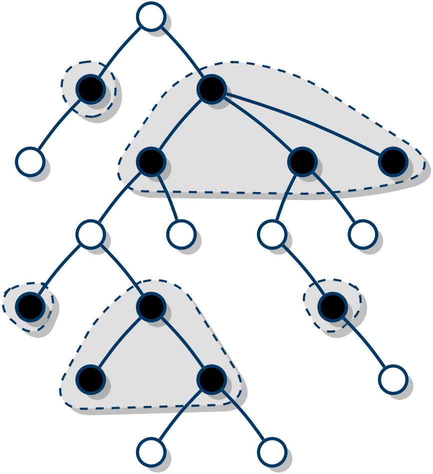

Figure: Input tree.

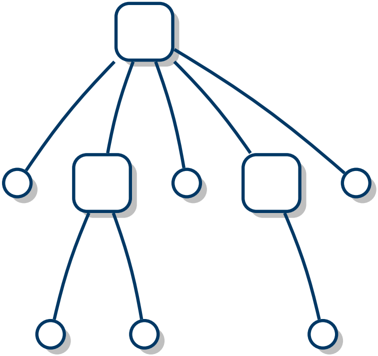

Figure: Input tree with heavy modules contracted.

Propositions

Proposition

Let T be a tree, c' an aligned cost function, $p \in \mathbb{N}$, k the box size, n the size of the original input tree, and d be the depth. There exists a `DPTimelinesCosts` procedure which calculates an optimal aligned decision tree D, running in $pn^{O(d)}$ time.

Proposition

Let T be a tree, c' be an aligned cost function, D_A be a decision tree for T and F_C be forest of decision trees for T with all heavy groups contracted, $p \in \mathbb{N}$ be a constant, $k \in \mathbb{R}_{>0}$ be the box size. There exists a polynomial time `MergeDTs` procedure which returns a decision tree of cost at most:

$$\texttt{COST}_D\left(T, c', k\right) = \texttt{OPT}'\left(T, c', k\right) + 2pk \cdot \texttt{COST}_{F_C}\left(T, 1\right).$$

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proc $\texttt{QPTAS}\,(T, \epsilon)$:

1. $p \leftarrow \lfloor 35/\epsilon \rfloor$,
2. $d \leftarrow p^2 \cdot (\lfloor \log n \rfloor + 1)$.
3. $k \leftarrow 0$.
4. Repeat the following steps until a decision tree D is found:
    1. $k \leftarrow k + \frac{1}{pn}$.
    2. For every $v \in V\,(T)$, if $c\,(v) > pk$, then $c'\,(v) \leftarrow \lceil c\,(v) \rceil_k$, $c'\,(v) \leftarrow \lceil c\,(v) \rceil_{\frac{1}{pn}}$ otherwise.
    3. $D_A \leftarrow \texttt{DPTimelinesCosts}\,(T, c', p, k, n, d)$
    4. If $D_A \neq \emptyset$:
        1. $T_C \leftarrow T$ with all heavy modules contracted.
        2. $D_C \leftarrow \texttt{RankingBasedDT}\,(T_C)$.
        3. $D \leftarrow \texttt{MergeDTs}\,(T, D_A, D_C)$.
5. Return D.

Quality guarantee

Let k' be the value of k for which D was found. Since: $k' \leq \frac{\mathtt{OPT}'(T,c',k')}{d} = \frac{\mathtt{OPT}'(T,c')}{p^2 \cdot (\lfloor \log n \rfloor + 1)}$, we have that:

$$\begin{aligned}
\mathtt{COST}_D\left(T, c'\right) &\leq \mathtt{OPT}'\left(T, c'\right) + 2pk' \cdot \left(\lfloor \log n \rfloor + 1\right) \\
&\leq \mathtt{OPT}'\left(T, c'\right) + 2p \cdot \left(\lfloor \log n \rfloor + 1\right) \cdot \frac{\mathtt{OPT}'\left(T, c'\right)}{p^2 \cdot \left(\lfloor \log n \rfloor + 1\right)} \\
&\leq \left(1 + \frac{2}{p}\right) \cdot \mathtt{OPT}'\left(T, c'\right) \\
&\leq \left(1 + \frac{2}{p}\right) \cdot \left(1 + \frac{2}{p}\right) \cdot \left(1 + \frac{3}{p}\right) \cdot \mathtt{OPT}\left(T, c\right) \\
&\leq \left(1 + \frac{35}{p}\right) \cdot \mathtt{OPT}\left(T, c\right) = \left(1 + \frac{35}{\left\lceil \frac{35}{\epsilon} \right\rceil}\right) \cdot \mathtt{OPT}\left(T, c\right) \\
&\leq \left(1 + \epsilon\right) \cdot \mathtt{OPT}\left(T, c\right)
\end{aligned}$$

proc `MergeDTs` (T, D_A):

1. If F_C is connected, $r = r(F_C)$.
2. Else, $r = r(D_A)$.
3. $D \leftarrow$ decision tree with root r.
4. For each $T' \in T - r$:
    1. Let forest F'_C be F_C restricted to T'_C
    2. Let D'_A be D restricted to T'.
    3. $D' \leftarrow \texttt{MergeDTs}(T', D'_A, F'_C)$
    4. Hang D' below r in D.
5. Return D.

Boxed decision tree

Definition

Boxed decision tree:

1. $D = (V(D), E(D), u, l)$, $V(D)$ - nodes of D, called *boxes*, $E(D)$ - edges of D, $u : V(T) \times V(D) \to \{0, 1/pn, 2/pn, \ldots, k\}$ - *usage* function and $l : V(D) \to \{0, 1/pn, 2/pn, \ldots, k\}$ - *load* function.
2. For every $b \in V(D)$, $Q(b) = \{v \in V(T) \,|\, u(v, b) > 0\}$ - *query assignment*, vertices of T, such that queries to them overlap with b.
3. All boxes containing v form a left path. Additionally, for rach interior box b, $l(b) = 0$ and $u(v, b) = k$.
4. For every $b \in V(D)$, $l(b) + \sum_{v \in V(T)} u(v, b) \leq k$ and for every $v \in V(T)$, either $\sum_{b \in V(D)} (v, b) = 0$ or $\sum_{b \in V(D)} (v, b) = c'(v)$.
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GDAŃSK UNIVERSITY OF TECHNOLOGY | **Boxed decision tree**

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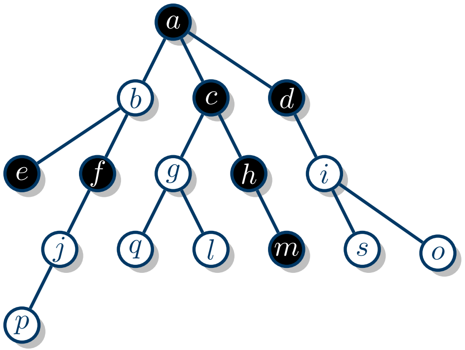

Figure: Input tree.

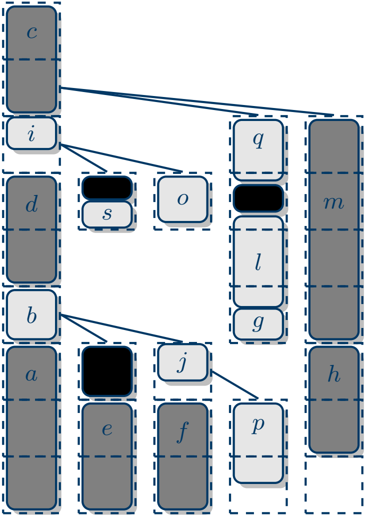

Figure: Boxed decision tree.

Boxline

Definition

Boxline: $B = \langle (b_1, \tau_1), (b_2, \tau_2), \dots, (b_d, \tau_d) \rangle$, b_j - box, such that $Q(b_j) = \emptyset$, τ_j - boolean flag.

Definition

Left box-path of D:

1. $B_D = \langle q_1, f_1, (q_2, f_2), \dots, (q_h, f_h) \rangle$, q_j - box f_j - boolean flag, obtained by traversing boxes of D towards left. For each such box b_j, $q_j = l(b) + \sum_{v \in Q(b)} q(v, b)$, whereas f_j denotes whether there exists a **transcending** query in $Q(b_j)$, i. e.: $v \in Q(b_j)$ such that $v \in Q(b_{j+1})$.
2. Decision tree$D$ with a left box-path B_D is **box-compatible** with boxline B $(h \leq d)$, if $l(q_j) \geq l(b_j)$ and $\tau_j \implies f_j$.

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Useful operations I

- Puting a query to vertex v at s-th slot of a box b:
  1. $\sigma(v) \leftarrow c(v)$:
  2. While $\sigma(v) > 0$:
     1. $u(v, b) \leftarrow \min\{k - s/pn, \sigma(v)\}$.
     2. $\sigma(v) \leftarrow \sigma(v) - u(v, b)$.
     3. $b \leftarrow$ left child of b.

If such operation violates the definition of D or query to v transcendents any box b_j, such that τ_j. we mark D as **conflicted**.

Useful operations II

- Bipartitioning of B. A bipartition of B consists of (B_1, B_2) such that:
  - $|B| = |B_1| = |B_2|$.
  - $l\left(b_{1,j}\right) + l\left(b_{2,j}\right) - k = l\left(b_j\right)$.
  - $\left(\tau_{1,j} \wedge \tau_{2,j} \iff \tau_j\right)$.
  - $\left(\tau_{1,j} \vee \tau_{2,j}\right)$.

Useful operations III

- Rotating a decision tree D around vertex v:
  1. $q_h \leftarrow$ the box containing the end of the query to v.
  2. Sort vertices whose queries start in $Q(q_h)$ according to c.
  3. Create box q.
  4. Move queries from $Q(q_h)$ to $Q(q)$, so that all queries after v are in $Q(q)$,
  5. Hang q'_h as a right child of q_h.
  6. Rehang left child of q_h as a left child of q.

Cases of DP

Figure: No children

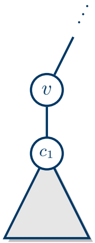

Figure: One child

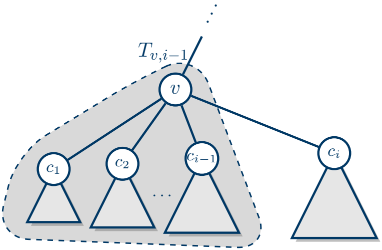

Figure: Many children

No children

1. For $1 \leq b \leq d$ and $0 \leq s \leq (k/pn$ **if** $c(v) > pk$ **else** $0)$:
   1. $D \leftarrow P$
   2. Try putting query to v at the s-th slot of q_b.
   3. If there are no conflicts in D.
      1. If $\texttt{COST}'_D(T_{v,i}, c, k) \leq dk$, then return D, else return $\emptyset$.

2. Return $\emptyset$.

One child

1. $\mathcal{D} \leftarrow \emptyset$.
2. For $1 \leq b \leq d$ and $0 \leq s \leq (k/pn$ **if** $c(v) > pk$ **else** $0)$:
    1. $D \leftarrow P$
    2. Try putting query to v at the s-th slot of q_b.
    3. If there are no conflicts in D.
        1. If $\texttt{COST}'_D(T_{v,i}, c, k) \leq dk$:
            1. $P' \leftarrow$ left box-path of D.
            2. $h \leftarrow$ index of the last box q_h occupied by the query to v.
            3. For $h < j \leq d$, $b'_j \leftarrow 0$, $t'_j \leftarrow$ False.
            4. $D' \leftarrow \texttt{DPTimelinesCosts}(T_{c_1}, c, P')$.
            5. Put query to v at he s-th slot of q_b.
            6. Rotate left path of D' around v.
            7. $\mathcal{D} \leftarrow \mathcal{D} \cup \{D$ and D' with their left paths aligned$\}$.
3. Return $\arg\min_{D \in \mathcal{D}} \{\texttt{COST}'_D(T_{v,i}, c, k)\}$.

Many children

1. $\mathcal{D} \leftarrow \emptyset$.
2. For each bipartition (B_1, B_2) of B:
   1. $D_1 \leftarrow \texttt{DPTimelinesCosts}\,(T_{v,i-1}, c, P_1)$.
   2. $h \leftarrow$ index the last box $q_{1,h}$ occupied by the query to v.
   3. For $h \leq j \leq d$, $b_{2,j} \leftarrow 0$, $t'_{2,j} \leftarrow$ False.
   4. $D_2 \leftarrow \texttt{DPTimelinesCosts}\,(T_{c_i}, c, P_2)$.
   5. Rotate left path of D' around v.
   6. $\mathcal{D} \leftarrow \mathcal{D} \cup \{D_1$ and $D_2$with their left paths aligned$\}$.
3. Return $\arg\min_{D \in \mathcal{D}} \{\texttt{COST}'_D\,(T_{v,i}, c, k)\}$.

Visualization

Figure: Boxline B

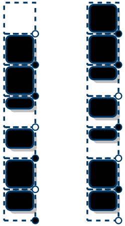

Figure: Bipartition of B

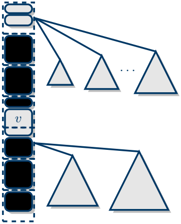

Figure: Decision tree D_1

Figure: Boxline B_2

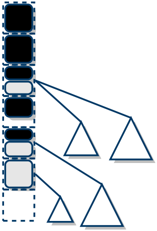

Figure: Decision tree D_2

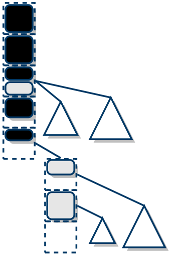

Figure: Rehanging step

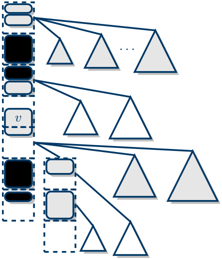

Figure: Resulting decision tree D