Analytical Solution of Asteroid ODEs

Eros433

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$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{-c_\mathrm{D}\,\rho_\mathrm{a}(z)}{2\,m} + g\sin(\theta) \\ \frac{\mathrm{d}m}{\mathrm{d}t} &= \frac{-c_\mathrm{H}\,\rho_\mathrm{a}(z)}{2\,Q} \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{g\cos(\theta)}{v} - \frac{c_\mathrm{L}\,\rho_{\mathrm{a}(z)}\,A\,v}{2\,m} - \frac{v\cos(\theta)}{R_\mathrm{P}+z} \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= -v\sin(\theta) \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{v\cos(\theta)}{1+\frac{z}{R_\mathrm{P}}} \\ \rho_a(z) &= \rho_0\,e^{-\frac{z}{H}} \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \left[\frac{7}{2}\,\alpha\frac{\rho_\mathrm{a}}{\rho_\mathrm{m}}\right]^{\frac{1}{2}} \\ \rho_\mathrm{a}(z)\,v^2 &= \sigma_0 \end{cases} \rightarrow \begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{-c_\mathrm{D}\,\rho_\mathrm{a}(z)}{2\,m} \\ \frac{\mathrm{d}m}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= -v\sin(\theta) \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= v\cos(\theta) \\ \rho_a(z) &= \rho_0\,e^{-\frac{z}{H}} \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= 0 \end{cases}$$

$\rightarrow m = \text{const.}, \theta = \text{const.}, r = \text{const.}$

$\{\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{-c_\mathrm{D}\,\rho_\mathrm{a}(z)\,A\,v^2}{2\,m}, \frac{\mathrm{d}z}{\mathrm{d}t} = -v\sin(\theta)\} \leftrightarrow \{\mathrm{d}t = \frac{2\,m}{-c_\mathrm{D}\,\rho_\mathrm{a}(z)\,A\,v^2}\mathrm{d}v, \mathrm{d}z = -v\sin(\theta)\mathrm{d}t\}$

$$\mathrm{d}z = \underbrace{\frac{2\,m\sin(\theta)}{c_\mathrm{D}\,A}}_{:=\zeta}\frac{1}{\rho_\mathrm{a}(z)}\frac{v}{v^2}\mathrm{d}v = \zeta\,\frac{\mathrm{d}v}{v}$$

$$\int\rho_\mathrm{a}(z)\mathrm{d}z = \zeta\int\frac{\mathrm{d}v}{v}$$

$$-H\,e^{-\frac{z}{H}} + \text{const.} = \zeta\ln(v)$$

$$\ln(v) = -\underbrace{\frac{H\,c_\mathrm{D}\,A}{2\,m\sin(\theta)}}_{=\text{const.}}\,e^{-\frac{z}{H}} + \text{const.}$$