Sec. 2.5 4(c,d), 28 , 36, 38

4. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) integers not divisible by 3

b) integers divisible by 5 but not by 7

c) the real numbers with decimal representations consisting of all 1s

d) the real numbers with decimal representations of all 1s or 9s

Key

4. **a)** This set is countable. The integers in the set are ± 1, ± 2, ± 4, ± 5, ± 7, and so on. We can list these numbers in the order $1, -1, 2, -2, 4, -4, 5, -5, 7, -7, \ldots$, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 1$, $2 \leftrightarrow -1$, $3 \leftrightarrow 2$, $4 \leftrightarrow -2$, $5 \leftrightarrow 4$, and so on.

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b) This is similar to part (a); we can simply list the elements of the set in order of increasing absolute value, listing each positive term before its corresponding negative: $5, -5, 10, -10, 15, -15, 20, -20, 25, -25, 30, -30, 40, -40, 45, -45, 50, -50, \ldots$.

c) This set is countable but a little tricky. We can arrange the numbers in a 2-dimensional table as follows:

$.\overline{1}$	.1	.11	.111	.1111	.11111	.111111	…
$1.\overline{1}$	1	1.1	1.11	1.111	1.1111	1.11111	…
$11.\overline{1}$	11	11.1	11.11	11.111	11.1111	11.11111	…
$111.\overline{1}$	111	111.1	111.11	111.111	111.1111	111.11111	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Thus we have shown that our set is the countable union of countable sets (each of the countable sets is one row of this table). Therefore by Exercise 27, the entire set is countable. For an explicit correspondence with the positive integers, we can zigzag along the positive-sloping diagonals as in Figure 3: $1 \leftrightarrow .\overline{1}$, $2 \leftrightarrow 1.\overline{1}$, $3 \leftrightarrow .1$, $4 \leftrightarrow .11$, $5 \leftrightarrow 1$, and so on.

d) This set is not countable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 5. All we need to do is choose $d_i = 1$ when $d_{ii} = 9$ and choose $d_i = 9$ when $d_{ii} = 1$ or d_{ii} is blank (if the decimal expansion is finite).

28. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable.

Key

28. We can think of $\mathbf{Z}^+ \times \mathbf{Z}^+$ as the countable union of countable sets, where the i^{th} set in the collection, for $i \in \mathbf{Z}^+$, is $\{(i, n) \mid n \in \mathbf{Z}^+\}$. The statement now follows from Exercise 27.

***36.** Show that there is a one-to-one correspondence from the set of subsets of the positive integers to the set real numbers between 0 and 1. Use this result and Exercises 34 and 35 to conclude that $\aleph_0 < |\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}|$. [*Hint:* Look at the first part of the hint for Exercise 35.]

Key

36. We can encode subsets of the set of positive integers as strings of, say, 5's and 6's, where the i^{th} symbol is a 5 if i is in the subset and a 6 otherwise. If we interpret this string as a real number by putting a 0 and a decimal point in front, then we have constructed a one-to-one function from $\mathcal{P}(\mathbf{Z}^+)$ to $(0,1)$. Also, we can construct a one-to-one function from $(0,1)$ to $\mathcal{P}(\mathbf{Z}^+)$ by sending the number whose binary expansion is $0.d_1d_2d_3\ldots$ to the set $\{i \mid d_i = 1\}$. Therefore by the Schröder-Bernstein theorem we have $|\mathcal{P}(\mathbf{Z}^+)| = |(0,1)|$. By Exercise 34, $|(0,1)| = |\mathbf{R}|$, so we have shown that $|\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}|$. (We already know from Cantor's diagonal argument that $\aleph_0 < |\mathbf{R}|$.) There is one technical point here. In order for our function from $(0,1)$ to $\mathcal{P}(\mathbf{Z}^+)$ to be well-defined, we must choose which of two equivalent expressions to represent numbers that have terminating binary expansions to use (for example, $0.10010\overline{1}$ versus $0.100110\overline{0}$); we can decide to always use the terminating form, i.e., the one ending in all 0's.)

***38.** Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable. [*Hint:* First set up a one-to-one correspondence between the set of real numbers between 0 and 1 and a subset of these functions. Do this by associating to the real number $0.d_1d_2\ldots d_n\ldots$ the function f with $f(n) = d_n$.]

Key

38. We know from Example 5 that the set of real numbers between 0 and 1 is uncountable. Let us associate to each real number in this range (including 0 but excluding 1) a function from the set of positive integers to the set $\{0,1,2,3,4,5,6,7,8,9\}$ as follows: If x is a real number whose decimal representation is $0.d_1d_2d_3\ldots$ (with ambiguity resolved by forbidding the decimal to end with an infinite string of 9's), then we associate to x the function whose rule is given by $f(n) = d_n$. Clearly this is a one-to-one function from the set of real numbers between 0 and 1 and a subset of the set of all functions from the set of positive integers to the set $\{0,1,2,3,4,5,6,7,8,9\}$. Two different real numbers must have different decimal representations, so the corresponding functions are different. (A few functions are left out, because of forbidding representations such as $0.239999\ldots$.) Since the set of real numbers between 0 and 1 is uncountable, the subset of functions we have associated with them must be uncountable. But the set of all such functions has at least this cardinality, so it, too, must be uncountable (by Exercise 15).

Sec. 2.6 27

27. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find

a) $\mathbf{A} \vee \mathbf{B}$. **b)** $\mathbf{A} \wedge \mathbf{B}$. **c)** $\mathbf{A} \odot \mathbf{B}$.

Key

27. a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Sec. 3.1 2, 4

2. Determine which characteristics of an algorithm described in the text (after Algorithm 1) the following procedures have and which they lack.

a) **procedure** *double*(n: positive integer)
while $n > 0$
  $n := 2n$

b) **procedure** *divide*(n: positive integer)
while $n \geq 0$
  $m := 1/n$
  $n := n - 1$

c) **procedure** *sum*(n: positive integer)
$sum := 0$
while $i < 10$
  $sum := sum + i$

d) **procedure** *choose*(a, b: integers)
$x :=$ either a or b

Key

2. **a)** This procedure is not finite, since execution of the **while** loop continues forever.

b) This procedure is not effective, because the step $m := 1/n$ cannot be performed when $n = 0$, which will eventually be the case.

c) This procedure lacks definiteness, since the value of i is never set.

d) This procedure lacks definiteness, since the statement does not tell whether x is to be set equal to a or to b.

4. Describe an algorithm that takes as input a list of n integers and produces as output the largest difference obtained by subtracting an integer in the list from the one following it.

Key

4. Set the answer to be $-\infty$. For i going from 1 through $n-1$, compute the value of the $(i+1)^{\text{st}}$ element in the list minus the i^{th} element in the list. If this is larger than the answer, reset the answer to be this value.

Sec. 3.2 8(c), 26(a), 54, 56

8. Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.

a) $f(x) = 2x^2 + x^3 \log x$
b) $f(x) = 3x^5 + (\log x)^4$
c) $f(x) = (x^4 + x^2 + 1)/(x^4 + 1)$
d) $f(x) = (x^3 + 5 \log x)/(x^4 + 1)$

Key

8. **a)** Since $x^3 \log x$ is not $O(x^3)$ (because the $\log x$ factor grows without bound as x increases), $n = 3$ is too small. On the other hand, certainly $\log x$ grows more slowly than x, so $2x^2 + x^3 \log x \le 2x^4 + x^4 = 3x^4$. Therefore $n = 4$ is the answer, with $C = 3$ and $k = 0$.

b) The $(\log x)^4$ is insignificant compared to the x^5 term, so the answer is $n = 5$. Formally we can take $C = 4$ and $k = 1$ as witnesses.

c) For large x, this fraction is fairly close to 1. (This can be seen by dividing numerator and denominator by x^4.) Therefore we can take $n = 0$; in other words, this function is $O(x^0) = O(1)$. Note that $n = -1$ will not do, since a number close to 1 is not less than a constant times n^{-1} for large n. Formally we can write $f(x) \le 3x^4/x^4 = 3$ for all $x > 1$, so witnesses are $C = 3$ and $k = 1$.

d) This is similar to the previous part, but this time $n = -1$ will do, since for large x, $f(x) \approx 1/x$. Formally we can write $f(x) \le 6x^3/x^3 = 6$ for all $x > 1$, so witnesses are $C = 6$ and $k = 1$.

26. Give a big-O estimate for each of these functions. For the function g in your estimate $f(x)$ is $O(g(x))$, use a simple function g of smallest order.

a) $(n^3 + n^2 \log n)(\log n + 1) + (17 \log n + 19)(n^3 + 2)$
b) $(2^n + n^2)(n^3 + 3^n)$
c) $(n^n + n2^n + 5^n)(n! + 5^n)$

key

26. The approach in these problems is to pick out the most rapidly growing term in each sum and discard the rest (including the multiplicative constants).

a) This is $O(n^3 \cdot \log n + \log n \cdot n^3)$, which is the same as $O(n^3 \cdot \log n)$.

b) Since 2^n dominates n^2, and 3^n dominates n^3, this is $O(2^n \cdot 3^n) = O(6^n)$.

c) The dominant terms in the two factors are n^n and $n!$, respectively. Therefore this is $O(n^n n!)$.

54. Show that $x^5y^3 + x^4y^4 + x^3y^5$ is $\Omega(x^3y^3)$.

Key

54. For all values of x and y greater than 1, each term of the given expression is greater than x^3y^3, so the entire expression is greater than x^3y^3. In other words, we take $C = k_1 = k_2 = 1$ in the definition given in the solution of Exercise 52.

56. Show that $\lceil xy \rceil$ is $\Omega(xy)$.

Key

56. For all positive values of x and y, we know that $\lceil xy \rceil \ge xy$ by definition (since the ceiling function value cannot be less than the argument). Thus $\lceil xy \rceil$ is $\Omega(xy)$ from the definition, taking $C = 1$ and $k_1 = k_2 = 0$. In fact, $\lceil xy \rceil$ is also $O(xy)$ (and therefore $\Theta(xy)$); this is easy to see since $\lceil xy \rceil \le (x+1)(y+1) \le (2x)(2y) = 4xy$ for all x and y greater than 1.

Sec. 3.3 7, 10

7. Suppose that an element is known to be among the first four elements in a list of 32 elements. Would a linear search or a binary search locate this element more rapidly?

Key

7. Linear

***10. a)** Show that this algorithm determines the number of 1 bits in the bit string S:

> **procedure** *bit count*(S: bit string)
> *count* := 0
> **while** $S \neq 0$
>   *count* := *count* + 1
> $S := S \wedge (S - 1)$
> **return** *count* {*count* is the number of 1s in S}

Here $S - 1$ is the bit string obtained by changing the rightmost 1 bit of S to a 0 and all the 0 bits to the right of this to 1s. [Recall that $S \wedge (S - 1)$ is the bitwise *AND* of S and $S - 1$.]

b) How many bitwise *AND* operations are needed to find the number of 1 bits in a string S using the algorithm in part (a)?

Key

10. a) By the way that $S - 1$ is defined, it is clear that $S \wedge (S - 1)$ is the same as S except that the rightmost 1 bit has been changed to a 0. Thus we add 1 to *count* for every one bit (since we stop as soon as $S = 0$, i.e., as soon as S consists of just 0 bits).

b) Obviously the number of bitwise *AND* operations is equal to the final value of *count*, i.e., the number of one bits in S.