The Electric Potential

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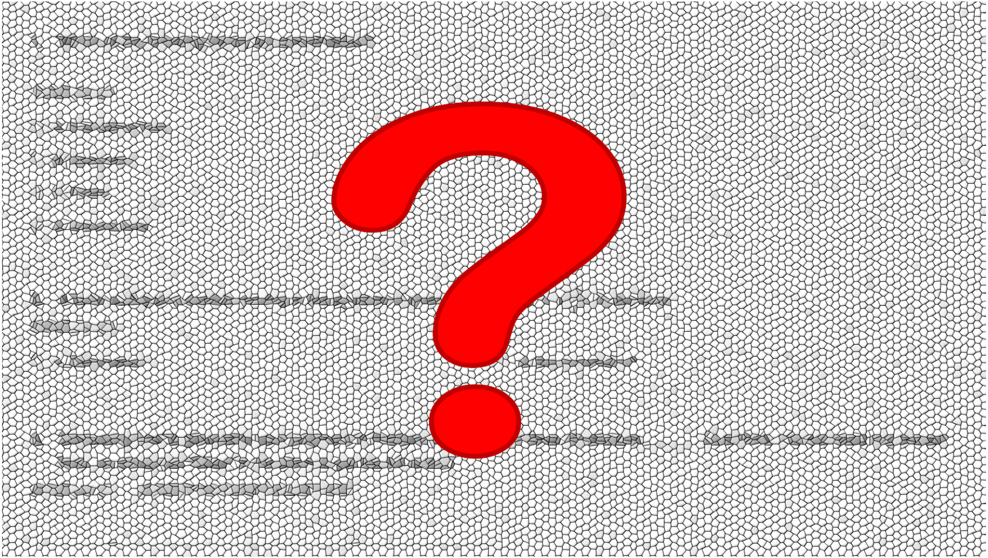
Lecture 3

Quiz 3-1



Motivation: Conservative Force

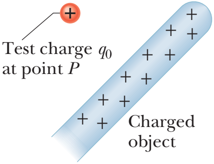
- The motivation for associating a potential energy with a force is that we can then apply the principle of the *conservation of mechanical energy* to closed systems involving the force.
- Experimentally, physicists discovered that the *electrostatic force is conservative* and thus has an associated electric potential energy.
  - How to define electric potential energy U?
  - How electric potential V is related to electric potential energy U?
  - How to calculate V for various arrangements of charged particles and objects?

Outline

- Potential Energy Difference and Electric Potential
- Electric Potential Due to Discrete Charges
- Electric Potential Due to Continuous Charge Distributions
- Obtaining the Value of Electric Field from Electric Potential

Electric Potential Energy

- Suppose we want to find the potential energy U associated with a positive test charge q_0 located at point P in the electric field of a charged rod.
  Hint: Recall how we defined gravitational potential energy last semester.



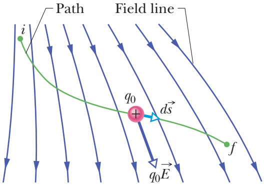
The rod sets up an electric potential, which determines the potential energy.

- First, we need a *reference configuration* for which $U = 0$. A reasonable choice is for the test charge to be infinitely far from the rod.

- Next, we bring the test charge in from infinity to point P. Along the way, we calculate the work done by the electric force on the test charge.
- The potential energy of the final configuration is then given by $U = -W$, where W is now the work done by the electric force.
- The work and thus the potential energy can be positive or negative depending on the sign of the rod's charge.

Potential Energy Difference

- Now consider an arbitrary electric field $\vec{E}(x, y, z)$, represented by the field lines, and a positive test charge q_0 that moves along the path from point i to point f.



- The differential work δW done on the charge by a force $\vec{F}$ during a displacement $d\vec{s}$ is given by the dot product of the force and the displacement:

$$\delta W = \vec{F} \cdot d\vec{s} = q_0 \vec{E} \cdot d\vec{s}.$$

- To find the total work W done on the charge by the field, we sum, via integration, the differential works done on the charge as it moves through all the displacements $d\vec{s}$ along the particular path from i to f (a line integral)

$$W = q_0 \int_i^f \vec{E} \cdot d\vec{s}.$$

- The change of electric potential energy is

$$\Delta U \equiv U_f - U_i = -W = -q_0 \int_i^f \vec{E} \cdot d\vec{s}.$$

Conservation of Energy

- If a charged particle moves through an electric field (with no force other than the electric force), the total work W done on the charge by the field leads to the kinetic energy change of the particle

$$\Delta K = K_f - K_i = W = -\Delta U = -(U_f - U_i).$$

- Then we can write down the *conservation of energy* of the particle that moves from point i to point f as

$$U_i + K_i = U_f + K_f.$$

Electric Potential

- In fact, it is more general to define the amount of electric potential energy per unit charge when a positive test charge is brought from i to f.
- We define the electric potential V at P in terms of the work done by the electric force and the resulting potential energy:

$$V = \frac{-W}{q_0} = \frac{U}{q_0}.$$

- For a particle with charge q placed at point P, we can immediately find the potential energy

$$U = qV.$$

- In the previous example, we thus have

$$V_f - V_i = \frac{-W}{q_0} = -\int_i^f \vec{E} \cdot d\vec{s}.$$

- Thus, the potential difference $V_f - V_i$ between any two points i and f in an electric field is equal to the negative of the line integral of $\vec{E} \cdot d\vec{s}$ from i to f.
- However, because the electric force is conservative, all paths yield the same result.
- If we set potential $V_i = 0$, then

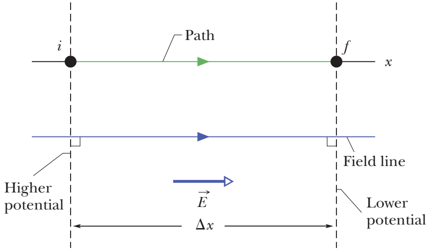
$$V = -\int_0^f \vec{E} \cdot d\vec{s}.$$

- For a uniform field $\vec{E}$,

$$V_f - V_i = -\vec{E} \cdot \int_i^f d\vec{s} = -E\Delta x,$$

where Δx is the separation between the two parallel equipotential surfaces.

- The electric field vector points from higher potential toward lower potential.



Properties of Electric Charge

Description	*Vector*	*Scalar*
Interaction between two charges	Force $\vec{F}$	Potential energy U
Effect of one charge or group of charges at a point in space	Field $\vec{E}$	Potential V

Equipotential Surface

- Adjacent points that have the same electric potential form an *equipotential surface*.
- No net work W is done on a charged particle by an electric field when the particle moves between two points on the same equipotential surface.

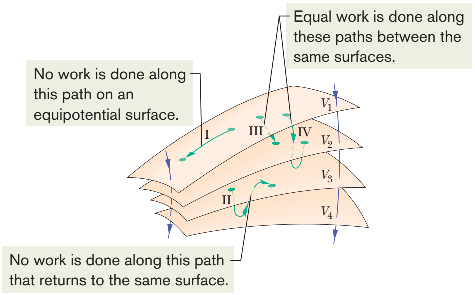


Figure 1: Family of equipotential surfaces.

- Equipotential surfaces often have symmetry according to that of the charge distribution or field.

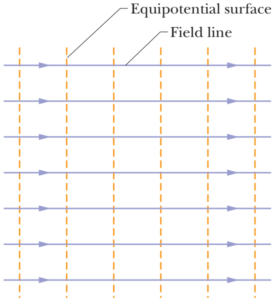


Figure 2: Uniform field, translational symmetry.

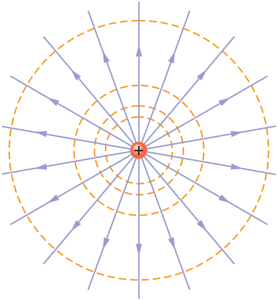


Figure 3: Point charge, spherical symmetry.

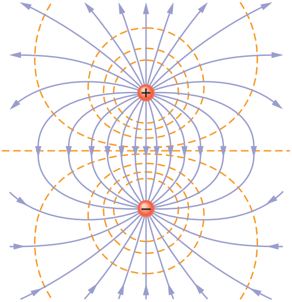
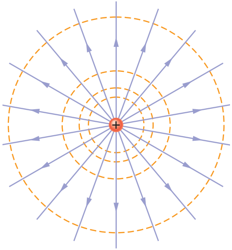
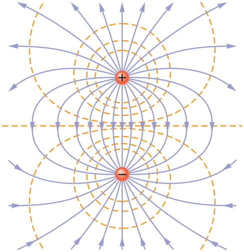


Figure 4: Dipole, cylindrical symmetry.

- In fact, equipotential surfaces are *always perpendicular* to electric field lines and thus to $\vec{E}$, which is always tangent to these lines.
  - If $\vec{E}$ were not perpendicular to an equipotential surface, it would have a component lying along that surface.
  - This component would then do work on a charged particle as it moved along the surface.
  - However, work cannot be done if the surface is truly an equipotential surface; the only possible conclusion is that $\vec{E}$ *must be everywhere perpendicular to the surface*.





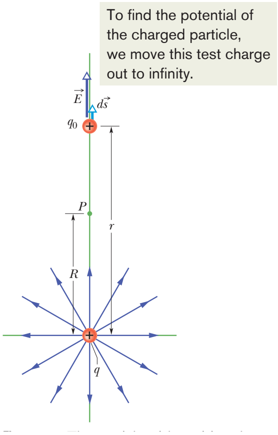
Potential due to Charged Particle(s)

- The electric potential V due to a particle of charge q at any radial distance r from the particle is

$$V = -\int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{s} = \frac{1}{4\pi\epsilon_0} \frac{q}{r}.$$

- For a group of n charges, the net potential is

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i}.$$

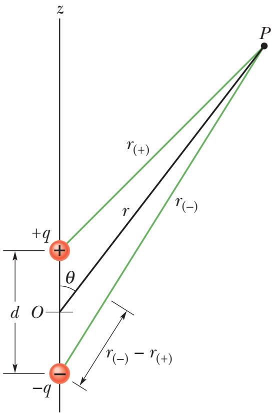


Potential due to an Electric Dipole

- At P, the positively charged particle (at distance $r_{(+)}$) sets up potential $V_{(+)}$ and the negatively charged particle (at distance $r_{(-)}$) sets up potential $V_{(-)}$. The net potential at P is given by

$$V = V_{(+)} + V_{(-)} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_{(+)}} + \frac{-q}{r_{(-)}}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{r_{(-)} - r_{(+)}}{r_{(-)}r_{(+)}}$$



- For points that are relatively far from the dipole ($r \gg d$), we can approximate the two lines to P as being parallel. Thus

$$r_{(-)} - r_{(+)} \approx d\cos\theta$$
$$r_{(-)} r_{(+)} \approx r^2$$

- We can now write V as

$$V = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3},$$

in which $\vec{p} = q\vec{d}$ is the magnitude of the electric dipole moment.

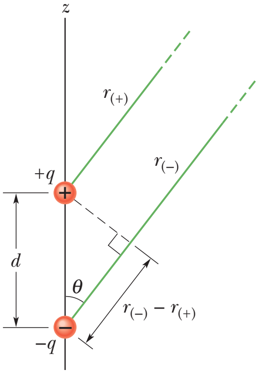
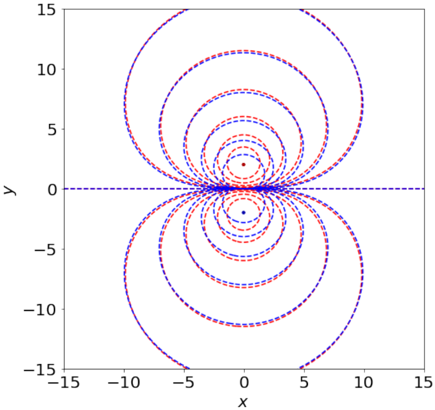


Figure 5: Dipole in the large-distance approximation.

- Here, we compare the exact result of the dipole potential (red) and its large-distance ($r \gg d$) approximation (blue).
- The equipotential lines are for $V =$ -0.5, -0.25, -0.125, -0.0625, -0.03125, -0.015625, 0, 0.015625, 0.03125, 0.0625, 0.125, 0.25, 0.5, respectively, in units of $p/(4\pi\epsilon_0)$.



Systems of Charged Particles

- We start with particle 1 fixed in place and others particles infinitely far away from particle 1 and from each other. The initial potential energy $U_1 = 0$.
- We bring particle 2 to its final position, and then the system's potential energy is

$$U_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}.$$

- Next we bring particle 3 to its final position, and then the system's potential energy increases by

$$U_{13} + U_{23} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right).$$

- Thus, the total potential energy is

$$U_{12} + U_{13} + U_{23} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right).$$

- The lesson here is this: If you are given an assembly of charged particles, you can find the potential energy of the assembly by finding the potential of every possible pair of the particles and then summing the results.

$$U_{\rm tot} = \sum_{i<j} U_{ij} = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}.$$

Continuous Charge Distribution: Rod

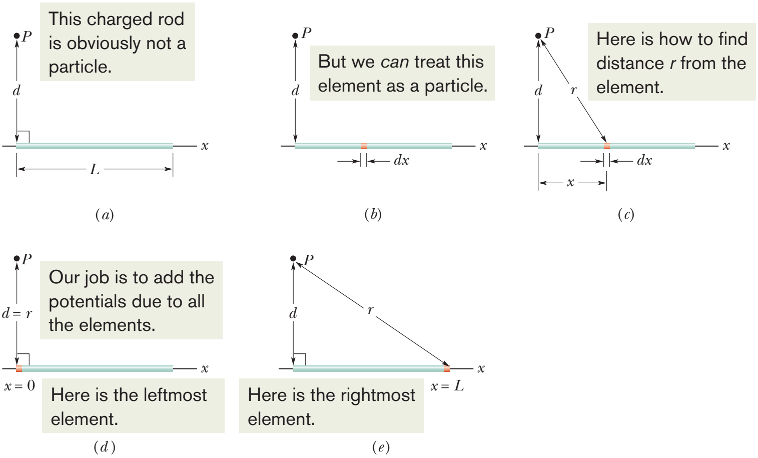


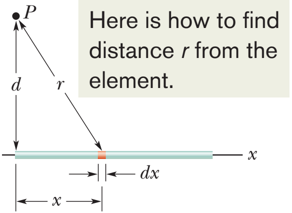
Figure 6: Integration over a continuous charge distribution.

- To find the total potential V at P with a perpendicular distance d from the left end of the thin, nonconducting rod, we integrate the potentials due to all the charge elements

$$V = \int dV = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r}.$$

- For the rod of length L with a positive charge of uniform linear density λ,

$$dq = \lambda dx,$$
$$r = \sqrt{x^2 + d^2}.$$

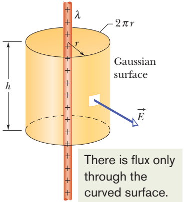


- The total potential V at point P can be obtained by integrating dV from $x = 0$ to $x = L$ to be

$$\begin{aligned} V = \int dV &= \int_0^L \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{\sqrt{x^2 + d^2}} \\ &= \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{dx}{\sqrt{x^2 + d^2}} \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[\ln(x + \sqrt{x^2 + d^2}) \right]_0^L \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[\ln(L + \sqrt{L^2 + d^2}) - \ln d \right] \\ &= \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{L + \sqrt{L^2 + d^2}}{d} \right] \end{aligned}$$

- In the previous lecture, we have applied Gauss' law to obtain

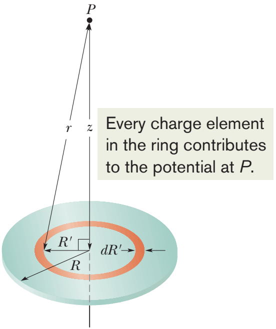
$$E = \frac{\lambda h}{\epsilon_0 (2\pi rh)} = \frac{\lambda}{2\pi\epsilon_0 r}.$$



- Can you relate that result with the present one on the electric potential?
- We also mentioned that the infinite-rod result also approximates the field of a finite rod of charge at points that are *not too near the ends*.
- Can you read the correction for the potential and understand what exactly we mean by saying not too near the ends?

Continuous Charge Distribution: Disk

- Consider the electric potential $V(z)$ along the central axis of a thin plastic disk of radius R that has a uniform charge density σ.
- Due to the circular symmetry, we can set up a 1-D integration. Consider a differential element consisting of a flat ring of radius R' and radial width dR'.



$$dq = \sigma(2\pi R')dR',$$

$$r = \sqrt{z^2 + R'^2}.$$

- We find the net potential at P by integrating the contributions of all the rings from $R' = 0$ to $R' = R$:

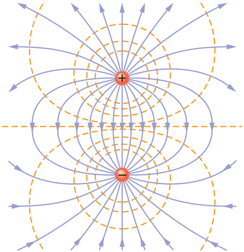
$$
\begin{aligned}
V &= \int_0^R \frac{1}{4\pi\epsilon_0} \frac{\sigma(2\pi R')dR'}{\sqrt{z^2 + R'^2}} \\
&= \frac{\sigma}{2\epsilon_0} \int_0^R \frac{R' dR'}{\sqrt{z^2 + R'^2}} \\
&= \frac{\sigma}{2\epsilon_0} (\sqrt{z^2 + R^2} - z).
\end{aligned}
$$

- Now, have you seen the following result before?

$$
E_z = \frac{\sigma}{2\epsilon_0} \left[1 - \frac{z}{(z^2 + R^2)^{1/2}} \right] = -\frac{dV}{dz}
$$

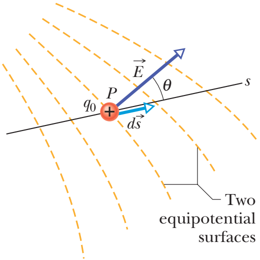
Calculating the Field from the Potential

- If we know the potential V at all points near an assembly of charges, we can draw a family of equipotential surfaces. The electric field lines, sketched perpendicular to those surfaces, reveal the variation of $\vec{E}$.



- What is the mathematical equivalence of this graphical procedure?

- Suppose that a positive test charge q_0 moves through a displacement $d\vec{s}$ from one equipotential surface to the adjacent surface.



- The work done by the electric field can be written as the scalar product

$$(q_0\vec{E}) \cdot d\vec{s} = q_0 E(\cos\theta)\delta s,$$

where δs is the length of the path sandwiched between the adjacent equipotential surfaces.

- The work the electric field does on the test charge during the move may also be written as $-q_0 \delta V$, where δV is the potential difference of the two surfaces.
- Equating these two expressions for the work yields

$$q_0 E(\cos\theta)\delta s = -q_0 \delta V,$$

or

$$E_s \equiv E\cos\theta = -\frac{\delta V}{\delta s} \equiv -\frac{\partial V}{\partial s}.$$

- The subscript to E and and the partial derivative symbols emphasize that the component of $\vec{E}$ along a specified axis (here called the s axis) equals the variation of V along that axis only.

- Therefore, if we know the function $V(x, y, z)$ — we can find the components of $\vec{E}$ (and thus $\vec{E}$ itself) at any point by taking partial derivatives:

$$E_x = -\frac{\partial V}{\partial x}; \quad E_y = -\frac{\partial V}{\partial y}; \quad E_z = -\frac{\partial V}{\partial z}$$

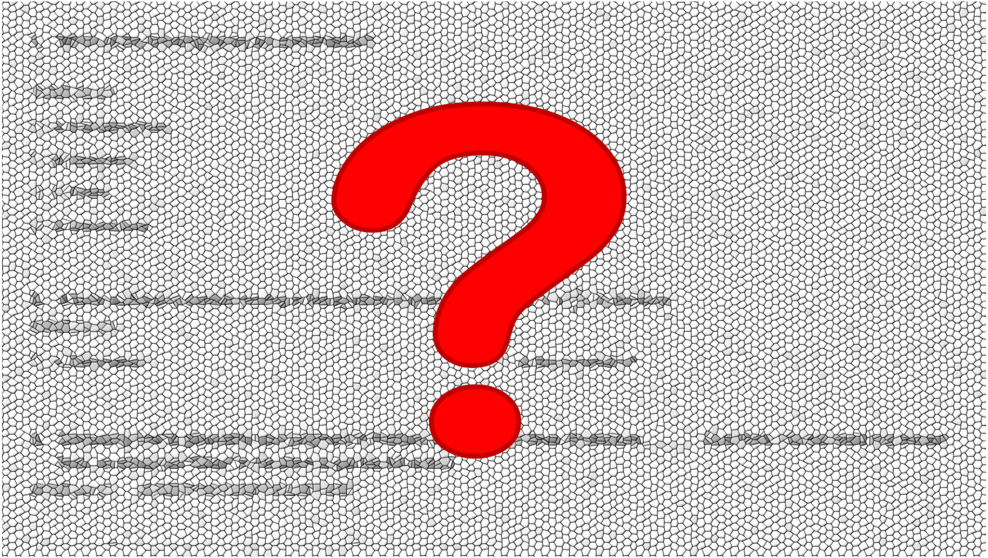
- In the next lecture, we will learn the vector form of the equations:

$$\vec{E} = -\nabla V,$$

  in which we define the vector operator ∇ as

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

Quiz 3-2



Summary

- Concepts: electric potential energy U, electric potential V, equipotential surface
- Key techniques:
  - Finding V from $\vec{E}$

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}.$$

  - Calculating $\vec{E}$ from V

$$E_x = -\frac{\partial V}{\partial x}; \quad E_y = -\frac{\partial V}{\partial y}; \quad E_z = -\frac{\partial V}{\partial z}$$

- Applications:
  - A charged particle
$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$
  - An electric dipole
$$V = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}.$$
  - A continuous charge distribution (e.g., rod and disk)
$$V = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}.$$
  - Electric potential energy of a system of charged particles
$$U_{\text{tot}} = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}.$$

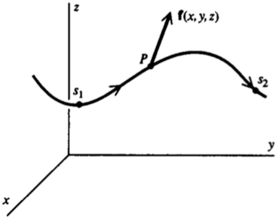
Reading

Halliday, Resnick & Krane:

- Chapter 28. Electric Potential Energy and Potential

Appendix 3A: Line Integrals

- We have learned the idea of line integral in electric potential energy.
- Suppose we have a directed curve C in three dimensions. We have a vector function $\vec{f}(x, y, z)$ defined everywhere on C. We define a line integral as



$$I = \int_C \vec{f}(x, y, z) \cdot d\vec{s} = \int_C \vec{f}(x, y, z) \cdot \hat{s}\, ds,$$

where $\hat{s}$ is a unit vector tangent to the curve.

- Suppose curve C is given, for $t_i \leq t \leq t_f$, by

$$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}.$$

- We have

$$ds = \sqrt{\left|\frac{dx}{dt}\right|^2 + \left|\frac{dy}{dt}\right|^2 + \left|\frac{dz}{dt}\right|^2} dt = \left|\frac{d\vec{r}}{dt}\right| dt,$$

$$\hat{s} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|}.$$

- Therefore, we obtain

$$I = \int_C \vec{f}(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt} dt,$$

which is usually easy to use.

- We can see that, for $\vec{f} = f_x \hat{x} + f_y \hat{y} + f_z \hat{z}$,

$$I = \int_C (f_x dx + f_y dy + f_z dz).$$

Note that this is a formal expression; it is often needed to restore ds along path segment(s) to evaluate the integral.