- Sec. 8.1: 24, 26, 42, 45, 48

24. Find a recurrence relation for the number of bit sequences of length n with an even number of 0s.

24. Let e_n be the number of bit sequences of length n with an even number of 0's. Note that therefore there are $2^n - e_n$ bit sequences with an odd number of 0's. There are two ways to get a bit string of length n with an even number of 0's. It can begin with a 1 and be followed by a bit string of length $n-1$ with an even number of 0's, and there are e_{n-1} of these; or it can begin with a 0 and be followed by a bit string of length $n-1$ with an odd number of 0's, and there are $2^{n-1} - e_{n-1}$ of these. Therefore $e_n = e_{n-1} + 2^{n-1} - e_{n-1}$, or simply $e_n = 2^{n-1}$. See also Exercise 35 in Section 6.4.

26. a) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. [*Hint:* Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]

b) What are the initial conditions for the recurrence relation in part (a)?

c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?

26. Let a_n be the number of coverings.

a) We follow the hint. If the right-most domino is positioned vertically, then we have a covering of the left-most $n-1$ columns, and this can be done in a_{n-1} ways. If the right-most domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first $n-2$ columns therefore will need to contain a covering by dominoes, and this can be done in a_{n-2} ways. Thus we obtain the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$.

b) Clearly $a_1 = 1$ and $a_2 = 2$.

c) The sequence we obtain is just the Fibonacci sequence, shifted by one. The sequence is thus 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, ..., so the answer to this part is 2584.

***41.** Show that if $R(n)$ is the number of moves used by the Frame–Stewart algorithm to solve the Reve's puzzle with n disks, where k is chosen to be the smallest integer with $n \le k(k+1)/2$, then $R(n)$ satisfies the recurrence relation $R(n) = 2R(n-k) + 2^k - 1$, with $R(0) = 0$ and $R(1) = 1$.

***42.** Show that if k is as chosen in Exercise 41, then $R(n) - R(n-1) = 2^{k-1}$.

***43.** Show that if k is as chosen in Exercise 41, then $R(n) = \sum_{i=1}^{k} i2^{i-1} - (t_k - n)2^{k-1}$.

***44.** Use Exercise 43 to give an upper bound on the number of moves required to solve the Reve's puzzle for all integers n with $1 \le n \le 25$.

***45.** Show that $R(n)$ is $O(\sqrt{n}2^{\sqrt{2n}})$.

42. To clarify the problem, we note that k is chosen to be the smallest nonnegative integer such that $n \le k(k+1)/2$. If $n-1 \ne k(k-1)/2$, then this same value of k applies to $n-1$ as well; otherwise the value for $n-1$ is $k-1$. If $n-1 \ne k(k-1)/2$, it also follows by subtracting k from both sides of the inequality that the smallest nonnegative integer m such that $n-k \le m(m+1)/2$ is $m = k-1$, so $k-1$ is the value selected by the Frame–Stewart algorithm for $n-k$. Now we proceed by induction, the basis steps being trivial. There are two cases for the inductive step. If $n-1 \ne k(k-1)/2$, then we have from the recurrence relation in Exercise 41 that $R(n) = 2R(n-k) + 2^k - 1$ and $R(n-1) = 2R(n-k-1) + 2^k - 1$. Subtracting yields $R(n) - R(n-1) = 2(R(n-k) - R(n-k-1))$. Since $k-1$ is the value selected for $n-k$, the inductive hypothesis tells us that this difference is $2 \cdot 2^{k-2} = 2^{k-1}$, as desired. On the other hand, if $n-1 = k(k-1)/2$, then $R(n) - R(n-1) = 2R(n-k) + 2^k - 1 - (2R(n-1-(k-1)) + 2^{k-1} - 1 = 2^{k-1}$.

Key for 42

the final term in the given expression accounts for. **45.** By Exercise 43, $R(n)$ is no larger than $\sum_{i=1}^{k} i2^{i-1}$. It can be shown that this sum equals $(k+1)2^k - 2^{k+1} + 1$, so it is no greater than $(k+1)2^k$. Because $n > k(k-1)/2$, the quadratic formula can be used to show that $k < 1 + \sqrt{2n}$ for all $n > 1$. Therefore, $R(n)$ is bounded above by $(1 + \sqrt{2n} + 1)2^{1+\sqrt{2n}} < 8\sqrt{n}2^{\sqrt{2n}}$ for all $n > 2$. Hence, $R(n)$ is $O(\sqrt{n}2^{\sqrt{2n}})$. **47. a)** 0 **b)** 0

Key for 45

Let $\{a_n\}$ be a sequence of real numbers. The **backward differences** of this sequence are defined recursively as shown next. The **first difference** ∇a_n is

$$\nabla a_n = a_n - a_{n-1}.$$

The **$(k+1)$st difference** $\nabla^{k+1} a_n$ is obtained from $\nabla^k a_n$ by

$$\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}.$$

48. Show that $a_{n-1} = a_n - \nabla a_n$.

48. This follows immediately (by algebra) from the definition.

Sec. 8.2 2,4(g), 20, 30, 35

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-2}$ **b)** $a_n = 3$
c) $a_n = a_{n-1}^2$ **d)** $a_n = a_{n-1} + 2a_{n-3}$
e) $a_n = a_{n-1}/n$
f) $a_n = a_{n-1} + a_{n-2} + n + 3$
g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

2. a) linear, homogeneous, with constant coefficients; degree 2
b) linear with constant coefficients but not homogeneous
c) not linear
d) linear, homogeneous, with constant coefficients; degree 3
e) linear and homogeneous, but not with constant coefficients
f) linear with constant coefficients, but not homogeneous
g) linear, homogeneous, with constant coefficients; degree 7

4. Solve these recurrence relations together with the initial conditions given.

a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$
e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$
f) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
g) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$

g) $r^2 + 4r - 5 = 0 \qquad r = -5, 1$

$$a_n = \alpha_1(-5)^n + \alpha_2 1^n = \alpha_1(-5)^n + \alpha_2$$

$$2 = \alpha_1 + \alpha_2$$

$$8 = -5\alpha_1 + \alpha_2$$

$$\alpha_1 = -1 \qquad \alpha_2 = 3$$

$$a_n = -(-5)^n + 3$$

20. Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.

20. This is a fourth degree recurrence relation. The characteristic polynomial is $r^4 - 8r^2 + 16$, and as we see in Exercise 27, $r = \pm 2$ are the only roots, each with multiplicity 2. Thus we can write down the general solution as usual: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3(-2)^n + \alpha_4 n \cdot (-2)^n$.

30. a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$.

b) Find the solution of this recurrence relation with $a_1 = 56$ and $a_2 = 278$.

30. a) The associated homogeneous recurrence relation is $a_n = -5a_{n-1} - 6a_{n-2}$. To solve it we find the characteristic equation $r^2 + 5r + 6 = 0$, find that $r = -2$ and $r = -3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha(-2)^n + \beta(-3)^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = c \cdot 4^n$. We plug this into our recurrence relation and obtain $c \cdot 4^n = -5c \cdot 4^{n-1} - 6c \cdot 4^{n-2} + 42 \cdot 4^n$. We divide through by 4^{n-2}, obtaining $16c = -20c - 6c + 42 \cdot 16$, whence with a little simple algebra $c = 16$. Therefore the particular solution we seek is $a_n^{(p)} = 16 \cdot 4^n = 4^{n+2}$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha(-2)^n + \beta(-3)^n + 4^{n+2}$.

b) We plug the initial conditions into our solution from part (a) to obtain $56 = a_1 = -2\alpha - 3\beta + 64$ and $278 = a_2 = 4\alpha + 9\beta + 256$. A little algebra yields $\alpha = 1$ and $\beta = 2$. So the solution is $a_n = (-2)^n + 2(-3)^n + 4^{n+2}$.

35. Find the solution of the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ and $a_1 = 4$.

35. $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + 1/8 + (39/8)3^n$

Sec. 8.3 22

22. Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

a) Find $f(16)$.

b) Find a big-O estimate for $f(n)$. [*Hint:* Make the substitution $m = \log n$.]

22. a) $f(16) = 2f(4) + 4 = 2(2f(2) + 2) + 4 = 2(2 \cdot 1 + 2) + 4 = 12$

b) Let $m = \log n$, so that $n = 2^m$. Also, let $g(m) = f(2^m)$. Then our recurrence becomes $f(2^m) = 2f(2^{m/2}) + m$, since $\sqrt{2^m} = (2^m)^{1/2} = 2^{m/2}$. Rewriting this in terms of g we have $g(m) = 2g(m/2) + m$. Theorem 2 (with $a = 2$, $b = 2$, $c = 1$, and $d = 1$ now tells us that $g(m)$ is $O(m \log m)$. Since $m = \log n$, this says that our function is $O(\log n \cdot \log \log n)$.