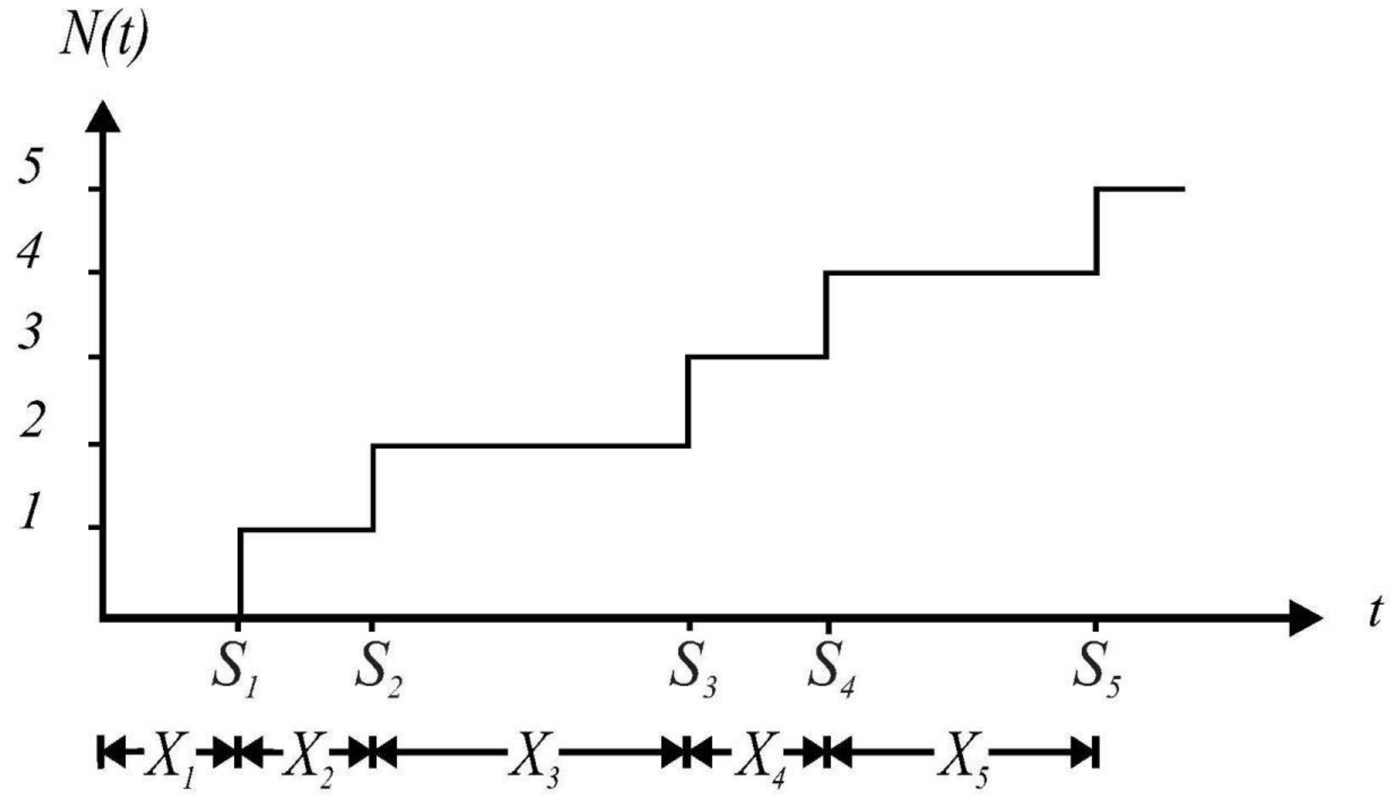
The Poisson Process

Definition 13.8 Counting Process

A stochastic process $N(t)$ *is a* counting process *if for every sample function,* $n(t, s) = 0$ *for* $t < 0$ *and* $n(t, s)$ *is integer-valued and nondecreasing with time.*

Figure 13.4



Sample path of a counting process.

Definition 13.9 Poisson Process

A counting process $N(t)$ *is a* Poisson process *of rate* λ *if*

(a) The number of arrivals in any interval $(t_0, t_1]$, $N(t_1) - N(t_0)$, *is a Poisson random variable with expected value* $\lambda(t_1 - t_0)$.

(b) For any pair of nonoverlapping intervals $(t_0, t_1]$ *and* $(t_0', t_1']$, *the number of arrivals in each interval,* $N(t_1) - N(t_0)$ *and* $N(t_1') - N(t_0')$, *respectively, are independent random variables.*

Theorem 13.2

For a Poisson process $N(t)$ of rate λ, the joint PMF of

$$\mathbf{N} = \left[N(t_1), \ldots, N(t_k)\right]'$$

for ordered time instances $t_1 < \cdots < t_k$ is

$$P_{\mathbf{N}}(\mathbf{n}) = \begin{cases} \frac{\alpha_1^{n_1} e^{-\alpha_1}}{n_1!} \frac{\alpha_2^{n_2-n_1} e^{-\alpha_2}}{(n_2-n_1)!} \cdots \frac{\alpha_k^{n_k-n_{k-1}} e^{-\alpha_k}}{(n_k-n_{k-1})!} & 0 \le n_1 \le \cdots \le n_k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_1 = \lambda t_1$, and for $i = 2, \ldots, k$, $\alpha_i = \lambda(t_i - t_{i-1})$.

Let $M_1 = N(t_1)$ and for $i > 1$, let $M_i = N(t_i) - N(t_{i-1})$. By the definition of the Poisson process, $M_1, \ldots, M_k$ is a collection of independent Poisson random variables such that $\mathrm{E}[M_i] = \alpha_i$.

$$\begin{aligned} P_{\mathbf{N}}(\mathbf{n}) &= P_{M_1, M_2, \ldots, M_k}(n_1, n_2 - n_1, \ldots, n_k - n_{k-1}) && (1) \\ &= P_{M_1}(n_1)\, P_{M_2}(n_2 - n_1) \cdots P_{M_k}(n_k - n_{k-1}). && (2) \end{aligned}$$

The theorem follows by substituting Equation (13.14) for $P_{M_i}(n_i - n_{i-1})$.

Theorem 13.3

For a Poisson process of rate λ, the interarrival times $X_1, X_2, \ldots$ are an iid random sequence with the exponential PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

作业（不交）：几何分布推指数分布

Given $X_1 = x_1, X_2 = x_2, \ldots, X_{n-1} = x_{n-1}$, arrival $n-1$ occurs at time

$$t_{n-1} = x_1 + \cdots + x_{n-1}. \tag{1}$$

For $x > 0$, $X_n > x$ if and only if there are no arrivals in the interval $(t_{n-1}, t_{n-1} + x]$. The number of arrivals in $(t_{n-1}, t_{n-1} + x]$ is independent of the past history described by $X_1, \ldots, X_{n-1}$. This implies

$$\mathrm{P}[X_n > x | X_1 = x_1, \ldots, X_{n-1} = x_{n-1}] = \mathrm{P}[N(t_{n-1} + x) - N(t_{n-1}) = 0] = e^{-\lambda x}.$$

Thus X_n is independent of $X_1, \ldots, X_{n-1}$ and has the exponential CDF

$$F_{X_n}(x) = 1 - \mathrm{P}[X_n > x] = \begin{cases} 1 - e^{-\lambda x} & x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

From the derivative of the CDF, we see that X_n has the exponential PDF $f_{X_n}(x) = f_X(x)$ in the statement of the theorem.

Theorem 13.4

A counting process with independent exponential (λ) interarrivals $X_1, X_2, \ldots$ is a Poisson process of rate λ.

Quiz 13.4

Data packets transmitted by a modem over a phone line form a Poisson process of rate 10 packets/sec. Using M_k to denote the number of packets transmitted in the kth hour, find the joint PMF of M_1 and M_2.

Quiz 13.4 Solution

The first and second hours are nonoverlapping intervals. Since one hour equals 3600 sec and the Poisson process has a rate of 10 packets/sec, the expected number of packets in each hour is $\mathrm{E}[M_i] = \alpha = 36,000$. This implies M_1 and M_2 are independent Poisson random variables each with PMF

$$P_{M_i}(m) = \begin{cases} \frac{\alpha^m e^{-\alpha}}{m!} & m = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since M_1 and M_2 are independent, the joint PMF of M_1 and M_2 is

$$P_{M_1,M_2}(m_1, m_2) = P_{M_1}(m_1)\, P_{M_2}(m_2) = \begin{cases} \frac{\alpha^{m_1+m_2} e^{-2\alpha}}{m_1! m_2!} & m_1 = 0, 1, \ldots; \\ & m_2 = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Section 13.5

Properties of the Poisson Process

Theorem 13.5

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes of rates λ_1 and λ_2. The counting process $N(t) = N_1(t) + N_2(t)$ is a Poisson process of rate $\lambda_1 + \lambda_2$.

Example 13.16 Problem

Cars, trucks, and buses arrive at a toll booth as independent Poisson processes with rates $\lambda_c = 1.2$ cars/minute, $\lambda_t = 0.9$ trucks/minute, and $\lambda_b = 0.7$ buses/minute. In a 10-minute interval, what is the PMF of N, the number of vehicles (cars, trucks, or buses) that arrive?

Example 13.16 Solution

By Theorem 13.5, the arrival of vehicles is a Poisson process of rate $\lambda = 1.2 + 0.9 + 0.7 = 2.8$ vehicles per minute. In a 10-minute interval, $\lambda T = 28$ and N has PMF

$$P_N(n) = \begin{cases} 28^n e^{-28}/n! & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Theorem 13.6

The counting processes $N_1(t)$ and $N_2(t)$ derived from a Bernoulli decomposition of the Poisson process $N(t)$ are independent Poisson processes with rates λp and $\lambda(1-p)$.

Example 13.17 Problem

A corporate Web server records hits (requests for HTML documents) as a Poisson process at a rate of 10 hits per second. Each page is either an internal request (with probability 0.7) from the corporate intranet or an external request (with probability 0.3) from the Internet. Over a 10-minute interval, what is the joint PMF of I, the number of internal requests, and X, the number of external requests?

Example 13.17 Solution

By Theorem 13.6, the internal and external request arrivals are independent Poisson processes with rates of 7 and 3 hits per second. In a 10-minute (600-second) interval, I and X are independent Poisson random variables with parameters $\alpha_I = 7(600) = 4200$ and $\alpha_X = 3(600) = 1800$ hits. The joint PMF of I and X is

$$
\begin{aligned}
P_{I,X}(i,x) &= P_I(i)\,P_X(x) \\
&= \begin{cases} \dfrac{(4200)^i e^{-4200}}{i!}\dfrac{(1800)^x e^{-1800}}{x!} & i,x \in \{0,1,\ldots\}, \\ 0 & \text{otherwise.} \end{cases} \qquad (1)
\end{aligned}
$$

Theorem 13.7

Let $N(t) = N_1(t) + N_2(t)$ be the sum of two independent Poisson processes with rates λ_1 and λ_2. Given that the $N(t)$ process has an arrival, the conditional probability that the arrival is from $N_1(t)$ is $\lambda_1/(\lambda_1 + \lambda_2)$.

Proof: Theorem 13.7

We can view $N_1(t)$ and $N_2(t)$ as being derived from a Bernoulli decomposition of $N(t)$ in which an arrival of $N(t)$ is labeled a type 1 arrival with probability $\lambda_1/(\lambda_1 + \lambda_2)$. By Theorem 13.6, $N_1(t)$ and $N_2(t)$ are independent Poisson processes with rate λ_1 and λ_2, respectively. Moreover, given an arrival of the $N(t)$ process, the conditional probability that an arrival is an arrival of the $N_1(t)$ process is also $\lambda_1/(\lambda_1 + \lambda_2)$.

Section 13.6

The Brownian Motion Process

Definition 13.10 Brownian Motion Process

A Brownian motion process $W(t)$ *has the property that* $W(0) = 0$*, and for* $\tau > 0$, $W(t+\tau) - W(t)$ *is a Gaussian* $(0, \sqrt{\alpha\tau})$ *random variable that is independent of* $W(t')$ *for all* $t' \leq t$.

Theorem 13.8

For the Brownian motion process $W(t)$, the PDF of

$$\mathbf{W} = \left[W(t_1), \ldots, W(t_k)\right]'$$

is

$$f_{\mathbf{W}}(\mathbf{w}) = \prod_{n=1}^{k} \frac{1}{\sqrt{2\pi\alpha(t_n - t_{n-1})}} e^{-(w_n - w_{n-1})^2/[2\alpha(t_n - t_{n-1})]}.$$

Proof: Theorem 13.8

Since $W(0) = 0$, $W(t_1) = X(t_1) - W(0)$ is a Gaussian random variable. Given time instants $t_1, \ldots, t_k$, we define $t_0 = 0$ and, for $n = 1, \ldots, k$, we can define the increments $X_n = W(t_n) - W(t_{n-1})$. Note that $X_1, \ldots, X_k$ are independent random variables such that X_n is Gaussian $(0, \sqrt{\alpha(t_n - t_{n-1})})$.

$$f_{X_n}(x) = \frac{1}{\sqrt{2\pi\alpha(t_n - t_{n-1})}} e^{-x^2/[2\alpha(t_n - t_{n-1})]}. \tag{1}$$

Note that $\mathbf{W} = \mathbf{w}$ if and only if $W_1 = w_1$ and for $n = 2, \ldots, k$, $X_n = w_n - w_{n-1}$. Although we omit some significant steps that can be found in Problem 13.6.5, this does imply

$$f_{\mathbf{W}}(\mathbf{w}) = \prod_{n=1}^{k} f_{X_n}(w_n - w_{n-1}). \tag{2}$$

The theorem follows from substitution of Equation (13.26) into Equation (13.27).

Quiz 13.6

Let $W(t)$ be a Brownian motion process with variance $\text{Var}[W(t)] = \alpha t$. Show that $X(t) = W(t)/\sqrt{\alpha}$ is a Brownian motion process with variance $\text{Var}[X(t)] = t$.

Quiz 13.6 Solution

First, we note that for $t > s$,

$$X(t) - X(s) = \frac{W(t) - W(s)}{\sqrt{\alpha}}. \tag{1}$$

Since $W(t) - W(s)$ is a Gaussian random variable, Theorem 4.13 states that $W(t) - W(s)$ is Gaussian with expected value

$$\mathrm{E}\left[X(t) - X(s)\right] = \frac{\mathrm{E}\left[W(t) - W(s)\right]}{\sqrt{\alpha}} = 0 \tag{2}$$

and variance

$$\mathrm{E}\left[(W(t) - W(s))^2\right] = \frac{\mathrm{E}\left[(W(t) - W(s))^2\right]}{\alpha} = \frac{\alpha(t-s)}{\alpha}. \tag{3}$$

Consider $s' \leq s < t$. Since $s \geq s'$, $W(t) - W(s)$ is independent of $W(s')$. This implies $[W(t) - W(s)]/\sqrt{\alpha}$ is independent of $W(s')/\sqrt{\alpha}$ for all $s \geq s'$. That is, $X(t) - X(s)$ is independent of $X(s')$ for all $s \geq s'$. Thus $X(t)$ is a Brownian motion process with variance $\mathrm{Var}[X(t)] = t$.

Section 13.7

Expected Value and Correlation

Definition 13.11 The Expected Value of a Process

The expected value of a stochastic process $X(t)$ *is the deterministic function*

$$\mu_X(t) = \mathsf{E}\left[X(t)\right].$$

Definition 13.12 Autocovariance

The autocovariance *function of the stochastic process* $X(t)$ *is*

$$\boxed{C_X(t,\tau)} = \text{Cov}\left[X(t), X(t+\tau)\right].$$

. . . .

The autocovariance *function of the random sequence* X_n *is*

$$\boxed{C_X[m,k]} = \text{Cov}\left[X_m, X_{m+k}\right].$$

$$\text{Cov}[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

Definition 13.13 Autocorrelation Function

The autocorrelation function *of the stochastic process* $X(t)$ *is*

$$R_X(t,\tau) = \mathsf{E}\left[X(t)X(t+\tau)\right].$$

. . . .

The autocorrelation *function of the random sequence* X_n *is*

$$R_X[m,k] = \mathsf{E}\left[X_m X_{m+k}\right].$$

Theorem 13.9

The autocorrelation and autocovariance functions of a process $X(t)$ satisfy

$$C_X(t, \tau) = R_X(t, \tau) - \mu_X(t)\mu_X(t + \tau).$$

. . . .

The autocorrelation and autocovariance functions of a random sequence X_n satisfy

$$C_X[n, k] = R_X[n, k] - \mu_X(n)\mu_X(n + k).$$

Quiz 13.7

$X(t)$ has expected value $\mu_X(t)$ and autocorrelation $R_X(t,\tau)$. We make the noisy observation $Y(t) = X(t) + N(t)$, where $N(t)$ is a random noise process independent of $X(t)$ with $\mu_N(t) = 0$ and autocorrelation $R_N(t,\tau)$. Find the expected value and autocorrelation of $Y(t)$.

Quiz 13.7 Solution

First we find the expected value

$$\mu_Y(t) = \mu_X(t) + \mu_N(t) = \mu_X(t). \tag{1}$$

To find the autocorrelation, we observe that since $X(t)$ and $N(t)$ are independent and since $N(t)$ has zero expected value,

$$\mathsf{E}[X(t)N(t')] = \mathsf{E}[X(t)]\,\mathsf{E}[N(t')] = 0.$$

Since $R_Y(t,\tau) = \mathsf{E}[Y(t)Y(t+\tau)]$, we have

$$\begin{aligned} R_Y(t,\tau) &= \mathsf{E}\left[(X(t)+N(t))\,(X(t+\tau)+N(t+\tau))\right] \\ &= \mathsf{E}\left[X(t)X(t+\tau)\right] + \mathsf{E}\left[X(t)N(t+\tau)\right] \\ &\qquad + \mathsf{E}\left[X(t+\tau)N(t)\right] + \mathsf{E}\left[N(t)N(t+\tau)\right] \\ &= R_X(t,\tau) + R_N(t,\tau). \end{aligned} \tag{2}$$