Stationary Processes

Definition 13.14 Stationary Process

A stochastic process $X(t)$ is stationary *if and only if for all sets of time instants* $t_1, \ldots, t_m$*, and any time difference* τ*,*

$$f_{X(t_1),\ldots,X(t_m)}(x_1, \ldots, x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1, \ldots, x_m).$$

. . . .

A random sequence X_n is stationary *if and only if for any set of integer time instants* $n_1, \ldots, n_m$*, and integer time difference* k*,*

$$f_{X_{n_1},\ldots,X_{n_m}}(x_1, \ldots, x_m) = f_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1, \ldots, x_m).$$

Example 13.21 Problem

Is the Brownian motion process with parameter α introduced in Section 13.6 stationary?

Example 13.21 Solution

For Brownian motion, $X(t_1)$ is the Gaussian $(0, \sqrt{\alpha\tau_1})$ random variable. Similarly, $X(t_2)$ is Gaussian $(0, \sqrt{\alpha\tau_2})$. Since $X(t_1)$ and $X(t_2)$ do not have the same variance, $f_{X(t_1)}(x) \neq f_{X(t_2)}(x)$, and the Brownian motion process is not stationary.

Theorem 13.10

Let $X(t)$ be a stationary random process. For constants $a > 0$ and b, $Y(t) = aX(t) + b$ is also a stationary process.

Theorem 13.11

For a stationary process $X(t)$, the expected value, the autocorrelation, and the autocovariance have the following properties for all t:

(a) $\mu_X(t) = \mu_X$,

(b) $R_X(t, \tau) = R_X(0, \tau) = R_X(\tau)$,

(c) $C_X(t, \tau) = R_X(\tau) - \mu_X^2 = C_X(\tau)$.

. . . .

For a stationary random sequence X_n the expected value, the autocorrelation, and the autocovariance satisfy for all n

(a) $\mathsf{E}[X_n] = \mu_X$,

(b) $R_X[n, k] = R_X[0, k] = R_X[k]$,

(c) $C_X[n, k] = R_X[k] - \mu_X^2 = C_X[k]$.

Basics

- Set Theory
- Conditional Probability
- Law of Total Probability
- Independence

Discrete random variable

- Probability mass function
- Cumulative distribution function
- Expected value and variance
- Families of discrete random variables (Bernoulli, Binomial, Geometric, Pascal, Uniform, Poisson)
- Derived variables

Continuous random variable

- Probability density function
- Cumulative distribution function
- Expected value and variance
- Families of continuous random variables (Uniform, Gaussian)

Joint Random Variables

- Joint Cumulative Distribution Function
- Joint Probability Mass Function
  - Marginal PMF
- Joint Probability Density Function
  - Marginal PDF
- Independence, Covariance and Correlation
- Expectation and Variance
- Bivariate Gaussian Variables

1. Random variables X and Y have the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} cxy & x = 1,2,3,4;\ y = 1,3 \\ 0 & otherwise \end{cases}$$

   a. What is the value of c?
   b. What is P[Y<X]?
   c. What is P[Y>X]?
   d. What is P[Y=X]?
   e. Find the marginal PMF $P_X(x)$ and $P_Y(y)$.
   f. Determine if X and Y independent. Justify your answer.
   g. Find the expected value of W=Y/X?
   h. Find the correlation $r_{X,Y} = E[XY]$
   i. Find covariance Cov[X,Y].
   j. Find the correlation coefficient, $\rho_{X,Y}$.
   k. Find the variance Var[X+Y].

a. $c\,(1\times1+1\times2+1\times3+1\times4+3\times1+3\times2+3\times3+3\times4)=1$

$c=\frac{1}{40}$

b. $P[Y<X]=c\,(1\times2+1\times3+1\times4+3\times4)$

$=\frac{21}{40}$

c $P[Y>X]=c\,(1\times3+2\times3)=\frac{9}{40}$

d $P[Y=X]=c\,(1\times1+3\times3)=\frac{1}{4}$

e)

$$P_X(x)\begin{cases} P_X(1)=P_{XY}(1,1)+P_{XY}(1,3)=c\,(1+3)=\frac{4}{40}=\frac{1}{10}\\ P_X(2)=P_{XY}(2,1)+P_{XY}(2,3)=c\,(2+6)=\frac{8}{40}=\frac{1}{5}\\ P_X(3)=P_{XY}(3,1)+P_{XY}(3,3)=c\,(3+9)=\frac{12}{40}=\frac{3}{10}\\ P_X(4)=P_{XY}(4,1)+P_{XY}(4,3)=c\,(4+12)=\frac{16}{40}=\frac{2}{5} \end{cases}$$

$$P_Y(y)\begin{cases} P_Y(1)=P_{XY}(1,1)+P_{XY}(2,1)+P_{XY}(3,1)+P_{XY}(4,1)\\ \quad =c\,(1+2+3+4)=\frac{10}{40}=\frac{1}{4}\\ P_Y(3)=P_{XY}(1,3)+P_{XY}(2,3)+P_{XY}(3,3)+P_{XY}(4,3)\\ \quad =c\,(3+6+9+12)=\frac{30}{40}=\frac{3}{4} \end{cases}$$

(f) Yes, X & Y are ind. because

$P_{XY}(x,y)=P_X(x)\cdot P_Y(y)\ \forall x,y$

(g) $E[W]=\sum_{x=1}^{4}\sum_{y=\{1,3\}}\frac{y}{x}P_{XY}(x,y)$

$=c\left(\frac{1}{1}\times1+\frac{3}{1}\times3+\frac{1}{2}\times2+\frac{3}{2}\times6\right.$

$\left.+\frac{1}{3}\times3+\frac{3}{3}\times9+\frac{1}{4}\times4+\frac{3}{4}\times12\right)$

$=\frac{1}{40}(1+9+1+9+1+9+1+9)=1$

(h) $r_{XY}=E[XY]=\sum_{x=1}^{4}\sum_{y\in\{1,3\}}xy\,\overset{cxy}{P_{XY}(x,y)}$

$=c\sum_{x=1}^{4}\sum_{y\in\{1,3\}}x^2y^2$

$=c\left(\sum_{x=1}^{4}x^2\right)\left(\sum_{y=\{1,3\}}y^2\right)=c\,(1+4+9+16)(1+9)$

$=\frac{1}{40}\times30\times10=\frac{15}{2}$

(i) $\mathrm{Cov}[X,Y] = r_{XY} - E[X]\cdot E[Y]$

$$E[X] = \sum_{x=1}^{4} x\,P_X(x) = 1\times\frac{1}{10} + 2\times\frac{1}{5} + 3\times\frac{3}{10} + 4\times\frac{2}{5}$$

$$= E[X^2] - (E[X])^2 + E[Y^2] - (E[Y])^2$$

$$= \frac{1}{10} + \frac{2}{5} + \frac{9}{10} + \frac{8}{5} = 3$$

$$E[Y] = \sum_{y=\{1,3\}} y\,P_Y(y) = 1\times\frac{1}{4} + 3\times\frac{3}{4} = \frac{5}{2}$$

$$\therefore\ \mathrm{Cov}[X,Y] = \frac{15}{2} - 3\times\frac{5}{2} = 0$$

(j) $$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X\,\sigma_Y} = 0$$

(k) Since X & Y are uncorrelated,

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$$

$$= E[X^2] - (E[X])^2 + E[Y^2] - (E[Y])^2$$

$$E[X^2] = \sum_{x=1}^{4} x^2 P_X(x) = 1\times\frac{1}{10} + 4\times\frac{1}{5} + 9\times\frac{3}{10} + 16\times\frac{2}{5}$$

$$= \frac{100}{10} = 10$$

$$E[Y^2] = \sum_{y=\{1,3\}} y^2 P_Y(y) = 1\times\frac{1}{4} + 9\times\frac{3}{4}$$

$$= 7$$

$$\therefore\ \mathrm{Var}[X+Y] = 10 - 3^2 + 7 - \left(\frac{5}{2}\right)^2$$

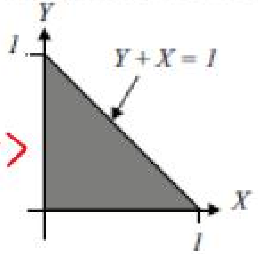
$$= 8 - \frac{25}{4} = \frac{7}{4}$$

2. Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c & x \geq 0, y \geq 0, x + y \leq 1 \\ 0 & otherwise \end{cases}$$

a. What is the value of the constant c?
b. What is $P[X \leq Y]$?
c. What is $P[X + Y \leq \frac{1}{2}]$?
d. Find the marginal PDF $f_X(x)$ and $f_Y(y)$
e. Are X and Y independent? Justify your answer.

(a) The joint PDF of X and Y is



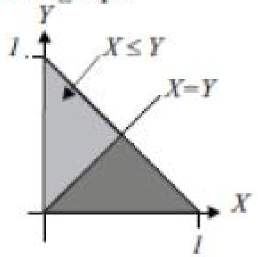
$$f_{X,Y}(x,y) = \begin{cases} c & x + y \le 1, x, y \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

To find the constant c we integrate over the region shown. This gives

$$\int_0^1 \int_0^{1-x} c\, dy\, dx = cx - \frac{cx}{2}\Big|_0^1 = \frac{c}{2} = 1. \tag{1}$$

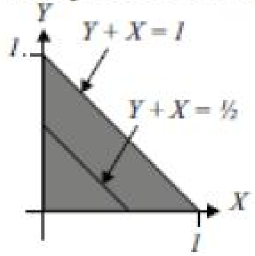
Therefore $c = 2$.

(b) To find the $\mathrm{P}[X \le Y]$ we look to integrate over the area indicated by the graph



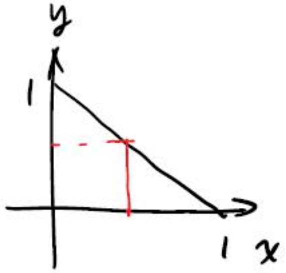
$$\begin{aligned} \mathrm{P}[X \le Y] &= \int_0^{1/2} \int_x^{1-x} dy\, dx \\ &= \int_0^{1/2} (2 - 4x)\, dx \\ &= 1/2. \end{aligned} \tag{2}$$

(c) The probability $\mathrm{P}[X + Y \le 1/2]$ can be seen in the figure. Here we can set up the following integrals



$$\begin{aligned} \mathrm{P}[X + Y \le 1/2] &= \int_0^{1/2} \int_0^{1/2-x} 2\, dy\, dx \\ &= \int_0^{1/2} (1 - 2x)\, dx \\ &= 1/2 - 1/4 = 1/4. \end{aligned} \tag{3}$$

(d) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$



$$= \int_{0}^{1-x} 2\, dy = 2(1-x);\ 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$$

$$= \int_{0}^{1-y} 2\, dy = 2(1-y);\ 0 < y < 1$$

(e) Since $f_{XY}(x, y) \neq f_X(x) f_Y(y)$

X & Y are NOT ind

3. Observe 100 independent flips of a fair coin. Let X equal the number of heads in the first 75 flips and Y equal the number heads in the remaining 25 flips. Find $P_X(x)$ and $P_Y(y)$. Are X and Y independent (i.e. is outcome of X impact outcome of Y?)? Find $P_{X,Y}(x,y)$.

3. Observe 100 independent flips of a fair coin. Let X equal the number of heads in the first 75 flips and Y equal the number heads in the remaining 25 flips. Find $P_X(x)$ and $P_Y(y)$. Are X and Y independent (i.e. is outcome of X impact outcome of Y?)? Find $P_{X,Y}(x,y)$.

Solution:

a. X is a Binomial random variable with n= 75 and p=0.5. Therefore, the PMF of X is

$$P_X(x) = \binom{75}{x} 0.5^x (1-0.5)^{75-x} = \binom{75}{x} 0.5^{75}$$

a. Y is a Binomial random variable with n= 25 and p=0.5. Therefore, the PMF of X is

$$P_Y(y) = \binom{25}{y} 0.5^y (1-0.5)^{25-y} = \binom{25}{y} 0.5^{25}$$

b. Because the outcome of X does not impact outcome of Y, therefore X & Y are independent.

c. Because X & Y are independent, therefore

$$P_{X,Y}(x,y) = P_X(x)P_Y(y) = \binom{75}{x}\binom{25}{y} 0.5^{100}$$

Sample Theorem

- Expected value of sums
- Central limit theorem
  - CDF of Gaussian
- Sample mean
  - Expected value and variance
  - Inequalities in Probability
    - Markov, Chebyshev
- Point estimates of model parameters
  - Mean Square Error

$$\mathsf{E}[W_n] = \mathsf{E}[X_1] + \mathsf{E}[X_2] + \cdots + \mathsf{E}[X_n]$$

$$\mathrm{Var}[W_n] = \sum_{i=1}^{n} \mathrm{Var}[X_i] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{Cov}\left[X_i, X_j\right].$$

$$M_n(X) = \frac{X_1 + \cdots + X_n}{n}.$$

$$\mathsf{E}[M_n(X)] = \mathsf{E}[X], \qquad \mathrm{Var}[M_n(X)] = \frac{\mathrm{Var}[X]}{n}.$$

$$e = \mathsf{E}\left[(\hat{R} - r)^2\right].$$