第7版 Sec. 6.2 10, 38, 40, 42

第8版 Sec. 6.2 12, 40, 42, 44

***12.** Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

12. The midpoint of the segment whose endpoints are (a, b) and (c, d) is $((a+c)/2, (b+d)/2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and c have the same parity (both odd or both even) and b and d have the same parity. Thus what matters in this problem is the parities of the coordinates. There are four possible pairs of parities: (odd, odd), (odd, even), (even, odd), and (even, even). Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

40. Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers. Justify your answer.

40. This is similar to Example 9. Label the computers C_1 through C_8, and label the printers P_1 through P_4. If we connect C_k to P_k for $k = 1, 2, 3, 4$ and connect each of the computers C_5 through C_8 to *all* the printers, then we have used a total of $4 + 4 \cdot 4 = 20$ cables. Clearly this is sufficient, because if computers C_1 through C_4 need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, since they are connected to all the printers. Now we must show that 19 cables are not enough. Since there are 19 cables and 4 printers, the average number of computers per printer is $19/4$, which is less than 5. Therefore some printer must be connected to fewer than 5 computers (the average of a set of numbers cannot be bigger than each of the numbers in the set). That means it is connected to 4 or fewer computers, so there are at least 4 computers that are not connected to it. If those 4 computers all needed a printer simultaneously, then they would be out of luck, since they are connected to at most the 3 other printers.

***42.** Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

42. Let $K(x)$ be the number of other people at the party that person x knows. The possible values for $K(x)$ are $0, 1, \ldots, n-1$, where $n \geq 2$ is the number of people at the party. We cannot apply the pigeonhole principle directly, since there are n pigeons and n pigeonholes. However, it is impossible for both 0 and $n-1$ to be in the range of K, since if one person knows everybody else, then nobody can know no one else (we assume that "knowing" is symmetric). Therefore the range of K has at most $n-1$ elements, whereas the domain has n elements, so K is not one-to-one, precisely what we wanted to prove.

43. An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 P.M., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.

***44.** Is the statement in Exercise 43 true if 24 is replaced by

a) 2? **b)** 23? **c)** 25? **d)** 30?

44. **a)** The solution of Exercise 43, with 24 replaced by 2 and 149 replaced by 127, tells us that the statement is true.

b) The solution of Exercise 43, with 24 replaced by 23 and 149 replaced by 148, tells us that the statement is true.

c) We begin in a manner similar to the solution of Exercise 43. Look at a_1, a_2, $\ldots$, a_{75}, $a_1 + 25$, $\ldots$, $a_{75} + 25$, where a_i is the total number of matches played up through and including hour i. Then $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$, and $26 \leq a_1 + 25 < a_2 + 25 < \cdots < a_{75} + 25 \leq 150$. Now either these 150 numbers are precisely all the number from 1 to 150, or else by the pigeonhole principle we get, as in Exercise 43, $a_i = a_j + 25$ for some i and j and we are done. In the former case, however, since each of the numbers $a_i + 25$ is greater than or equal to 26, the numbers $1, 2, \ldots, 25$ must all appear among the a_i's. But since the a_i's are increasing, the only way this can happen is if $a_1 = 1$, $a_2 = 2$, $\ldots$, $a_{25} = 25$. Thus there were exactly 25 matches in the first 25 hours.

d) We need a different approach for this part, an approach, incidentally, that works for many numbers besides 30 in this setting. Let a_1, a_2, $\ldots$, a_{75} be as before, and note that $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$. By the pigeonhole principle two of the numbers among a_1, a_2, $\ldots$, a_{31} are congruent modulo 30. If they differ by 30, then we have our solution. Otherwise they differ by 60 or more, so $a_{31} \geq 61$. Similarly, among a_{31} through a_{61}, either we find a solution, or two numbers must differ by 60 or more; therefore we can assume that $a_{61} \geq 121$. But this means that $a_{66} \geq 126$, a contradiction.

第7版 Sec 6.3: 20, 42, 44

第8版 Sec 6.3: 20, 44, 46

20. How many bit strings of length 10 have
a) exactly three 0s?
b) more 0s than 1s?
c) at least seven 1s?
d) at least three 1s?

20. **a)** There are $C(10, 3)$ ways to choose the positions for the 0's, and that is the only choice to be made, so the answer is $C(10, 3) = 120$.

b) There are more 0's than 1's if there are fewer than five 1's. Using the same reasoning as in part (a), together with the sum rule, we obtain the answer $C(10, 0) + C(10, 1) + C(10, 2) + C(10, 3) + C(10, 4) = 1 + 10 + 45 + 120 + 210 = 386$. Alternatively, by symmetry, half of all cases in which there are not five 0's have more 0's than 1's; therefore the answer is $(2^{10} - C(10, 5)/2 = (1024 - 252)/2 = 386$.

c) We want the number of bit strings with 7, 8, 9, or 10 1's. By the same reasoning as above, there are $C(10, 7) + C(10, 8) + C(10, 9) + C(10, 10) = 120 + 45 + 10 + 1 = 176$ such strings.

d) If a string does not have at least three 1's, then it has 0, 1, or 2 1's. There are $C(10, 0) + C(10, 1) + C(10, 2) = 1 + 10 + 45 = 56$ such strings. There are $2^{10} = 1024$ strings in all. Therefore there are $1024 - 56 = 968$ strings with at least three 1's.

44. Find a formula for the number of ways to seat r of n people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.

44. The only difference between this problem and the problem solved in Exercise 43 is a factor of 2. Each seating under the rules here corresponds to two seatings under the original rules, because we can change the order of people around the table from clockwise to counterclockwise. Therefore we need to divide the formula there by 2, giving us $n!/(2r(n - r)!)$. This assumes that $r \geq 3$. If $r = 1$ then the problem is trivial (there are n choices under both sets of rules). If $r = 2$, then we do not introduce the extra factor of 2, because clockwise order and counterclockwise order are the same. In this case, both answers are just $n!/(2(n - 2)!)$, which is $C(n, 2)$, as one would expect.

***46.** How many ways are there for a horse race with four horses to finish if ties are possible? [*Note:* Any number of the four horses may tie.]

46. We can solve this problem by breaking it down into cases depending on the number of ties. There are five cases. (1) If there are no ties, then there are clearly $P(4,4) = 24$ possible ways for the horses to finish. (2) Assume that there are two horses that tie, but the others have distinct finishes. There are $C(4,2) = 6$ ways to choose the horses to be tied; then there are $P(3,3) = 6$ ways to determine the order of finish for the three groups (the pair and the two single horses). Thus there are $6 \cdot 6 = 36$ ways for this to happen. (3) There might be two groups of two horses that are tied. There are $C(4,2) = 6$ ways to choose the winners (and the other two horses are the losers). (4) There might be a group of three horses all tied. There are $C(4,3) = 4$ ways to choose which these horses will be, and then two ways for the race to end (the tied horses win or they lose), so there are $4 \cdot 2 = 8$ possibilities. (5) There is only one way for all the horses to tie. Putting this all together, the answer is $24 + 36 + 6 + 8 + 1 = 75$.

第7版 Sec.6.4: 14, 22, 26, 30

第8版 Sec.6.4: 18, 26, 30, 34

18. Show that if n is a positive integer, then $1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1$.

18. Using the factorial formulae for computing binomial coefficients, we see that $\binom{n}{k-1} = \frac{k}{n-k+1}\binom{n}{k}$. If $k \le n/2$, then $\frac{k}{n-k+1} < 1$, so the "less than" signs are correct. Similarly, if $k > n/2$, then $\frac{k}{n-k+1} > 1$, so the "greater than" signs are correct. The middle equality is Corollary 2 in Section 6.3, since $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. The equalities at the ends are clear.

26. Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r \le n$ and $k \le r$,

a) using a combinatorial argument.

b) using an argument based on the formula for the number of r-combinations of a set with n elements.

26. a) Suppose that we have a set with n elements, and we wish to choose a subset A with k elements and another, disjoint, subset with $r - k$ elements. The left-hand side gives us the number of ways to do this, namely the product of the number of ways to choose the r elements that are to go into one or the other of the subsets and the number of ways to choose which of *these* elements are to go into the first of the subsets. The right-hand side gives us the number of ways to do this as well, namely the product of the number of ways to choose the first subset and the number of ways to choose the second subset from the elements that remain.

b) On the one hand,

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!} \cdot \frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!},$$

and on the other hand

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{(r-k)!(n-r)!} = \frac{n!}{k!(n-r)!(r-k)!}.$$

29. Let n be a positive integer. Show that

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}/2.$$

***30.** Let n and k be integers with $1 \le k \le n$. Show that

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1} = \binom{2n+2}{n+1}/2 - \binom{2n}{n}.$$

30. First, use Exercise 29 to rewrite the right-hand side of this identity as $\binom{2n}{n+1}$. We give a combinatorial proof, showing that both sides count the number of ways to choose from collection of n men and n women, a subset that has one more man than woman. For the left-hand side, we note that this subset must have k men and $k-1$ women for some k between 1 and n, inclusive. For the (modified) right-hand side, choose any set of $n+1$ people from this collection of n men and n women; the desired subset is the set of men chosen and the women left behind.

***34.** Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$. [*Hint:* Count in two ways the number of ways to select a committee, with n members from a group of n mathematics professors and n computer science professors, such that the chairperson of the committee is a mathematics professor.]

34. We follow the hint. The number of ways to choose this committee is the number of ways to choose the chairman from among the n mathematicians (n ways) times the number of ways to choose the other $n-1$ members of the committee from among the other $2n-1$ professors. This gives us $n\binom{2n-1}{n-1}$, the expression on the right-hand side. On the other hand, for each k from 1 to n, we can have our committee consist of k mathematicians and $n-k$ computer scientists. There are $\binom{n}{k}$ ways to choose the mathematicians, k ways to choose the chairman from among these, and $\binom{n}{n-k}$ ways to choose the computer scientists. Since this last quantity equals $\binom{n}{k}$, we obtain the expression on the left-hand side of the identity.

第七版 Sec. 6.5 10, 16, 26, 32, 46, 50, 54, 61

第八版 Sec. 6.5 10, 16, 28, 34, 48, 52, 56, 63

10. A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose

a) a dozen croissants?
b) three dozen croissants?
c) two dozen croissants with at least two of each kind?
d) two dozen croissants with no more than two broccoli croissants?
e) two dozen croissants with at least five chocolate croissants and at least three almond croissants?
f) two dozen croissants with at least one plain croissant, at least two cherry croissants, at least three chocolate croissants, at least one almond croissant, at least two apple croissants, and no more than three broccoli croissants?

10. a) $C(6+12-1,12) = C(17,12) = 6188$ **b)** $C(6+36-1,36) = C(41,36) = 749{,}398$

c) If we first pick the two of each kind, then we have picked $2 \cdot 6 = 12$ croissants. This leaves one dozen left to pick without restriction, so the answer is the same as in part (a), namely $C(6+12-1,12) = C(17,12) = 6188$.

d) We first compute the number of ways to violate the restriction, by choosing at least three broccoli croissants. This can be done in $C(6+21-1,21) = C(26,21) = 65780$ ways, since once we have picked the three broccoli croissants there are 21 left to pick without restriction. Since there are $C(6+24-1,24) = C(29,24) = 118755$ ways to pick 24 croissants without any restriction, there must be $118755 - 65780 = 52{,}975$ ways to choose two dozen croissants with no more than two broccoli.

e) Eight croissants are specified, so this problem is the same as choosing $24-8=16$ croissants without restriction, which can be done in $C(6+16-1,16) = C(21,16) = 20{,}349$ ways.

f) First let us include all the lower bound restrictions. If we choose the required 9 croissants, then there are $24-9=15$ left to choose, and if there were no restriction on the broccoli croissants then there would be $C(6+15-1,15) = C(20,15) = 15504$ ways to make the selections. If in addition we were to violate the broccoli restriction by choosing at least four broccoli croissants, there would be $C(6+11-1,11) = C(16,11) = 4368$ choices. So the number of ways to make the selection without violating the restriction is $15504-4368 = 11{,}136$.

16. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where x_i, $i = 1, 2, 3, 4, 5, 6$, is a nonnegative integer such that

a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?
b) $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5$, and $x_6 \geq 6$?
c) $x_1 \leq 5$?
d) $x_1 < 8$ and $x_2 > 8$?

16. **a)** We require each $x_i \geq 2$. This uses up 12 of the 29 total required, so the problem is the same as finding the number of solutions to $x_1' + x_2' + x_3' + x_4' + x_5' + x_6' = 17$ with each x_i' a nonnegative integer. The number of solutions is therefore $C(6+17-1,17) = C(22,17) = 26{,}334$.

b) The restrictions use up 22 of the total, leaving a free total of 7. Therefore the answer is $C(6+7-1,7) = C(12,7) = 792$.

c) The number of solutions without restriction is $C(6+29-1,29) = C(34,29) = 278256$. The number of solution violating the restriction by having $x_1 \geq 6$ is $C(6+23-1,23) = C(28,23) = 98280$. Therefore the answer is $278256 - 98280 = 179{,}976$.

d) The number of solutions with $x_2 \geq 9$ (as required) but without the restriction on x_1 is $C(6+20-1,20) = C(25,20) = 53130$. The number of solution violating the additional restriction by having $x_1 \geq 8$ is $C(6+12-1,12) = C(17,12) = 6188$. Therefore the answer is $53130 - 6188 = 46{,}942$.

28. How many positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13?

28. We can model this problem by letting x_i be the i^{th} digit of the number for $i = 1,2,3,4,5,6$, and asking for the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 13$, where each x_i is between 0 and 8, inclusive, except that one of them equals 9. First, there are 6 ways to decide which of the digits is 9. Without loss of generality assume that $x_6 = 9$. Then the number of ways to choose the remaining digits is the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 4$ (note that the restriction that each $x_i \leq 8$ was moot, since the sum was only 4). By Theorem 2 there are $C(5+4-1,4) = C(8,4) = 70$ solutions. Therefore the answer is $6 \cdot 70 = 420$.

34. How many different strings can be made from the letters in *AARDVARK*, using all the letters, if all three *A*s must be consecutive?

34. We can treat the 3 consecutive A's as one letter. Thus we have 6 letters, of which 2 are the same (the two R's), so by Theorem 3 the answer is $6!/2! = 360$.

48. A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen? [*Hint:* Represent the books that are chosen by bars and the books not chosen by stars. Count the number of sequences of five bars and seven stars so that no two bars are adjacent.]

48. We follow the hint. There are 5 bars (chosen books), and therefore there are 6 places where the 7 stars (nonchosen books) can fit (before the first bar, between the first and second bars, ..., after the fifth bar). Each of the second through fifth of these slots must have at least one star in it, so that adjacent books are not chosen. Once we have placed these 4 stars, there are 3 stars left to be placed in 6 slots. The number of ways to do this is therefore $C(6+3-1,3) = C(8,3) = 56$.

52. How many ways are there to distribute five distinguishable objects into three indistinguishable boxes?

52. This is actually a problem about partitions of sets. Let us call the set of 5 objects $\{a,b,c,d,e\}$. We want to partition this set into three pairwise disjoint subsets (some possibly empty). We count in a fairly ad hoc way. First, we could put all five objects into one subset (i.e., all five objects go into one box, with the other two boxes empty). Second, we could put four of the objects into one subset and one into another, such as $\{a,b,c,d\}$ together with $\{e\}$. There are 5 ways to do this, since each of the five objects can be the singleton. Third, we could put three of the objects into one set (box) and two into another; there are $C(5,2) = 10$ ways to do this, since there are that many ways to choose which objects are to be the doubleton. Similarly, there are 10 ways to distribute the elements so that three go into one set and one each into the other two sets (for example, $\{a,b,c\}$, $\{d\}$, and $\{e\}$). Finally, we could put two items into one set, two into another, and one into the third (for example, $\{a,b\}$, $\{c,d\}$, and $\{e\}$). Here we need to choose the singleton (5 ways), and then we need to choose one of the 3 ways to separate the remaining four elements into pairs; this gives a total of 15 partitions. In all we have 41 different partitions.

This can also be solved by using the formulae given in the text in a discussion of Stirling numbers of the second kind (this follows Example 10):

$$S(5,1) = \frac{1}{1!}\left(\binom{1}{0}1^5\right) = \frac{1}{1!}(1) = 1$$

$$S(5,2) = \frac{1}{2!}\left(\binom{2}{0}2^5 - \binom{2}{1}1^5\right) = \frac{1}{2!}(32-2) = 15$$

$$S(5,3) = \frac{1}{3!}\left(\binom{3}{0}3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5\right) = \frac{1}{3!}(243-96+3) = 25$$

$$\sum_{j=1}^{3} S(5,j) = 1 + 15 + 25 = 41$$

56. How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes?

56. We are asked for the partitions of 5 into at most 3 parts; notice that we are not required to use all three boxes. We can easily list these partitions explicitly: $5 = 5$, $5 = 4 + 1$, $5 = 3 + 2$, $5 = 3 + 1 + 1$, and $5 = 2 + 2 + 1$. Therefore the answer is 5.

***63.** Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X, the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?

63. 90,720 $= \underbrace{(C_{10}^2 - 9)}_{\text{X的排列}} \cdot \underbrace{(C_8^2 C_6^2 C_4^2 \cdot C_2^2)}_{\text{剩余4个地方的排列}}$

Sec. 6.6 6(f), 7, 9

6. Find the next larger permutation in lexicographic order after each of these permutations.

a) 1342 **b)** 45321 **c)** 13245
d) 612345 **e)** 1623547 **f)** 23587416

6. These can be done using Algorithm 1 or Example 2. This will be explained in detail for part (a); the others are similar. In the last four parts of this exercise, the next permutation exchanges only the last two elements.
a) The last pair of integers a_j and a_{j+1} where $a_j < a_{j+1}$ is $a_2 = 3$ and $a_3 = 4$. The least integer to the right of 3 that is greater than 3 is 4. Hence 4 is placed in the second position. The integers 2 and 3 are then placed in order in the last two positions, giving the permutation 1423.

b) 51234 c) 13254 d) 612354 e) 1623574 f) 23587461

7. Use Algorithm 1 to generate the 24 permutations of the first four positive integers in lexicographic order.

9. Use Algorithm 3 to list all the 3-combinations of $\{1, 2, 3, 4, 5\}$.

CCC2 **5. a)** 2134 **b)** 54132 **c)** 12534 **d)** 45312 **7.** 1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321 **9.** $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\}$