- 第7版：Sec. 5.3 6(a,d), 14, 29(a)
- 第8版：Sec. 5.3 6(a,d), 14, 31(a)

6. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer and prove that your formula is valid.

a) $f(0) = 1, f(n) = -f(n-1)$ for $n \geq 1$

b) $f(0) = 1,\ f(1) = 0,\ f(2) = 2,\ f(n) = 2f(n-3)$ for $n \geq 3$

c) $f(0) = 0, f(1) = 1, f(n) = 2f(n+1)$ for $n \geq 2$

d) $f(0) = 0, f(1) = 1, f(n) = 2f(n-1)$ for $n \geq 1$

e) $f(0) = 2$, $f(n) = f(n-1)$ if n is odd and $n \geq 1$ and $f(n) = 2f(n-2)$ if $n \geq 2$

key

6. a) This is valid, since we are provided with the value at $n = 0$, and each subsequent value is determined by the previous one. Since all that changes from one value to the next is the sign, we conjecture that $f(n) = (-1)^n$. This is true for $n = 0$, since $(-1)^0 = 1$. If it is true for $n = k$, then we have $f(k+1) = -f(k+1-1) = -f(k) = -(-1)^k$ by the inductive hypothesis, whence $f(k+1) = (-1)^{k+1}$.

b) This is valid, since we are provided with the values at $n = 0$, 1, and 2, and each subsequent value is determined by the value that occurred three steps previously. We compute the first several terms of the sequence: 1, 0, 2, 2, 0, 4, 4, 0, 8, ... We conjecture the formula $f(n) = 2^{n/3}$ when $n \equiv 0 \pmod 3$, $f(n) = 0$ when $n \equiv 1 \pmod 3$, $f(n) = 2^{(n+1)/3}$ when $n \equiv 2 \pmod 3$. To prove this, first note that in the base cases we have $f(0) = 1 = 2^{0/3}$, $f(1) = 0$, and $f(2) = 2 = 2^{(2+1)/3}$. Assume the inductive hypothesis that the formula is valid for smaller inputs. Then for $n \equiv 0 \pmod 3$ we have $f(n) = 2f(n-3) = 2 \cdot 2^{(n-3)/3} = 2 \cdot 2^{n/3} \cdot 2^{-1} = 2^{n/3}$, as desired. For $n \equiv 1 \pmod 3$ we have $f(n) = 2f(n-3) = 2 \cdot 0 = 0$, as desired. And for $n \equiv 2 \pmod 3$ we have $f(n) = 2f(n-3) = 2 \cdot 2^{(n-3+1)/3} = 2 \cdot 2^{(n+1)/3} \cdot 2^{-1} = 2^{(n+1)/3}$, as desired.

c) This is invalid. We are told that $f(2)$ is defined in terms of $f(3)$, but $f(3)$ has not been defined.

d) This is invalid, because the value at $n = 1$ is defined in two conflicting ways—first as $f(1) = 1$ and then as $f(1) = 2f(1-1) = 2f(0) = 2 \cdot 0 = 0$.

e) This appears syntactically to be not valid, since we have conflicting instruction for odd $n \geq 3$. On the one hand $f(3) = f(2)$, but on the other hand $f(3) = 2f(1)$. However, we notice that $f(1) = f(0) = 2$ and $f(2) = 2f(0) = 4$, so these apparently conflicting rules tell us that $f(3) = 4$ on the one hand and $f(3) = 2 \cdot 2 = 4$ on the other hand. Thus we got the same answer either way. Let us show that in fact this definition is valid because the rules coincide.

We compute the first several terms of the sequence: 2, 2, 4, 4, 8, 8, ... We conjecture the formula $f(n) = 2^{\lceil (n+1)/2 \rceil}$. To prove this inductively, note first that $f(0) = 2 = 2^{\lceil (0+1)/2 \rceil}$. For larger values we have for n odd using the first part of the recursive step that $f(n) = f(n-1) = 2^{\lceil (n-1+1)/2 \rceil} = 2^{\lceil n/2 \rceil} = 2^{\lceil (n+1)/2 \rceil}$, since $n/2$ is not an integer. For $n \geq 2$, whether even or odd, using the second part of the recursive step we have $f(n) = 2f(n-2) = 2 \cdot 2^{\lceil (n-2+1)/2 \rceil} = 2 \cdot 2^{\lceil (n+1)/2 \rceil - 1} = 2 \cdot 2^{\lceil (n+1)/2 \rceil} \cdot 2^{-1} = 2^{\lceil (n+1)/2 \rceil}$, as desired.

***14.** Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

Key

14. The basis step ($n = 1$) is clear, since $f_2 f_0 - f_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1$. Assume the inductive hypothesis. Then we have

$$\begin{aligned} f_{n+2}f_n - f_{n+1}^2 &= (f_{n+1} + f_n)f_n - f_{n+1}^2 \\ &= f_{n+1}f_n + f_n^2 - f_{n+1}^2 \\ &= -f_{n+1}(f_{n+1} - f_n) + f_n^2 \\ &= -f_{n+1}f_{n-1} + f_n^2 \\ &= -(f_{n+1}f_{n-1} - f_n^2) \\ &= -(-1)^n = (-1)^{n+1}. \end{aligned}$$

31. Give a recursive definition of each of these sets of ordered pairs of positive integers. Use structural induction to prove that the recursive definition you found is correct. [*Hint:* To find a recursive definition, plot the points in the set in the plane and look for patterns.]

a) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, \text{ and } a + b \text{ is even}\}$

b) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, \text{ and } a \text{ or } b \text{ is odd}\}$

c) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, a + b \text{ is odd, and } 3 \mid b\}$

Key for 31(a)

and $a + 2 \leq 2(b + 1)$. **31. a)** Define S by $(1, 1) \in S$, and if $(a, b) \in S$, then $(a + 2, b) \in S$, $(a, b + 2) \in S$, and $(a+1, b+1) \in S$. All elements put in S satisfy the condition, because $(1, 1)$ has an even sum of coordinates, and if (a, b) has an even sum of coordinates, then so do $(a + 2, b)$, $(a, b + 2)$, and $(a + 1, b + 1)$. Conversely, we show by induction on the sum of the coordinates that if $a + b$ is even, then $(a, b) \in S$. If the sum is 2, then $(a, b) = (1, 1)$, and the basis step put (a, b) into S. Otherwise the sum is at least 4, and at least one of $(a-2, b)$, $(a, b-2)$, and $(a-1, b-1)$ must have positive integer coordinates whose sum is an even number smaller than $a+b$, and therefore must be in S. Then one application of the recursive step shows that $(a, b) \in S$. **b)** Define S by $(1, 1)$,

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29. Devise a recursive algorithm to find the nth term of the sequence defined by $a_0 = 1, a_1 = 2$, and $a_n = a_{n-1} \cdot a_{n-2}$, for $n = 2, 3, 4, \ldots$.

Key

29. procedure a(n: nonnegative integer)
  if $n = 0$ **then return** 1
  else if $n = 1$ **then return** 2
  else return $a(n-1) \cdot a(n-2)$

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41. A **palindrome** is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

Key: If n is even, $2^{n/2}$; if n is odd, $2^{(n+1)/2}$.

58. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)

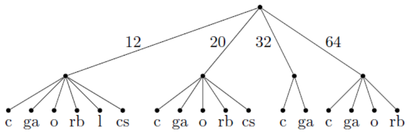
58. We need to compute the number of variable names of length i for $i = 1, 2, \ldots, 8$, and add. A variable name of length i is specified by choosing a first character, which can be done in 53 ways ($2 \cdot 26$ letters and 1 underscore to choose from), and $i - 1$ other characters, each of which can be done in $53 + 10 = 63$ ways. Therefore the answer is

$$\sum_{i=1}^{8} \underset{53}{\cancel{52}} \cdot 63^{i-1} = \underset{53}{\cancel{52}} \cdot \frac{63^8 - 1}{63 - 1} \approx 2.1 \times 10^{14}.$$

70. a) Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles, and all but lemonade and cream soda are available in 64-ounce bottles?

b) Answer the question in part (a) using counting rules.

70. a) It is more convenient to branch on bottle size first. Note that there are a different number of branches coming off each of the nodes at the second level. The number of leaves in the tree is 17, which is the answer.



b) We can add the number of different varieties for each of the sizes. The 12-ounce bottle has 6, the 20-ounce bottle has 5, the 32-once bottle has 2, and the 64-ounce bottle has 4. Therefore $6+5+2+4=17$ different types of bottles need to be stocked.