- 第七版 Sec. 9.1 7(a,c,h), 26, 32, 47, 51
- 第八版 Sec. 9.1 7(a,c,h), 26, 32, 49, 53

7. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x \neq y$. **b)** $xy \geq 1$.
c) $x = y + 1$ or $x = y - 1$.
d) $x \equiv y \pmod 7$. **e)** x is a multiple of y.
f) x and y are both negative or both nonnegative.
g) $x = y^2$. **h)** $x \geq y^2$.

7. a) Symmetric **b)** Symmetric, transitive **c)** Symmetric **d)** Reflexive, symmetric, transitive **e)** Reflexive, transitive **f)** Reflexive, symmetric, transitive **g)** Antisymmetric **h)** Antisymmetric, transitive **9.** Each of the three properties

Let R be a relation from a set A to a set B. The **inverse relation** from B to A, denoted by R^{-1}, is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** $\overline{R}$ is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$.

26. Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find

a) R^{-1}. **b)** $\overline{R}$.

26. a) $R^{-1} = \{(b, a) \mid (a, b) \in R\} = \{(b, a) \mid a < b\} = \{(a, b) \mid a > b\}$

b) $\overline{R} = \{(a, b) \mid (a, b) \notin R\} = \{(a, b) \mid a \not< b\} = \{(a, b) \mid a \geq b\}$

32. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$.

32. Since $(1, 2) \in R$ and $(2, 1) \in S$, we have $(1, 1) \in S \circ R$. We use similar reasoning to form the rest of the pairs in the composition, giving us the answer $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

***49.** How many relations are there on a set with n elements that are

a) symmetric? **b)** antisymmetric?
c) asymmetric? **d)** irreflexive?
e) reflexive and symmetric?
f) neither reflexive nor irreflexive?

47. a) 65,536 **b)** 32,768 **49. a)** $2^{n(n+1)/2}$ **b)** $2^n 3^{n(n-1)/2}$ **c)** $3^{n(n-1)/2}$ **d)** $2^{n(n-1)}$ **e)** $2^{n(n-1)/2}$ **f)** $2^{n^2} - 2 \cdot 2^{n(n-1)}$ **51.** There

53. Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$, where R^{-1} is the inverse relation.

may be no such b. **53.** If R is symmetric and $(a, b) \in R$, then $(b, a) \in R$, so $(a, b) \in R^{-1}$. Hence, $R \subseteq R^{-1}$. Similarly, $R^{-1} \subseteq R$. So $R = R^{-1}$. Conversely, if $R = R^{-1}$ and $(a, b) \in R$, then $(a, b) \in R^{-1}$, so $(b, a) \in R$. Thus, R is symmetric.

Sec. 9.3 13,14,31

13. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing

a) R^{-1}. **b)** $\overline{R}$. **c)** R^2.

13. a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

14. Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$. **c)** $R_2 \circ R_1$.
d) $R_1 \circ R_1$. **e)** $R_1 \oplus R_2$.

14. a) The matrix for the union is formed by taking the join: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

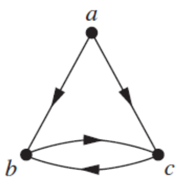
b) The matrix for the intersection is formed by taking the meet: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

c) The matrix is the Boolean product $M_{R_1} \odot M_{R_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

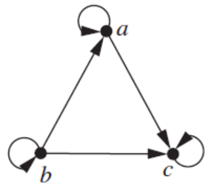
d) The matrix is the Boolean product $M_{R_1} \odot M_{R_1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

e) The matrix is the entrywise XOR: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

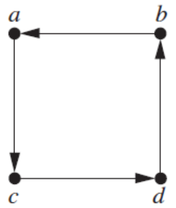
23.



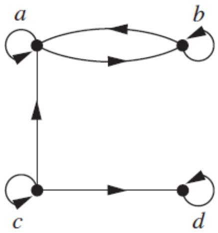
24.



25.



26.



31. Determine whether the relations represented by the directed graphs shown in Exercises 23–25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

length 2. **31.** Exercise 23: irreflexive. Exercise 24: reflexive, antisymmetric, transitive. Exercise 25: irreflexive, antisymmetric. **33.** Reverse the direction on every edge in the

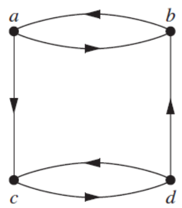
Sec.9.4 2, 6, 9(6), 11(6), 20, 28(a), 29

2. Let R be the relation $\{(a, b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

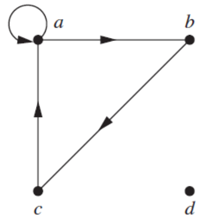
2. When we add all the pairs (x, x) to the given relation we have all of $\mathbf{Z} \times \mathbf{Z}$; in other words, we have the relation that always holds.

In Exercises 5–7 draw the directed graph of the reflexive closure of the relations with the directed graph shown.

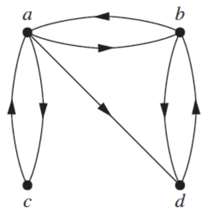
5.



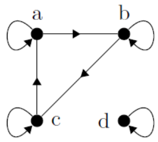
6.



7.

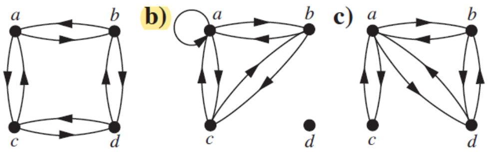


6. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



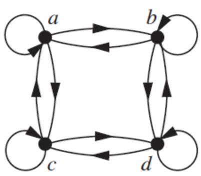
9. Find the directed graphs of the symmetric closures of the relations with directed graphs shown in Exercises 5–7.

9. **a)**

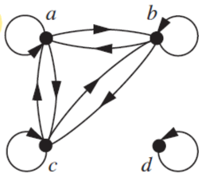


11. Find the directed graph of the smallest relation that is both reflexive and symmetric that contains each of the relations with directed graphs shown in Exercises 5–7.

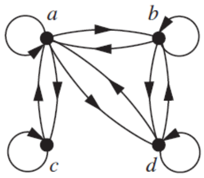
11. **a)**



b)



c)



20. Let R be the relation that contains the pair (a, b) if a and b are cities such that there is a direct nonstop airline flight from a to b. When is (a, b) in

a) R^2? **b)** R^3? **c)** R^*?

20. **a)** The pair (a, b) is in R^2 precisely when there is a city c such that there is a direct flight from a to c and a direct flight from c to b—in other words, when it is possible to fly from a to b with a scheduled stop (and possibly a plane change) in some intermediate city.

b) The pair (a, b) is in R^3 precisely when there are cities c and d such that there is a direct flight from a to c, a direct flight from c to d, and a direct flight from d to b—in other words, when it is possible to fly from a to b with two scheduled stops (and possibly a plane change at one or both) in intermediate cities.

c) The pair (a, b) is in R^* precisely when it is possible to fly from a to b.

26. Use Algorithm 1 to find the transitive closures of these relations on $\{a, b, c, d, e\}$.

a) $\{(a, c), (b, d), (c, a), (d, b), (e, d)\}$

b) $\{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$

c) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$

d) $\{(a, e), (b, a), (b, d), (c, d), (d, a), (d, c), (e, a), (e, b), (e, c), (e, e)\}$

26. **a)** We show the various matrices that are involved. First,

$$A = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix}, \quad A^{[2]} \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0 \end{bmatrix}, \text{ and } \quad A^{[3]} = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix} = A.$$

It follows that $A^{[4]} = A^{[2]}$ and $A^{[5]} = A^{[3]}$. Therefore the answer B, the meet of all the A's, is $A \vee A^{[2]}$, namely

$$\begin{bmatrix} 1&0&1&0&0\\ 0&1&0&1&0\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&1&0&1&0 \end{bmatrix}.$$

28. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 26.

28. We compute the matrices W_i for $i = 0, 1, 2, 3, 4, 5$, and then W_5 is the answer.

a) $$W_0 = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix} \quad W_2 = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&0&1&0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 1&0&1&0&0\\ 0&0&0&1&0\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&0&1&0 \end{bmatrix} \quad W_4 = \begin{bmatrix} 1&0&1&0&0\\ 0&1&0&1&0\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&1&0&1&0 \end{bmatrix} = W_5$$

29. Find the smallest relation containing the relation $\{(1, 2), (1, 4), (3, 3), (4, 1)\}$ that is

a) reflexive and transitive.

b) symmetric and transitive.

c) reflexive, symmetric, and transitive.

27. Answers same as for Exercise 25. **29.** **a)** $\{(1, 1), (1, 2), (1, 4), (2, 2), (3, 3), (4, 1), (4, 2), (4, 4)\}$ **b)** $\{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (3, 3), (4, 1), (4, 2), (4, 4)\}$ **c)** $\{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (3, 3), (4, 1), (4, 2), (4, 4)\}$

- Sec. 9.5 3, 10, 16, 36(b), 39, 41

3. Which of these relations on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(f, g) \mid f(1) = g(1)\}$
b) $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$
c) $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbf{Z}\}$
d) $\{(f, g) \mid \text{ for some } C \in \mathbf{Z}, \text{ for all } x \in \mathbf{Z}, f(x) - g(x) = C\}$
e) $\{(f, g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$

not transitive **3. a)** Equivalence relation **b)** Not transitive **c)** Not reflexive, not symmetric, not transitive **d)** Equivalence relation **e)** Not reflexive, not transitive **5.** Many answers are

10. Suppose that A is a nonempty set and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$.

10. The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

36. What is the congruence class $[4]_m$ when m is

a) 2? **b)** 3? **c)** 6? **d)** 8?

36. In each case, the equivalence class of 4 is the set of all integers congruent to 4, modulo m.
a) $\{4 + 2n \mid n \in \mathbf{Z}\} = \{\ldots, -2, 0, 2, 4, \ldots\}$ b) $\{4 + 3n \mid n \in \mathbf{Z}\} = \{\ldots, -2, 1, 4, 7, \ldots\}$
c) $\{4 + 6n \mid n \in \mathbf{Z}\} = \{\ldots, -2, 4, 10, 16, \ldots\}$ d) $\{4 + 8n \mid n \in \mathbf{Z}\} = \{\ldots, -4, 4, 12, 20, \ldots\}$

15. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

39. a) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 15?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 15. [*Hint:* Look at the difference $a - b$ corresponding to (a, b).]

39. a) $[(1, 2)] = \{(a, b) \mid a - b = -1\} = \{(1, 2), (3, 4), (4, 5), (5, 6), \ldots\}$ **b)** Each equivalence class can be interpreted as an integer (negative, positive, or zero); specifically, $[(a, b)]$ can be interpreted as $a - b$.

41. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$

b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$

c) $\{2, 4, 6\}, \{1, 3, 5\}$ **d)** $\{1, 4, 5\}, \{2, 6\}$

41. a) No b) Yes c) Yes d) No

Sec. 9.6 5, 10, 23(a),(c), 32, 44, 66

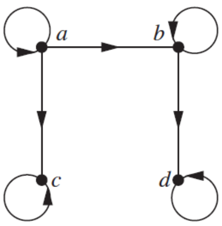
5. Which of these are posets?

a) $(\mathbf{Z}, =)$ **b)** $(\mathbf{Z}, \neq)$ **c)** $(\mathbf{Z}, \geq)$ **d)** $(\mathbf{Z}, \nmid)$

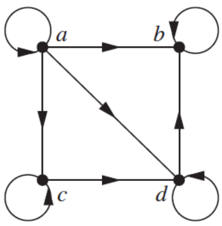
Y N Y N

In Exercises 9–11 determine whether the relation with the directed graph shown is a partial order.

9.



10.



11.

10. This relation is not transitive (there is no arrow from c to b), so it is not a partial order.

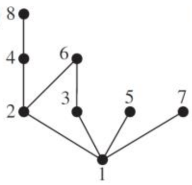
23. Draw the Hasse diagram for divisibility on the set

a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$. **b)** $\{1, 2, 3, 5, 7, 11, 13\}$.

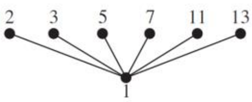
c) $\{1, 2, 3, 6, 12, 24, 36, 48\}$.

d) $\{1, 2, 4, 8, 16, 32, 64\}$.

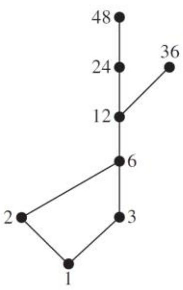
23. a)



b)

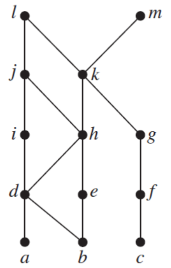


c)



d)

32. Answer these questions for the partial order represented by this Hasse diagram.



a) Find the maximal elements.
b) Find the minimal elements.
c) Is there a greatest element?
d) Is there a least element?
e) Find all upper bounds of $\{a, b, c\}$.
f) Find the least upper bound of $\{a, b, c\}$, if it exists.
g) Find all lower bounds of $\{f, g, h\}$.
h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

32. a) The maximal elements are the ones with no other elements above them, namely l and m.

b) The minimal elements are the ones with no other elements below them, namely a, b, and c.

c) There is no greatest element, since neither l nor m is greater than the other.

d) There is no least element, since neither a nor b is less than the other.

e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l, and m are the elements fitting this description.

f) Since k is less than both l and m, it is the least upper bound of a, b, and c.

g) No element is less than both f and h, so there are no lower bounds.

h) Since there are no lower bounds, there can be no greatest lower bound.

44. Determine whether these posets are lattices.

a) $(\{1, 3, 6, 9, 12\}, |)$ **b)** $(\{1, 5, 25, 125\}, |)$

c) $(\mathbf{Z}, \geq)$

d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S

N Y Y Y

44. In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.

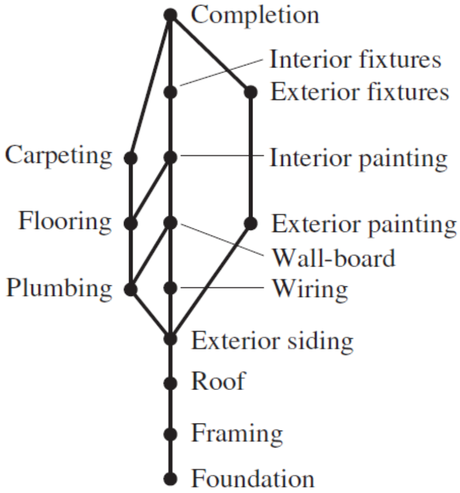
a) This is not a lattice, since the elements 6 and 9 have no upper bound (no element in our set is a multiple of both of them).

b) This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

c) Again, this is a lattice because it is a linear order. The least upper bound of two numbers in the list is the smaller number (since here "greater" really means "less"!), and the greatest lower bound is the larger of the two numbers.

d) This is similar to Example 24, with the roles of subset and superset reversed. Here the g.l.b. of two subsets A and B is $A \cup B$, and their l.u.b. is $A \cap B$.

66. Schedule the tasks needed to build a house, by specifying their order, if the Hasse diagram representing these tasks is as shown in the figure.



66. There are many compatible total orders here. We just need to work from the bottom up. One answer is to take Foundation $\prec$ Framing $\prec$ Roof $\prec$ Exterior siding $\prec$ Wiring $\prec$ Plumbing $\prec$ Flooring $\prec$ Wall $-$ board $\prec$ Exterior painting $\prec$ Interior painting $\prec$ Carpeting $\prec$ Interior fixtures $\prec$ Exterior fixtures $\prec$ Completion.