Sec 2.1 13、20、24、26、34(a)(c)

13. Determine whether each of these statements is true or false.

a) $x \in \{x\}$ **b)** $\{x\} \subseteq \{x\}$ **c)** $\{x\} \in \{x\}$
d) $\{x\} \in \{\{x\}\}$ **e)** $\emptyset \subseteq \{x\}$ **f)** $\emptyset \in \{x\}$

Key

a) True b) True c) False d) True e) True f) False

20. Find two sets A and B such that $A \in B$ and $A \subseteq B$.

key

20. Since the empty set is a subset of every set, we just need to take a set B that contains $\emptyset$ as an element. Thus we can let $A = \emptyset$ and $B = \{\emptyset\}$ as the simplest example.

24. Can you conclude that $A = B$ if A and B are two sets with the same power set?

key

24. The union of all the sets in the power set of a set X must be exactly X. In other words, we can recover X from its power set, uniquely. Therefore the answer is yes.

26. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$ **b)** $\{\emptyset, \{a\}\}$
c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ **d)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Key

26. a) The power set of every set includes at least the empty set, so the power set cannot be empty. Thus $\emptyset$ is not the power set of any set.

b) This is the power set of $\{a\}$.

c) This set has three elements. Since 3 is not a power of 2, this set cannot be the power set of any set.

d) This is the power set of $\{a, b\}$.

34. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$. **b)** $C \times B \times A$.
c) $C \times A \times B$. **d)** $B \times B \times B$.

Key

34. In each case the answer is a set of 3-tuples.

a) $\{(a,x,0),(a,x,1),(a,y,0),(a,y,1),(b,x,0),(b,x,1),(b,y,0),(b,y,1),(c,x,0),(c,x,1),(c,y,0),(c,y,1)\}$

b) $\{(0,x,a),(0,x,b),(0,x,c),(0,y,a),(0,y,b),(0,y,c),(1,x,a),(1,x,b),(1,x,c),(1,y,a),(1,y,b),(1,y,c)\}$

c) $\{(0,a,x),(0,a,y),(0,b,x),(0,b,y),(0,c,x),(0,c,y),(1,a,x),(1,a,y),(1,b,x),(1,b,y),(1,c,x),(1,c,y)\}$

d) $\{(x,x,x),(x,x,y),(x,y,x),(x,y,y),(y,x,x),(y,x,y),(y,y,x),(y,y,y)\}$

Sec 2.2 19、54、63(c)

19. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

a) by showing each side is a subset of the other side.

b) using a membership table.

Key

19. a) $x \in \overline{A \cap B \cap C} \equiv x \notin A \cap B \cap C \equiv x \notin A \vee x \notin B \vee x \notin C \equiv x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C} \equiv x \in \overline{A} \cup \overline{B} \cup \overline{C}$

b)

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

54. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

key

54. We note that these sets are increasing, that is, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Therefore, the union of any collection of these sets is just the one with the largest subscript, and the intersection is just the one with the smallest subscript.

a) $A_n = \{\ldots, -2, -1, 0, 1, \ldots, n\}$ b) $A_1 = \{\ldots, -2, -1, 0, 1\}$

63. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, t, u, x, y\}$.

a) $A \cup B$ **b)** $A \cap B$
c) $(A \cup D) \cap (B \cup C)$ **d)** $A \cup B \cup C \cup D$

Key

and is 0 otherwise. **63. a)** 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 = 11 1110 1000 0000 0100 0101 0000, representing $\{a, b, c, d, e, g, p, t, v\}$ **b)** 11 1110 0000 0000 0000 0000 0000 $\wedge$ 01 1100 1000 0000 0100 0101 0000 = 01 1100 0000 0000 0000 0000 0000, representing $\{b, c, d\}$ **c)** (11 1110 0000 0000 0000 0000 0000 $\vee$ 00 0110 0110 0001 1000 0110 0110) $\wedge$ (01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111) = 11 1110 0110 0001 1000 0110 0110 $\wedge$ 01 1110 1010 0000 1100 0111 0111 = 01 1110 0010 0000 1000 0110 0110, representing $\{b, c, d, e, i, o, t, u, x, y\}$ **d)** 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111 $\vee$ 00 0110 0110 0001 1000 0110 0110 = 11 1110 1110 0001 1100 0111 0111, representing $\{a,b,c,d,e,g,h,i,n,o,p,t,u,v,x,y,z\}$ **65. a)** $\{1, 2, 3, \{1, 2, 3\}\}$

Sec 2.3 22(c), 36, 42(a) 74, 76(c,d)

22. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = -3x + 4$
b) $f(x) = -3x^2 + 7$
c) $f(x) = (x + 1)/(x + 2)$
d) $f(x) = x^5 + 1$

Key

22. If we can find an inverse, the function is a bijection. Otherwise we must explain why the function is not on-to-one or not onto.

a) This is a bijection since the inverse function is $f^{-1}(x) = (4 - x)/3$.

b) This is not one-to-one since $f(17) = f(-17)$, for instance. It is also not onto, since the range is the interval $(-\infty, 7]$. For example, 42548 is not in the range.

c) This function is a bijection, but not from $\mathbf{R}$ to $\mathbf{R}$. To see that the domain and range are not $\mathbf{R}$, note that $x = -2$ is not in the domain, and $x = 1$ is not in the range. On the other hand, f is a bijection from $\mathbf{R} - \{-2\}$ to $\mathbf{R} - \{1\}$, since its inverse is $f^{-1}(x) = (1 - 2x)/(x - 1)$.

d) It is clear that this continuous function is increasing throughout its entire domain ($\mathbf{R}$) and it takes on both arbitrarily large values and arbitrarily small (large negative) ones. So it is a bijection. Its inverse is clearly $f^{-1}(x) = \sqrt[5]{x - 1}$.

* **36.** If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

Key

36. To clarify the setting, suppose that $g : A \to B$ and $f : B \to C$, so that $f \circ g : A \to C$. We will prove that if $f \circ g$ is one-to-one, then g is also one-to-one, so not only is the answer to the question "yes," but part of the hypothesis is not even needed. Suppose that g were not one-to-one. By definition this means that there are distinct elements a_1 and a_2 in A such that $g(a_1) = g(a_2)$. Then certainly $f(g(a_1)) = f(g(a_2))$, which is the same statement as $(f \circ g)(a_1) = (f \circ g)(a_2)$. By definition this means that $f \circ g$ is not one-to-one, and our proof is complete.

42. Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

a) $f(S \cup T) = f(S) \cup f(T)$.
b) $f(S \cap T) \subseteq f(S) \cap f(T)$.

Key

42. a) This really has two parts. First suppose that b is in $f(S \cup T)$. Thus $b = f(a)$ for some $a \in S \cup T$. Either $a \in S$, in which case $b \in f(S)$, or $a \in T$, in which case $b \in f(T)$. Thus in either case $b \in f(S) \cup f(T)$. This shows that $f(S \cup T) \subseteq f(S) \cup f(T)$. Conversely, suppose $b \in f(S) \cup f(T)$. Then either $b \in f(S)$ or $b \in f(T)$. This means either that $b = f(a)$ for some $a \in S$ or that $b = f(a)$ for some $a \in T$. In either case, $b = f(a)$ for some $a \in S \cup T$, so $b \in f(S \cup T)$. This shows that $f(S) \cup f(T) \subseteq f(S \cup T)$, and our proof is complete.

b) Suppose $b \in f(S \cap T)$. Then $b = f(a)$ for some $a \in S \cap T$. This implies that $a \in S$ and $a \in T$, so we have $b \in f(S)$ and $b \in f(T)$. Therefore $b \in f(S) \cap f(T)$, as desired.

74. Suppose that f is a function from A to B, where A and B are finite sets with $|A| = |B|$. Show that f is one-to-one if and only if it is onto.

Key

74. If f is one-to-one, then every element of A gets sent to a different element of B. If in addition to the range of A there were another element in B, then $|B|$ would be at least one greater than $|A|$. This cannot happen, so we conclude that f is onto. Conversely, suppose that f is onto, so that every element of B is the image of some element of A. In particular, there is an element of A for each element of B. If two or more elements of A were sent to the same element of B, then $|A|$ would be at least one greater than the $|B|$. This cannot happen, so we conclude that f is one-to-one.

76. Prove or disprove each of these statements about the floor and ceiling functions.

a) $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ for all real numbers x.
b) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y.
c) $\lceil \lceil x/2 \rceil /2 \rceil = \lceil x/4 \rceil$ for all real numbers x.
d) $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor$ for all positive real numbers x.
e) $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$ for all real numbers x and y.

Key

76. a) This is true. Since $\lceil x \rceil$ is already an integer, $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$.

b) A little experimentation shows that this is not always true. To disprove it we need only produce a counterexample, such as $x = y = \frac{3}{4}$. In this case the left-hand side is $\lfloor 3/2 \rfloor = 1$, while the right-hand side is $0 + 0 = 0$.

c) A little trial and error fails to produce a counterexample, so maybe this is true. We look for a proof. Since we are dividing by 4, let us write $x = 4n + k$, where $0 \leq k < 4$. In other words, write x in terms of how much it exceeds the largest multiple of 4 not exceeding it. There are three cases. If $k = 0$, then x is already a multiple of 4, so both sides equal n. If $0 < k \leq 2$, then $\lceil x/2 \rceil = 2n + 1$, so the left-hand side is

$\lceil n + \frac{1}{2} \rceil = n + 1$. Of course the right-hand side is $n + 1$ as well, so again the two sides agree. Finally, suppose that $2 < k < 4$. Then $\lceil x/2 \rceil = 2n + 2$, and the left-hand side is $\lceil n + 1 \rceil = n + 1$; of course the right-hand side is still $n + 1$, as well. Since we proved that the two sides are equal in all cases, the proof is complete.

d) For $x = 8.5$, the left-hand side is 3, whereas the right-hand side is 2.

e) This is true. Write $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers and ϵ and δ are nonnegative real numbers less than 1. The left-hand side is $n + m + (n + m)$ or $n + m + (n + m + 1)$, the latter occurring if and only if $\epsilon + \delta \geq 1$. The right-hand side is the sum of two quantities. The first is either $2n$ (if $\epsilon < \frac{1}{2}$) or $2n + 1$ (if $\epsilon \geq \frac{1}{2}$). The second is either $2m$ (if $\delta < \frac{1}{2}$) or $2m + 1$ (if $\delta \geq \frac{1}{2}$). The only way, then, for the left-hand side to exceed the right-hand side is to have the left-hand side be $2n + 2m + 1$ and the right-hand side be $2n + 2m$. This can occur only if $\epsilon + \delta \geq 1$ while $\epsilon < \frac{1}{2}$ and $\delta < \frac{1}{2}$. But that is an impossibility, since the sum of two numbers less than $\frac{1}{2}$ cannot be as large as 1. Therefore the right-hand side is always at least as large as the left-hand side.

Sec. 2.4 25(a,b,c)

25. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …
b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …
c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …
d) 3, 6, 12, 24, 48, 96, 192, …
e) 15, 8, 1, −6, −13, −20, −27, …
f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …
g) 2, 16, 54, 128, 250, 432, 686, …
h) 2, 3, 7, 25, 121, 721, 5041, 40321, …

Key

25. **a)** One 1 and one 0, followed by two 1s and two 0s, followed by three 1s and three 0s, and so on; 1, 1, 1 **b)** The positive integers are listed in increasing order with each even positive integer listed twice; 9, 10, 10. **c)** The terms in odd-numbered locations are the successive powers of 2; the terms in even-numbered locations are all 0; 32, 0, 64. **d)** $a_n = 3 \cdot 2^{n-1}$; 384, 768, 1536 **e)** $a_n = 15 - 7(n - 1) = 22 - 7n$; −34, −41, −48 **f)** $a_n = (n^2 + n + 4)/2$; 57, 68, 80 **g)** $a_n = 2n^3$; 1024, 1458, 2000 **h)** $a_n = n! + 1$; 362881, 3628801, 39916801 **27.** Among the integers 1, 2, … , a_n,