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***46.** Prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

Key.

46. This proof will be similar to the proof in Example 10. The basis step is clear, since for $n = 3$, the set has exactly one subset containing exactly three elements, and $3(3-1)(3-2)/6 = 1$. Assume the inductive hypothesis, that a set with n elements has $n(n-1)(n-2)/6$ subsets with exactly three elements; we want to prove that a set S with $n+1$ elements has $(n+1)n(n-1)/6$ subsets with exactly three elements. Fix an element a in S, and let T be the set of elements of S other than a. There are two varieties of subsets

of S containing exactly three elements. First there are those that do not contain a. These are precisely the three-element subsets of T, and by the inductive hypothesis, there are $n(n-1)(n-2)/6$ of them. Second, there are those that contain a together with two elements of T. Therefore there are just as many of these subsets as there are two-element subsets of T. By Exercise 45, there are exactly $n(n-1)/2$ such subsets of T; therefore there are also $n(n-1)/2$ three-element subsets of S containing a. Thus the total number of subsets of S containing exactly three elements is $(n(n-1)(n-2)/6) + n(n-1)/2$, which simplifies algebraically to $(n+1)n(n-1)/6$, as desired.

In Exercises 47 and 48 we consider the problem of placing towers along a straight road, so that every building on the road receives cellular service. Assume that a building receives cellular service if it is within one mile of a tower.

47. Devise a greedy algorithm that uses the minimum number of towers possible to provide cell service to d buildings located at positions $x_1, x_2, \ldots, x_d$ from the start of the road. [*Hint:* At each step, go as far as possible along the road before adding a tower so as not to leave any buildings without coverage.]

***48.** Use mathematical induction to prove that the algorithm you devised in Exercise 47 produces an optimal solution, that is, that it uses the fewest towers possible to provide cellular service to all buildings.

Key for 47.

completes the inductive proof. **47.** Reorder the locations if necessary so that $x_1 \le x_2 \le x_3 \le \cdots \le x_d$. Place the first tower at position $t_1 = x_1 + 1$. Assume tower k has been placed at position t_k. Then place tower $k+1$ at position $t_{k+1} = x + 1$, where x is the smallest x_i greater than $t_k + 1$. **49.** The two

Key for 48

48. We will show that any minimum placement of towers can be transformed into the placement produced by the algorithm. Although it does not strictly have the form of a proof by mathematical induction, the spirit is the same. Let $s_1 < s_2 < \cdots < s_k$ be an optimal locations of the towers (i.e., so as to minimize k), and let $t_1 < t_2 < \cdots < t_l$ be the locations produced by the algorithm from Exercise 47. In order to serve the first building, we must have $s_1 \le x_1 + 1 = t_1$. If $s_1 \ne t_1$, then we can move the first tower in the optimal solution to position t_1 without losing cell service for any building. Therefore we can assume that $s_1 = t_1$. Let x_j be smallest location of a building out of range of the tower at s_1; thus $x_j > s_1 + 1$. In order to serve that building there must be a tower s_i such that $s_i \le x_j + 1 = t_2$. If $i > 2$, then towers at positions s_2 through s_{i-1} are not needed, a contradiction. As before, it then follows that we can move the second tower from s_2 to t_2. We continue in this manner for all the towers in the given minimum solution; thus $k = l$. This proves that the algorithm produces a minimum solution.

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8. Suppose that a store offers gift certificates in denominations of 25 dollars and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.

先除以5计算，最后再乘以5

key

8. Since both 25 and 40 are multiples of 5, we cannot form any amount that is not a multiple of 5. So let's determine for which values of n we can form $5n$ dollars using these gift certificates, the first of which provides 5 copies of \$5, and the second of which provides 8 copies. We can achieve the following values of n: $5 = 5$, $8 = 8$, $10 = 5+5$, $13 = 8+5$, $15 = 5+5+5$, $16 = 8+8$, $18 = 8+5+5$, $20 = 5+5+5+5$, $21 = 8+8+5$, $23 = 8+5+5+5$, $24 = 8+8+8$, $25 = 5+5+5+5+5$, $26 = 8+8+5+5$, $28 = 8+5+5+5+5$, $29 = 8+8+8+5$, $30 = 5+5+5+5+5+5$, $31 = 8+8+5+5+5$, $32 = 8+8+8+8$. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form total amounts of the form $5n$ for all $n \geq 28$ using these gift certificates. (In other words, \$135 is the largest multiple of \$5 that we cannot achieve.)

To prove this by strong induction, let $P(n)$ be the statement that we can form $5n$ dollars in gift certificates using just 25-dollar and 40-dollar certificates. We want to prove that $P(n)$ is true for all $n \geq 28$. From our work above, we know that $P(n)$ is true for $n = 28, 29, 30, 31, 32$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $28 \leq j \leq k$, where k is a fixed integer greater than or equal to 32. We want to show that $P(k+1)$ is true. Because $k - 4 \geq 28$, we know that $P(k-4)$ is true, that is, that we can form $5(k-4)$ dollars. Add one more \$25-dollar certificate, and we have formed $5(k+1)$ dollars, as desired.

***18.** Use strong induction to show that when a simple polygon P with consecutive vertices $v_1, v_2, \ldots, v_n$ is triangulated into $n-2$ triangles, the $n-2$ triangles can be numbered $1, 2, \ldots, n-2$ so that v_i is a vertex of triangle i for $i = 1, 2, \ldots, n-2$.

key

18. We prove something slightly stronger: If a convex n-gon whose vertices are labeled consecutively as v_m, v_{m+1}, $\ldots$, v_{m+n-1} is triangulated, then the triangles can be numbered from m to $m+n-3$ so that v_i is a vertex of triangle i for $i = m, m+1, \ldots, m+n-3$. (The statement we are asked to prove is the case $m = 1$.) The basis step is 1 $n = 3$, and there is nothing to prove. For the inductive step, 2 assume the inductive hypothesis that the statement is true for polygons with fewer than n vertices, and consider any triangulation of 3 a convex n-gon whose vertices are labeled consecutively as v_m, v_{m+1}, $\ldots$, v_{m+n-1}. One of the diagonals in the triangulation must have either v_{m+n-1} or v_{m+n-2} as an endpoint (otherwise, the region containing v_{m+n-1} would not be a triangle). So there are two cases. If the triangulation uses diagonal $v_k v_{m+n-1}$, then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-1} as v_{k+1} in the polygon that contains v_m. This gives us the desired numbering of the triangles, with numbers v_m through v_{k-1} in the first polygon and numbers v_k through v_{m+n-3} in the second polygon. If the triangulation uses diagonal $v_k v_{m+n-2}$, then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-2} as v_{k+1} and v_{m+n-1} as v_{k+2} in the polygon that contains v_{m+n-1}, and renumbering all the vertices by adding 1 to their indices in the other polygon. This gives us the desired numbering of the triangles, with numbers v_m through v_k in the first polygon and numbers v_{k+1} through v_{m+n-3} in the second polygon. Note that we did not need the convexity of our polygons.

39. Can you use the well-ordering property to prove the statement: "Every positive integer can be described using no more than fifteen English words"? Assume the words come from a particular dictionary of English. [*Hint:* Suppose that there are positive integers that cannot be described using no more than fifteen English words. By well ordering, *the smallest positive integer that cannot be described using no more than fifteen English words* would then exist.]

key

39. This is a paradox caused by self-reference. The answer is "no." There are a finite number of English words, so only a finite number of strings of 15 words or fewer; therefore, only a finite number of positive integers can be so described, not all of them. **41.** Suppose that the well-ordering prop-

- 第7版：Sec. 5.3 6(a,d), 14, 29(a)
- 第8版：Sec. 5.3 6(a,d), 14, 31(a)

6. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer and prove that your formula is valid.

a) $f(0) = 1, f(n) = -f(n-1)$ for $n \geq 1$

b) $f(0) = 1,\ f(1) = 0,\ f(2) = 2,\ f(n) = 2f(n-3)$ for $n \geq 3$

c) $f(0) = 0, f(1) = 1, f(n) = 2f(n+1)$ for $n \geq 2$

d) $f(0) = 0, f(1) = 1, f(n) = 2f(n-1)$ for $n \geq 1$

e) $f(0) = 2$, $f(n) = f(n-1)$ if n is odd and $n \geq 1$ and $f(n) = 2f(n-2)$ if $n \geq 2$

key

6. a) This is valid, since we are provided with the value at $n = 0$, and each subsequent value is determined by the previous one. Since all that changes from one value to the next is the sign, we conjecture that $f(n) = (-1)^n$. This is true for 1. $n = 0$, since $(-1)^0 = 1$. If it 2. is true for $n = k$, then we have $f(k+1) = -f(k+1-1) =$ 3. $-f(k) = -(-1)^k$ by the inductive hypothesis, whence $f(k+1) = (-1)^{k+1}$.

b) This is valid, since we are provided with the values at $n = 0$, 1, and 2, and each subsequent value is determined by the value that occurred three steps previously. We compute the first several terms of the sequence: 1, 0, 2, 2, 0, 4, 4, 0, 8, ... We conjecture the formula $f(n) = 2^{n/3}$ when $n \equiv 0 \pmod 3$, $f(n) = 0$ when $n \equiv 1 \pmod 3$, $f(n) = 2^{(n+1)/3}$ when $n \equiv 2 \pmod 3$. To prove this, first note that in the base cases we have $f(0) = 1 = 2^{0/3}$, $f(1) = 0$, and $f(2) = 2 = 2^{(2+1)/3}$. Assume the inductive hypothesis that the formula is valid for smaller inputs. Then for $n \equiv 0 \pmod 3$ we have $f(n) = 2f(n-3) = 2 \cdot 2^{(n-3)/3} = 2 \cdot 2^{n/3} \cdot 2^{-1} = 2^{n/3}$, as desired. For $n \equiv 1 \pmod 3$ we have $f(n) = 2f(n-3) = 2 \cdot 0 = 0$, as desired. And for $n \equiv 2 \pmod 3$ we have $f(n) = 2f(n-3) = 2 \cdot 2^{(n-3+1)/3} = 2 \cdot 2^{(n+1)/3} \cdot 2^{-1} = 2^{(n+1)/3}$, as desired.

c) This is invalid. We are told that $f(2)$ is defined in terms of $f(3)$, but $f(3)$ has not been defined.

d) This is invalid, because the value at $n = 1$ is defined in two conflicting ways—first as $f(1) = 1$ and then as $f(1) = 2f(1-1) = 2f(0) = 2 \cdot 0 = 0$. $f(1) = 2f(0) = 0 \neq 1$

e) This appears syntactically to be not valid, since we have conflicting instruction for odd $n \geq 3$. On the one hand $f(3) = f(2)$, but on the other hand $f(3) = 2f(1)$. However, we notice that $f(1) = f(0) = 2$ and $f(2) = 2f(0) = 4$, so these apparently conflicting rules tell us that $f(3) = 4$ on the one hand and $f(3) = 2 \cdot 2 = 4$ on the other hand. Thus we got the same answer either way. Let us show that in fact this definition is valid because the rules coincide.

We compute the first several terms of the sequence: 2, 2, 4, 4, 8, 8, ... We conjecture the formula $f(n) = 2^{\lceil (n+1)/2 \rceil}$. To prove this inductively, note first that $f(0) = 2 = 2^{\lceil (0+1)/2 \rceil}$. For larger values we have for n odd using the first part of the recursive step that $f(n) = f(n-1) = 2^{\lceil (n-1+1)/2 \rceil} = 2^{\lceil n/2 \rceil} = 2^{\lceil (n+1)/2 \rceil}$, since $n/2$ is not an integer. For $n \geq 2$, whether even or odd, using the second part of the recursive step we have $f(n) = 2f(n-2) = 2 \cdot 2^{\lceil (n-2+1)/2 \rceil} = 2 \cdot 2^{\lceil (n+1)/2 \rceil - 1} = 2 \cdot 2^{\lceil (n+1)/2 \rceil} \cdot 2^{-1} = 2^{\lceil (n+1)/2 \rceil}$, as desired.

f_n is the nth Fibonacci number, 0, 1, 1, 2, 3, 5, ...

*14. Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

Key

14. The basis step ($n = 1$) is clear, since $f_2 f_0 - f_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1$. Assume the inductive hypothesis. Then we have

$$\begin{aligned} f_{n+2}f_n - f_{n+1}^2 &= (f_{n+1} + f_n)f_n - f_{n+1}^2 \\ &= f_{n+1}f_n + f_n^2 - f_{n+1}^2 \\ &= -f_{n+1}(f_{n+1} - f_n) + f_n^2 \\ &= -f_{n+1}f_{n-1} + f_n^2 \\ &= -(f_{n+1}f_{n-1} - f_n^2) \\ &= -(-1)^n = (-1)^{n+1}. \end{aligned}$$

* 31. Give a recursive definition of each of these sets of ordered pairs of positive integers. Use structural induction to prove that the recursive definition you found is correct. [*Hint:* To find a recursive definition, plot the points in the set in the plane and look for patterns.]

a) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, \text{ and } a + b \text{ is even}\}$

b) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, \text{ and } a \text{ or } b \text{ is odd}\}$

c) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, a + b \text{ is odd, and } 3 \mid b\}$

Key for 31(a)

and $a + 2 \le 2(b + 1)$. **31. a)** Define S by $(1, 1) \in S$, and if $(a, b) \in S$, then $(a + 2, b) \in S$, $(a, b + 2) \in S$, and $(a + 1, b + 1) \in S$. All elements put in S satisfy the condition, because $(1, 1)$ has an even sum of coordinates, and if (a, b) has an even sum of coordinates, then so do $(a + 2, b)$, $(a, b + 2)$, and $(a + 1, b + 1)$. Conversely, we show by induction on the sum of the coordinates that if $a + b$ is even, then $(a, b) \in S$. If the sum is 2, then $(a, b) = (1, 1)$, and the basis step put (a, b) into S. Otherwise the sum is at least 4, and at least one of $(a-2, b)$, $(a, b-2)$, and $(a-1, b-1)$ must have positive integer coordinates whose sum is an even number smaller than $a+b$, and therefore must be in S. Then one application of the recursive step shows that $(a, b) \in S$. **b)** Define S by $(1, 1)$,

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29. Devise a recursive algorithm to find the nth term of the sequence defined by $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1} \cdot a_{n-2}$, for $n = 2, 3, 4, \ldots$.

Key

29. procedure $a(n$: nonnegative integer)
if $n = 0$ **then return** 1
else if $n = 1$ **then return** 2
else return $a(n-1) \cdot a(n-2)$