10.1 Parameter Nonlinear Models

Application: Determining subjective thresholds/indifference regions

Person class	Time Alternative 1 [min]	Time Alternative 2 [min]	Choice Alt. 1	Choice Alt. 2
1	25	30	11	10
2	30	30	10	10
3	35	30	10	10
4	40	30	9	11
5	45	30	5	15
6	50	30	2	15
7	55	30	1	15
8	60	30	0	15

Modelling the threshold

The reverse: Increased sensitivity at reference point

Person class	Time Alternative 1 [min]	Time Alternative 2 [min]	Choice Alt. 1	Choice Alt. 2
1	25	30	16	7
2	30	30	10	10
3	35	30	7	20
4	40	30	3	20
5	45	30	3	25
6	50	30	2	30
7	55	30	1	17
8	60	30	2	50

Such increased sensitivity at the reference (here: equal trip times) is proposed by the **Prospect Theory** of Kahneman/Twersky in certain situations

Four further models applied to the threshold data

figsDiscr/nonlinUtility_3param2param.png

10.2 GEV and Nested Logit Models

Motivation
When taking decisions, the available options are often coupled in a way that i.i.d. random utilities are not applicable:

▶ Destination and mode choice

▶ Destination city and job offers when about to moving

▶ Expansion of a company: Creating a new branch office and if so, where?

In these cases, a decision involves taking two or more sub-decisions with nearly fixed random utilities in the associated alternative sets, so the total random utility is correlated

⇒ **Red-Bus-Blue-Bus problem.**

⇒ How to model this while retaining explicit expressions for the choice probabilities?

The Red-Bus-Blue-Bus Problem

The Red-Bus-Blue-Bus Problem

Problem solved: 100% correlated random utilities

Nontrivial nested decision: partial correlations

The general GEV generating function

All the GEV models are defined via a **Generating function $G(y) = G(y_1, ..., y_I)$**
satisfying following formal conditions:

▶ Not negative: $G(\boldsymbol{y}) \geq 0$ for all $\boldsymbol{y}$,

▶ Asymptotics: $G \to \infty$ if any $y_i \to \infty$,

▶ Sign of derivatives:

$$
\begin{aligned}
G_i &\equiv \frac{\partial G}{\partial y_i} \geq 0, \\[4pt]
G_{ij} &\equiv \frac{\partial^2 G}{\partial y_i \, \partial y_j} \leq 0 \text{ if } i \neq j, \\[4pt]
G_{ijk} &\geq 0 \text{ and so on,}
\end{aligned}
$$

▶ Homogeneity of degree 1: $G(\alpha \boldsymbol{y}) = \alpha G(\boldsymbol{y})$

The Nobel-Price winning result of McFadden et. al.

Any GEV function $G(\boldsymbol{y})$ satisfying the above four conditions

- generates a random vector $\boldsymbol{\epsilon}$ satisfying a generalized extreme-value distribution with the distribution function

$$F(\boldsymbol{e}) = P(\epsilon_1 \leq e_1, ..., \epsilon_I \leq e_I) = e^{-G(\boldsymbol{y})} \text{ with } y_i = e^{-e_i}$$

- has analytic choice probabilities when maximizing the total utilities $U_i = V_i + \epsilon_i$:

$$P_i = \frac{y_i G_i(\boldsymbol{y})}{G(\boldsymbol{y})} \text{ with } G_i = \frac{\partial G}{\partial y_i}, \ y_i = e^{+V_i}$$

? Check why the above conditions for $G(\boldsymbol{y})$ must be true

Question: Check the conditions for $G(\boldsymbol{y})$

? Why $G(\boldsymbol{y}) \geq 0$ for all $\boldsymbol{y}$?

! Because a distribution function $F = e^{-G}$ must be ≤ 1 (the condition $F \geq 0$ is satisfied automatically)

? Why $G \to \infty$ if any $y_i \to \infty$?

! If $y_i \to \infty$ then the argument $e_i = -\ln y_i$ of the distribution function tends to $-\infty$. Since the corresponding random variable ϵ_i is always $> -\infty$, we have $F = e^{-G} = 0$, hence $G \to \infty$

? Sign of derivatives of G?

! We check only the first derivative $G_i = \frac{\partial G}{\partial y_i}$. We have $P_i = y_i G_i / G$ with P_i, $y_i = e^{-e_i}$ and G because of the first requirement all ≥ 0. Hence $G_i \geq 0$. The other conditions follow from the non-negativity of the distribution functions

? Homogeneity $G(\alpha \boldsymbol{y}) = \alpha G(\boldsymbol{y})$ for any $\alpha > 0$?

! Because of $P_i = y_i G_i / G$ and the scaling invariance $P(\epsilon_1 < e_1) = P(\lambda \epsilon_1 < \lambda e_1)$ with $\alpha = e^{\lambda}$

Special Case I: Multinomial-Logit

▶ Generating function:

$$G(\boldsymbol{y})^{\mathsf{MNL}} = \sum_{j=1}^{I} y_j$$

▶ Distribution function of the random utilities (RUs):

$$
\begin{aligned}
F(\boldsymbol{e}) &= \exp\left[-G\left(e^{-e_1}, ...\right)\right] = \exp\left(-\sum_j e^{-e_j}\right) \\
&= \prod_j \exp\left(-e^{-e_j}\right) \Rightarrow \epsilon_i \sim \text{ i.i.d. Gumbel}
\end{aligned}
$$

▶ Choice probabilities:

$$
\begin{aligned}
G_i &= \frac{\partial G}{\partial y_i} = 1, \\
P_i &= \frac{y_i}{\sum_{j=1}^{I} y_j} = \frac{\exp(V_i)}{\sum_{j=1}^{I} \exp(V_j)}
\end{aligned}
$$

Special Case II: Two-level Nested Logit model

▶ Hierarchical decision: $i = (l, m)$, l: top-level alternatives, m second-level alternatives depending on l

▶ Generating GEV function:

$$G^{\mathsf{NL}}(\boldsymbol{y}) = \sum_{l=1}^{L} \left(\sum_{m=1}^{M_l} y_{lm}^{1/\lambda_l} \right)^{\lambda_l}$$

where $\lambda_l \in [0, 1]$ determine the correlations of the RUs in "nest" l:

  ▶ $\lambda_l \to 1$: Limit of MNL, zero correlation $\Rightarrow$ check it!
  ▶ $\lambda_l \to 0$: no RUs inside the nests, correlation=1: **sequential model: blue and red buses**

▶ Distribution of the RUs:

$$
\begin{aligned}
F(\boldsymbol{e}) &= \exp\left[-\sum_l \left(\sum_m e^{-e_{lm}/\lambda_l} \right)^{\lambda_l} \right] = \prod_l \exp\left[-\left(\sum_m e^{-e_{lm}/\lambda_l} \right)^{\lambda_l} \right] \\
&= \prod_l F_l(\boldsymbol{e}_l) \Rightarrow \text{independent at top level}
\end{aligned}
$$

Nested Logit choice probabilities

Insert $G^{\mathsf{NL}}(\boldsymbol{y})$ into the general expression $P_i = y_i G_i / G$:

$$P_i = P_{lm} = P_l P_{m|l} = \frac{e^{V_{lm}/\lambda_l} \left(\sum_{m'} e^{V_{lm'}/\lambda_l}\right)^{\lambda_l - 1}}{\sum_{l'} \left(\sum_{m'} e^{V_{l'm'}/\lambda_{l'}}\right)^{\lambda_{l'}}}$$

A more intuitive form of the NL choice probabilities

▶ Set/assume $\quad V_{lm} = W_l + \tilde{V}_{lm}$
  - ▶ W_l: top-level contributions not appearing inside the nests
  - ▶ $\tilde{V}_{lm}$: inner contributions of alternative m in nest l

▶ Then, the NL choice probabilities can be formulated as

$$P_{lm} = P_l P_{m|l}, \quad P_l = \frac{e^{W_l + \lambda_l I_l}}{\sum_{l'} e^{W_{l'} + \lambda_{l'} I_{l'}}}, \quad P_{m|l} = \frac{e^{\tilde{V}_{lm}/\lambda_l}}{\sum_{m'} e^{\tilde{V}_{lm'}/\lambda_l}}$$

with the **inclusion values** $\quad I_l = \ln \left(\sum_m e^{\tilde{V}_{lm}/\lambda_l} \right)$

? Argue that the outer nest decision is a normal MNL with the *effective nest utilities* given by $\lambda_l I_l$. Because for these assumptions P_l has the normal MNL form

? Show that $\lambda_l I_l$ is at least as high as the utility $\tilde{V}_{lm_l^*}$ of the best alternative within the nest and that

$\lambda_l I = \tilde{V}_{lm_l^*}$ for $\lambda_l \to 0$. All contributions of the sum inside the log are exponentials and thus positive. Hence, $\lambda_l I_l$ is larger than any single $\tilde{V}_{lm}$ including the maximum. For $\lambda_l \to 0$, only the maximum contributes to the sum

? Argue that the (potential) selection within a nest is independent from the outer decision and obeys a normal MNL Independent because $P_{lm} = P_l P_{m|l}$, MNL for the utilities $\tilde{V}_{lm}/\lambda_l$ for fixed l

10.2.3 Example: Combined Destination and Mode Choice

figsDiscr/NL-Beispiel_eng.png

Combined destination and mode choice: the data

Per-son group	T [min] Emma, PT	T [min] Emma, car	T [min] superm, PT	T [min] superm, car	Fridge fill level F	y_{11}	y_{12}	y_{21}	y_{22}
1	25	15	25	20	0.9	1	2	0	0
2	25	30	40	30	0.8	3	0	0	1
3	20	20	30	30	0.7	2	1	1	1
4	25	10	25	10	0.6	0	3	0	2
5	15	5	30	20	0.5	1	2	0	2
6	15	15	25	20	0.4	1	1	0	1
7	15	20	45	45	0.3	3	1	0	1
8	15	15	15	15	0.2	1	0	2	3
9	25	15	40	30	0.1	1	1	0	1
10	25	10	25	20	0.0	0	1	1	3

Conditional modal splits

Top-level choice of the type of shop

Final combined probabilities

Counter check: normal MNL

10.3 Mixed-Logit models

if time allows, see German script, Sec. 4.14

10.4 Models for Reliability

[if time allows, see German script, Sec. 4.15]