2d equations of motion for vessels in matrix notation

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1 Overview

A vessel/ship essentially is a rigid body obeying classical Newtonian mechanics with various external forces. As such, it has six degrees of freedom, three positional ones and three orientations. Thus, the complete *state vector* (position/orientation and the associated rates of change or generalized moments) has 12 components:

- The 3d positional vector,
- the associated 3d velocity vector,
- three orientations (the *Euler angles*),
- and associated rates of change (rotation) components.

In 2d, we only have two positional coordinates and associated velocity components and one orientation degree of freedom, the *heading* with the associated *yaw* movement (rotation about the vertical axis, German *gieren*). Notice that, trivially, this 2d approach does *not* contain the *squat* (German *Absunk*) of the vessels letting the vessel sink a certain amount (approx $v_{\text{rel}}^2/(2g)$ in deep water) due to their relative speed through the water mediated by the coordinate transformation $p = p_0 - \frac{1}{2}\rho v_{\text{rel}}^2$ (Bernoulli effect) of the pressure.

It is convenient to define the dynamics of the *positional coordinates* and the heading – essentially the kinematic definition of the speed and the rotation rate – in an earth-fixed or *Eulerian* frame of reference since the geometry of the waterway and various external forces caused, e.g., by the water velocity field, wind, or waves, are given in this system.

The dynamics of the *generalized moments* governed by the Second Newton's law for rigid bodies, however, is simpler in the body-bound *Lagrangian* frame of reference where the origin follows a certain fixed point inside the rigid body (vessel), and the orientation of the axes follows the three axes of the body (vessel). Why? The control forces $\boldsymbol{\tau}$ (naming convention as in Cheng-Zang (2017) which I chose since this paper has the clearest representation) due to the engine (longitudinal thrust) and rudder (yaw) as well as the unknown disturbance forces $\boldsymbol{\tau}_w$ and the tensor of the water friction forces $\mathbf{D}$ in non-moving water act in this system.[1] Moreover, the generalized mass tensor $\mathbf{M}$ (containing the actual mass, the tensor of the moments of inertia and some dynamic masses due to the induced water motion) is only constant in the Lagrangian frame.

Using a non-inertial frame of reference, however, gives rise to additional inertial forces (in German erroneously called *Scheinkräfte*) in form of centrifugal (not centripetal) and coriolis terms that are summarized in the $\mathbf{C}$ matrix.

In summary, the complete 2d state in terms of Eulerian positions/heading and Lagrangian generalized moments/velocity vector is given as follows:

$$\begin{aligned} \text{Position vector (Eulerian)}: \quad & \boldsymbol{\eta} = (x, y, \Psi)' \quad &= (\text{pos}_x, \text{pos}_y, \text{heading})', \\ \text{Velocity vector (Lagrangian)}: \quad & \boldsymbol{v} = (u, v, r)' \quad &= (\text{surge, sway, yaw})' \end{aligned}$$

where

- surge: velocity component parallel to the vessels's longitudinal (principal) axis,
- sway: horizontal motion normal to this axis ("slip-sliding")
- yaw: change of the heading

2 Explaining some of the terms in the matrix formulation of the vessel dynamics

2.1 The kinematic equation for the generalized positions

If the generalized velocities were given in the Eulerian frame $(v_x, v_y, \omega_z)'$, we would simply have

$$\dot{x} = v_x, \quad \dot{y} = v_y, \quad \dot{\psi} = \omega_z$$

However, we want the Lagrangian frame for the generalized velocity vector. The transformation matrix rotates the velocity components around the z axis while $\dot{\psi}$ is unchanged, $r = \omega_z$ since there is only one rotation degree of freedom (things get more complicated in 3d):

$$\begin{aligned} \dot{x} &= \cos\psi u - \sin\psi v, \\ \dot{y} &= -\sin\psi u + \cos\psi v, \\ \dot{\psi} &= r, \end{aligned}$$

or

$$\dot{\boldsymbol{\eta}} = \mathbf{R}(\psi)\boldsymbol{v}$$

with the transformation matrix $\mathbf{R}$ given as in Cheng/Zhang (2017).

[1]Notice that the sway can only be controlled directly if the vessel has "Seitenstrahlruder" which is not assumed here.

2.2 The dynamic equation for the generalized velocities

This is just second Newton's law applied to moving rigid bodies in the Lagrangian coordinates.

2.2.1 Pure rotation

For pure rotation, this is done by the Euler's equations for a rigid body

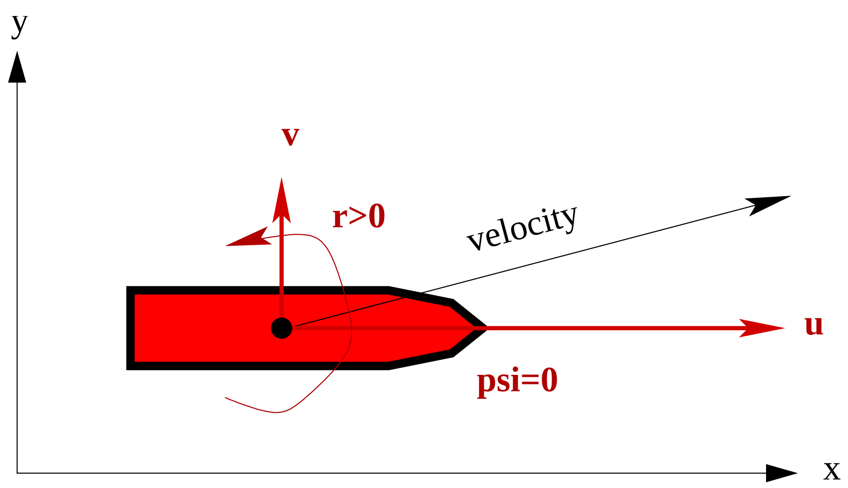
$$\dot{\boldsymbol{L}} = \boldsymbol{T} + \boldsymbol{\omega} \times \boldsymbol{L}, \quad \boldsymbol{L} = \mathbf{J}\,\boldsymbol{\omega}$$

where L is the angular momentum and $\boldsymbol{T}$ the torque in the Lagrangian system. For 2d, the cross product is zero since sufficient symmetries of the vessel are assumed such that not only $\boldsymbol{\omega} = (0, 0, r)$ points into the z direction but also $\boldsymbol{L}$. [2]

2.2.2 Translation and couplings

With added translation, there is also the second Newton's law for linear motion $\dot{\boldsymbol{p}} = \boldsymbol{F}$ (the rate of change of the linear momentum is equal to the linear force) and a wide variety of couplings of the translational and rotational degrees of freedom by *coriolis* and *centrifugal* inertial terms given in the matrix $\mathbf{C}_{\mathrm{RB}}$. Euler's equations and the linear Newton's equations, together with their couplings, are often referred to as the *Newton-Euler equations.*

Example of the translational-rotational coupling



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[2]This is not generally so; there are asymmetric vessels where none of the three principal inertia axes points in the z direction.

A vessel with $\psi = 0$ (heading in x direction), surge $u = \dot{x} > 0$ (moving in the x direction), sway $v = \dot{y} > 0$ (drifting sideways in positive y direction) and yaw $r = \dot{\psi} > 0$ (moving its "nose" to the left) will increase its surge and decrease its sway even if the displacement rate $(\dot{x}, \dot{y})$ remains constant (the vessel moves in a straight line at an angle ψ to the x-axis) because the sway component is transformed into the surge component by the yaw action:

$$(\dot{u}, \dot{v})'_{\text{yaw}} = (vr, -ur)'$$

This is represented by the basic equations of motion (2)-(10) of Cheng-Zang (2017) for the special case $x_g = 0$ (the origin of the Lagrangian coordinates is at the center of gravity), without inertia added by the induced water motion ($X_{\dot{u}} = Y_{\dot{v}} = N_{\dot{t}} = Y_{\dot{r}} = 0$), without damping ($\mathbf{D}$ tensor is zero), and without external and non-modelled forces/torques ($\boldsymbol{\tau} = \boldsymbol{\tau}_w = \boldsymbol{g} = 0$). Then, the dynamics is just given by

$$\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_z \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} 0 & 0 & +mv \\ 0 & 0 & -mu \\ -mv & +mu & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ r \end{pmatrix}.$$

Notice that there should be no coriolis/centrifugal forces acting on the rotation r itself, so the third line of the negative $\mathbf{C}$ matrix (the matrix on the rhs) should be given in a way that $\dot{r} = 0$. This is satisfied.

2.2.3 Added masses by the induced water motion

The water not only introduces frictional forces but is also set into motion itself: The longitudinal u motion gives rise to a u motion of the water (added dynamical mass $X_{\dot{u}}$), and the v motion and rotation to a v and rotation motion of the water (added masses $Y_{\dot{v}}$ and $N_{\dot{r}}$, respectively). Furthermore, the rotation induces another v movement of the water (added mass $Y_{\dot{r}}$). These masses are added to the vessel masses/moment of inertia diag(m, m, J) as an additional *added-mass tensor* $\mathbf{M}_{\text{A}}$. This water motion is also subject to coriolis/centrifugal forces giving rise to an additional matrix $\mathbf{C}_{\text{A}}$.

2.2.4 Added friction

This includes linear friction for laminar fluid motion, quadratic ones for turbulent flows, and higher-order ones for shallow water or if the vessel reaches her terminal velocity.

Linear friction Additional linear friction constants are introduced: X_u (friction for surge=longitudinal motion), Y_v (sway=lateral motion), N_r (friction slowing the rotation rate if she rotates around herself), and Y_r and N_v (cross coupling between sway and rotational friction). no coupling between surge and rotational friction or between surge and sway is assumed. This makes up the constant part of the friction tensor $\mathbf{D}$.

Quadratic and higher-order ("nonlinear") friction Vessels nearly always operate in the turbulant hydrodynamic regime, so the nonlinear friction coefficients increasing with the square of $\boldsymbol{u}$ components (classical turbulent hydrodynamics) and higher orders (in shallow water or if the vessel reaches her terminal speed) are added with additional forces X and Y and additional torques N with two (quadratic) or more (higher-order) subscripts. This makes up the $\boldsymbol{v}$ dependent part of the matrix $\mathbf{D}$. However, I miss a coupling between u and v which clearly exists in the nonlinear regime: As a cyclist, I feel more air drag at 90 degrees side wind than without wind. Putting aside anecdotal evidence, a rotation-symmetric object moving through motionless air or water feels a quadradic turbulent drag vector proportional to $-\sqrt{u^2+v^2}(u,v)'$ which could be realized by replacing $X_{|u|u}|u|$ and $Y_{|v|v}|v|$ by $X_{|u|u}|(u,v)'|$ and $Y_{|v|v}|(u,v)'|$, respectively (absolute value of the velocity vector rather than from a component).

Origin of the Lagrangian coordinate system is not at the center of gravity The Euler-Newton equations are simplest when the origin of the Lagrange system is at the center of gravity (CG) since this is the only point which does not change in the Euler system under pure rotations. However, the water and wind forces as well as the rudder often have other pivotal points and sometimes one wants to describe vessels from this perspective. None of these points is fixed in the vessel coordinate system for all times, not even the CG (if one loads asymmetrically), so it is a matter of taste which origin to choose.

Assuming symmetry in the xz-plane, all sensible other points are shifted from the CG in the u direction (longitudinal vessel axis)[3]. If

$$x_g = u_0 - u_{\mathrm{CG}}$$

is nonzero,[4] the pivot point "wobbles" under rotations. This gives rise to two categories of additional terms:

(i) Additional masses. Assume $x_g = u_0 - u_{\mathrm{CG}} > 0$ and a standstill vessel headed in the x direction ($x = y = \psi = r = 0$) starting to rotate anticlockwise ($\dot{r} > 0$). Then, in this frame, the center of gravity is accelerated in the negative $y = v$ direction,

$$\dot{\boldsymbol{r}}_{\mathrm{CG}} = -x_g \dot{r} \boldsymbol{e}_v = -x_g \dot{r} \boldsymbol{e}_y$$

($\boldsymbol{e}$ are unit vectors; the second equation sign is only valid at the beginning where $\psi = 0$). This means, there is a force and an associated torque

$$\boldsymbol{f}_{\mathrm{inertia}} = -m x_g \dot{r} \boldsymbol{e}_v, \; \boldsymbol{T}_{\mathrm{inertia}} = -m x_g \dot{v} \boldsymbol{e}_z$$

giving rise to the entries $-(-m x_g)$ as the 23 and 32 components of the dynamic mass tensor $\mathbf{M}_{\mathrm{RB}}$.

(ii) Of course, the motion of the CG under rotations also gives rise to additional centrifugal and coriolis terms appearing in the tensor $\mathbf{C}_{\mathrm{RB}}$.

[3]The coordinate origin u_0 is sometimes referred to as CO but I hate multi-letter variables

[4]The variable name x_g is misleading, it should be called u_g since it is defined in the Lagrangian system

3 Summary

The approach seems to be consistent with only a few misleading variable names and missing terms. Some, e.g., the linear friction terms, may be irrelevant for actual vessel traffic since they typically operate in the turbulent regime. Some other more important terms are missing.

Pros

1. Derived from basic physical laws, particularly Second Newton's law
2. In spite of the apparent complexity, simpler than the equations used in the BAWs simulator; due to the matrix formulation, faster to calculate
3. Completely documented and reproducible

Cons

1. Does not include the vertical dimension, even parametrically, since there is no z component (needs to be done in the Eulerian frame). Unlike ocean ships, this is relevant for inland waterway vessels because of the squat (*Absunk*)
2. Other effects of shallow water such as the increased drag or the reduced length-dependent terminal speed of a vessel may be described by the mentioned (third-order) nonlinear friction terms but the explicit expressions need to be taken from the BAW and adapted to the formalism here
3. The authors assume standstill waters. Any effects of water currents (changed friction, additional gravitational surge, additional torques by cross currents/current gradients) needs to be taken from the BAW and adapted
4. The authors assume extended 2d water surface, not channels/waterways. The restricted space in the latter will introduce additional kinetic and friction terms (change the $\mathbf{M}_{\mathrm{A}}$, $\mathbf{C}_{\mathrm{A}}$, and $\mathbf{D}$ matrices and make some of their coefficients dynamic) particularly if two vessels cross each other at a close distance (due to the Bernoulli effect, they will be "sucked together")

Of cours, points 2-4 of the Cons may be hidden in the non-modelled alibi term $\boldsymbol{g}(\boldsymbol{v})$ which, however, then would read $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{\eta})$ since these points depend explicitely on the waterway geometry. Still, being aware of these limitations, the approach seems to be a good starting point (particularly if some changes of the matrices due to restricted waterways and currents are proposed).