STA 360/602L: Module 4.4

Multivariate normal model IV

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Reading example: posterior computation

- Recall that we have

$$\pi(\boldsymbol{\theta}|\Sigma, \boldsymbol{Y}) = \mathcal{N}_2(\boldsymbol{\mu}_n, \Lambda_n)$$

where

$$\Lambda_n = \left[\Lambda_0^{-1} + n\Sigma^{-1}\right]^{-1}$$

$$\boldsymbol{\mu}_n = \Lambda_n \left[\Lambda_0^{-1}\boldsymbol{\mu}_0 + n\Sigma^{-1}\bar{\boldsymbol{y}}\right],$$

- For our reading example,

$$\boldsymbol{\mu}_0 = (\mu_{0(1)}, \mu_{0(2)})^T = (50, 50)^T$$

$$\Lambda_0 = \begin{pmatrix} 156 & 78 \\ 78 & 156 \end{pmatrix}$$

Reading example: posterior computation

- We also have

$$\pi(\Sigma|\boldsymbol{\theta}, \boldsymbol{Y}) = \mathcal{IW}_2(\nu_n, \boldsymbol{S}_n)$$

or using the notation in the book, $\mathcal{IW}_2(\nu_n, \boldsymbol{S}_n^{-1})$, where

$$\begin{aligned}\nu_n &= \nu_0 + n\\ \\ \boldsymbol{S}_n &= [\boldsymbol{S}_0 + \boldsymbol{S}_\theta]\\ &= \left[\boldsymbol{S}_0 + \sum_{i=1}^{n}(\boldsymbol{y}_i - \boldsymbol{\theta})(\boldsymbol{y}_i - \boldsymbol{\theta})^T\right].\end{aligned}$$

- Again, for our reading example,

$$\nu_0 = p + 2 = 4$$

$$\Sigma_0 = \begin{pmatrix} 625 & 312.5 \\ 312.5 & 625 \end{pmatrix}$$

Posterior computation

```
#Data summaries
n <- nrow(Y)
ybar <- apply(Y,2,mean)

#Hyperparameters for the priors
mu_0 <- c(50,50)
Lambda_0 <- matrix(c(156,78,78,156),nrow=2,ncol=2)
nu_0 <- 4
S_0 <- matrix(c(625,312.5,312.5,625),nrow=2,ncol=2)

#Initial values for Gibbs sampler
#No need to set initial value for theta, we can simply sample it first
Sigma <- cov(Y)

#Set null matrices to save samples
THETA <- SIGMA <- NULL
```

Next, the code for the Gibbs sampler.

Posterior computation

```
#Now, to the Gibbs sampler
#library(mvtnorm) for multivariate normal
#library(MCMCpack) for inverse-Wishart

#first set number of iterations and burn-in, then set seed
n_iter <- 10000; burn_in <- 0.3*n_iter
set.seed(1234)

for (s in 1:(n_iter+burn_in)){
##update theta using its full conditional
Lambda_n <- solve(solve(Lambda_0) + n*solve(Sigma))
mu_n <- Lambda_n %*% (solve(Lambda_0)%*%mu_0 + n*solve(Sigma)%*%ybar)
theta <- rmvnorm(1,mu_n,Lambda_n)

#update Sigma
S_theta <- (t(Y)-c(theta))%*%t(t(Y)-c(theta))
S_n <- S_0 + S_theta
nu_n <- nu_0 + n
Sigma <- riwish(nu_n, S_n)

#save results only past burn-in
if(s > burn_in){
  THETA <- rbind(THETA,theta)
  SIGMA <- rbind(SIGMA,c(Sigma))
  }
}
colnames(THETA) <- c("theta_1","theta_2")
colnames(SIGMA) <- c("sigma_11","sigma_12","sigma_21","sigma_22") #symmetry in sigma
```

Note that the text also has a function to sample from the Wishart distribution.

Diagnostics

```r
#library(coda)
THETA.mcmc <- mcmc(THETA,start=1); summary(THETA.mcmc)
```

```
##
## Iterations = 1:10000
## Thinning interval = 1
## Number of chains = 1
## Sample size per chain = 10000
##
## 1. Empirical mean and standard deviation for each variable,
##    plus standard error of the mean:
##
##          Mean    SD Naive SE Time-series SE
## theta_1 47.30 2.956  0.02956        0.02956
## theta_2 53.69 3.290  0.03290        0.03290
##
## 2. Quantiles for each variable:
##
##          2.5%   25%   50%   75% 97.5%
## theta_1 41.55 45.35 47.36 49.23 53.08
## theta_2 47.08 51.53 53.69 55.82 60.13
```

```r
effectiveSize(THETA.mcmc)
```

```
## theta_1 theta_2
##   10000   10000
```

Diagnostics

```
SIGMA.mcmc <- mcmc(SIGMA,start=1); summary(SIGMA.mcmc)
```

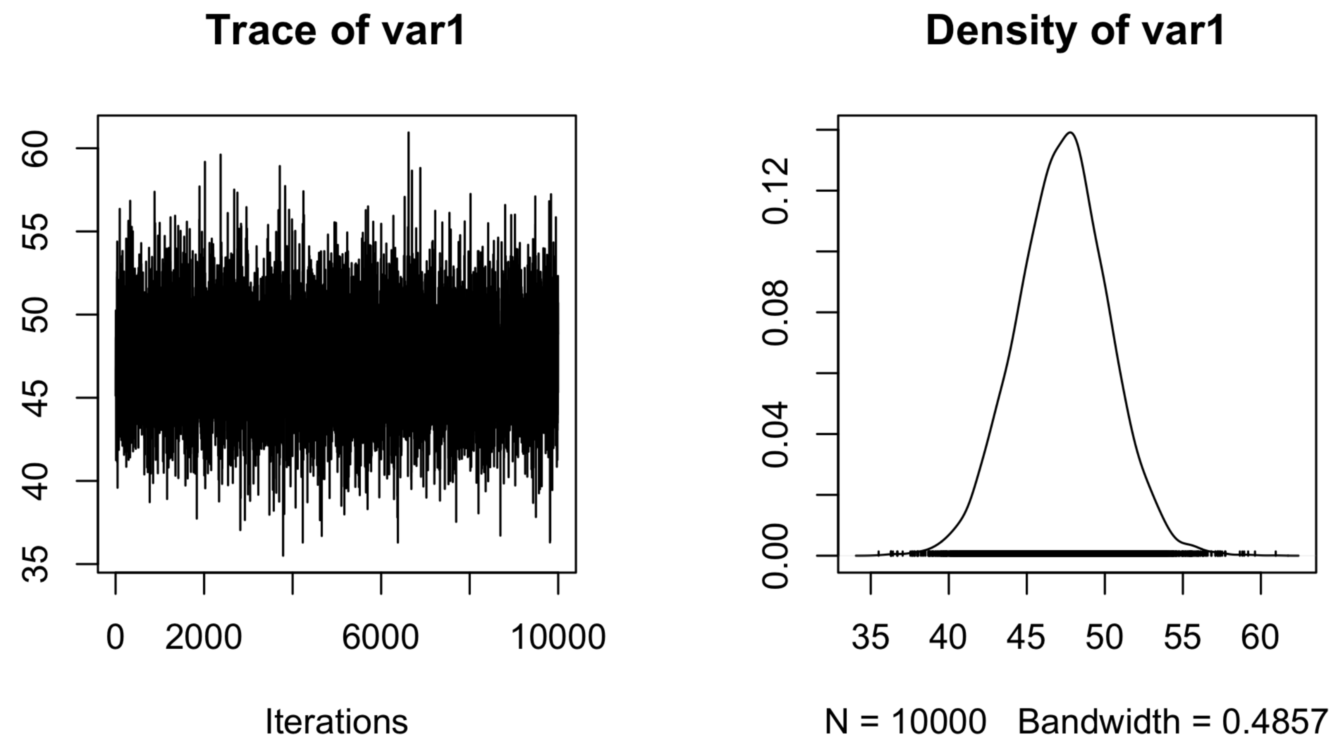
```
##
## Iterations = 1:10000
## Thinning interval = 1
## Number of chains = 1
## Sample size per chain = 10000
##
## 1. Empirical mean and standard deviation for each variable,
##    plus standard error of the mean:
##
##           Mean    SD Naive SE Time-series SE
## sigma_11 202.3 63.39   0.6339         0.6511
## sigma_12 155.3 60.92   0.6092         0.6244
## sigma_21 155.3 60.92   0.6092         0.6244
## sigma_22 260.1 81.96   0.8196         0.8352
##
## 2. Quantiles for each variable:
##
##            2.5%   25%   50%   75% 97.5%
## sigma_11 113.50 158.2 190.8 234.8 357.3
## sigma_12  67.27 113.2 144.7 186.5 305.4
## sigma_21  67.27 113.2 144.7 186.5 305.4
## sigma_22 145.84 203.2 244.6 300.9 461.0
```

```
effectiveSize(SIGMA.mcmc)
```

```
## sigma_11 sigma_12 sigma_21 sigma_22
## 9478.710 9517.989 9517.989 9629.352
```

Diagnostics: trace plots

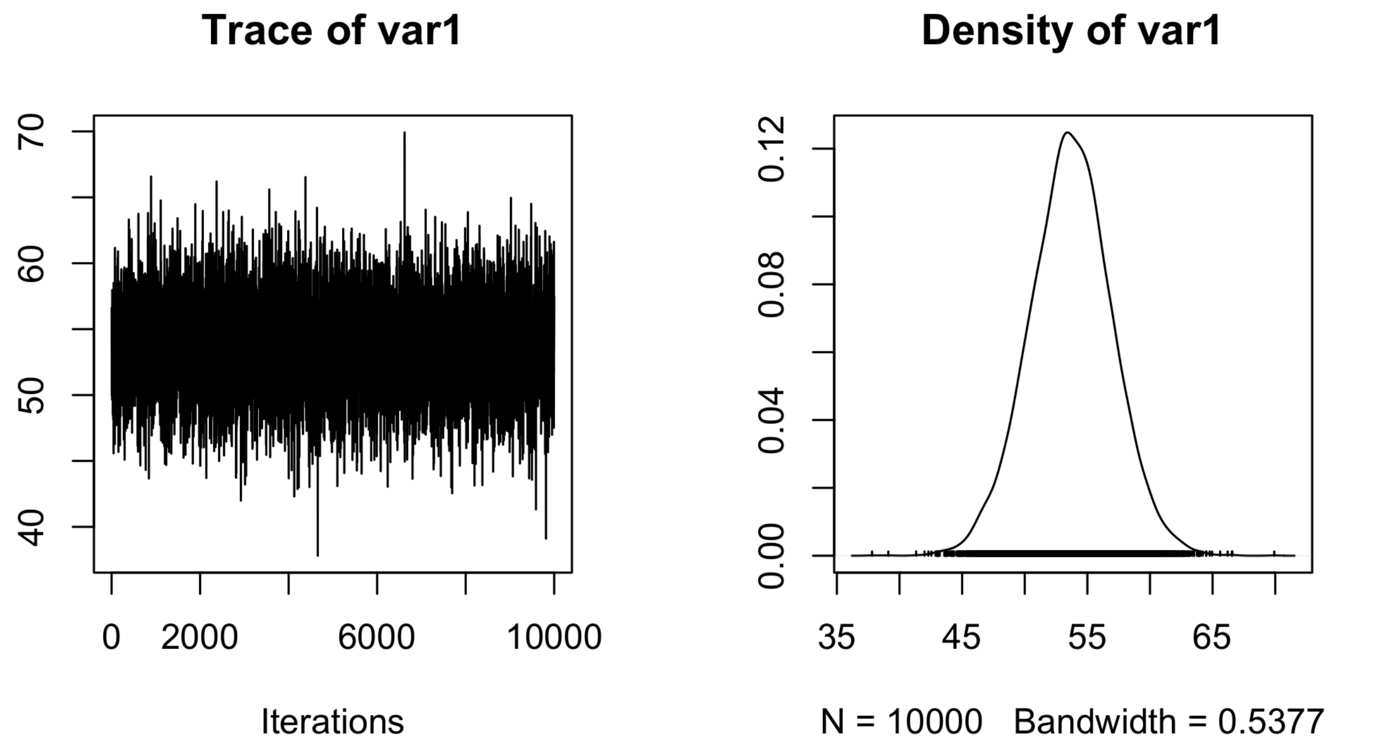
```
plot(THETA.mcmc[,"theta_1"])
```



Looks good!

Diagnostics: trace plots

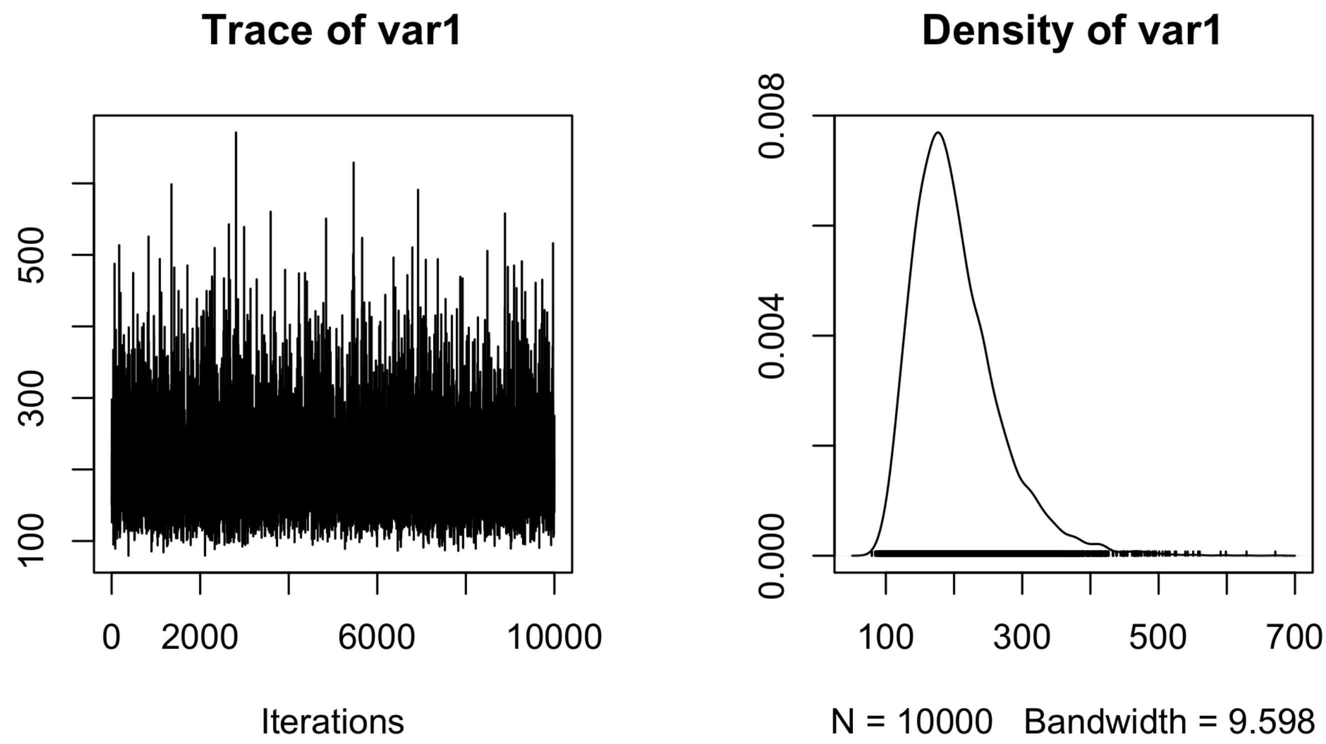
```
plot(THETA.mcmc[,"theta_2"])
```



Looks good!

Diagnostics: trace plots

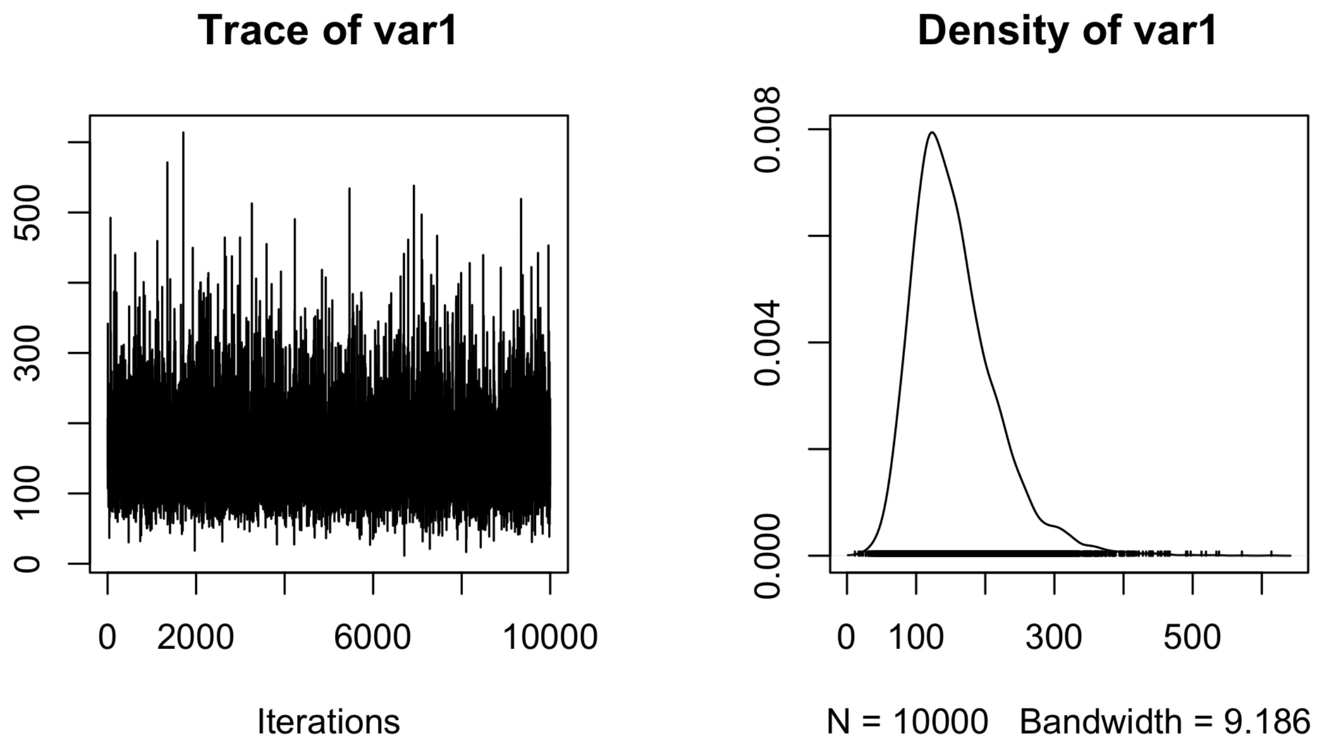
```
plot(SIGMA.mcmc[,"sigma_11"])
```



Looks good!

Diagnostics: trace plots

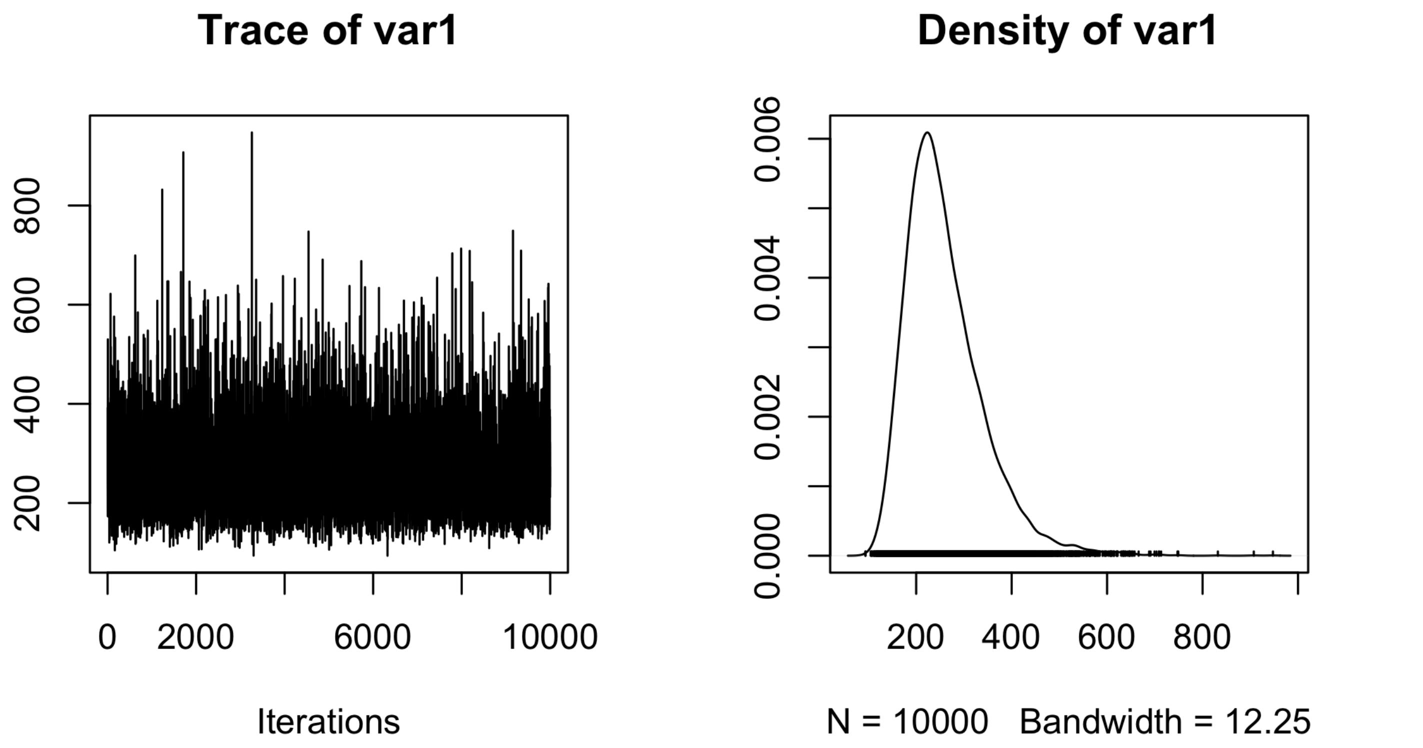
```
plot(SIGMA.mcmc[,"sigma_12"])
```



Looks good!

Diagnostics: trace plots

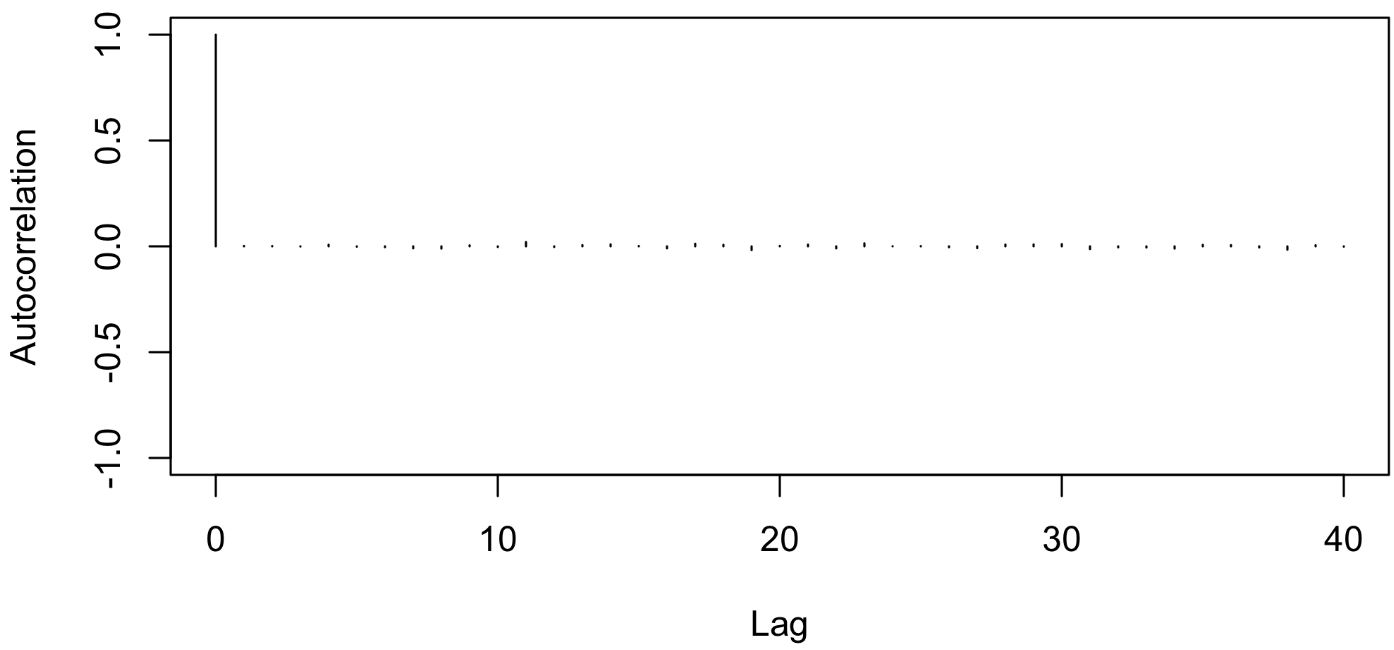
```
plot(SIGMA.mcmc[,"sigma_22"])
```



Looks good!

Diagnostics: autocorrelation

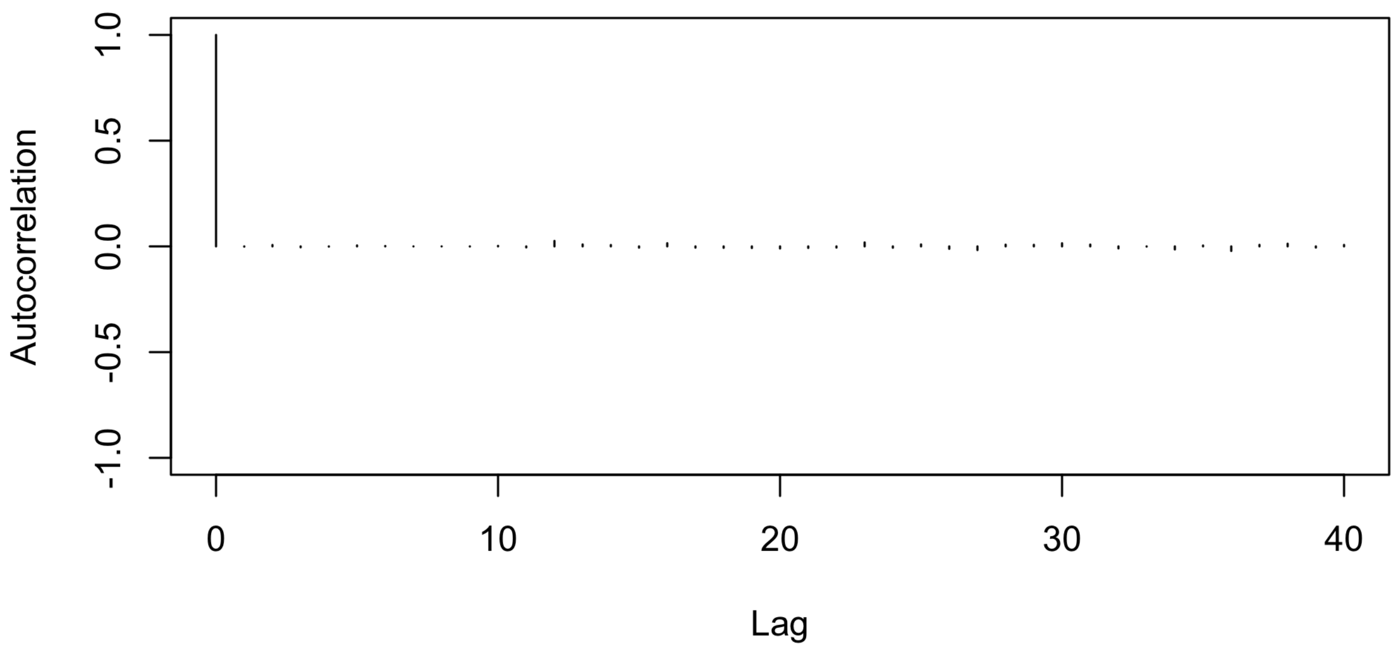
```
autocorr.plot(THETA.mcmc[,"theta_1"])
```



Looks good!

Diagnostics: autocorrelation

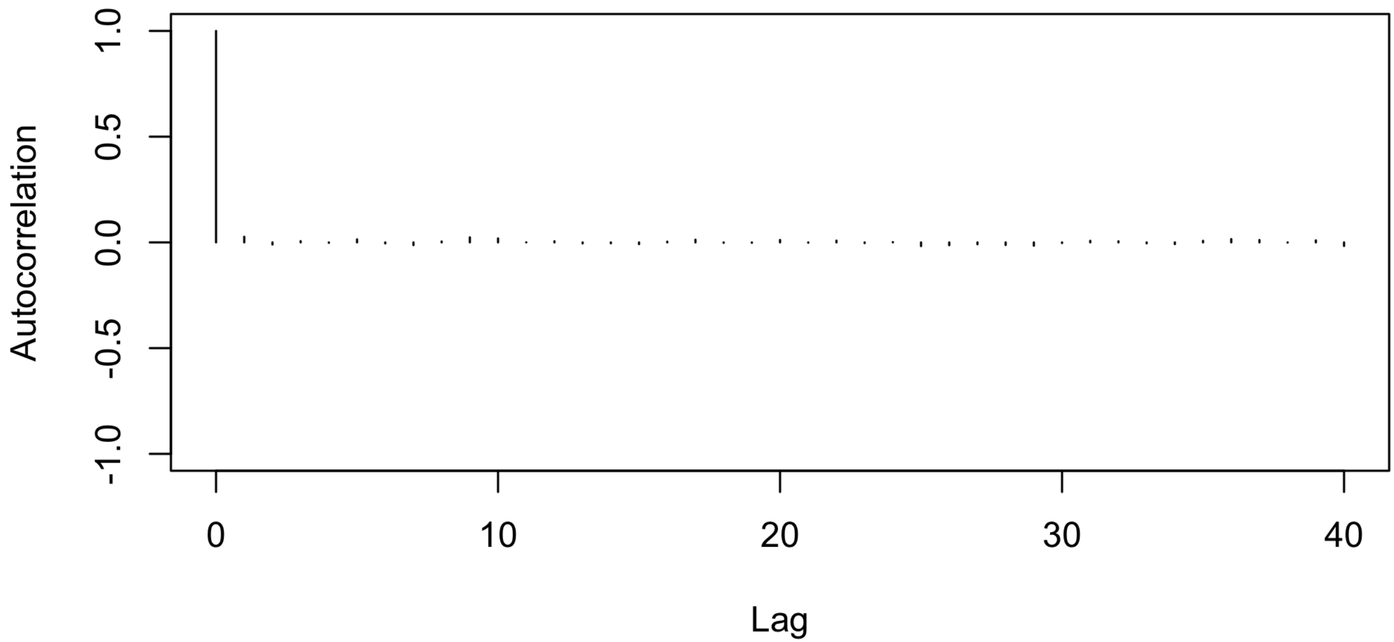
```
autocorr.plot(THETA.mcmc[,"theta_2"])
```



Looks good!

Diagnostics: autocorrelation

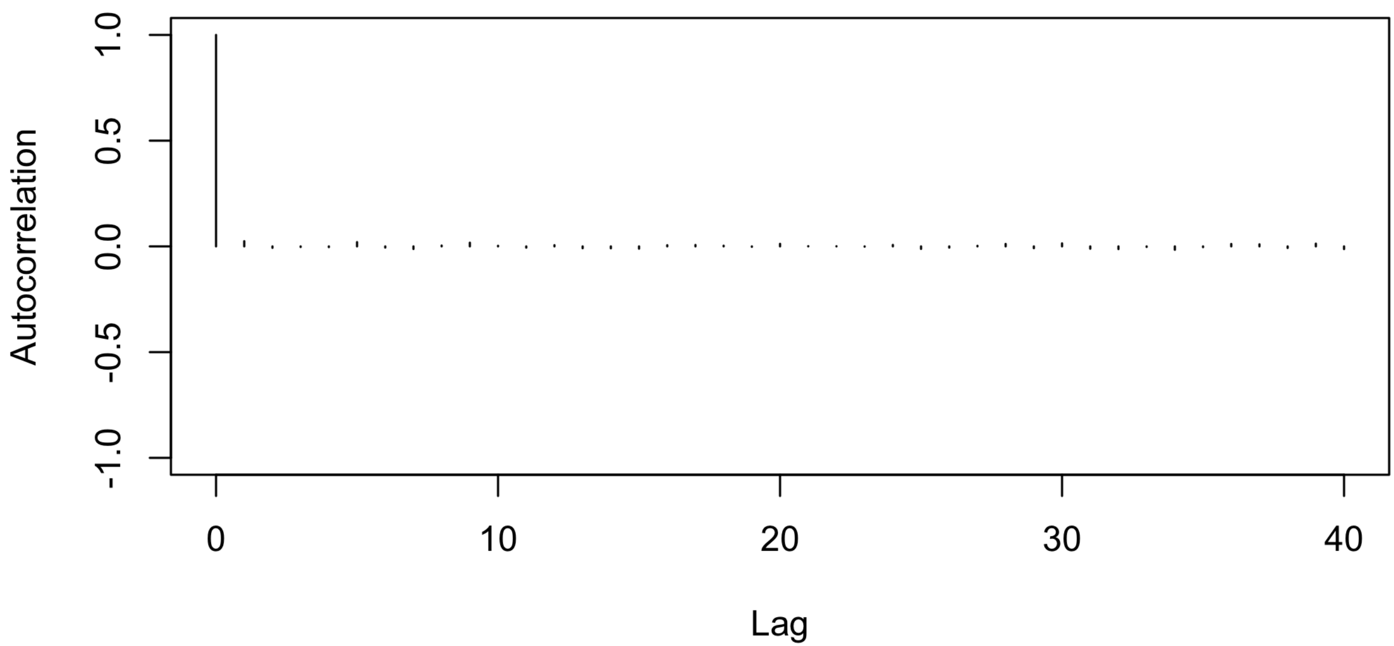
```
autocorr.plot(SIGMA.mcmc[,"sigma_11"])
```



Looks good!

Diagnostics: autocorrelation

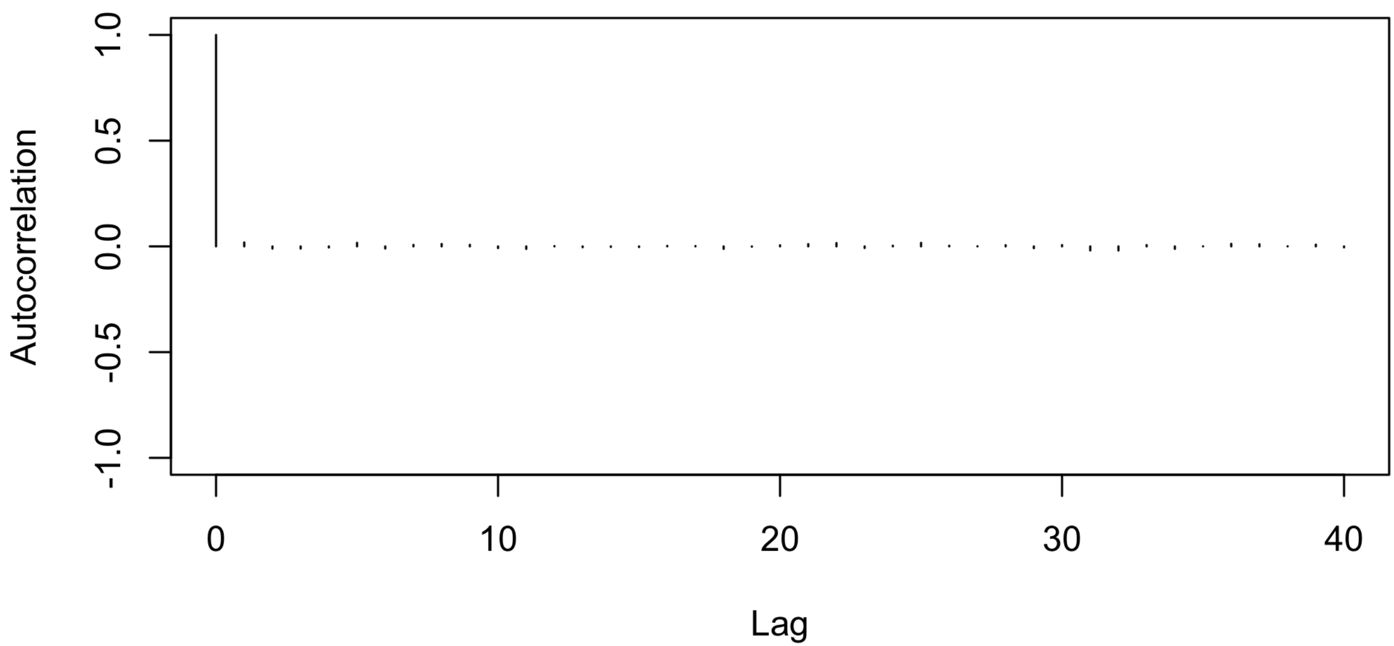
```r
autocorr.plot(SIGMA.mcmc[,"sigma_12"])
```



Looks good!

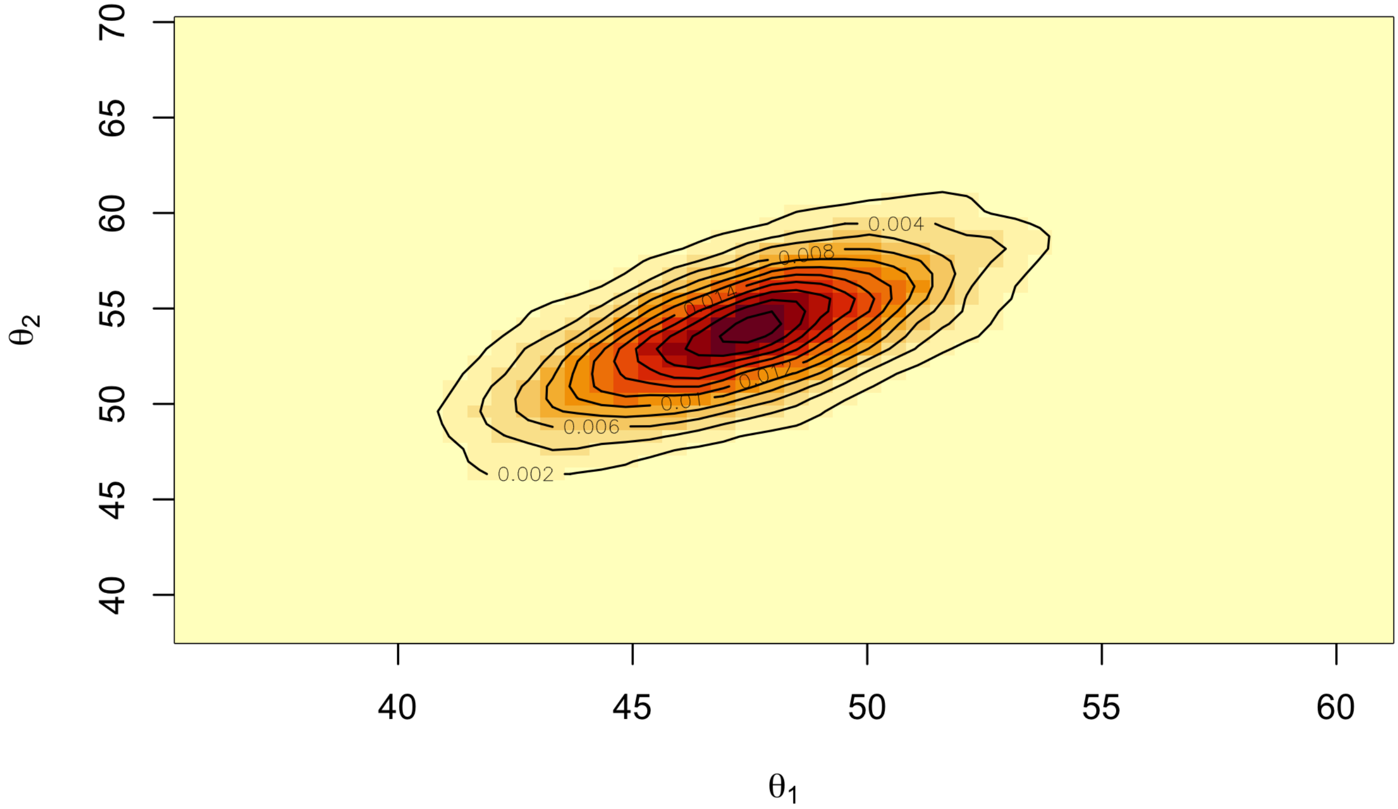
Diagnostics: autocorrelation

```
autocorr.plot(SIGMA.mcmc[,"sigma_22"])
```



Looks good!

Posterior distribution of the mean



Answering questions of interest

- Questions of interest:
  - Do students improve in reading comprehension on average?
- Need to compute $\Pr[\theta_2 > \theta_1 | \boldsymbol{Y}]$. In R,

```
mean(THETA[,2]>THETA[,1])
```

```
## [1] 0.992
```

- That is, posterior probability > 0.99 and indicates strong evidence that test scores are higher in the second administration.

Answering questions of interest

- Questions of interest:
    - If so, by how much?
- Need posterior summaries of $\pi[\theta_2 - \theta_1|\boldsymbol{Y}]$. In R,

```
mean(THETA[,2] - THETA[,1])
```

```
## [1] 6.385515
```

```
quantile(THETA[,2] - THETA[,1], prob=c(0.025, 0.5, 0.975))
```

```
##      2.5%       50%     97.5%
##  1.233154  6.385597 11.551304
```

- Mean (and median) improvement is ≈ 6.39 points with 95% credible interval (1.23, 11.55).

Answering questions of interest

- Questions of interest:
  - How correlated (positively) are the post-test and pre-test scores?
- We can compute $\Pr[\sigma_{12} > 0|\boldsymbol{Y}]$. In R,

```
mean(SIGMA[,2]>0)
```

```
## [1] 1
```

- Posterior probability that the covariance between them is positive is basically 1.

Answering questions of interest

- Questions of interest:
  - How correlated (positively) are the post-test and pre-test scores?
- We can also look at the distribution of ρ instead. In R,

```
CORR <- SIGMA[,2]/(sqrt(SIGMA[,1])*sqrt(SIGMA[,4]))
quantile(CORR,prob=c(0.025, 0.5, 0.975))
```

```
##      2.5%       50%     97.5%
## 0.4046817 0.6850218 0.8458880
```

- Median correlation between the 2 scores is 0.69 with a 95% quantile-based credible interval of (0.40, 0.85)
- Because density is skewed, we may prefer the 95% HPD interval, which is (0.45, 0.88).

```
#library(hdrcde)
hdr(CORR,prob=95)$hdr
```

```
##          [,1]      [,2]
## 95% 0.4468522 0.8761174
```

Jeffreys' prior

- Clearly, there's a lot of work to be done in specifying the hyperparameters (two of which are $p \times p$ matrices).
- What if we want to specify the priors so that we put in as little information as possible?
- We already know how to do that somewhat with Jeffreys' priors.
- For the multivariate normal model, turns out that the Jeffreys' rule for generating a prior distribution on $(\boldsymbol{\theta}, \Sigma)$ gives

$$\pi(\boldsymbol{\theta}, \Sigma) \propto |\Sigma|^{-\frac{(p+2)}{2}}.$$

- Can we derive the full conditionals under this prior?
- **To be done during discussion session.**

Jeffreys' prior

- We will leverage previous work. For the likelihood we have both

$$L(\boldsymbol{Y};\boldsymbol{\theta},\Sigma) \propto \exp\left\{-\frac{1}{2}\boldsymbol{\theta}^T(n\Sigma^{-1})\boldsymbol{\theta} + \boldsymbol{\theta}^T(n\Sigma^{-1}\bar{\boldsymbol{y}})\right\}$$

and

$$L(\boldsymbol{Y};\boldsymbol{\theta},\Sigma) \propto |\Sigma|^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\boldsymbol{S}_\theta\Sigma^{-1}\right]\right\},$$

where $\boldsymbol{S}_\theta = \sum_{i=1}^n (\boldsymbol{y}_i - \boldsymbol{\theta})(\boldsymbol{y}_i - \boldsymbol{\theta})^T$.

- Also, we can rewrite any $\mathcal{N}_p(\boldsymbol{\mu}_0, \Lambda_0)$ as

$$p(\boldsymbol{\theta}) \propto \exp\left\{-\frac{1}{2}\boldsymbol{\theta}^T\Lambda_0^{-1}\boldsymbol{\theta} + \boldsymbol{\theta}^T\Lambda_0^{-1}\boldsymbol{\mu}_0\right\}.$$

- Finally, $\Sigma \sim \mathcal{IW}_p(\nu_0, \boldsymbol{S}_0)$,

$$\Rightarrow \quad p(\Sigma) \ \propto \ |\Sigma|^{\frac{-(\nu_0+p+1)}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}(\boldsymbol{S}_0\Sigma^{-1})\right\}.$$

What's next?

Move on to the readings for the next module!