# Hierarchical models II

Dr. Olanrewaju Michael Akande

March 25, 2020

## Announcements

- Review changes to syllabus.
- Any concerns from the lab meetings?
- Going forward, there will be 5 minute breaks (roughly) halfway through each class meeting.

## Outline

- Hierarchical modeling of means recap
- Hierarchical modeling of means and variances
- Gibbs sampler
- ELS data

# Regular univariate normal model

- Recall that if we assume

$$y_i \sim \mathcal{N}(\mu, \sigma^2), \;\; i = 1, \ldots, n,$$

and set our priors to be

$$\pi(\mu) = \mathcal{N}\left(\mu_0, \gamma_0^2\right).$$
$$\pi(\sigma^2) = \mathcal{IG}\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right),$$

then we have

$$\pi(\mu, \sigma^2|Y) \propto \left\{\prod_{i=1}^{n} p(y_i|\mu, \sigma^2)\right\} \cdot \pi(\mu) \cdot \pi(\sigma^2).$$

# Full conditionals

- So that

$$\pi(\mu|\sigma^2, Y) = \mathcal{N}\left(\mu_n, \gamma_n^2\right).$$

where

$$\gamma_n^2 = \frac{1}{\dfrac{n}{\sigma^2} + \dfrac{1}{\gamma_0^2}}; \qquad \mu_n = \gamma_n^2 \left[\frac{n}{\sigma^2}\bar{y} + \frac{1}{\gamma_0^2}\mu_0\right],$$

- and

$$\pi(\sigma^2|\mu, Y) = \mathcal{IG}\left(\frac{\nu_n}{2}, \frac{\nu_n \sigma_n^2}{2}\right),$$

where

$$\nu_n = \nu_0 + n; \qquad \sigma_n^2 = \frac{1}{\nu_n}\left[\nu_0\sigma_0^2 + \sum_{i=1}^{n}(y_i - \mu)^2\right].$$

# Hierarchical modeling of means recap

- We've looked at the hierarchical normal model of the form

$$
\begin{aligned}
y_{ij}|\theta_j, \sigma^2 &\sim \mathcal{N}\left(\theta_j, \sigma_j^2\right); \quad i = 1, \ldots, n_j \\
\theta_j|\mu, \tau^2 &\sim \mathcal{N}\left(\mu, \tau^2\right); \quad j = 1, \ldots, J.
\end{aligned}
$$

- The model gives us an extra hierarchy through the prior on the means, leading to sharing of information across the groups, when estimating the group-specific means.

- As before, first set $\sigma_j^2 = \sigma^2$ for all groups, to simplify posterior inference. We will revisit this today.

- Thus, we only have two variance terms, $\sigma^2$ and $\tau^2$, to inform us on the within-group variation and between-group variation respectively.

# Hierarchical normal model recap

- Standard semi-conjugate priors as before:

$$\pi(\mu) = \mathcal{N}\left(\mu_0, \gamma_0^2\right)$$
$$\pi(\sigma^2) = \mathcal{IG}\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right)$$
$$\pi(\tau^2) = \mathcal{IG}\left(\frac{\eta_0}{2}, \frac{\eta_0\tau_0^2}{2}\right).$$

with

- $\mu_0$: best guess of average of school averages
- $\gamma_0^2$: set based on plausible ranges of values of $\mu$
- $\tau_0^2$: best guess of the (scaled) variance of school averages
- $\eta_0$: set based on how tight prior for $\tau^2$ is around $\tau_0^2$
- $\sigma_0^2$: best guess of the (scaled) variance of individual test scores around respective school means
- $\nu_0$: set based on how tight prior for $\sigma^2$ is around $\sigma_0^2$.

# Posterior inference recap

- The resulting posterior is therefore:

$$
\begin{aligned}
\pi(\theta_1, \ldots, \theta_J, \mu, \sigma^2, \tau^2 | Y) \propto\ & p(y|\theta_1, \ldots, \theta_J, \mu, \sigma^2, \tau^2) \\
& \times p(\theta_1, \ldots, \theta_J | \mu, \sigma^2, \tau^2) \\
& \times \pi(\mu, \sigma^2, \tau^2) \\
\\
=\ & p(y|\theta_1, \ldots, \theta_J, \sigma^2) \\
& \times p(\theta_1, \ldots, \theta_J | \mu, \tau^2) \\
& \times \pi(\mu) \cdot \pi(\sigma^2) \cdot \pi(\tau^2) \\
\\
=\ & \left\{ \prod_{j=1}^{J} \prod_{i=1}^{n_j} p(y_{ij}|\theta_j, \sigma^2) \right\} \\
& \times \left\{ \prod_{j=1}^{J} p(\theta_j|\mu, \tau^2) \right\} \\
& \times \pi(\mu) \cdot \pi(\sigma^2) \cdot \pi(\tau^2)
\end{aligned}
$$

# Full conditional for grand mean

## recap

- $$\pi(\mu|\theta_1,\ldots,\theta_J,\sigma^2,\tau^2,Y) \propto \left\{\prod_{j=1}^{J} p(\theta_j|\mu,\tau^2)\right\} \cdot \pi(\mu).$$

- This looks like the full conditional distribution from the one-sample normal case, so that

$$\pi(\mu|\theta_1,\ldots,\theta_J,\sigma^2,\tau^2,Y) = \mathcal{N}\left(\mu_n,\gamma_n^2\right) \quad \text{where}$$

$$\gamma_n^2 = \frac{1}{\dfrac{J}{\tau^2}+\dfrac{1}{\gamma_0^2}}; \qquad \mu_n = \gamma_n^2\left[\frac{J}{\tau^2}\bar{\theta}+\frac{1}{\gamma_0^2}\mu_0\right]$$

and $\bar{\theta} = \frac{1}{J}\sum_{j=1}^{J}\theta_j$.

STA 602L

# Full conditionals for group means

## recap

- $$\pi(\theta_j|\theta_{-j},\mu,\sigma^2,\tau^2,Y) \propto \left\{\prod_{i=1}^{n_j} p(y_{ij}|\theta_j,\sigma^2)\right\} \cdot p(\theta_j|\mu,\tau^2)$$

- Those terms include a normal density for $\theta_j$ multiplied by a product of normal densities in which $\theta_j$ is the mean, again mirroring the one-sample case, so you can show that

$$\pi(\theta_j|\theta_{-j},\mu,\sigma^2,\tau^2,Y) = \mathcal{N}\left(\mu_j^\star,\tau_j^\star\right) \quad \text{where}$$

$$\tau_j^\star = \frac{1}{\dfrac{n_j}{\sigma^2} + \dfrac{1}{\tau^2}}; \qquad \mu_j^\star = \tau_j^\star\left[\frac{n_j}{\sigma^2}\bar{y}_j + \frac{1}{\tau^2}\mu\right]$$

# Full conditionals for within-group variance recap

- $$\pi(\sigma^2|\theta_1,\ldots,\theta_J,\mu,\tau^2,Y) \propto \left\{\prod_{j=1}^{J}\prod_{i=1}^{n_j} p(y_{ij}|\theta_j,\sigma^2)\right\} \cdot \pi(\sigma^2)$$

- We can take advantage of the one-sample normal problem, so that our full conditional posterior is

$$\pi(\sigma^2|\theta_1,\ldots,\theta_J,\mu,\tau^2,Y) = \mathcal{IG}\left(\frac{\nu_n}{2},\frac{\nu_n\sigma_n^2}{2}\right) \quad \text{where}$$

$$\nu_n = \nu_0 + \sum_{j=1}^{J} n_j; \qquad \sigma_n^2 = \frac{1}{\nu_n}\left[\nu_0\sigma_0^2 + \sum_{j=1}^{J}\sum_{i=1}^{n_j}(y_{ij}-\theta_j)^2\right].$$

# Full conditionals for across-group variance recap

- $$\pi(\tau^2|\theta_1,\ldots,\theta_J,\mu,\sigma^2,Y) \propto \left\{\prod_{j=1}^{J} p(\theta_j|\mu,\tau^2)\right\} \cdot \pi(\tau^2)$$

- Again, we have

$$\pi(\tau^2|\theta_1,\ldots,\theta_J,\mu,\sigma^2,Y) = \mathcal{IG}\left(\frac{\eta_n}{2}, \frac{\eta_n\tau_n^2}{2}\right) \quad \text{where}$$

$$\eta_n = \eta_0 + J; \qquad \tau_n^2 = \frac{1}{\eta_n}\left[\eta_0\tau_0^2 + \sum_{j=1}^{J}(\theta_j - \mu)^2\right].$$

# Hierarchical modeling of means and variances

- Often researchers emphasize differences in means. However, variances can be very important.
- If we think means vary across groups, why shouldn't we worry about variances also varying across groups?
- In that case, we have the model

$$y_{ij} | \theta_j, \sigma^2 \sim \mathcal{N}\left(\theta_j, \sigma_j^2\right); \quad i = 1, \ldots, n_j$$
$$\theta_j | \mu, \tau^2 \sim \mathcal{N}\left(\mu, \tau^2\right); \quad j = 1, \ldots, J,$$

- However, now we also need a prior on all the $\sigma_j^2$'s that lets us borrow information about across groups.

# Full conditionals

- Notice that our prior won't affect the full conditions for $\mu$ and $\tau^2$ since those have nothing to do with all the $\sigma_j^2$'s.

- The full conditional for each $\theta_j$, we have

$$\pi(\theta_j | \theta_{-j}, \mu, \sigma_1^2, \ldots, \sigma_J^2, \tau^2, Y) \propto \left\{ \prod_{i=1}^{n_j} p(y_{ij} | \theta_j, \sigma_j^2) \right\} \cdot p(\theta_j | \mu, \tau^2)$$

with the only change from before being $\sigma_j^2$.

- That is, those terms still include a normal density for $\theta_j$ multiplied by a product of normals in which $\theta_j$ is the mean, again mirroring the previous case, so you can show that

$$\pi(\theta_j | \theta_{-j}, \mu, \sigma_1^2, \ldots, \sigma_J^2, \tau^2, Y) = \mathcal{N}\left(\mu_j^\star, \tau_j^\star\right) \quad \text{where}$$

$$\tau_j^\star = \frac{1}{\dfrac{n_j}{\sigma_j^2} + \dfrac{1}{\tau^2}}; \qquad \mu_j^\star = \tau_j^\star \left[ \frac{n_j}{\sigma_j^2} \bar{y}_j + \frac{1}{\tau^2} \mu \right]$$

# How about within-group variances?

- Now we need to find a semi-conjugate prior for the $\sigma_j^2$'s. Before, with one $\sigma^2$, we had

$$\pi(\sigma^2) = \mathcal{IG}\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right),$$

  which was nicely semi-conjugate.

- That suggests that maybe we should start with.

$$\sigma_1^2, \ldots, \sigma_J^2 | \nu_0, \sigma_0^2 \sim \mathcal{IG}\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right)$$

- However, if we just fix the hyperparameters $\nu_0$ and $\sigma_0^2$ in advance, the prior on the $\sigma_j^2$'s does not allow borrowing of information across other values of $\sigma_j^2$, to aid in estimation.

- Thus, we actually need to treat $\nu_0$ and $\sigma_0^2$ as parameters in a hierarchical model for both means and variances.

# How about within-group variances?

- Before we get to the choice of the priors for $\nu_0$ and $\sigma_0^2$, we have enough to derive the full conditional for each $\sigma_j^2$. This actually takes a similar form to what we had before we indexed by $j$, that is,

$$\pi(\sigma_j^2|\sigma_{-j}^2, \theta_1, \ldots, \theta_J, \mu, \tau^2, \nu_0, \sigma_0^2, Y) \propto \left\{ \prod_{i=1}^{n_j} p(y_{ij}|\theta_j, \sigma_j^2) \right\} \cdot \pi(\sigma_j^2|\nu_0, \sigma_0^2)$$

- This still looks like what we had before, that is, products of normals and one inverse-gamma, so that

$$\pi(\sigma_j^2|\sigma_{-j}^2, \theta_1, \ldots, \theta_J, \mu, \tau^2, \nu_0, \sigma_0^2, Y) = \mathcal{IG}\left( \frac{\nu_j^\star}{2}, \frac{\nu_j^\star \sigma_j^{2(\star)}}{2} \right) \quad \text{where}$$

$$\nu_j^\star = \nu_0 + n_j; \qquad \sigma_j^{2(\star)} = \frac{1}{\nu_j^\star} \left[ \nu_0 \sigma_0^2 + \sum_{i=1}^{n_j} (y_{ij} - \theta_j)^2 \right].$$

# Remaining hyper-priors

- Now we can get back to priors for $\nu_0$ and $\sigma_0^2$. Turns out that a semi-conjugate prior for $\sigma_0^2$ (see question 2 on homework 2) is a gamma distribution. That is, if we set

$$\pi(\sigma_0^2) = \mathcal{G}a\left(a, b\right),$$

then,

$$\begin{aligned}\pi(\sigma_0^2|\theta_1, \ldots, \theta_J, \sigma_1^2, \ldots, \sigma_J^2, \mu, \tau^2, \nu_0, Y) &\propto \left\{\prod_{j=1}^{J} p(\sigma_j^2|\nu_0, \sigma_0^2)\right\} \cdot \pi(\sigma_0^2)\\ &\propto \mathcal{IG}\left(\sigma_j^2; \frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right) \cdot \mathcal{G}a\left(\sigma_0^2; a, b\right)\end{aligned}$$

- Recall that
  - $\mathcal{G}a(y; a, b) \equiv \dfrac{b^a}{\Gamma(a)} y^{a-1} e^{-by}$, and
  - $\mathcal{IG}(y; a, b) \equiv \dfrac{b^a}{\Gamma(a)} y^{-(a+1)} e^{-\frac{b}{y}}$.

# REMAINING HYPER-PRIORS

- So $\pi(\sigma_0^2|\theta_1,\ldots,\theta_J,\sigma_1^2,\ldots,\sigma_J^2,\mu,\tau^2,\nu_0,Y)$

$$\propto \left\{\prod_{j=1}^{J} p(\sigma_j^2|\nu_0,\sigma_0^2)\right\}\cdot\pi(\sigma_0^2)$$

$$\propto \mathcal{IG}\left(\sigma_j^2;\frac{\nu_0}{2},\frac{\nu_0\sigma_0^2}{2}\right)\cdot\mathcal{G}a\left(\sigma_0^2;a,b\right)$$

$$=\left[\prod_{j=1}^{J}\frac{\left(\frac{\nu_0\sigma_0^2}{2}\right)^{\left(\frac{\nu_0}{2}\right)}}{\Gamma\left(\frac{\nu_0}{2}\right)}(\sigma_j^2)^{-\left(\frac{\nu_0}{2}+1\right)}e^{-\frac{\nu_0\sigma_0^2}{2(\sigma_j^2)}}\right]\cdot\left[\frac{b^a}{\Gamma(a)}(\sigma_0^2)^{a-1}e^{-b\sigma_0^2}\right]$$

$$\propto\left[\prod_{j=1}^{J}(\sigma_0^2)^{\left(\frac{\nu_0}{2}\right)}e^{-\frac{\nu_0\sigma_0^2}{2(\sigma_j^2)}}\right]\cdot\left[(\sigma_0^2)^{a-1}e^{-b\sigma_0^2}\right]$$

$$\propto\left[(\sigma_0^2)^{\left(\frac{J\nu_0}{2}\right)}e^{-\sigma_0^2\left[\frac{\nu_0}{2}\sum_{j=1}^{J}\frac{1}{\sigma_j^2}\right]}\right]\cdot\left[(\sigma_0^2)^{a-1}e^{-b\sigma_0^2}\right]$$

# Remaining hyper-priors

- That is, the full conditional is

$$
\begin{aligned}
\pi(\sigma_0^2 | \cdots\cdots) &\propto \left[ \left(\sigma_0^2\right)^{\left(\frac{J\nu_0}{2}\right)} e^{-\sigma_0^2 \left[\frac{\nu_0}{2} \sum_{j=1}^{J} \frac{1}{\sigma_j^2}\right]} \right] \cdot \left[ (\sigma_0^2)^{a-1} e^{-b\sigma_0^2} \right] \\
&\propto \left[ \left(\sigma_0^2\right)^{\left(a + \frac{J\nu_0}{2} - 1\right)} e^{-\sigma_0^2 \left[b + \frac{\nu_0}{2} \sum_{j=1}^{J} \frac{1}{\sigma_j^2}\right]} \right] \\
&\equiv \mathcal{G}a\left(\sigma_0^2; a_n, b_n\right),
\end{aligned}
$$

where

$$
a_n = a + \frac{J\nu_0}{2}; \quad b_n = b + \frac{\nu_0}{2} \sum_{j=1}^{J} \frac{1}{\sigma_j^2}.
$$

# Remaining hyper-priors

- Ok that leaves us with one parameter to go, i.e., $\nu_0$. Turns out there is no simple conjugate/semi-conjugate prior for $\nu_0$.

- Common practice is to restrict $\nu_0$ to be an integer (which makes sense when we think of it as being degrees of freedom, which also means it cannot be zero). With the restriction, we need a discrete distribution as the prior with support on $\nu_0 = 1, 2, 3, \ldots$.

- **Poll question: Can we use either a binomial or a Poisson prior on for $\nu_0$?**

- A popular choice is the geometric distribution with pmf $p(\nu_0) = (1-p)^{\nu_0 - 1} p$.

- However, we will rewrite the kernel as $\pi(\nu_0) \propto e^{-\alpha \nu_0}$. How did we get here from the geometric pmf and what is $\alpha$?

# Final full conditional

- With this prior, $\pi(\nu_0|\theta_1,\ldots,\theta_J,\sigma_1^2,\ldots,\sigma_J^2,\mu,\tau^2,\sigma_0^2,Y)$

$$\propto \left\{\prod_{j=1}^{J} p(\sigma_j^2|\nu_0,\sigma_0^2)\right\}\cdot\pi(\nu_0)$$

$$\propto \mathcal{IG}\left(\sigma_j^2;\frac{\nu_0}{2},\frac{\nu_0\sigma_0^2}{2}\right)\cdot e^{-\alpha\nu_0}$$

$$= \left[\prod_{j=1}^{J}\frac{\left(\frac{\nu_0\sigma_0^2}{2}\right)^{\left(\frac{\nu_0}{2}\right)}}{\Gamma\left(\frac{\nu_0}{2}\right)}\left(\sigma_j^2\right)^{-\left(\frac{\nu_0}{2}+1\right)}e^{-\frac{\nu_0\sigma_0^2}{2(\sigma_j^2)}}\right]\cdot e^{-\alpha\nu_0}$$

$$\propto \left[\left(\frac{\left(\frac{\nu_0\sigma_0^2}{2}\right)^{\left(\frac{\nu_0}{2}\right)}}{\Gamma\left(\frac{\nu_0}{2}\right)}\right)^{J}\cdot\left(\prod_{j=1}^{J}\frac{1}{\sigma_j^2}\right)^{\left(\frac{\nu_0}{2}-1\right)}\cdot e^{-\nu_0\left[\frac{\sigma_0^2}{2}\sum_{j=1}^{J}\frac{1}{\sigma_j^2}\right]}\right]\cdot e^{-\alpha\nu_0}$$

# Final full conditional

- That is, the full conditional is

$$\pi(\nu_0 | \cdots\cdots) \propto \left[ \left( \frac{\left(\frac{\nu_0 \sigma_0^2}{2}\right)^{\left(\frac{\nu_0}{2}\right)}}{\Gamma\left(\frac{\nu_0}{2}\right)} \right)^J \cdot \left( \prod_{j=1}^{J} \frac{1}{\sigma_j^2} \right)^{\left(\frac{\nu_0}{2} - 1\right)} \cdot e^{-\nu_0 \left[ \alpha + \frac{\sigma_0^2}{2} \sum_{j=1}^{J} \frac{1}{\sigma_j^2} \right]} \right],$$

which is not a known kernel and is thus unnormalized (i.e., does not integrate to 1 in its current form).

- This sure looks like a lot, but it will be relatively easy to compute in R.
- Now, technically, the support is $\nu_0 = 1, 2, 3, \ldots$, however, we can compute this to compute the unnormalized distribution across a grid of $\nu_0$ values, say, $\nu_0 = 1, 2, 3, \ldots$ for some large $K$, and then sample.

# Final full conditional

- One more thing, computing these probabilities on the raw scale can be problematic particularly because of the product inside. Good idea to transform to the log scale instead.

- That is,

$$\pi(\nu_0|\cdots\cdots) \propto \left[ \left( \frac{\left(\frac{\nu_0 \sigma_0^2}{2}\right)^{\left(\frac{\nu_0}{2}\right)}}{\Gamma\left(\frac{\nu_0}{2}\right)} \right)^J \cdot \left( \prod_{j=1}^J \frac{1}{\sigma_j^2} \right)^{\left(\frac{\nu_0}{2}-1\right)} \cdot e^{-\nu_0\left[\alpha + \frac{\sigma_0^2}{2}\sum_{j=1}^J \frac{1}{\sigma_j^2}\right]} \right]$$

$$\begin{aligned}
\Rightarrow \ln\pi(\nu_0|\cdots\cdots) \propto & \left(\frac{J\nu_0}{2}\right) \ln\left(\frac{\nu_0\sigma_0^2}{2}\right) - J\ln\left[\Gamma\left(\frac{\nu_0}{2}\right)\right] \\
& + \left(\frac{\nu_0}{2} - 1\right) \left( \sum_{j=1}^J \ln\left[\frac{1}{\sigma_j^2}\right] \right) \\
& - \nu_0 \left[ \alpha + \frac{\sigma_0^2}{2} \sum_{j=1}^J \frac{1}{\sigma_j^2} \right]
\end{aligned}$$

# ELS DATA

- Finally, enough math and some data!
- We have data from the 2002 Educational Longitudinal Survey (ELS). This survey includes a random sample of 100 large urban public high schools, and 10th graders randomly sampled within these high schools.

```
Y <- as.matrix(dget("http://www2.stat.duke.edu/~pdh10/FCBS/Inline/Y.school.mathscore"))
dim(Y)
```

```
## [1] 1993    2
```

```
head(Y)
```

```
##      school mathscore
## [1,]      1     52.11
## [2,]      1     57.65
## [3,]      1     66.44
## [4,]      1     44.68
## [5,]      1     40.57
## [6,]      1     35.04
```

```
length(unique(Y[,"school"]))
```

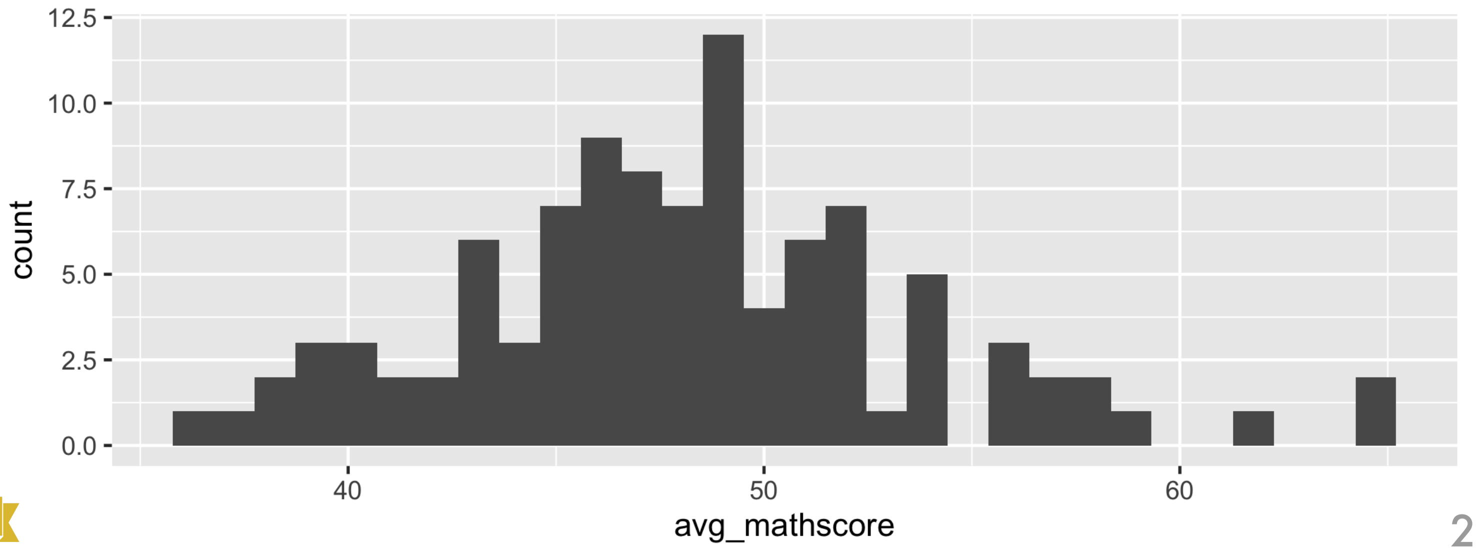
```
## [1] 100
```

# ELS DATA

First, some EDA:

```
Data <- as.data.frame(Y)
Data$school <- as.factor(Data$school)
Data %>%
  group_by(school) %>%
  na.omit()%>%
  summarise(avg_mathscore = mean(mathscore)) %>%
  dplyr::ungroup() %>%
  ggplot(aes(x = avg_mathscore))+
  geom_histogram()
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```

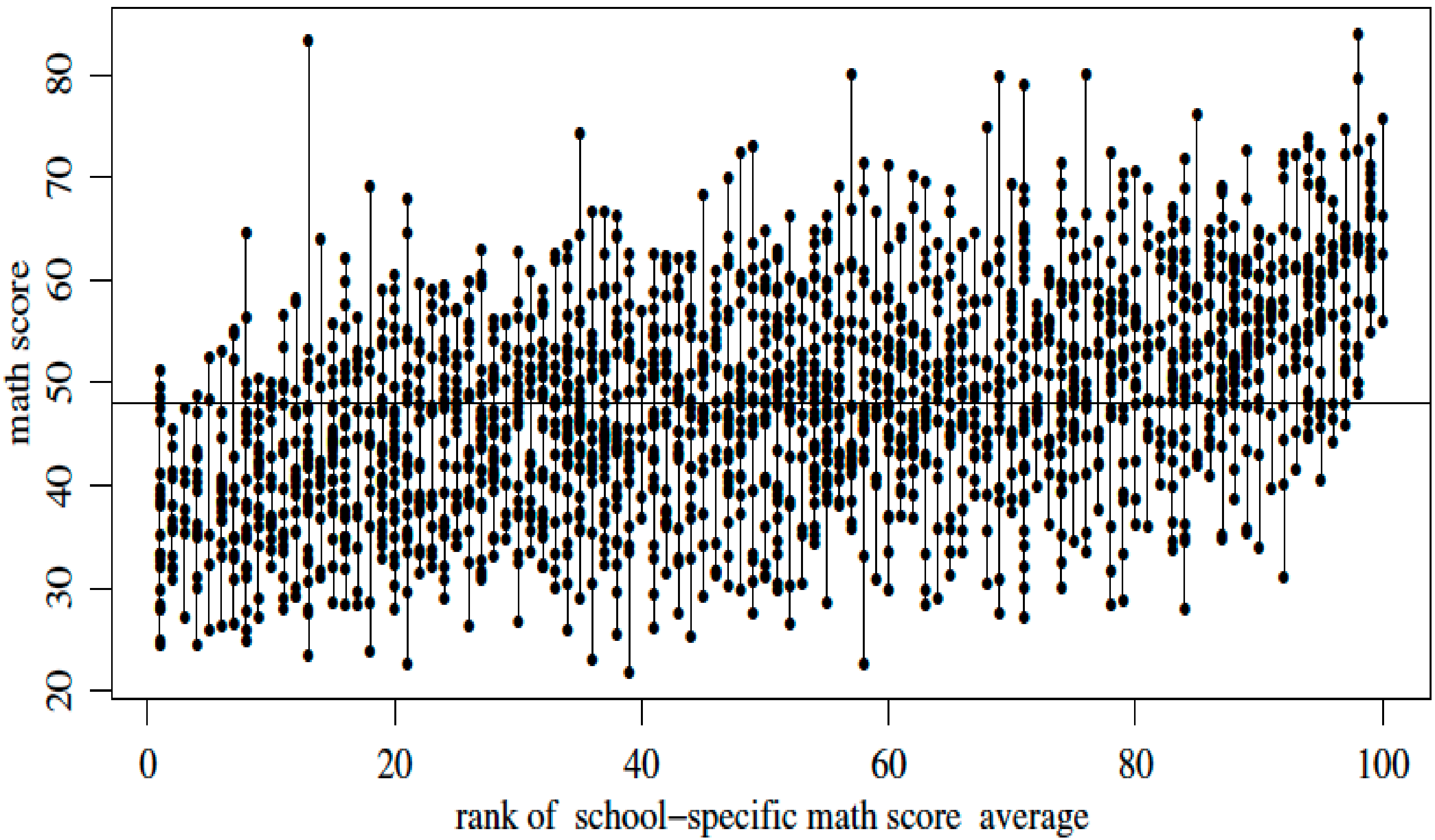


# ELS DATA

There does appear to be school-related differences in means and in variances, some of which are actually related to the sample sizes.

# ELS DATA

Consider the math scores of these children:



# ELS HYPOTHESES

- Investigators may be interested in the following:
  - Differences in mean scores across schools
  - Differences in school-specific variances
- How do we evaluate these questions in a statistical model?

# Hierarchical model

- We can write out the full model we've been describing as follows.

$$y_{ij}|\theta_j, \sigma^2 \sim \mathcal{N}\left(\theta_j, \sigma_j^2\right); \quad i = 1, \ldots, n_j$$

$$\theta_j|\mu, \tau^2 \sim \mathcal{N}\left(\mu, \tau^2\right); \quad j = 1, \ldots, J$$

$$\sigma_1^2, \ldots, \sigma_J^2|\nu_0, \sigma_0^2 \sim \mathcal{IG}\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right)$$

$$\mu \sim \mathcal{N}\left(\mu_0, \gamma_0^2\right)$$

$$\tau^2 \sim \mathcal{IG}\left(\frac{\eta_0}{2}, \frac{\eta_0\tau_0^2}{2}\right).$$

$$\pi(\nu_0) \propto e^{-\alpha\nu_0}$$

$$\sigma_0^2 \sim \mathcal{G}a\left(a, b\right).$$

- Now, we need to specify hyperparameters. That should be fun!

# Prior specification

- This math exam was designed to have a national mean of 50 and standard deviation of 10. Suppose we don't have any other information besides that.
- Thus, we can let

$$\mu \sim \mathcal{N}\left(\mu_0 = 50, \gamma_0^2 = 25\right)$$

$$\tau^2 \sim \mathcal{IG}\left(\frac{\eta_0}{2} = \frac{1}{2}, \frac{\eta_0 \tau_0^2}{2} = \frac{100}{2}\right).$$

$$\pi(\nu_0) \propto e^{-\alpha \nu_0} \propto e^{-\nu_0}$$

$$\sigma_0^2 \sim \mathcal{G}a\left(a = 1, b = \frac{1}{100}\right).$$

- Are these prior distributions overly informative?

# Full conditionals (recap)

- $$\pi(\theta_j | \cdots\cdots) = \mathcal{N}\left(\mu_j^\star, \tau_j^\star\right) \quad \text{where}$$
  $$\tau_j^\star = \frac{1}{\dfrac{n_j}{\sigma_j^2} + \dfrac{1}{\tau^2}}; \qquad \mu_j^\star = \tau_j^\star \left[\frac{n_j}{\sigma_j^2}\bar{y}_j + \frac{1}{\tau^2}\mu\right]$$

- $$\pi(\sigma_j^2 | \cdots\cdots) = \mathcal{IG}\left(\frac{\nu_j^\star}{2}, \frac{\nu_j^\star \sigma_j^{2(\star)}}{2}\right) \quad \text{where}$$
  $$\nu_j^\star = \nu_0 + n_j; \qquad \sigma_j^{2(\star)} = \frac{1}{\nu_j^\star}\left[\nu_0\sigma_0^2 + \sum_{i=1}^{n_j}(y_{ij} - \theta_j)^2\right].$$

- $$\pi(\mu | \cdots\cdots) = \mathcal{N}\left(\mu_n, \gamma_n^2\right) \quad \text{where}$$
  $$\gamma_n^2 = \frac{1}{\dfrac{J}{\tau^2} + \dfrac{1}{\gamma_0^2}}; \qquad \mu_n = \gamma_n^2 \left[\frac{J}{\tau^2}\bar{\theta} + \frac{1}{\gamma_0^2}\mu_0\right]$$

# Full conditionals (recap)

- $$\pi(\tau^2|\cdots\cdots) = \mathcal{IG}\left(\frac{\eta_n}{2}, \frac{\eta_n\tau_n^2}{2}\right) \quad \text{where}$$
  $$\eta_n = \eta_0 + J; \qquad \tau_n^2 = \frac{1}{\eta_n}\left[\eta_0\tau_0^2 + \sum_{j=1}^{J}(\theta_j - \mu)^2\right].$$
- $$\begin{aligned}\ln\pi(\nu_0|\cdots\cdots) \propto & \left(\frac{J\nu_0}{2}\right)\ln\left(\frac{\nu_0\sigma_0^2}{2}\right) - J\ln\left[\Gamma\left(\frac{\nu_0}{2}\right)\right] \\ & + \left(\frac{\nu_0}{2} - 1\right)\left(\sum_{j=1}^{J}\ln\left[\frac{1}{\sigma_j^2}\right]\right) \\ & - \nu_0\left[\alpha + \frac{\sigma_0^2}{2}\sum_{j=1}^{J}\frac{1}{\sigma_j^2}\right]\end{aligned}$$
- $$\pi(\sigma_0^2|\cdots\cdots) = \mathcal{G}a\left(\sigma_0^2; a_n, b_n\right) \quad \text{where}$$
  $$a_n = a + \frac{J\nu_0}{2}; \quad b_n = b + \frac{\nu_0}{2}\sum_{j=1}^{J}\frac{1}{\sigma_j^2}.$$
- Obviously, as you have seen in the lab, we can simply use Stan (or Jags) to fit these models without needing to do any of this ourselves. The point here is to learn all the details.

# Gibbs sampler

```r
#Data summaries
J <- length(unique(Y[,"school"]))
ybar <- c(by(Y[,"mathscore"],Y[,"school"],mean))
s_j_sq <- c(by(Y[,"mathscore"],Y[,"school"],var))
n <- c(table(Y[,"school"]))

#Hyperparameters for the priors
mu_0 <- 50
gamma_0_sq <- 25
eta_0 <- 1
tau_0_sq <- 100
alpha <- 1
a <- 1
b <- 1/100

#Grid values for sampling nu_0_grid
nu_0_grid<-1:5000

#Initial values for Gibbs sampler
theta <- ybar
sigma_sq <- s_j_sq
mu <- mean(theta)
tau_sq <- var(theta)
nu_0 <- 1
sigma_0_sq <- 100
```

# Gibbs sampler

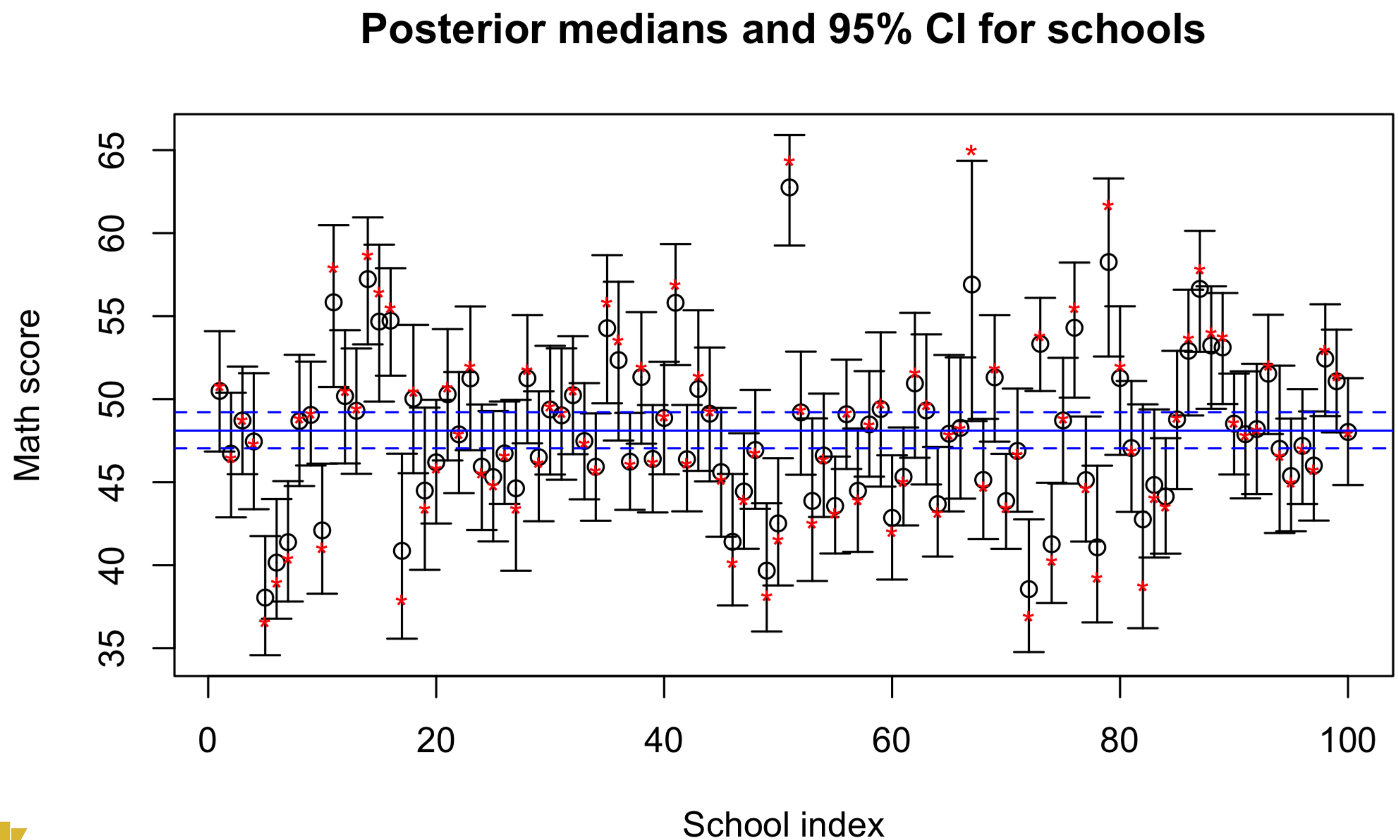
```
#first set number of iterations and burn-in, then set seed
n_iter <- 10000; burn_in <- 0.3*n_iter
set.seed(1234)

#Set null matrices to save samples
SIGMA_SQ <- THETA <- matrix(nrow=n_iter, ncol=J)
OTHER_PAR <- matrix(nrow=n_iter, ncol=4)

#Now, to the Gibbs sampler
for(s in 1:(n_iter+burn_in)){

  #update the theta vector (all the theta_j's)
  tau_j_star <- 1/(n/sigma_sq + 1/tau_sq)
  mu_j_star <- tau_j_star*(ybar*n/sigma_sq + mu/tau_sq)
  theta <- rnorm(J,mu_j_star,sqrt(tau_j_star))

  #update the sigma_sq vector (all the sigma_sq_j's)
  nu_j_star <- nu_0 + n
  theta_long <- rep(theta,n)
  nu_j_star_sigma_j_sq_star <-
    nu_0*sigma_0_sq + c(by((Y[,"mathscore"] - theta_long)^2,Y[,"school"],sum))
  sigma_sq <- 1/rgamma(J,(nu_j_star/2),(nu_j_star_sigma_j_sq_star/2))
  #update mu
  gamma_n_sq <- 1/(J/tau_sq + 1/gamma_0_sq)
  mu_n <- gamma_n_sq*(J*mean(theta)/tau_sq + mu_0/gamma_0_sq)
  mu <- rnorm(1,mu_n,sqrt(gamma_n_sq))
```

# GIBBS SAMPLER

```
  #update tau_sq
  eta_n <- eta_0 + J
  eta_n_tau_n_sq <- eta_0*tau_0_sq + sum((theta-mu)^2)
  tau_sq <- 1/rgamma(1,eta_n/2,eta_n_tau_n_sq/2)

  #update sigma_0_sq
  sigma_0_sq <- rgamma(1,(a + J*nu_0/2),(b + nu_0*sum(1/sigma_sq)/2))

  #update nu_0
  log_prob_nu_0 <- (J*nu_0_grid/2)*log(nu_0_grid*sigma_0_sq/2) -
    J*lgamma(nu_0_grid/2) +
    (nu_0_grid/2-1)*sum(log(1/sigma_sq)) -
    nu_0_grid*(alpha + sigma_0_sq*sum(1/sigma_sq)/2)
  nu_0 <- sample(nu_0_grid,1, prob = exp(log_prob_nu_0 - max(log_prob_nu_0)) )
  #this last step substracts the maximum logarithm from all logs
  #it is a neat trick that throws away all results that are so negative
  #they will screw up the exponential
  #note that the sample function will renormalize the probabilities internally


  #save results only past burn-in
  if(s > burn_in){
    THETA[(s-burn_in),] <- theta
    SIGMA_SQ[(s-burn_in),] <- sigma_sq
    OTHER_PAR[(s-burn_in),] <- c(mu,tau_sq,sigma_0_sq,nu_0)
  }
}
colnames(OTHER_PAR) <- c("mu","tau_sq","sigma_0_sq","nu_0")
```

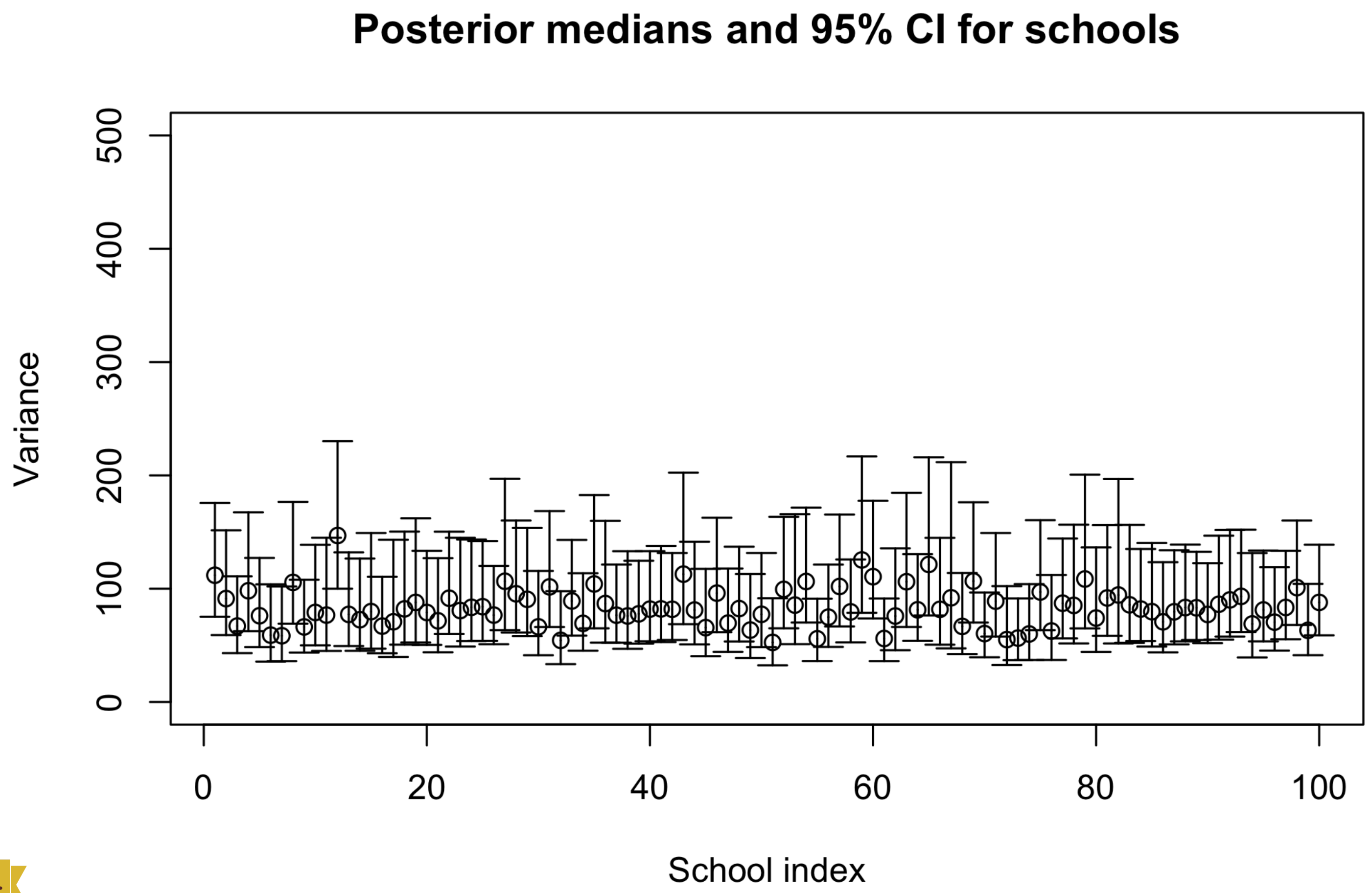
# Posterior inference

The blue lines indicate the posterior median and a 95% for $\mu$. The red asterisks indicate the data values $\bar{y}_j$.



# Posterior inference

Posterior summaries of $\sigma_j^2$.



# Posterior inference

## Shrinkage as a function of sample size

```
##    n Sample group mean Post. est. of group mean Post. est. of overall mean
## 1 31          50.81355                 50.49363                   48.10549
## 2 22          46.47955                 46.71544                   48.10549
## 3 23          48.77696                 48.71578                   48.10549
## 4 19          47.31632                 47.44935                   48.10549
## 5 21          36.58286                 38.04669                   48.10549
```

```
##     n Sample group mean Post. est. of group mean Post. est. of overall mean
## 15 12          56.43083                 54.67213                   48.10549
## 16 23          55.49609                 54.72904                   48.10549
## 17  7          37.92714                 40.86290                   48.10549
## 18 14          50.45357                 50.03007                   48.10549
```

```
##     n Sample group mean Post. est. of group mean Post. est. of overall mean
## 67  4          65.01750                 56.90436                   48.10549
## 68 19          44.74684                 45.13522                   48.10549
## 69 24          51.86917                 51.31079                   48.10549
## 70 27          43.47037                 43.86470                   48.10549
## 71 22          46.70455                 46.88374                   48.10549
## 72 13          36.95000                 38.55704                   48.10549
```

# How about non-normal models?

- Suppose we have $y_{ij} \in \{0, 1, \ldots\}$ being a count for subject $i$ in group $j$.
- For count data, it is natural to use a Poisson likelihood, that is,

  $$y_{ij} \sim \text{Poisson}(\theta_j)$$

  where each $\theta_j = \mathbb{E}[y_{ij}]$ is a group specific mean.
- When there are limited data within each group, it is natural to borrow information.
- How can we accomplish this with a hierarchical model?
- See homework 6!