

STA 360/602L: MODULE 7.4

METROPOLIS WITHIN GIBBS

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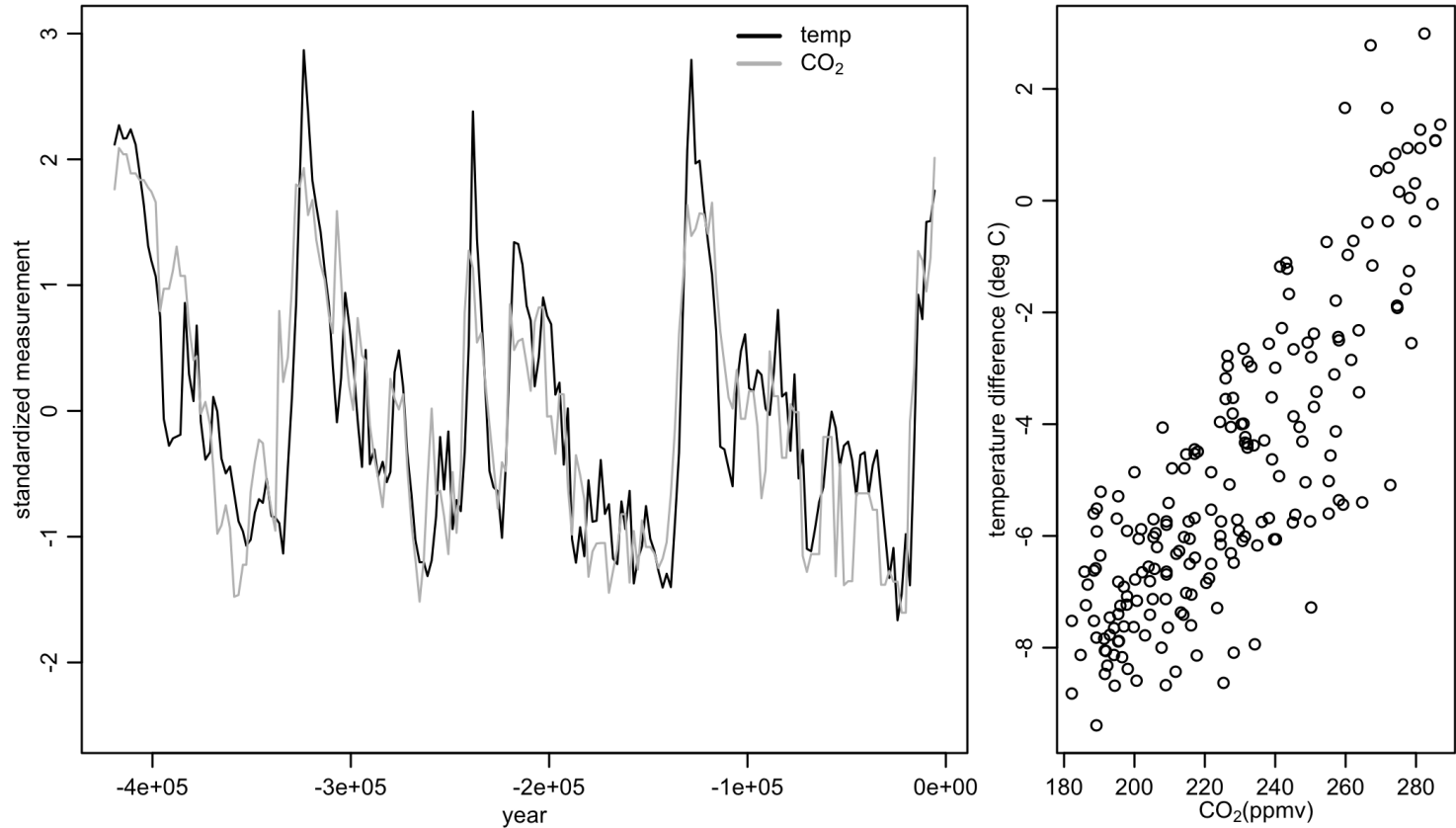
COMBINING METROPOLIS AND GIBBS

- It is often the case that full conditionals are available for some parameters but not all.
- Very useful trick is to combine Gibbs and Metropolis.
- We will illustrate this by analyzing time series data on global warming.

CARBON DIOXIDE AND TEMPERATURE

- Data are from analysis of ice cores from East Antarctica
- Temperature (recorded in terms of difference from current present temp in degrees *C*) and CO₂ (measured in ppm by volume) are standardized to have mean 0 and variance 1.
- 200 values, each roughly 2000 years apart.
- CO₂ values matched with temperature values roughly 1000 years later.

DATA



CO₂ and temperature follow similar patterns over time.

INFERENCE

- Interest lies in predicting temperature as a function of CO₂.
- In these data, the error terms are temporally correlated so that a reasonable model for temperature is

$$\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \Sigma),$$

where \mathbf{X} contains a column for the intercept plus a column for CO₂, and Σ has a first-order autoregressive structure so that:

$$\Sigma = \sigma^2 \mathbf{C}_\rho = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{n-1} \\ \rho & 1 & \rho & \dots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \dots & 1 \end{bmatrix}$$

- The covariance model assumes constant variance but a decreasing correlation as the time between temperature measures is greater.

POSTERIOR INFERENCE

- We need to specify prior distributions for β , σ^2 and ρ .
- If we assume

$$\pi(\beta) = \mathcal{N}_p(\boldsymbol{\mu}_0, \Lambda_0),$$

then

$$\pi(\beta|\mathbf{y}, \mathbf{X}, \sigma^2, \rho) = \mathcal{N}_p(\boldsymbol{\mu}_n, \Lambda_n),$$

where

$$\Lambda_n = \left[\Lambda_0^{-1} + \frac{1}{\sigma^2} \mathbf{X}^T \mathbf{C}_\rho^{-1} \mathbf{X} \right]^{-1}$$
$$\boldsymbol{\mu}_n = \Lambda_n \left[\Lambda_0^{-1} \boldsymbol{\mu}_0 + \frac{1}{\sigma^2} \mathbf{X}^T \mathbf{C}_\rho^{-1} \mathbf{y} \right].$$

POSTERIOR INFERENCE

- If we assume

$$\pi(\sigma^2) = \mathcal{IG} \left(\frac{\nu_0}{2}, \frac{\nu_0 \sigma_0^2}{2} \right),$$

then

$$\pi(\sigma^2 | \mathbf{y}, \mathbf{X}, \boldsymbol{\beta}, \rho) = \mathcal{IG} \left(\frac{\nu_n}{2}, \frac{\nu_n \sigma_n^2}{2} \right),$$

where

$$\nu_n = \nu_0 + n$$

$$\sigma_n^2 = \frac{1}{\nu_n} [\nu_0 \sigma_0^2 + (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T \mathbf{C}_\rho^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})] = \frac{1}{\nu_n} [\nu_0 \sigma_0^2 + \text{SSR}(\boldsymbol{\beta}, \rho)].$$

- Therefore, given ρ , we can use Gibbs sampling to cycle through the full conditionals for $\boldsymbol{\beta}$ and σ^2 .

POSTERIOR INFERENCE

- Next, we need a prior for the correlation ρ . There is no semi-conjugate option here.
- Since we expect ρ to be positive, we could use $\pi(\rho) = \text{Unif}(0, 1)$.
- Unfortunately, this does not lead to a standard full conditional.
- However, we can use Metropolis-Hastings for the resulting full conditional for ρ . Actually, if we could come up with a symmetric proposal for ρ , we can just use the Metropolis algorithm.
- So, technically, we have a Gibbs sampler since we will cycle through full conditionals. However, the sampling step for ρ will rely on Metropolis.
- Therefore, we have a **Metropolis within Gibbs** sampler.

POSTERIOR INFERENCE

- Update for ρ (Metropolis) at iteration $(s + 1)$:
 1. Generate a candidate value $\rho^* \sim \text{Unif}(\rho^{(s)} - \delta, \rho^{(s)} + \delta)$. If $\rho^* < 0$, reassign as $|\rho^*|$. If $\rho^* > 1$, reassign as $2 - \rho^*$.

I leave the proof that this "reflecting random walk" is symmetric to you.

2. Compute the acceptance ratio

$$r = \frac{p(\mathbf{y}|\mathbf{X}, \boldsymbol{\beta}^{(s+1)}, \sigma^{2(s+1)}, \rho^*) \cdot \pi(\theta^*)}{p(\mathbf{y}|\mathbf{X}, \boldsymbol{\beta}^{(s+1)}, \sigma^{2(s+1)}, \rho^{(s)}) \cdot \pi(\theta^{(s)})}.$$

3. Set

$$\rho^{(s+1)} = \begin{cases} \rho^* & \text{if } u < r \\ \rho^{(s)} & \text{if otherwise} \end{cases}.$$

- So, for each iteration, we first sample from the full conditionals for $\boldsymbol{\beta}$ and σ^2 , and then use this step to update ρ .

MOVE TO THE R SCRIPT **HERE.**

WHAT'S NEXT?

MOVE ON TO THE READINGS FOR THE NEXT MODULE!