Laboratory Exercise 6: Advection equation

OBJECTIVES: To experiment with an upstream scheme for integrating the linear advection equation

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$$

on a periodic domain $0 \leq x \leq 1$. We will consider two different initial conditions:

$$\phi(x,0) = \begin{cases} \frac{1}{2}\left\{1+\cos\left[4\pi\left(x-1/2\right)\right]\right\} & \text{if } 1/4 \leq x \leq 3/4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\phi(x,0) = \begin{cases} 1 & \text{if } 1/4 \leq x \leq 3/4 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Upstream scheme

Write a python program to integrate the linear advection equation using a forward in time, backward in space discretization. Take $u = 1.0$ to be a constant, use $N = 40$ equally spaced gridpoints, and take $\Delta t = 0.01$. You might find it helpful to follow the structure of `diffuse.py` and use the same tricks for coping with the periodic boundary condition.

Run your procedure for a total of 1 time unit, so that the true solution should have gone once around the domain and back to its starting position. Plot the true solution and the FTBS solution on the same set of axes. What kind of errors do you get for the initial condition (1) and the inititial condition (2)?

Now set $u = -1.0$ and run your procedure for a few steps, plotting the solution at each step. What happens? Why?

Adapt your procedure so that it checks the sign of u and uses FTBS for $u \geq 0$ and FTFS for $u < 0$. Check that it now works and gives stable solutions for both $u = 1.0$ and $u = -1.0$.

Increase the number of grid points to $N = 150$ and run your procedure for a few steps, plotting the solution at each step. Is it more accurate than before? Why?

Use the theory given in lectures to calculate the maximum stable timestep at this resolution. Modify the procedure to use a stable timestep and run it again. Now is it more accurate than before?

Optional - CTCS scheme

Write a new program (or modify the one you've just created) to solve the linear advection equation using a CTCS scheme. Take $u = 1.0$ to be a constant, use $N = 40$ equally spaced gridpoints, and take $\Delta t = 0.01$. You will need to introduce an extra array to hold $\phi^{(n-1)}$, and think carefully about how to rename the data at the end of each step. Use FTCS for the first time step when $\phi^{(n-1)}$ is not available.

Run your procedure for a total of 1 time unit, so that the true solution should have gone once around the domain and back to its starting position. Plot the true solution and the CTCS solution on the same set of axes. What kind of errors do you get for the initial condition (1) and the inititial condition (2)?

Now set $u = -1.0$ and re-run your procedure. Does it work? Why?

For the initial condition (1), set $\phi_j^{(1)} = -\phi_j^{(0)}\ \forall j$ instead of using FTCS for the first step. Run your procedure for a few steps, plotting the solution at each step; how can you tell that what you are looking at is almost entirely the computational mode?