Design of Observer Controller Pair

MTRN3020 MODELLING AND CONTROL OF MECHATRONIC SYSTEMS

Abstract

This provides an example where both the controller and the predictive estimator is described. The question is extracted from John Dorsey's Continuous and Discrete Time Control Systems.

1 The Problem

Design a state feedback controller for the system,

$$G_p(s) = \frac{4}{s(s+4)}$$

with dominant close loop poles corresponding to $\omega_n = 5$ rad/sec and the damping ration of $\zeta = 0.8$.

Also design a prediction state estimator with error system dynamics to be critically damped and the speed of response to be 5 times faster than the controller's speed of response.

Use a sampling frequency of $f_s = 10$ Hz.

2 Obtaining State Space Model

Let the input to the plant be $u(t)$ and the output of the plant be $y(t)$. Therefore,

$$\frac{Y(s)}{U(s)} = \frac{4}{s(s+4)} \tag{1}$$

In differential equation form,

$$\ddot{y}(t) + 4\dot{y}(t) = 4u(t) \tag{2}$$

Introduce the state variables,

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned}$$

Hence, the system equations are,

$$\dot{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \mathrm{x} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} u(t) \tag{3}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathrm{x} \tag{4}$$

Thus,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \quad (5)$$

$$B = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad (6)$$

3 Discretization of the State Space Model

The discritization of the system matrices can be carried out by using the following equations.

$$\begin{aligned}
G &= e^{AT} \\
H &= \left\{ \int_0^T e^{A\lambda} d\lambda \right\} B \\
e^{AT} &= \mathcal{L}^{-1} \left[sI - A\right]^{-1} |_{t=T} \\
[sI - A] &= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} s & -1 \\ 0 & s+4 \end{pmatrix} \\
[sI - A]^{-1} &= \frac{1}{s(s+4)} \begin{pmatrix} (s+4) & 1 \\ 0 & s \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{s} & \frac{1}{s(s+4)} \\ 0 & \frac{1}{(s+4)} \end{pmatrix} \\
\therefore \mathcal{L}^{-1} \left[sI - A\right]^{-1} |_{t=T} &= \begin{pmatrix} 1 & \frac{1}{4}(1 - e^{-4T}) \\ 0 & e^{-4T} \end{pmatrix}
\end{aligned}$$

For a sampling frequency of 10 Hz, T = 0.1.

$$G = \begin{pmatrix} 1 & 0.0824 \\ 0 & 0.6703 \end{pmatrix}$$

$$\begin{aligned}
H &= \int_0^T \begin{pmatrix} 1 & \frac{1}{4}(1 - e^{-4\lambda}) \\ 0 & e^{-4\lambda} \end{pmatrix} d\lambda \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} (T + \frac{1}{4}e^{-4T} - \frac{1}{4}) \\ (1 - e^{-4T}) \end{pmatrix} \\
&= \begin{pmatrix} 0.0176 \\ 0.3297 \end{pmatrix}
\end{aligned}$$

The descretized state equations are,

$$\begin{aligned}
\mathrm{x}(k+1) &= G\mathrm{x}(k) + Hu(k) \\
y(k) &= C\mathrm{x}(k)
\end{aligned}$$

4 State Feedback Controller

For ease of simplification, let

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}; \quad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}; \quad K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$$

where K is the state feedback matrix. In a regulator problem, $u(k) = -Kx(k)$. Substituting for $u(k)$ in the state equations, we get,

$$\mathrm{x}(k+1) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \mathrm{x}(k) - \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} \mathrm{x}(k) \tag{7}$$

which can be written in the z-domain as,

$$\left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} \right] X(z) = 0 \tag{8}$$

The characteristic equation of this system is,

$$\left| \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} \right| = 0 \tag{9}$$

which when expanded,

$$\begin{vmatrix} z - g_{11} + h_1 k_1 & -g_{12} + h_1 k_2 \\ -g_{21} + h_2 k_1 & z - g_{22} + h_2 k_2 \end{vmatrix} = 0 \tag{10}$$

when simplified,

$$(z - g_{11} + h_1 k_1)(z - g_{22} + h_2 k_2) - (h_2 k_1 - g_{21})(h_1 k_2 - g_{12}) = 0 \tag{11}$$

After further simplification,

$$\begin{aligned} & z^2 + z(h_1 k_1 + h_2 k_2 - g_{11} - g_{22}) \\ & \qquad + g_{11} g_{22} - g_{12} g_{21} + h_2 k_1 g_{12} + h_1 k_2 g_{21} - h_1 k_1 g_{22} - h_2 k_2 g_{11} = 0 \end{aligned} \tag{12}$$

We can now extract the coefficients of z and the constant term of (12). The coefficient of z is,

$$h_1 k_1 + h_2 k_2 - \text{Trace}[G] \tag{13}$$

The constant term is,

$$\text{Det}[G] + (h_2 g_{12} - h_1 g_{22}) k_1 + (h_1 k_2 g_{21} - h_2 g_{11}) k_2 \tag{14}$$

The controller pole locations are at, $s = -4 \pm j3$. Hence the desired pole locations on the z-plane are at,

$$z = e^{(-4 \pm j3)T} = 0.6404 \pm j0.1981$$

The resulting characteristic equation is,

$$z^2 - 1.28076z + 0.449329 = 0 \tag{15}$$

By substituting the values for G and H to (13) and (14) and equating the corresponding coefficients we get,

$$\begin{aligned} -1.6703 + 0.0176k_1 + 0.3297k_2 &= -1.28076 & (16) \\ 0.6703 + 0.01537k_1 - 0.3297k_2 &= 0.449329 & (17) \end{aligned}$$

By solving these two equations, we get,

$$K = [5.11273 \quad 0.908564]$$

5 Design of a Prediction Observer

The discrete time state equations are,

$$\mathrm{x}(k+1) = G\mathrm{x}(k) + Hu(k) \tag{18}$$

The estimator equations are,

$$\hat{\mathrm{x}}(k+1) = G\hat{\mathrm{x}}(k) + Hu(k) \tag{19}$$

Subtracting (19) from (18) will only give us an error equation, the dynamics of which will be governed by G of which we have no control. As such the error $\mathbf{e}$ (see definition below) will evolve as dictated by G which might eventually grow bigger. That does not serve the purpose of an observer. The aim of an observer is to drive the error to zero. This can only be achieved by making the error dynamics to behave the way we desire. Hence we must influence the error dynamics with an opportunity for us to choose the way we want the error to converge to zero, hence the matrix L which we can choose. The quantities that force the error to zero are clumped together and is given the name "forcing function" in (21). This also means that (19) modifies to,

$$\hat{\mathrm{x}}(k+1) = G\hat{\mathrm{x}}(k) + Hu(k) - L[y(k) - \hat{y}(k)] \tag{20}$$

The error dynamics can now be obtained by subtracting (20) from (18) as follows.

$$\begin{aligned} \mathrm{e}(k+1) &= G\mathrm{e}(k) + \overbrace{Ly(k) - L\hat{y}(k)}^{\text{forcing function}} && (21) \\ &= (G + LC)\mathrm{e}(k) && (22) \end{aligned}$$

where,

$$\mathrm{e}(k) = \mathrm{x}(k) - \hat{\mathrm{x}}(k)$$

The characteristic equation is,

$$|zI - G - LC| = 0 \tag{23}$$

For ease of simplification, let,

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}; \quad L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Therefore, the characteristic equation is,

$$\left| \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} - \begin{pmatrix} l_1 & 0 \\ l_2 & 0 \end{pmatrix} \right| = 0$$

when simplified,

$$\left| \begin{pmatrix} z - g_{11} - l_1 & -g_{12} \\ -g_{21} - l_2 & z - g_{22} \end{pmatrix} \right| = 0 \tag{24}$$

Thus the characteristic equation in quadratic form is,

$$z^2 - zg_{11} - g_{12}g_{21} - zg_{22} + g_{11}g_{22} - zl_1 + g_{22}l_1 - g_{12}l_2 = 0 \tag{25}$$

The coefficient of z is

$$-g_{11} - g_{22} - l_1$$

and the constant term is,

$$-g_{12}g_{21} + g_{11}g_{22} + g_{22}l_1 - g_{12}l_2$$

If we place the estimator poles at a location that is 5 time faster, then that would be at $s = -20$, which corresponds to $z = e^{-20T} = e^{-2} = 0.1353$. The desired characteristic equation is, therefore,

$$(z - 0.1353)^2 = 0 \tag{26}$$

When expanded,

$$z^2 - 0.2706z + 0.0183061 = 0 \tag{27}$$

By equating coefficients,

$$\begin{aligned} -g_{11} - g_{22} - l_1 &= -0.2706 &(28)\\ -g_{12}g_{21} + g_{11}g_{22} + g_{22}l_1 - g_{12}l_2 &= 0.0183061 &(29) \end{aligned}$$

Substituting numerical values,

$$\begin{aligned} -1.6703 - l_1 &= -0.2706 &(30)\\ 0.6703 + 0.6703l_1 - 0.0824l_2 &= 0.0183061 &(31) \end{aligned}$$

Solving the above equations we get,

$$l_1 = -1.39963, \quad l_2 = -3.47315.$$

or

$$L = \begin{pmatrix} -1.39963 \\ -3.47315 \end{pmatrix}$$

Hence, the complete observer equation is,

$$\begin{aligned} \hat{\mathrm{x}}(k+1) &= G\hat{\mathrm{x}}(k) - HK\hat{\mathrm{x}}(k) - Ly(k) + LC\hat{\mathrm{x}}(k) &(32)\\ &= [G - HK + LC]\hat{\mathrm{x}}(k) - Ly(k) &(33) \end{aligned}$$

where all matrices G, H, L and K are known. Hence, in numerical form,

$$\hat{\mathrm{x}}(k+1) = \begin{pmatrix} -0.4896 & 0.0664 \\ -5.1588 & 0.3707 \end{pmatrix} \hat{\mathrm{x}}(k) + \begin{pmatrix} 1.39963 \\ 3.47315 \end{pmatrix} y(k) \tag{34}$$