Modeling and Control of Mechatronic Systems

Exercise 2: Mathematical Modeling and Simulation of an Inverted Pendulum

Dr Jayantha Katupitiya

Exercise 2

Modeling and Control of an Inverted Pendulum

The aim of this exercise is to model an inverted pendulum mounted on a cart subjected to a 1-D force $\boldsymbol{u(t)}$. The arrangement is shown in Fig. 2.1. The controller designed must be capable of positioning the cart to a desired location, while maintaining the vertical position of the pendulum. First, we will model the complete system. We will be using two approaches, (i) Euler-Lagrange approach and (ii) Newton-Euler approach. In the former, we will be using the Lagrangian, and in the latter we will be using the free body diagrams. As a first approach, we will design the so-called state feedback controller that will drive the *state vector* to zero. This will ensure the pendulum resting upright while the cart is positioned at zero. In the final solution, we will augment the states of the system so that instead of the state vector, the *error vector* will be driven to zero. This will allow us to move the cart to any desired position while the pendulum remains upright.

2.1 Derivation of Dynamic Equations

2.1.1 Euler-Lagrange Approach

The Lagrangian is defined as the difference between the kinetic energy and potential energy. The kinetic energy $\boldsymbol{K}$ is,

$$\boldsymbol{K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\frac{d}{dt}(x + l\sin\theta)\right]^2 + \frac{1}{2}m\left[\frac{d}{dt}(l\cos\theta)\right]^2} \tag{2.1}$$

The potential energy $\boldsymbol{P}$ is,

$$\boldsymbol{P = mgl\cos\theta} \tag{2.2}$$

Therefore, Lagrangian is,

$$\boldsymbol{L = K - P} \tag{2.3}$$

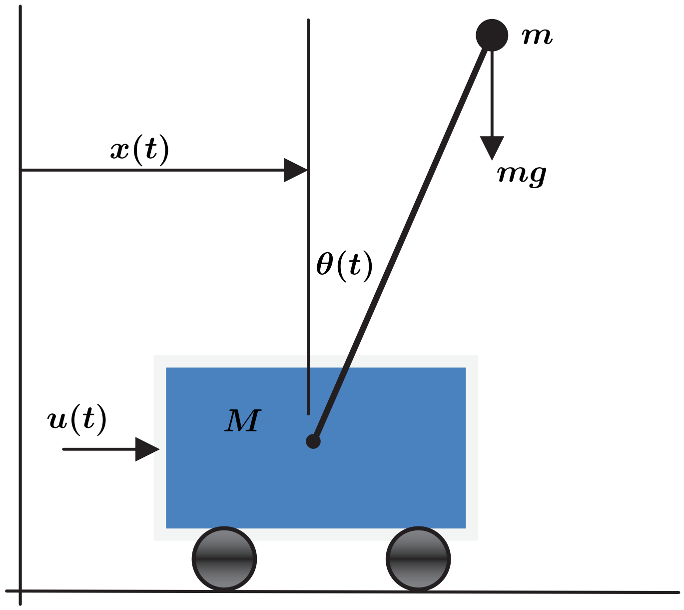


Figure 2.1: Inverted pendulum mounted on a cart

The dynamic equations can be obtained by using

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau} \tag{2.4}$$

where $\mathbf{q}$ are the so-called *generalized coordinates* and $\boldsymbol{\tau}$ are the so-called *generalized forces*. In our case $\mathbf{q} = \{x, \theta\}^T$ and the corresponding $\boldsymbol{\tau} = \{u, 0\}^T$. Thus, the two dynamic equations can be obtained by,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= u \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \end{aligned} \tag{2.5}$$

Applying the first equation of (2.5), the following can be obtained.

$$M\ddot{x} + m\ddot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = u \tag{2.6}$$

Applying the second equation of (2.5),

$$ml^2\ddot{\theta} + ml\cos\theta\ddot{x} - lmg\sin\theta = 0 \tag{2.7}$$

Obviously, these equations are non-linear time-varying. However, given the problem, a very good linear time invariant problem can be carved out of this non-linear problem. Note that the pendulum is not expected to deviate very much from zero and the pendulum velocities can also be considered small. Then the two equations can be simplified to give,

$$\begin{aligned} (M+m)\ddot{x} + ml\ddot{\theta} &= u \\ ml\ddot{\theta} + m\ddot{x} - mg\theta &= 0 \end{aligned} \tag{2.8}$$

2.1.2 Newton-Euler Approach

The same dynamic equations can be obtained by resolving forces in horizontal direction and taking moments of forces acting on the pendulum alone about the pendulum pivot point. Resolving forces horizontally for the entire system gives,

$$M\ddot{x} + m\frac{d^2}{dt^2}(x + l\sin\theta) = u \tag{2.9}$$

Taking moments about the pivot point for the pendulum gives,

$$m\left\{\frac{d^2}{dt^2}(x + l\sin\theta)\right\} l\cos\theta - \left\{\frac{d^2}{dt^2}(l\cos\theta)\right\} l\sin\theta = mgl\sin\theta \tag{2.10}$$

When (2.9) and (2.10) are simplified, we obtain (2.6) and (2.7), respectively. They can then be simplified to obtain (2.8).

2.2 Transfer Function Approach

By re-arranging the two equations in (2.8) we obtain,

$$\begin{aligned} Ml\ddot{\theta} &= (M+m)g\theta - u \\ M\ddot{x} &= -mg\theta + u \end{aligned} \tag{2.11}$$

which are in fact,

$$\ddot{\theta} = \frac{(M+m)g}{Ml}\theta - \frac{1}{Ml}u \tag{2.12}$$

$$\ddot{x} = -\frac{mg}{M}\theta + \frac{1}{M}u \tag{2.13}$$

Note that (2.12) has no x in it. Therefore, we can design a controller that will give stable θ response. This may or may not give the desired response of x.

Task 1

(a). Generate a transfer function out of (2.12). Use $M = 2$ kg., $m = 0.1$ kg., $l = 0.5$ m and $g = 9.81$.

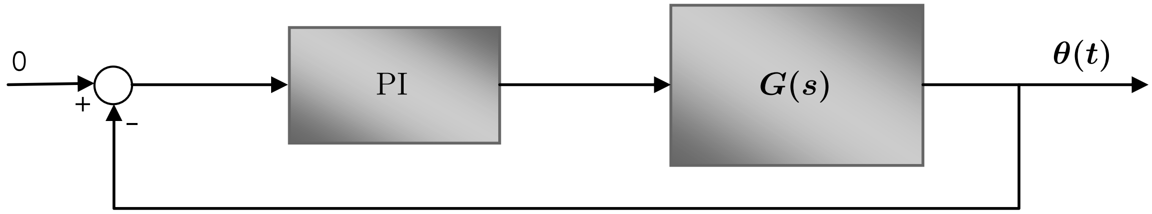


Figure 2.2: PI controller for pendulum angle only

(b). Plot root-locus.

(c). Design a controller that will give stable θ response.

(d). Plot the response of θ for a desired value of 0 degrees. Assume an initial condition of 2 degrees for the pendulum and $x = 0$ for the cart with all velocities zero. Also plot the response of the cart.

(e). Design a PI controller using root-locus and repeat the plots mentioned in item (d) above. Plot the responses of items (d) and (e) so that comparisons can be made.

Task 1 - Solution

(a). Let $\Theta(s)$ be the Laplace transform of the pendulum position and $U(s)$ be the Laplace transform of the force applied to the cart. Using (2.12),

$$\frac{\Theta(s)}{U(s)} = \frac{-\dfrac{1}{Ml}}{\left(s^2 - \dfrac{(M+m)g}{Ml}\right)}$$

when the numerical values are substituted, this becomes,

$$\frac{\Theta(s)}{U(s)} = \frac{-1}{(s^2 - 20.601)}$$

(b). If we assume that a proportional controller with gain $K > 0$ can be used, the root locus will be as shown in Fig. 2.3. As can be seen a proportional controller that will give a stable response cannot be designed. The reason is that, no matter which point you will choose on the root locus, there will always be a closed loop pole (i.e. a root of the characteristic equation) on the right half plane.

(c). To be able to design a statable controller we can consider using $K < 0$. The root locus is shown in Fig. 2.4. While this situation is better than the situation for $K > 0$, it can at best be marginally stable. That is, there will either be a closed loop system pole in the right half plane

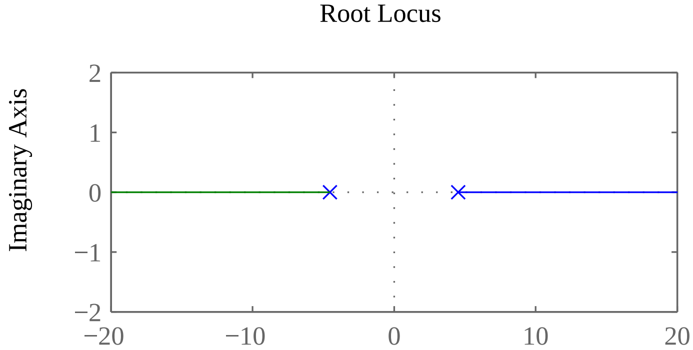


Figure 2.3: The root-locus for a proportional controller with $K > 0$.

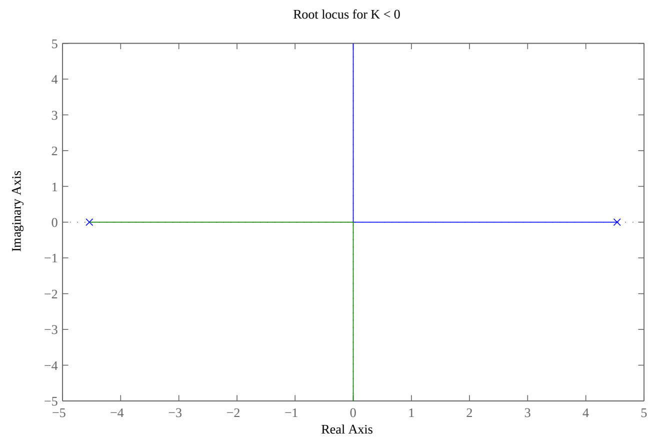


Figure 2.4: The root-locus for $K < 0$.

or there will be roots on the imaginary axis. For a system to be stable, we must have all roots on the left half plane, not on the imaginary axis.

Having seen that just a K is not going to work, we move on to introduce a controller of the form,

$$G_c(s) = K\frac{(a+\alpha)}{(s+\beta)}$$

In Fig. 2.4, the poles are at $s = 4.5388$ and $s = 4.5388$. We can use a controller of this form to move the breakaway point to a point we desire. This needs to be ascertained based on the design specifications. If we choose the break away point to be at $s = -1$ for example, and if we are to choose the breakaway point as the operating point, then that amounts to a controller design specification with, "no oscillations and an exponential decay of errors at a rate of e^{-t}".

To achieve this, we can place the zero of the controller at -4.5388 and the pole of the controller at -6.5388. The controller will then be,

$$G_c(s) = K\frac{(s+4.5388)}{(s+6.5388)} \quad K < 0.$$

The root locus corresponding to this situation is shown in Fig. 2.5. What remains to be done is

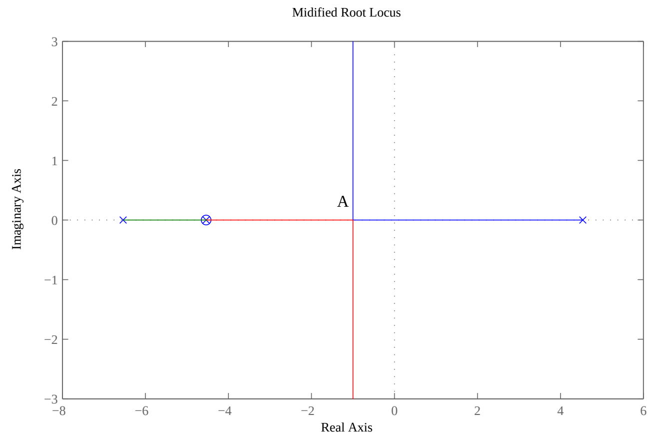


Figure 2.5: The modified root-locus

the determination of K. The characteristic equation of the close loop system is,

$$\frac{(s+4.5388)}{(s+6.5388)(s-4.5388)(s+4.5388)} = \frac{1}{K}$$

Taking magnitudes only,

$$\frac{|(s+4.5388)|}{|(s+6.5388)||(s-4.5388)||(s+4.5388)|} = \frac{1}{|K|}$$

Given that the operating point is at $s = -1$,the above equation becomes,

$$\frac{|(-1+4.5388)|}{|(-1+6.5388)||(-1-4.5388)||(-1+4.5388)|} = \frac{1}{|K|}$$

which give $|K| = 30.678$.

(d). To plot the response we need to generate the simulation files. The controller also needs to be represented in state space form so that we can put it in a model file in MATLAB. To achieve that, we can use `tf2ss` function. By executing the following line, you can obtain the Ac, Bc, Cc, Dc matrices that represent the designed controller in the controllable cannonical form.

```
>> numc = -30.678*[1 4.5388];
>> denc = [1 6.5388];
>> [Ac,Bc,Cc,Dc] = tf2ss(numc,denc)
Ac =
    -6.5388
Bc =
```

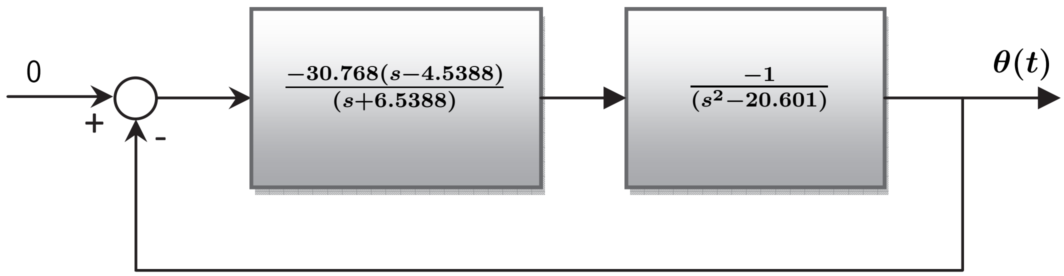


Figure 2.6: The stable controller

```
       1
Cc =
   61.3560
Dc =
  -30.6780
```

The entire model file can now be written as,

```
function dy = mtrn9211_Ex2_model(t,y)
% Initialisation
setangle = 0;
e_t = setangle-y(1);
% State variables
% [x1 x2 x3]
% x1 - Theta, the pendulum angle in radians.
% x2 - \dot{Theta}, the pendulum angular velocity.
% x3 - x, the linear position of the cart
% x4 - \dot{x}, the linear velocity of the cart
% x5 - The state variable of the controller (no physical meaning)
% inputs
% e_t - input to the controller
% m_t - output of the controller = the input to the system

dy = zeros(5,1);

% update state variables

x1 = y(1);
x2 = y(2);
```

```
x3 = y(3);
x4 = y(4);
x5 = y(5);

m_t = 61.3560*x5--30.6780*e_t;
dy(1) = x2;
dy(2) = 20.601*x1 - m_t;
dy(3) = x4;
dy(4) = -0.491*x1+0.5*m_t;
dy(5) = -6.5388*x5+e_t;
return;
```

The corresponding simulation file is,

```
% This file simulates the model described in mtrn9211_Ex2_model.m
options = odeset('RelTol',1e-4,'AbsTol',[1e-5 1e-5 1e-5 1e-5 1e-5]);
[t,y] = ode45(@mtrn9211_Ex2_model,[0 10],[0.035 0 0 0 0], options);
figure(1);
plot(t,y(:,1),'b');
%axis equal;
title('Pendulum Angular Position');
xlabel('time (sec.)');
ylabel('angle (rad.)');
grid
figure(2);
plot(t,y(:,3),'k');
%axis equal;
title('Cart Linear Position');
xlabel('time (sec.)');
ylabel('position (m.)');
grid
return;
```

The pendulum and cart positions are shown in Fig. 2.7.

2.3 State Space Approach

Introduce the state variables as follows.

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$
$$x_3 = x$$
$$x_4 = \dot{x}$$

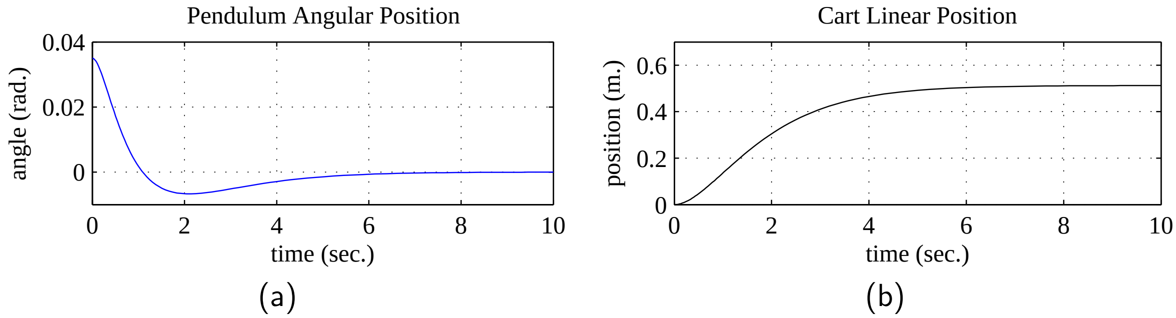


Figure 2.7: (a). The angular position of the pendulum and (b). The linear position of the cart

Then (2.12) and (2.13) can be put into matrix form as follows,

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{M+m}{M}\frac{g}{l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M}g & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{pmatrix} u \tag{2.14}$$

The output equation is,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.15}$$

The state equations can now be represented as,

$$\dot{\mathrm{x}} = \mathbf{A}\mathrm{x} + \mathbf{B}u \tag{2.16}$$

$$\mathrm{y} = \mathbf{C}\mathrm{x} \tag{2.17}$$

2.3.1 State Feedback Controller Design

In a state feedback controller, we feedback *all states* to generate the control input. This obviously means that we assume that all the states are available for feedback. However, in practice, it is impossible to measure all the states. Even in this pendulum case it is possible for us to put encoders to measure the cart position and the pendulum position, however, although possible, mounting sensors to measure the cart velocity and the pendulum velocity is an over kill. In such situations we will assume that a state estimator can be designed so that all states can be estimates and hence can be fed back.

The design task we undertake here is to drive the state vector to $\mathbf{0}$. Hence, this type of controllers are called *regulators* in which the set point does not change. Let the desired state vector be x_d and the state feedback matrix be $\mathbf{K}$. Therefore the error signal is $(\mathrm{x}_d - \mathrm{x})$. Therefore, the control input is $u = \mathbf{K}(\mathrm{x}_d - \mathrm{x})$. If $\mathrm{x}_d = \mathbf{0}$,

$$u = -\mathbf{K}\mathrm{x} \tag{2.18}$$

with $\boldsymbol{K} = \{\boldsymbol{k_1} \quad \boldsymbol{k_2} \quad \boldsymbol{k_3} \quad \boldsymbol{k_4}\}$. Substituting into (2.16),

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} - \mathbf{BKx} \\ &= (\mathbf{A} - \mathbf{BK})\mathbf{x} \end{aligned} \tag{2.19}$$

Taking Laplace transforms of both sides,

$$[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})]\,\mathbf{X}(s) = 0 \tag{2.20}$$

Recall the scalar equations we have seen earlier in the course, for example,

$$\boldsymbol{D(s)X(s) = 0}$$

$\boldsymbol{D(s) = 0}$, represented the characteristic equation and for the system to be stable the roots of the characteristic equation must have negative real parts. In a matrix equation such as (2.20), for the system to be stable the matrix $[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})]$ must have eigen values with negative real parts. The characteristic polynomial of (2.20) is,

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{BK})| = 0 \tag{2.21}$$

If the desired characteristic equation is given, then by equating the coefficients of (2.21) to those of the desired characteristic equation, $\boldsymbol{k_1}$, $\boldsymbol{k_2}$, $\boldsymbol{k_3}$ and $\boldsymbol{k_4}$ can be found.

Task 2

(a). Generate the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ out of (2.16) and (2.17). Use $\boldsymbol{M = 2}$ kg., $\boldsymbol{m = 0.1}$ kg., $\boldsymbol{l = 0.5}$ m and $\boldsymbol{g = 9.81}$.

(b). Design a state feedback controller that will place the closed loop poles at $\boldsymbol{-1.25 \pm j5.0}$, $\boldsymbol{-4.5396}$ and $\boldsymbol{-2.9748}$.

(c). Implement the controller and plot the response of $\boldsymbol{\theta}$ for a desired value of $\boldsymbol{0}$ degrees. Assume an initial condition of $\boldsymbol{2}$ degrees for the pendulum and $\boldsymbol{x = 0}$ for the cart with all velocities zero. Also plot the response of cart position.

(d). Compare the plots you obtained with those you obtained under **Task 1**

Task 2 - Solution

(a).

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.491 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{pmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(b). For a state feedback control input of

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$$

the characteristic equation is,

$$|s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})| = \boldsymbol{0}$$

By solving this equation to ensure the closed loop poles at the desired locations, the feedback gain matrix $\boldsymbol{K}$ is,

$$\boldsymbol{K} = [-\mathbf{97.7377} \quad -\mathbf{21.9091} \quad -\mathbf{36.5677} \quad -\mathbf{23.7894}]$$

(c). The model file for the simulation is,

```
function dy = mtrn9211_Ex2_statefeedback_model(t,y)
% Initialisation
% Set point is assumed to be 0
% State variables
% [x1 x2 x3 x4]
% x1 - Theta, the pendulum angle in radians.
% x2 - \dot{Theta}, the pendulum angular velocity.
% x3 - x, the linear position of the cart
% x4 - \dot{x}, the linear velocity of the cart


dy = zeros(4,1);

x1 = y(1);
x2 = y(2);
x3 = y(3);
x4 = y(4);

u_k = -( -97.7377*x1  -21.9091*x2  -36.5677*x3  -23.7894*x4);

dy(1)=x2;
dy(2)=20.601*x1 - u_k;
dy(3) = x4;
dy(4) = -0.491*x1+0.5*u_k;
return;
```

The simulation file is,

```
% This file simulates the model described in mtrn9211_statefeedback_model.m
options = odeset('RelTol',1e-4,'AbsTol',[1e-5 1e-5 1e-5 1e-5]);
[t,y] = ode45(@mtrn9211_Ex2_statefeedback_model,[0 5],[0.035 0 0 0], options);
```

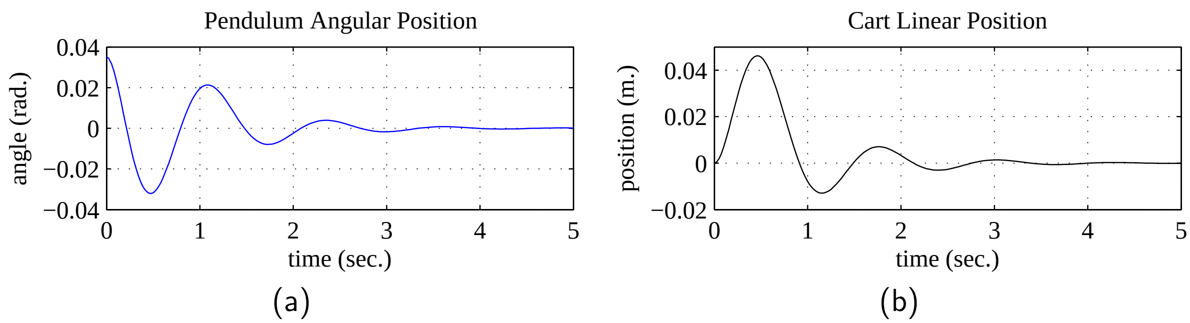


Figure 2.8: State feedback control, (a). The angular position of the pendulum and (b). The linear position of the cart

```
figure(1);
plot(t,y(:,1),'b');
%axis equal;
title('Pendulum Angular Position');
xlabel('time (sec.)');
ylabel('angle (rad.)');
grid
figure(2);
plot(t,y(:,3),'k');
%axis equal;
title('Cart Linear Position');
xlabel('time (sec.)');
ylabel('position (m.)');
grid
return;
```

The pendulum position and cart position are shown in Fig. 2.8.

(d). Compare the responses in Fig. 2.7 with those shown in Fig. 2.8. Note that as the state vector is driven to zero, the cart position does not settle at an arbitrary position. Instead it always goes to zero and it cannot be positioned anywhere else.

2.3.2 Cart Position Control Through Scaling

Note that in the above solution procedure, the cart position was moved to $\mathbf{0}$. If the cart position was non-zero, then the there must be a new set point that reflects this fact. Let us substitute the numerical values to the state equation so we obtain,

$$\dot{\mathrm{x}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{pmatrix} \mathrm{x} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{pmatrix} u \qquad (2.22)$$
$$y_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \mathrm{x} + (0)u = \mathrm{C}'\mathrm{x} + \mathrm{D}'u$$

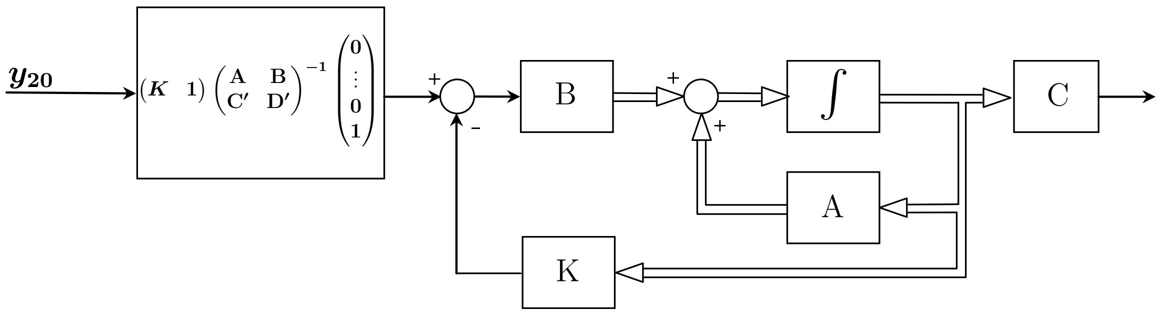


Figure 2.9: Scaling reference input to take into account desired cart position

Note that the output equation corresponds only to the cart position. The equations can be re-written as one matrix equation.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ u \end{pmatrix} \qquad (2.23)$$

Above equation can be re-written as,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ y_2 \end{pmatrix} \qquad (2.24)$$

We can use (2.24) to determine the contribution of steady state $\dot{\mathrm{x}}$s and y_2 on the steady state xs and u. We know that under steady state conditions, all $\dot{\mathrm{x}}$s must be zero and y_2 may be some non-zero y_{20}. Hence, using subscript $_f$ to signify final,

$$\begin{pmatrix} x_{f1} \\ x_{f2} \\ x_{f3} \\ x_{f4} \\ u_f \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ y_{20} \end{pmatrix} \qquad (2.25)$$

Therefore, the value that must be fed into the summing point is,

$$\mathbf{K}\mathbf{x}_f + u_f$$

and $\mathbf{x}_f$ and u_f are found from,

$$\begin{pmatrix} \mathbf{x}_f \\ u_f \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}' & \mathbf{D}' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} y_{20} \tag{2.26}$$

This approach re-scales the input to the summing point to suit the set point. However this approach cannot cater for disturbances acting on the cart position. The following approach gives better disturbance rejection.

2.3.3 Controlling Cart Position Through Error Dynamics

Instead of dynamic equations that represent states, we need to develop dynamic equations that represent error. Then we can drive error to zero just as we drove state to zero in the case we discussed earlier. Obviously for errors to be zero we need to incorporate an integrator for systems that do not have built-in integrators. In the case of the inverted pendulum system, there is no built-in integrator and hence we need to incorporate an integrator. Our approach is to keep the state feedback as it was and then to implement integral control to eliminate the steady state error in the cart position. Define a new state equation,

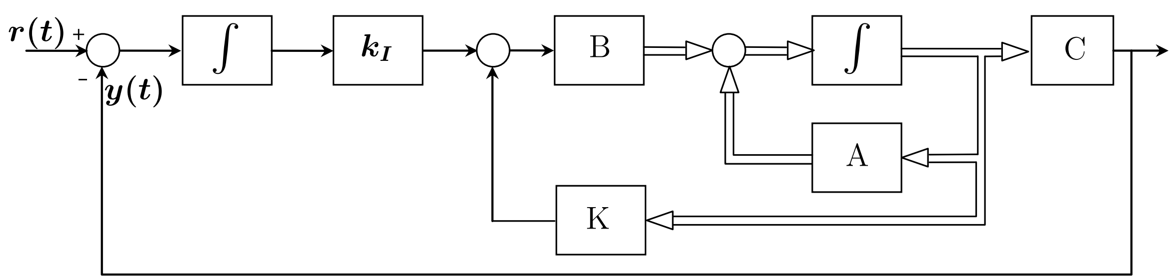


Figure 2.10: State feedback controller with error dynamics

$$\dot{\xi} = r(t) - y(t) \tag{2.27}$$

where $r(t)$ is the reference input and $y(t)$ is the cart position. Therefore, this equation represents error in the cart position. Then $\xi(t)$, is the integral of the cart position error. By summing up the state feedback control effort and the integral control effort we get,

$$u = -\mathbf{K}\mathbf{x} + k_I \xi \tag{2.28}$$

where $\boldsymbol{k_I}$ is the integral constant. The system state equations are,

$$\begin{aligned}\dot{\mathrm{x}} &= \mathbf{A}\mathrm{x} + \mathbf{B}\boldsymbol{u} \\ \boldsymbol{y} &= \mathbf{C}\mathrm{x} + \mathbf{D}\boldsymbol{u}\end{aligned} \tag{2.29}$$

Note that as $\boldsymbol{y(t)}$ represents cart position only, $\mathbf{C} = [\mathbf{0\ 0\ 1\ 0}]$ and $\mathbf{D} = \mathbf{0}$. Combining the above equation and (2.27) we can write,

$$\begin{aligned}\dot{\mathrm{x}} &= \mathbf{A}\mathrm{x} + \mathbf{B}\boldsymbol{u} \\ \dot{\boldsymbol{\xi}} &= -\mathbf{C}\mathrm{x} + \boldsymbol{r(t)}\end{aligned} \tag{2.30}$$

A combined equation can be formed as,

$$\begin{pmatrix}\dot{\mathrm{x}}(t) \\ \dot{\xi}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathrm{x}(t) \\ \xi(t)\end{pmatrix} + \begin{pmatrix}\mathbf{B} \\ \mathbf{0}\end{pmatrix} u(t) + \begin{pmatrix}0 \\ 1\end{pmatrix} r(t) \tag{2.31}$$

Substituting steady state values,

$$\begin{pmatrix}\dot{\mathrm{x}}(\infty) \\ \dot{\xi}(\infty)\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathrm{x}(\infty) \\ \xi(\infty)\end{pmatrix} + \begin{pmatrix}\mathbf{B} \\ \mathbf{0}\end{pmatrix} u(\infty) + \begin{pmatrix}0 \\ 1\end{pmatrix} r(\infty) \tag{2.32}$$

Subtracting (2.32) from (2.31)and noting that $\boldsymbol{r(\infty) = r(t)}$ (for a step change in cart position)

$$\begin{pmatrix}\dot{\mathrm{x}}_e(t) \\ \dot{\xi}_e(t)\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathrm{x}_e(t) \\ \xi_e(t)\end{pmatrix} + \begin{pmatrix}\mathbf{B} \\ \mathbf{0}\end{pmatrix} u_e(t) \tag{2.33}$$

where the subscript $\boldsymbol{e}$ refer to error in each quantity. With the error vector defined as,

$$\mathrm{e}(t) = \begin{pmatrix}\dot{\mathrm{x}}_e(t) \\ \xi_e(t)\end{pmatrix}$$

a new error dynamics equation can be written as,

$$\dot{\mathrm{e}} = \mathbf{A}'\mathrm{e} + \mathbf{B}'u_e \tag{2.34}$$

Choosing,

$$u_e = -\mathbf{K}'\mathrm{e} \tag{2.35}$$

The error dynamics become,

$$\dot{\mathrm{e}} = [\mathbf{A}' - \mathbf{B}'\mathbf{K}']\mathrm{e} \tag{2.36}$$

The matrix $\mathbf{K}'$ can be determined by equating the coefficients of the characteristic equation of error dynamics to the desired characteristic equation. Note that $\mathbf{K}'$ is,

$$\mathbf{K}' = [\mathbf{K} \quad \vdots - k_I] \tag{2.37}$$

Therefore, the control input is,

$$u(t) = -\mathbf{K}\mathrm{x}(t) + k_I\xi(t) \tag{2.38}$$

Task 3

(a). Generate the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ out of (2.29). Use $M = 2$ kg., $m = 0.1$ kg., $l = 0.5$ m and $g = 9.81$.

(b). Design a state feedback controller that will place the closed loop poles at $-1.25 \pm j5.0$, double pole at -4.5396 and a single pole at -2.9748.

(c). Implement the controller and plot the response of θ for a desired value of 0 degrees and a cart position of 1.0 m. Assume an initial condition of 2 degrees for the pendulum and $x = 0$ for the cart with all velocities zero. Also plot the response of the cart.

(d). Compare the plots you obtained with those you obtained under **Task 2**

Task 3 - Solution

(a).

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.491 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The modified A' and B' matrices are,

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.491 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad B' = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0.5 \\ 0 \end{pmatrix}.$$

(b). The state feedback matrix corresponding to the given pole locations is,

$$K = [-197.1963 \quad -44.2972 \quad -144.5621 \quad -59.4864 \quad -166.0027].$$

(c). The model file is,

```
function dy = mtrn9211_Ex2_errordynamics_model(t,y)
% Initialisation
% Set point is assumed to be 0
set_cart_position = 1;
% State variables
% [x1 x2 x3 x4 x5]
% x1 - Theta, the pendulum angle in radians.
% x2 - \dot{Theta}, the pendulum angular velocity.
```

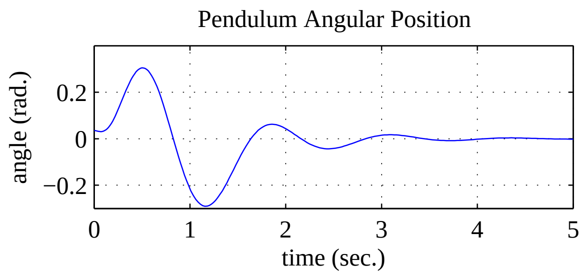
```
% x3 - x, the linear position of the cart
% x4 - \dot{x}, the linear velocity of the cart
% x5 - Cart position error

dy = zeros(5,1);

% update state variables (refer to eq (3.12) and fill in the state
% equations below

x1 = y(1);
x2 = y(2);
x3 = y(3);
x4 = y(4);
x5 = y(5);

u_k = -(-197.1963*x1  -44.2972*x2 -144.5621*x3  -59.4864*x4) -166.0027*x5;

dy(1)=x2;
dy(2)=20.601*x1 - u_k;
dy(3) = x4;
dy(4) = -0.491*x1+0.5*u_k;
dy(5) = set_cart_position - y(3);
return;
```

The simulation program is,

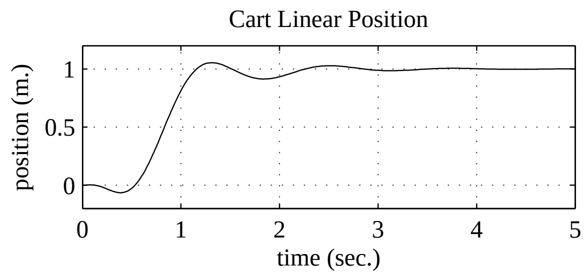
```
% This file simulates the model described in mtrn9211_errordynamics_model.m
options = odeset('RelTol',1e-4,'AbsTol',[1e-5 1e-5 1e-5 1e-5 1e-5]);
[t,y] = ode45(@mtrn9211_Ex2_errordynamics_model,[0 5],[0.035 0 0 0 0], options);
figure(1);
plot(t,y(:,1),'b');
%axis equal;
title('Pendulum Angular Position');
xlabel('time (sec.)');
ylabel('angle (rad.)');
grid
figure(2);
plot(t,y(:,3),'k');
%axis equal;
title('Cart Linear Position');
xlabel('time (sec.)');
ylabel('position (m.)');
grid
```

```
        return;
```

The response plots are shown in Fig. 2.11.



(a)



(b)

Figure 2.11: State feedback control of error dynamics, (a). The angular position of the pendulum and (b). The linear position of the cart

(d). Compare Fig. 2.11 with Fig. 2.8. Note that now the cart can be positioned anywhere.

Kouchibouguac