8 d) $2a+b-c=0 \Rightarrow c=2a+b$

$W=\left\{\begin{bmatrix}a\\b\\c\end{bmatrix} \mid 2a+b-c=0\right\} \subseteq \mathbb{R}^3$, $\begin{bmatrix}a\\b\\c\end{bmatrix} \in W \Rightarrow \begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}a\\b\\2a+b\end{bmatrix}$

$\Rightarrow \begin{bmatrix}a\\b\\2a+b\end{bmatrix}=a\begin{bmatrix}1\\0\\2\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix} \Rightarrow \left\langle \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\rangle = W.$

$\mathcal{B}=\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ is linearly independent because if

$a\begin{bmatrix}1\\0\\2\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, then $a=b=0$.

So $\mathcal{B}$ is a basis for W.

e) $W=\{at^3+bt^2+ct+d \mid c=a-2d,\ b=5a+3d\} \subseteq P_3$

$p(t) \in W \Rightarrow p(t)=at^3+(5a+3d)t^2+(a-2d)t+d$

$= a(\underbrace{t^3+5t^2+t}_{q_1(t)})+d(\underbrace{3t^2-2t+1}_{q_2(t)}) \in \langle q_1(t), q_2(t)\rangle$

$\Rightarrow W=\langle q_1(t), q_2(t)\rangle.$

$c_1q_1(t)+c_2q_2(t)=0 \Rightarrow c_1(t^3+5t^2+t)+c_2(3t^2-2t+1)=0$

$\overset{\text{poly. equality}}{\Rightarrow} c_1=0,\ 5c_1+3c_2=0,\ c_1-2c_2=0,\ c_2=0$

$\Rightarrow c_1=c_2=0.$

$\therefore \{q_1(t), q_2(t)\}$ is linearly independent.

$\therefore$ 〃 is a basis for W.

9)
$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 2 & 2 & -1 & 2 & 0 \\ 1 & 0 & 3 & 3 & 0 \end{array}\right] \xrightarrow[-2R_1+R_2]{-R_1+R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -2 & 1 & -4 & 0 \\ 0 & -2 & 4 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -2 & 1 & -4 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[2R_3+R_2]{-2R_3+R_1} \left[\begin{array}{cccc|c} 1 & 0 & -5 & 3 & 0 \\ 0 & 0 & 5 & -4 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -5 & 3 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -4 & 0 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -4 & 0 \end{array}\right]$$

$$\Rightarrow \quad \begin{aligned} x_1 - x_4 &= 0 \\ x_2 + 2x_3 &= 0 \\ 5x_3 - 4x_4 &= 0. \end{aligned} \qquad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -8/5\, t \\ 4/5\, t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -8/5 \\ 4/5 \\ 1 \end{bmatrix}$$

$x_4 = t \Rightarrow x_3 = \frac{4}{5}t$

$x_1 = x_4 = t$.

$x_2 = -2x_3 = -\frac{8}{5}t$

$\therefore$ Solution space is $W = \left\langle \begin{bmatrix} 1 \\ -8/5 \\ 4/5 \\ 1 \end{bmatrix} \right\rangle$, $\dim W = 1$ since any nonzero vector is lin. independent.

10) $\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{\alpha_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}}_{\alpha_2} \} \subseteq \mathbb{R}^3$. Since $\dim \mathbb{R}^3 = 3$, we have to add a vector to the given set. Consider the natural basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ and $\{\alpha_1, \alpha_2, e_1, e_2, e_3\}$. Find lin. independent vectors of this set.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -3 & 1 \end{bmatrix}$$

So $\{\alpha_1, \alpha_2, e_1\}$ is lin. independent and contains 3 vectors, hence it is a basis for $\mathbb{R}^3$ containing α_1, α_2.

11) $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d$ is a linear transformation: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

$L(A+B) = L\left(\begin{bmatrix} a+x & b+y \\ c+z & d+t \end{bmatrix}\right) = (a+x) + (d+t)$ by definition of L.

$L(A) + L(B) = (a+d) + (x+t) = a+x+d+t$. Hence $L(A+B) = L(A) + L(B)$.

Let $x \in \mathbb{R}$. $L(xA) = L\left(\begin{bmatrix} xa & xb \\ xc & xd \end{bmatrix}\right) = xa + xd = x(a+d) = x\,L(A)$.

12) $L: P_1 \to P_1$, $L(t+1) = 2t+3$, $L(t-1) = 3t-2$.

a) $L(6t-4) = ?$

$S = \{t+1, t-1\}$ is a basis for P_1 : $a(t+1) + b(t-1) = 0 \Rightarrow$

$$\begin{array}{l} a+b=0 \\ a-b=0 \\ \hline a=b=0 \end{array}$$

$\dim P_1 = 2$ and S is lin. independent. So S is a basis for P_1.

$6t-4$ is a linear combination of $t+1$ and $t-1$:

$$6t-4 = c_1(t+1) + c_2(t-1) \Rightarrow \begin{array}{l} c_1 + c_2 = 6 \\ c_1 - c_2 = -4 \\ \hline 2c_1 = 2 \Rightarrow c_1 = 1 , \; c_2 = 5. \end{array}$$

$$L(6t-4) = c_1 L(t+1) + c_2 L(t-1)$$
$$= 1 \cdot (2t+3) + 5(3t-2) = 17t - 7 \,//.$$

b) $L(at+b) = ?$

$$at+b = c_1(t+1) + c_2(t-1) \Rightarrow \begin{array}{l} c_1 + c_2 = a \\ c_1 - c_2 = b \\ \hline 2c_1 = a+b \Rightarrow c_1 = \frac{a+b}{2} \end{array}$$

$$c_2 = a - \frac{a+b}{2} = \frac{a-b}{2}.$$

$$\Rightarrow at+b = \frac{a+b}{2}(t+1) + \frac{a-b}{2}(t-1)$$

$$\Rightarrow L(at+b) = \frac{a+b}{2} L(t+1) + \frac{a-b}{2} L(t-1)$$
$$= \frac{a+b}{2}(2t+3) + \frac{a-b}{2}(3t-2).$$

13) ${}_B[L]_B = \begin{bmatrix} \boxed{\begin{matrix}1\\1\\0\end{matrix}} & \begin{matrix}3\\2\\1\end{matrix} & \begin{matrix}1\\0\\1\end{matrix} \end{bmatrix}$, $B = \{e_1, e_2, e_3\}$ (natural basis) $\Rightarrow$

$L(e_1) = 1e_1 + 1e_2 + 0e_3 = e_1 + e_2$

$L(e_2) = 3e_1 + 2e_2 + e_3$

$L(e_3) = e_1 + 0e_2 + e_3 = e_1 + e_3$.

$$\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix} = e_2 + 2e_3 \Rightarrow L\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = L(e_2 + 2e_3)$$
$$= L(e_2) + 2L(e_3)$$
$$= 3e_1 + 2e_2 + e_3 + 2(e_1 + e_3)$$
$$= 5e_1 + 2e_2 + 3e_3 = \begin{bmatrix}5\\2\\3\end{bmatrix} //.$$

15) $\alpha_1 = (-1,2,1)$, $\alpha_2 = (2,0,1)$, $\alpha_3 = (0,1,1) \in \mathbb{R}_3$ (basis).

a) Let $(a,b,c) \in \mathbb{R}_3$ and write as a lin. combination of α_i's.

$(a,b,c) = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 \Rightarrow \begin{cases} -c_1 + 2c_2 = a \\ 2c_1 + c_2 = b \end{cases} \Rightarrow 5c_2 = 2a+b$, $c_2 = \frac{2a+b}{5}$

$c_1 + c_2 + c_3 = c.$

$c_1 = 2c_2 - a = \frac{-a+2b}{5}$

$c_3 = c - c_1 - c_2 = c - \left(\frac{-a+2b}{5}\right) - \left(\frac{2a+b}{5}\right)$

$= c - \frac{a}{5} - \frac{3b}{5}$

$\Rightarrow L(a,b,c) = c_1 L(\alpha_1) + c_2 L(\alpha_2) + c_3 L(\alpha_3)$

$= \frac{-a+2b}{5} \cdot (-4) + \frac{2a+b}{5} \cdot (1) + \left(c - \frac{a}{5} - \frac{3b}{5}\right)(-2).$

b) $T = \{2\}$, $L: \mathbb{R}_3 \to \mathbb{R}$

$L(\alpha_1) = -4 = (-2).2 \in \langle T \rangle$

$L(\alpha_2) = 1 = \frac{1}{2}.2 \in \langle T \rangle$

$L(\alpha_3) = -2 = (-1).2 \in \langle T \rangle$.

$[L]_{S\,T} = \begin{bmatrix} -2 & \frac{1}{2} & -1 \end{bmatrix}_{1\times 3}$ //

c) $B_1 = \{e_1, e_2, e_3\} \subseteq \mathbb{R}_3$, $B_2 = \{1\} \subseteq \mathbb{R}$ are natural bases.

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1) \Rightarrow$ Find $L(e_i)$s.

By (a), $L(e_1) = L(1,0,0) = \frac{4}{5} + \frac{2}{5} + \frac{2}{5} = \frac{8}{5}.1$

$L(e_2) = L(0,1,0) = \frac{-8}{5} + \frac{1}{5} - \frac{6}{5} = \frac{-13}{5}.1$

$L(e_3) = L(0,0,1) = -2.1$

$\Rightarrow [L]_{B_1 B_2} = \begin{bmatrix} 8/5 & -13/5 & -2 \end{bmatrix}$ //

14) $[L]_{S\,T} = \begin{bmatrix} -3 & 7 \\ -1 & 1 \\ -3 & 6 \end{bmatrix} \Rightarrow L(1,-1) = -3(1,0,1) - (2,-1,3) - 3(-2,0,-3)$

$= (1,1,3)$

$L(-2,3) = 7(1,0,1) + (2,-1,3) + 6(-2,0,-3)$

$= (-3,-1,-8)$.

$(0,1) \in \langle S \rangle = \mathbb{R}_2 \Rightarrow (0,1) = c_1(1,-1) + c_2(-2,3)$

$\Rightarrow \begin{array}{l} c_1 - 2c_2 = 0 \\ -c_1 + 3c_2 = 1 \end{array}$

$c_2 = 1$, $c_1 = 2$.

$\Rightarrow L(0,1) = c_1 L(1,-1) + c_2 L(-2,3) = (1,1,3) + 2(-3,-1,-8) = (-5,-1,-13)$ //