H.Ü.	MAT 123-01-10-14 Midterm Exam 02.12.2019	
Surname : Name :	ID : Instructor :	Signature
5 questions, 2 pages	Duration : 90 min	100 points

Q1. [20 pts] Show that the funtion $f(x) = \begin{cases} x\cos\left(\frac{1}{x^3}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is continuous but not differentiable at $x = 0$.

Since $-1 \le \cos(\frac{1}{x^3}) \le 1$, $-x \le x\cos(\frac{1}{x^3}) \le x$ if $x>0$ and $x \le x\cos(\frac{1}{x^3}) \le -x$ if $x<0$. Since $\lim_{x\to 0} x = \lim_{x\to 0} -x = 0$, by the Sandwich Theorem, we have $\lim_{x\to 0} x\cos(\frac{1}{x^3}) = 0 = f(0)$; thus f is continuous at $x=0$.

$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x\cos(1/x^3)}{x} = \lim_{x\to 0} \cos(1/x^3)$ does not exist because $\cos(1/x^3) = 1$ for $x = \sqrt[3]{1/2k\pi}$ $(k=1,2,\dots)$ and $\cos(1/x^3) = -1$ for $x = \sqrt[3]{1/(2k+1)\pi}$ $(k=1,2,\dots)$, i.e. $\cos(1/x^3)$ takes on the values 1 and -1 infinitely many times as x approaches 0.

Q2. [15 pts] Find the tangent line to the curve $\ln(x^2+y^2) = \arcsin(\sqrt{x^3})$ at the point $(0,1)$.

Differentiating both sides with respect to x gives

$$\frac{2x + 2y\frac{dy}{dx}}{x^2+y^2} = \frac{3}{2}x^{1/2}\frac{1}{\sqrt{1-x^3}}.$$

Substitution gives

$$\frac{2.0 + 2.1.\frac{dy}{dx}\Big|_{(0,1)}}{0^2+1^2} = \frac{3}{2}.0^{1/2}\frac{1}{\sqrt{1-0^3}},$$

and so $\frac{dy}{dx}\Big|_{(0,1)} = 0$.

Thus the tangent line is the line $y=1$.

Q3. [20 pts] All dimensions of a right circular cylinder are changing. When the voulme is 150 cm^3, it is increasing at the rate of $5cm^3$/min and at the same moment, the radius is 5cm and is increasing at the rate of 3cm/min. At what rate the height of the cylinder is changing at the moment when the volume of the cyclinder is 150 cm^3? Is it increasing or decreasing?

$\frac{dV}{dt}\Big|_{V=150} = 5\ cm^3/min$ $\quad \frac{dh}{dt}\Big|_{V=150} = ?$

$\frac{dr}{dt}\Big|_{V=150} = 3\ cm/min$

$V = \pi r^2 h$

When $V=150$, $r=5$ and so $h = \frac{V}{\pi r^2} = \frac{150}{\pi . 25} = \frac{6}{\pi}$ cm.

Differentiating, we get $\frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$. Then

$$5 = \pi\left(\frac{180}{\pi} + 25\frac{dh}{dt}\Big|_{V=150}\right) \Rightarrow -175 = 25\pi\frac{dh}{dt}\Big|_{V=150} \Rightarrow$$

$\frac{dh}{dt}\Big|_{V=150} = \frac{-7}{\pi}$ cm/min. So the height is <u>decreasing</u>!

Q4. [25 pts] Let $f(x) = \dfrac{4-x^3}{x^2}$. **(a)** Find the domain of f. [2pts] **(b)** Determine the horizonral, vertical and oblique asymptotes of f. [5 pts] **(c)** Find the critical points of f and determine the intervals on which f is increasing and decreasing. Find the local minimum and maximum values. [8 pts] **(d)** Determine the intervals on which f is concave upward and concave downward, and find the inflection points of f. [5 pts] **(e)** Sketch the graph of f.

(a) $D(f) = \mathbb{R} - \{0\}$ (2pts)

(b) $\lim\limits_{x\to\pm\infty} \dfrac{4-x^3}{x^2} = \mp\infty$ (2pts)

no horizontal asymptotes.

$\lim\limits_{x\to 0} \dfrac{4-x^3}{x^2} = \infty$ (1 pt)

$x=0$ is the only vertical asymptote.

$\lim\limits_{x\to\pm\infty} f(x)+x = \lim\limits_{x\to\pm\infty} \dfrac{4}{x^2} = 0$ (2pts)

$y=-x$ is an oblique asymptote.

(c) $f'(x) = \dfrac{-3x^4 - 2x(4-x^3)}{x^4} = \dfrac{-3x^4 - 8x + 2x^4}{x^4}$

$= -\dfrac{x^3+8}{x^3}$ (3pts)

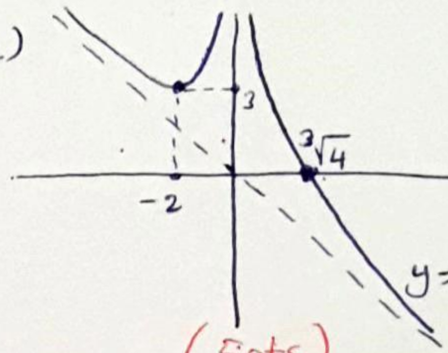
		-2		0	
f'	$-$		$+$		$-$
$y=f(x)$	↘		↗		↘

$f(-2)=3$ is a local min. value. (5pts)

(d) $f''(x) = -\dfrac{3x^5 - 3x^2(x^3+8)}{x^6} = \dfrac{24}{x^4}$ (3pts)

Since $f''(x)>0$ for all x in the domain of f. f is always concave upward and there are no inflection points (2pts)

(e)



(5pts)

Q5. **(a)** [10 pts] Evaluate the limit $\lim\limits_{x\to\infty} \left(\dfrac{x}{x-1}\right)^{2x}$.

$\lim\limits_{x\to\infty} \ln\left(\dfrac{x}{x-1}\right)^{2x} = \lim\limits_{x\to\infty} 2x\ln\left(\dfrac{x}{x-1}\right) = \lim\limits_{x\to\infty} \dfrac{2\ln\left(\frac{x}{x-1}\right)}{\frac{1}{x}} \overset{\text{l'Hospital's Rule}}{=}$

$\lim\limits_{x\to\infty} \dfrac{2\cdot\left(\frac{x-1-x}{(x-1)^2}\right)\Big/\frac{x}{x-1}}{\frac{-1}{x^2}} = \lim\limits_{x\to\infty} \dfrac{2\ \frac{-1}{(x-1)^2}\Big/\frac{x}{x-1}}{\frac{-1}{x^2}} = \lim\limits_{x\to\infty} \dfrac{2\cdot\frac{1}{x(x-1)}}{\frac{1}{x^2}}$

$= \lim\limits_{x\to\infty} \dfrac{2x}{x-1} = 2$. It follows that $\lim\limits_{x\to\infty}\left(\dfrac{x}{x-1}\right)^{2x} = e^2$.

(b) [10 pts] If $a>0$ show that the equation $x^3+ax-1=0$ has EXACTLY ONE real solution.

Let $f(x) = x^3+ax-1$. Since $f(0)=-1<0$ and $f(1)=a>0$, there exists a solution of $f(x)=0$ between 0 and 1 by the IVT. If there are two different real solutions c_1 and c_2 of the equation $f(x)=0$, then by Rolle's Theorem, there exists a real number c between c_1 and c_2 such that $f'(c)=0$. But $f'(c)=3c^2+a$ is a positive number and cannot be zero, which is a contradiction. It follows that the equation $f(x)=0$ has exactly one real solution.