MAT 254 Fundamentals of Linear Algebra

2020–2021 – Spring

07.04.2021

Midterm Exam Key

① Determine the values of a for which the following system has a) no solution b) infinitely many solutions c) a unique solution.

$$
\begin{aligned}
x+y+z+t&=4\\
x+ay+z+t&=4\\
x+y+az+(3-a)t&=6\\
2x+2y+2z+at&=6
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4\\ 1 & a & 1 & 1 & 4\\ 1 & 1 & a & 3-a & 6\\ 2 & 2 & 2 & a & 6 \end{array}\right]
\xrightarrow[-2R_1+R_4]{\substack{-R_1+R_2\\ -R_1+R_3}}
\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4\\ 0 & a-1 & 0 & 0 & 0\\ 0 & 0 & a-1 & 2-a & 2\\ 0 & 0 & 0 & a-2 & -2 \end{array}\right]
$$

a) The system has no solution if $a=2$. Because in this case the last row of the reduced matrix will be a bad row.

b) $a=1 \Rightarrow$
$$
\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 2\\ 0 & 0 & 0 & -1 & -2 \end{array}\right]
$$
So there are four unknowns and two leading terms. Thus the system has infinitely many solutions.

c) The system has a unique solution if $a\neq 1$ and $a\neq 2$. That is $a \in \mathbb{R}-\{1,2\}$.
Because in this case there will be three unknowns and three leading terms.

② a)

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 4 & 14 & 0 & 0 & 6 \\ 0 & 0 & 2 & 6 & 0 & 4 \\ 0 & 0 & 0 & 2 & 8 & 0 \\ 0 & 0 & -6 & -18 & 2 & -12 \\ -2 & -4 & -6 & 0 & -2 & 1 \end{bmatrix} \xrightarrow[3R_3+R_5]{2R_1+R_6} \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 4 & 14 & 0 & 0 & 6 \\ 0 & 0 & 2 & 6 & 0 & 4 \\ 0 & 0 & 0 & 2 & 8 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b) $|A| = 1 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 32$

c) $|(2A)^{-1}| = |\frac{1}{2} A^{-1}| = \frac{1}{2^6} \cdot \frac{1}{32} = \frac{1}{2^{11}}$

d) $A \xrightarrow{R_1 \leftrightarrow R_3} B \xrightarrow{-5R_3+R_1} C \xrightarrow{\frac{1}{4}R_2} D$

$|A| = 32$

$|B| = -|A|$ so $|B| = -32$

$|C| = |B|$ so $|C| = -32$

$|D| = \frac{1}{4} \cdot |C|$ so $|D| = -8$

e) $|B^T D^{-1}| = |B^T| \cdot |D^{-1}| = |B| \cdot \frac{1}{|D|} = (-32) \cdot \frac{1}{(-8)} = 4$

③ $A = \begin{bmatrix} a-b & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & a+b & 1 \\ 0 & 0 & -1 \end{bmatrix}$

a) If $AX = 0$ has a nontrivial solution, then

$|A| = 0$.

$|A| = (a-b) . 1 . (-1) . 1 . (a+b) . (-1) = a^2 - b^2$

So $a^2 - b^2 = 0$. In this case $a = b$ or $a = -b$.

b) $a = 1$ and $b = 0 \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

so $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 7 \\ 3 & -2 & 11 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 2 & -1 & 7 & 0 & 1 & 0 \\ 3 & -2 & 11 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow[-R_3+R_2]{-4R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 5 & -4 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} 3 & 5 & -4 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

4 a) Determine whether the set $W = \{ f \in P_3(x) \mid \deg f = 3 \}$ is a subspace of $P_3(x)$ or not. NO

Because W is not closed under addition.

For example $f(x) = -x^3 + x^2 + 1$ and $g(x) = x^3 + x^2 + 1$ are elements of W but

$f(x) + g(x) = 2x^2 + 2$ is not an element of W.

b) Determine whether the set

$U = \{ A \in M_{2\times 2} \mid A_{ij} = 0 \text{ if } j-i-1 \text{ is divisible by } 2 \}$

is a subspace of $M_{2\times 2}$ or not. YES

An arbitrary element of U is of the form

$$\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

a) U is non empty

b) U is closed under addition: Let $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ and $\begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$ be two elemets of U.

Then $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}+B_{11} & 0 \\ 0 & A_{22}+B_{22} \end{bmatrix} \in U.$

c) U is closed under scalar multiplication:

Let $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \in U$ and $c \in \mathbb{R}$ then

$$c \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} cA_{11} & 0 \\ 0 & cA_{22} \end{bmatrix} \in U.$$