Question 1. Identify the conic section with the given equation

$$4x^2 - 9y^2 - 16x + 54y - 101 = 0$$

and find its focus or foci.

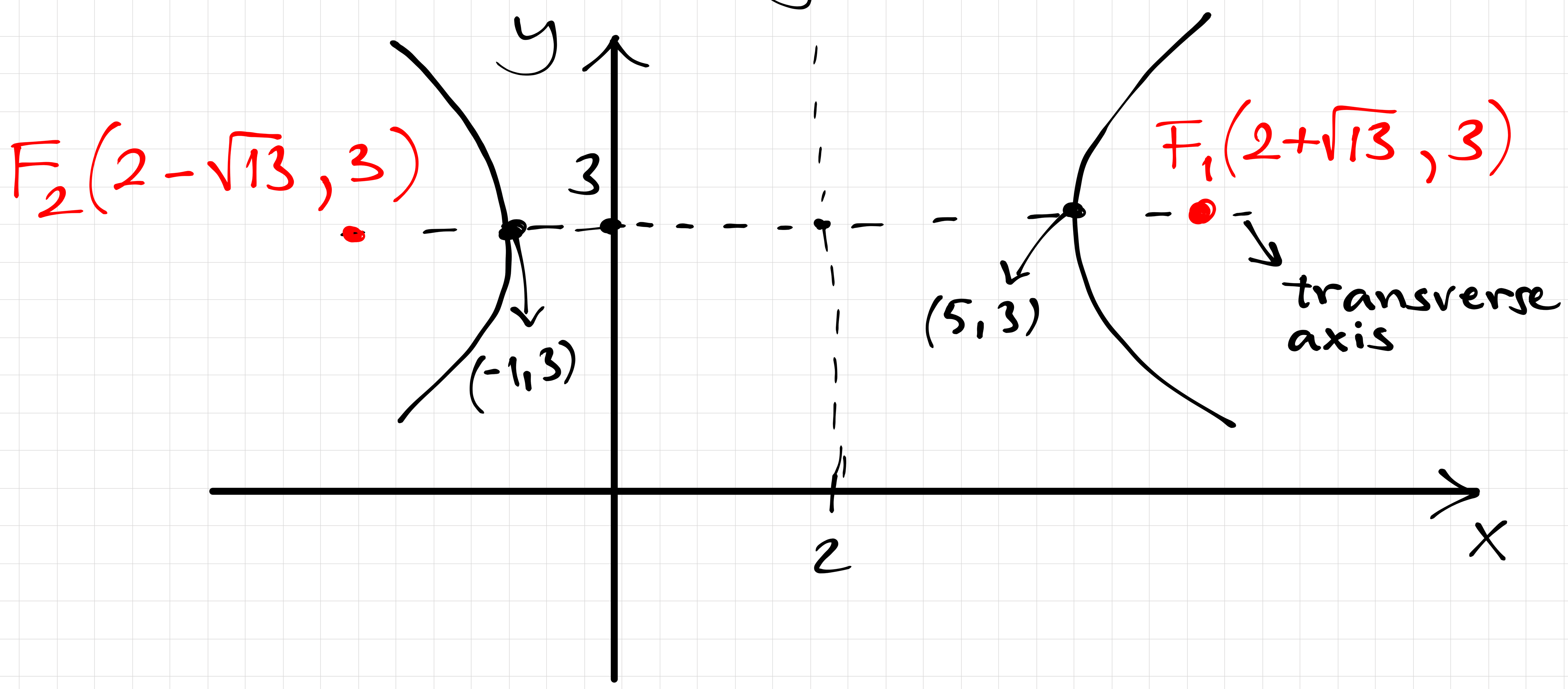
SOLUTION. We use the technique of completing the squares:

$$4(x^2 - 4x + 4) - 9(y^2 - 6y + 9) = 101 + 16 - 81$$

$$\Rightarrow \; 4(x-2)^2 - 9(y-3)^2 = 36$$

$$\Rightarrow \; \frac{(x-2)^2}{9} - \frac{(y-3)^2}{4} = 1 \qquad (5pt)$$

It follows that the given equation represents a hyperbola. To sketch its graph it is enough to shift the hyperbola given by $\frac{x^2}{9} - \frac{y^2}{4} = 1$ 2 units to the right and 3 units upward.



(5pts)

$$c = \sqrt{9+4} = \sqrt{13} \qquad (5pts)$$

The foci are $F_1(2+\sqrt{13}, 3)$ and $F_2(2-\sqrt{13}, 3)$.

Question 2. Determine if the plane given by $-x+2z=10$ and the line given by $\vec{r}(t)=5i+(2-t)j+(10+4t)k$ are orthogonal, parallel or neither.

SOLUTION. The given plane has normal vectors parallel to the vector $v=-i+2k$. Also, the given line is parallel to the vector $u=-j+4k$ since it can also be represented by the parametric equations

$$x=5$$
$$y=2-t$$
$$z=10+4t.$$

Since $u.v=0.(-1)+(-1).0+2\cdot 4=8\neq 0$, u and v are not perpendicular. Thus we conclude that the plane and the line are not parallel.
On the other hand, u and v are not parallel since u is parallel to the yz-plane while v is parallel to the xz-plane; but they are not parallel to the z-axis. This can also be verified by observing that the corresponding components of u and v are not proportional.
This shows that the line is not perpendicular to the plane.

In summary, we have seen that the plane and the line are neither orthogonal nor parallel.

Question 3. Determine whether the following limits exist and calculate if it exists.

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^3+y^3}$.

SOLUTION.

1. Using the substitutions $x=r\cos\theta$, $y=r\sin\theta$, $r^2=x^2+y^2$ (noting that $r\geq 0$), we get

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{\cancel{r^2}\sin\theta\cos\theta}{\cancel{r}}$$

(10 pts)

$$= \lim_{r\to 0} r\sin\theta\cos\theta = 0.$$

2. Along the y-axis (the line $x=0$)

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{xy}{x^3+y^3} = \lim_{y\to 0} \frac{0}{y^3} = 0.$$

(5pts)

On the other hand, along the parabola $y=x^2$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{xy}{x^3+y^3} = \lim_{x\to 0} \frac{x^3}{x^3+x^6}$$

(5pts)

$$= \lim_{x\to 0} \frac{1}{1+x^3} = 1.$$

Since limit varies depending on the way in which (x,y) approaches $(0,0)$, the limit does not exists.

Question 4.

(a) Use the Chain Rule to find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ when $u=3$, $v=-1$ if $w = xe^{y-z^2}$, where $x=2uv$, $y=u-v$, $z=u+v$

(b) If $f(x,y)=ye^{-x}$, find the rate of change of f at the point $P(0,1)$ in the direction from P to $Q(1,1/2)$. In what direction does f increase most rapidly at $P(0,1)$?

SOLUTION.

(a)

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= \left(e^{y-z^2}\right)2v + \left(xe^{y-z^2}\right).1 + \left(-2zxe^{y-z^2}\right).1$$

$$= e^{y-z^2}\left(2v + x - 2xz\right)$$

When $u=3$, $v=-1$ we have $x=-6$, $y=4$, $z=2$, and so

$$\boxed{\left.\frac{\partial w}{\partial u}\right|_{\substack{u=3\\v=-1}} = 16.}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$= e^{y-z^2}(2u) + \left(xe^{y-z^2}\right)(-1) + \left(-2zxe^{y-z^2}\right)(1)$$

$$= e^{y-z^2}\left(2u - x - 2xz\right)$$

So,

$$\boxed{\left.\frac{\partial w}{\partial v}\right|_{\substack{u=3\\v=-1}} = 36}$$

(b) Let v be the vector from $P(0,1)$ to $Q(1, 1/2)$. Then $v = i - \frac{1}{2} j$, and so the unit vector u in the direction of v is

$$u = \frac{v}{|v|} = \frac{2}{\sqrt{5}} i - \frac{1}{\sqrt{5}} j.$$

Also since $\nabla f(x,y) = -ye^{-x} i + e^{-x} j$, we have $\nabla f(0,1) = -i + j$. Thus

$$D_u f(0,1) = \nabla f(0,1) \cdot u = -\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}} = -\frac{3}{\sqrt{5}}$$

Moreover; f increases most rapidly at $P(0,1)$ in the same direction as $\nabla f(0,1) = -i + j$.

Question 5. Find the tangent line to the curve that is the intersection of the surfaces $xy+yz+zx-3=0$ and $\sin(xyz)=x-3y+2z$ at the point $(3,1,0)$.

SOLUTION. The given surfaces are level surfaces of the functions

$$F(x,y,z)=xy+yz+zx-3$$

and

$$G(x,y,z)=\sin(xyz)-x+3y-2z.$$

Then the tangent line is parallel to the vector

$$\nabla F(3,1,0)\times\nabla G(3,1,0).$$

Now,

$$\nabla F=(y+z)i+(x+z)j+(y+x)k \to \nabla F(3,1,0)=i+3j+4k$$

$$\nabla G=[yz\cos(xyz)-1]i+[xz\cos(xyz)+3]j+[xy\cos(xyz)-2]k$$

$$\downarrow$$

$$\nabla G(3,1,0)=-i+3j+k$$

$$\nabla F(3,1,0)\times\nabla G(3,1,0)=\begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ -1 & 3 & 1 \end{vmatrix}=-9i-5j+6k$$

Thus the tangent line is given by the equations

$$\boxed{\begin{aligned} x&=3-9t \\ y&=1-5t \\ z&=6t \end{aligned}}$$

Question 6. Find the points on the surface $xy^2z^4 = \frac{1}{4}$ that are closest to the origin.

SOLUTION. The problem is equivalent to the one below:

"minimize the function $f(x,y,z) = x^2+y^2+z^2$ subject to the constraint $g(x,y,z) = xy^2z^4 - \frac{1}{4} = 0$.

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 2x = \lambda y^2 z^4 \\ 2y = 2\lambda x y z^4 \\ 2z = 4\lambda x y^2 z^3 \end{cases}$$

$$\frac{x}{y} = \frac{y}{2x} \rightarrow \boxed{y^2 = 2x^2}$$

$$\frac{x}{z} = \frac{z}{4x} \rightarrow \boxed{z^2 = 4x^2}$$

Since $xy^2z^4 = \frac{1}{4}$, we have $x(2x^2)(16x^4) = \frac{1}{4}$, and so $x^7 = \frac{1}{2^7}$; hence $x = \frac{1}{2}$.

This gives that $y = \pm \frac{1}{\sqrt{2}}$, $z = \pm 1$.

Therefore, the points $\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm 1\right), \left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \mp 1\right)$ are the points on the surface $xy^2z^4 = \frac{1}{4}$ that are closest to the origin.