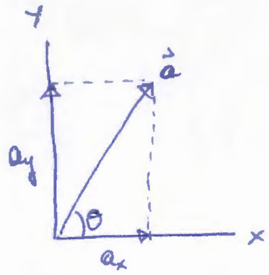
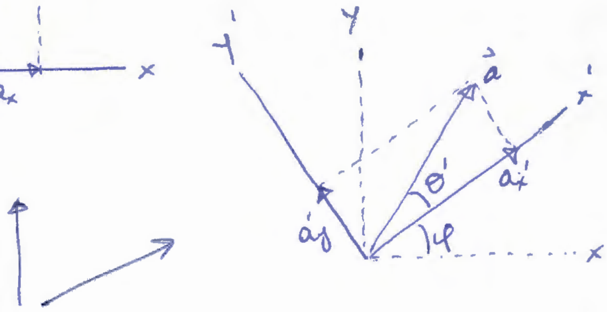
Vectors and Laws of Physics



← this looks ok but we could just as well rotate our axes by an angle φ while keeping the vector $\vec{a}$ fixed



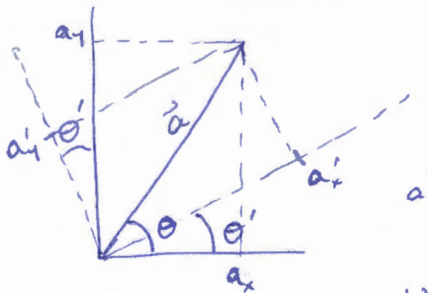
They are all equally valid because the relations among the vectors do not depend on the location of the origin or on the orientation of axis.

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2} \quad ; \quad \theta = \theta' + \varphi$$

This is also true of the relations of Physics:

⇒ They are all independent of the choice of the coordinate system.

Ex:



$|\vec{a}| = 17m$

$\theta = 56°$

$\sin 56 = 0.83$ $\quad$ $\sin 38 = 0.62$

$\cos 56 = 0.56$ $\quad$ $\cos 38 = 0.79$

$\sin 18 = 0.31$

$\cos 18 = 0.95$

a.) $a_x = ?$

$a_y = ?$

b) $\theta' = 18°$

$a'_x = ?$

$a'_y = ?$

a.) $a_x = a \cos 56 = 9.52m$

$a_y = a \sin 56 = 14.11m$ $\quad \Big) \sqrt{a_x^2 + a_y^2} \cong 17m$

b.) $a'_x = a \cdot \cos(56-18) = 17 \cdot 0{,}79 = 13{,}40m$

$a'_y = a \cdot \sin(56-18) = 17 \cdot 0{,}62 = 10{,}47m$

$\sqrt{a_x'^2 + a_y'^2} \cong 17m$

MULTIPLYING VECTORS

There are three kinds of multiplications with vectors involved and none of them is exactly like the usual algebraic multiplication.

* Multiplying a vector by a scalar

$s.\vec{a}$ = a new vector, same/opposite direction, its magnitude is the product of the magnitude of $\vec{a}$ and absolute value of s

s → scalar, $\vec{a}$ → vector

direction: $s > 0 \rightarrow$ same direction; $s < 0 \rightarrow$ opposite direction

* Multiplying a vector by a vector

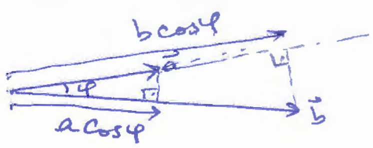
There are two ways – one way produces a scalar (scalar product)
– the other produces a vector (vector product)

SCALAR PRODUCT

$\vec{a} \cdot \vec{b} = ab\cos\varphi$ → $\hat{ab}$ or $\hat{ba}$ $(360-\varphi)$ since cos is an even function ↳ $\cos(\alpha) = \cos($

"a dot b" → dot product

product of the magnitude of one vector with the scalar component of the second along the direction of the first vector.



$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\vec{a} \cdot \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$

$= a_x b_x + a_y b_y + a_z b_z$

$\left.\begin{array}{l} \hat{i} \perp \hat{j}, \hat{k} \\ \hat{j} \perp \hat{i}, \hat{k} \\ \hat{k} \perp \hat{i}, \hat{j} \end{array}\right\} \rightarrow \begin{array}{l} \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \end{array}$

VECTOR PRODUCT

$\vec{a} \times \vec{b}$ ("a cross b"), cross product

produces a third vector $\vec{c}$ with magnitude:

$$c = ab\underbrace{\sin\varphi}$$

φ → the smaller of the two angles between $\vec{a}$ and $\vec{b}$

direction of $\vec{c}$ is perpendicular both to $\vec{a}$ and $\vec{b}$

↕ plane that contains $\vec{a}$ and $\vec{b}$

ORder is important

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$$

$$\vec{a} \times \vec{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \times (b_x \hat{i} + b_y \hat{j} + b_z \hat{k})$$

$$a_x \hat{i} \times b_x \hat{i} = a_x b_x \underbrace{(\hat{i} \times \hat{i})}_{0} = 0$$

$$a_x \hat{i} \times b_y \hat{j} = a_x b_y \underbrace{(\hat{i} \times \hat{j})}_{\hat{k}} = a_x b_y \hat{k}$$

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{i} + (a_z b_x - b_z a_x)\hat{j} + (a_x b_y - b_x a_y)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

MOTION ALONG A STRAIGHT LINE

Motion of objects → How fast they move
→ How far they move in a given amount of time

Race cars, tectonic plate motion, Blood flow

⇒ Special case: Movement along a single axis, i.e. one-dimensional motion.

Motion

The world, and everything on it moves.

Earth's rotation, its orbit around the Sun,

Sun's orbit around the center of Milky Way galaxy,

Galaxy's motion with respect to other galaxies.

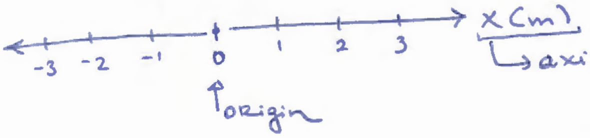
The $\underbrace{\text{classification and comparison}}_{\text{analysis}}$ of motions: kinematics.

We'll restrict our studies in three ways:

1) The motion is along a straight line only.
2) Forces (push/pull) cause motion but for now we'll assume it "just" moves and will be interested in motion itself and changes in the motion like slow down, speed up, stop, reverse direction
   → if it's changing, how is the time involved in this change?
3) The moving object is either a particle (point like object) or an object that moves like a particle (every portion moves in the same direction and at the same rate (rigid)).

Position and Displacement

To locate an object means to find its position relative to some reference point (like an origin)



axis name is always on the positive side of the origin

A change from position x_1 to position x_2 is called a **displacement** (Δx)

$$\Delta x = x_2 - x_1$$

"delta" (uppercase greek letter delta)

Represents a change in a quantity

final − initial value

zeta iota

α β γ δ ε ζ η θ ι κ λ μ

ν ξ ο π ρ σ τ υ φ χ ψ ω

xi omicron upsilon chi

Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ

Ν Ξ Ο Π Ρ Σ Τ Υ Φ Χ Ψ Ω

→ a displacement in the positive direction is always positive (and vice versa)

- $x_1 = 5m \longrightarrow x_2 = 12m$: $\Delta x = (12m) - (5m) = +7m$
- $x_1 = 5m \longrightarrow x_2 = 1m$: $\Delta x = (1m) - (5m) = -4m$

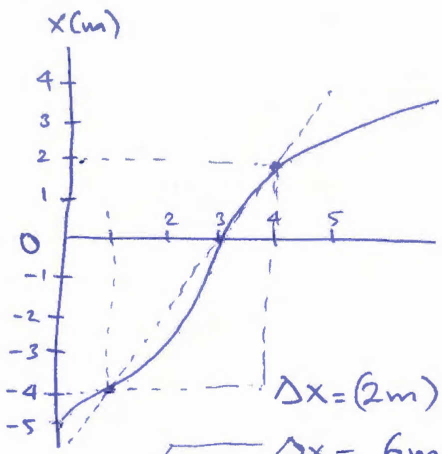
The path is irrelevant $x = 5m \longrightarrow x = 200m \longrightarrow x = 5m$

$\Delta x = 5m - 5m = \underline{\underline{0}}$

Displacement is an example of a vector quantity.

AVERAGE VELOCITY AND AVERAGE SPEED

A compact way to describe position is with a graph of position x plotted as a function of time – a graph of x(t)



(this graph also reveals "how fast")

$\Delta x = (2m) - (-4m)$

$\Delta x = 6m$

$\Delta t = 4s - 1s = 3s$

$V_{avg} = \frac{\Delta x}{\Delta t} = \frac{6m}{3s} = 2m/s$

→ average velocity : $V_{avg} = \frac{\Delta x}{\Delta t}$

$$V_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$\frac{x_f - x_i}{t_f - t_i}$$

$[V] = m/s$

On a x(t) graph, V_{avg} is the <u>slope</u> of the straight line that connects two particular positions on the x(t) curve.

It is also a vector quantity.

$V > 0$ ↗ ; $V < 0$ ↘

Average velocity V_{avg} always has the same sign as the displacement Δx (WHY? → because Δt is always positive)

Average Speed: Total Distance covered independent of direction.

not a vector → hence, not signed

$$S_{avg} = \frac{\text{total distance}}{\Delta t}$$

INSTANTENOUS VELOCITY & SPEED

We have already seen two ways to describe how fast something moves.

Both average velocity and speed are measured over a time interval Δt

"But...", the phrase "how fast" more commonly refers to how fast a particle is moving at a given instant

$\longrightarrow$ its instantenous velocity (or simply velocity) v.

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

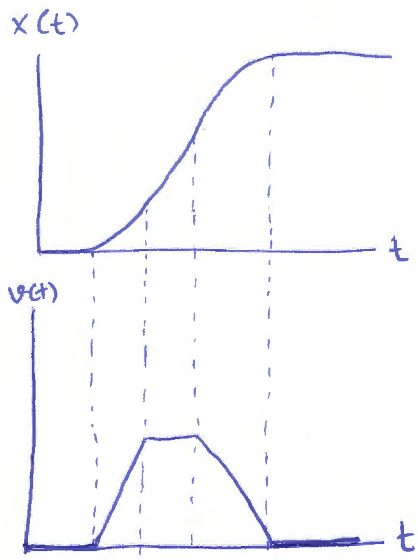
Velocity is a vector

and speed is its magnitude.

v is the slope of the position-time curve at the point representing that instant.

v is the derivative of x with respect to t.

Example: An elevator, initially stationery, then moves upward (+x direction) then stops. Plot $x(t)$, $v(t)$.



Acceleration

When a particle's velocity changes, the particle is said to undergo acceleration.

average acceleration $a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}$

Instantenous acceleration $= a = \frac{dv}{dt}$
(or simply acceleration)

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x}$$

$[a] = ?$ (m/s^2)

also a vector

Our bodies behave like an accelerometer but not like a speedometer.

→ We feel accelerations.

car 90 km/h
plane 900 km/h] if $a = 0$ → we can't tell the difference.

$1g = 9.8\ m/s^2$ (g unit)

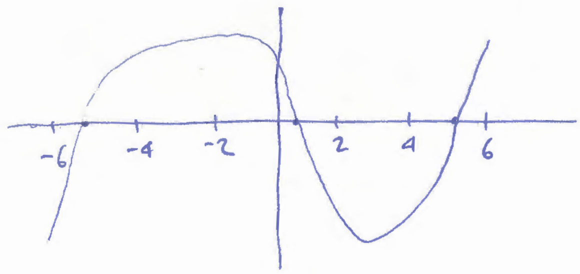
Example: A particle's position is given by:

$$x = 4 - 27t + t^3 \longleftrightarrow \text{roots: } t \simeq \begin{cases} -5.269\,s \\ 0.148\,s \\ 5.120\,s \end{cases}$$

a.) $v(t), a(t) = ?$

$v(t) = -27 + 3t^2$

$a(t) = 6t$



b) Is there a time when $v=0$?

$0 = -27 + 3t^2 \longrightarrow t = \pm 3s$

c.) Describe the particle's motion for $t \geq 0$

- $t=0$: $x(0) = 4m$

  $v(0) = -27 m/s \longrightarrow$ moving in $-x$ direction

  $a(0) = 0$

- $0 < t < 3$: $v < 0 \longrightarrow$ it continues to move in negative x direction

  however, now $a > 0 \rightarrow$ since v and a are of opposite signs, the particle must be slowing.

- $t = 3$: $v = 0 \longrightarrow$ particle stops momentarily

  $x(3) = -50m$

  $a > 0$

- $t > 3$: particle moves right on the axis

  $v > 0$, $a > 0$ $\longrightarrow$ it is increasing its speed in the positive x-direction.

Constant Acceleration: A special Case

In many types of motion, the acceleration is either constant, or approximately so.

When the acceleration is constant:

average and instantenous accelerations are equal.

$$a = a_{avg} = \frac{v - v_0}{t - 0} \longrightarrow \boxed{v = v_0 + at} \quad (1)$$

$$v_{avg} = \frac{x - x_0}{t - 0}$$

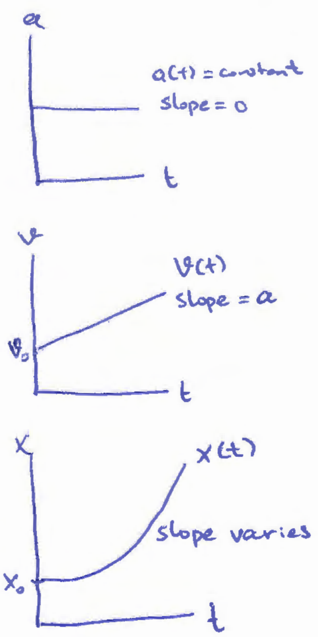
$$x = x_0 + v_{avg}\, t$$

$$v_{avg} = \frac{1}{2}(v_0 + v) = \frac{1}{2}(v_0 + v_0 + at)$$

$$\rightarrow v_{avg} = v_0 + \frac{1}{2}at$$

$$x = x_0 + \left(v_0 + \frac{1}{2}at\right)t$$

$$\boxed{x - x_0 = v_0 t + \frac{1}{2}at^2} \quad (2)$$



(1), (2) $\longrightarrow$ Basic equations for constant acceleration.

$$\underbrace{x - x_0,\ v,\ t,\ a,\ v_0}$$

five quantities that might be involved.

(1) and (2) can be used to solve any constant acceleration problem.

* eliminate t:

$$\boxed{v^2 = v_0^2 + 2a(x - x_0)} \quad (3)$$

$$(1) \longrightarrow \frac{v - v_0}{a} = t$$

$$x - x_0 = v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2$$

$$x - x_0 = \frac{\cancel{2v_0 v} - \cancel{2}v_0^2 + v^2 + \cancel{v_0^2} - \cancel{2vv_0}}{2a}$$

$$2a(x - x_0) = v^2 - v_0^2 \Rightarrow v^2 = v_0^2 + 2a(x - x_0)$$

* eliminate a:

$$\boxed{x - x_0 = \frac{1}{2}(v_0 + v)t} \quad (4)$$

$(1) \rightarrow \quad a = \dfrac{v - v_0}{t}$

$(2): \quad x - x_0 = v_0 t + \frac{1}{2}\left(\frac{v - v_0}{t}\right)t^2$

$= v_0 t + \frac{1}{2} v t - \frac{1}{2} v_0 t$

$x - x_0 = \frac{1}{2}(v_0 + v)t$

Equations of Motion (Constant acceleration)

1) $v = v_0 + at$

2) $x - x_0 = v_0 t + \frac{1}{2} a t^2$

3) $v^2 = v_0^2 + 2a(x - x_0)$

4) $x - x_0 = \frac{1}{2}(v_0 + v)t$

5) $x - x_0 = vt - \frac{1}{2} a t^2$

* Eliminate v_0:

$$\boxed{x - x_0 = vt - \frac{1}{2} a t^2} \quad (5)$$

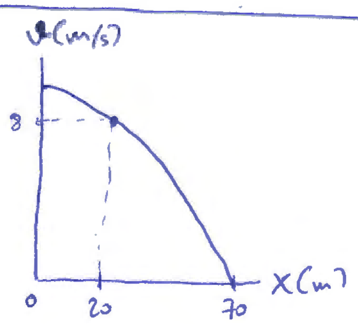
$(1) \rightarrow \quad v_0 = v - at$

$(2): \quad x - x_0 = (v - at)t - \frac{1}{2} a t^2$

$= vt - at^2 - \frac{1}{2} a t^2$

$x - x_0 = vt - \frac{1}{2} a t^2$

Ex:



Constant acceleration.

What is its velocity at $x = 0$?

$(3): \quad v^2 = v_0^2 + 2a(x - x_0)$

$(x = 20\,m, \; v = 8\,m/s)$

$(x = 70\,m, \; v = 0\,m/s)$

i) $(8\,m/s)^2 = v_0^2 + 2a(20m - 0)$ -or- ii) $0 = (8\,m/s)^2 + 2a(70m - 20m)$

$0^2 = v_0^2 + 2a(70m - 0)$

$0 = 64\,m^2/s^2 + (100\,m)a$

$\rightarrow \boxed{a = -0.64\,m/s^2}$

$64\,m^2/s^2 = v_0^2 + (40m)a$

$0 = v_0^2 + (140m)a$

$\rightarrow -64\,m^2/s^2 = (100m)a$

$\rightarrow \boxed{a = -0.64\,m/s^2}$

$\Rightarrow \quad 0^2 + v_0^2 + 2(-0.64\,m/s^2).(70m)$

$0 = v_0^2 - 89.6\,m^2/s^2$

$v_0 = \sqrt{89.6}\,m/s = \underline{\underline{9.4657\,m/s}}$

Another look at Constant acceleration

$$a = \frac{dv}{dt} \longrightarrow dv = a\,dt$$

$$\int dv = \int a\,dt$$

if $a = \text{const.} \longrightarrow \int dv = a \int dt$

$$\rightarrow \boxed{v = at + C}$$

$C = ?$: $t = 0 \rightarrow v(t=0) = v_0$

$$\Rightarrow v_0 = (a)(0) + C = C$$

$$C = v_0$$

$$\longrightarrow \underline{v = v_0 + at}$$

$dx = v\,dt$

$$\int dx = \int v\,dt = \int (v_0 + at)\,dt = v_0 \int dt + a \int t\,dt \Rightarrow x = v_0 t + \frac{1}{2} a t^2 + c'$$

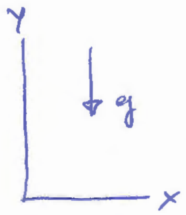
$$t = 0 \rightarrow x(t=0) = x_0 \Longrightarrow c' = x_0$$

$$\Rightarrow \boxed{x = x_0 + v_0 t + \frac{1}{2} a t^2}$$

FREE-FALL ACCELERATION

g (varies slightly with lattitude & elevation)

sea-level : $9.8\ m/s^2$



$$a = -g = -9.8\ m/s^2$$

free-fall acceleration

magnitude is $g = 9.8\,m/s^2$

Felix Baumgartner 2012

39000m , 1342 km/h

$$v_t = \sqrt{\frac{2mg}{\rho A C_d}}$$

$C_d \rightarrow$ drag coefficient

$A \rightarrow$ projected Area

$\rho \rightarrow$ density of the medium

$v_t \sim 56\,m/s$ (200km/h)

Skydivers reach at $\sim 12s$

4' 20" Felix

4' 36" Joe Kittinger

Ex: A ball is thrown upwards with $v_0 = 12\,m/s$.

a.) How long does the Ball take to Reach max height?

$a = -g$, v, a, v_0 are known, t is unknown

$$v = v_0 + at \rightarrow t = \frac{v - v_0}{a} = \frac{0 - 12\,m/s}{-9.8\,m/s^2} = \underline{1.2\,s}$$

b) What is the Ball's maximum Height reached?

$$Y = Y_0 + v_0 t + \frac{1}{2} a t^2 \Longrightarrow h_{max} = \underbrace{(12\,m/s)(1.2\,s)}_{14.4\,m} + \underbrace{\frac{1}{2}(-9.8\,m/s^2)(1.2\,s)^2}_{-7.056\,m}$$

$$= 7.3440\,m$$

alternatively:

$$v^2 = v_0^2 + 2a(y - y_0)$$

$$y = \frac{v^2 - v_0^2}{2a} = \frac{0 - (12\,m/s)^2}{2(-9.8\,m/s^2)} = 7.3440\,m$$

c) How long does it take to reach a point 5m above the ground?

$$y = v_0 t - \frac{1}{2} g t^2$$

$$5m = (12\,m/s)t - \frac{1}{2}(9.8\,m/s^2)t^2$$

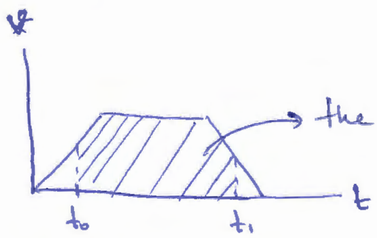
$$4.9t^2 - 12t + 5 = 0$$

$$t_{1,2} = \frac{12 \pm \sqrt{12^2 - 4 \cdot 4.9 \cdot 5}}{2 \cdot 4.9}\,s \quad \begin{cases} t_1 = 0.53\,s \\ t_2 = 1.92\,s \end{cases}$$

GRAPHICAL INTEGRATION IN MOTION ANALYSIS

$$v_1 - v_0 = \int_{t_0}^{t_1} a\,dt = \left(\text{area between acceleration curve and time axis, from } t_0 \text{ to } t_1 \right.$$

$$x_1 - x_0 = \int_{t_0}^{t_1} v\,dt = \left(\text{area between the velocity curve and the time axis, from } t_0 \text{ to } t_1 \right.$$



the area yields the change in the position.