Name:

BBM 205, Spring 2015 11.3.2015

Midterm I

SOLUTIONS

(2 pts)
1. The notation $\exists! x\, P(x)$ denotes

"There exists a unique x such that P(x) is true."

What are the truth values of these statements?

a) $\exists! x\, P(x) \rightarrow \exists x\, P(x)$ $\boxed{\text{True}}$

b) $\forall x\, P(x) \rightarrow \exists! x\, P(x)$ $\boxed{\text{False}}$

(2 pts)
2. Prove that if n is a positive integer, then n is even if and only if $7n+4$ is even.

n, even $\rightarrow$ $7n+4$, even:

Since $n=2k$ for some k, integer

$7n+4 = 7 \cdot 2k + 4 = 2(7k+2)$ also even.

$7n+4$ even $\rightarrow$ n, even

$7n+4 = 2m$ where m, integer

$7n = 2m-4 = 2(m-2)$, even

Since $2 \nmid 7$, $2 \mid n$. n is even.

(2 pts)

3. Let p, q and r be the propositions

p: You get an A on the final exam

q: You do every exercise in this book

r: You get an A in this class

Write these propositions using p, q and r and logical connectives.

a) You get an A in this class, but you do not do every exercise in this book.

b) You get an A on the final exam, you do every exercise in this book, and you get an A in this class.

c) To get an A in this class, it is necessary for you to get an A on the final exam.

d) Getting an A on the final exam and doing every exercise in this book is sufficient for getting an A in this class.

a) $r \wedge \bar{q}$

c) $r \rightarrow p$

b) $p \wedge q \wedge r$

d) $(p \wedge q) \rightarrow r$

(2 pts)

4. Show that this conditional statement is a tautology by using truth table: $(p \wedge q) \rightarrow (p \rightarrow q)$

p	q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
0	0	0	1	1
1	0	0	0	1
0	1	0	1	1
1	1	1	1	1

(1 pt)

5. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

If $x = \frac{a}{b}$ for two integers a, b with $\gcd(a, b) = 1$,

then $\frac{1}{x} = \frac{b}{a}$ also rational.

(3 pts)

6. Determine the truth value of the statement $\forall x \exists y (xy = 1)$ if the domain for the variables consists of

a) the nonzero real numbers

b) the nonzero integers

c) the positive real numbers.

a) $\forall x$, $y = \frac{1}{x}$ so that $xy = 1$. True

b) If $x = 2$, $y = \frac{1}{2}$ so that $xy = 1$. But $y \notin \mathbb{Z}^+$

False

c) $\forall x$, $y = \frac{1}{x}$ so that $xy = 1$. True

(3pts)

7. State the converse, contrapositive, and inverse of each of these conditional statements.

a) When $\underbrace{\text{I stay up late}}_{p}$, it is necessary that $\underbrace{\text{I sleep until noon}}_{q}$. $\} \; p \to q$

b) A $\overbrace{\text{positive integer is a prime}}^{p}$ only if it has $\underbrace{\text{no divisors other than 1 and itself}}_{q}$. $\} \; p \to q$

$\boxed{\text{converse}}$ $(q \to p)$
a) If I sleep... then I stay up late.
b) If no divisors other than 1... then prime.

$\boxed{\text{contrapositive}}$ $(\bar{q} \to \bar{p})$
a) If I do not sleep... then I do not stay up late.
b) If there is any divisor other than 1 and itself, then a positive integer is not a prime.

$\boxed{\text{Inverse}}$ $(\bar{p} \to \bar{q})$
a) If I do not stay up late, then I do not sleep....
b) If a positive integer is not a prime, then it has some divisor other than 1 and itself.

(3pts)

8. Let $Q(x)$ be the statement "$x+1 > 2x$". If the domain consists of all integers, what are these truth values?

a) $Q(-1)$ b) $Q(1)$ c) $\forall x \, \neg Q(x)$

d) $\exists x \, Q(x)$ e) $\forall x \, Q(x)$ f) $\exists x \, \neg Q(x)$

a) $-1+1 = 0 > -2$ True

b) $1+1 \not> 2 \cdot 1$ False

c) As in (a), $\exists x \, Q(x)$ where $x = -1$, So, false

d) $Q(-1)$ is true as in (a). True

e) $Q(1)$ is false as in (b). False

f) $Q(1)$ is false as in (b). True

(2 pts)

9. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.

a) $\forall x\ (x^2 \neq x)$ b) $\forall x\ (|x| > 0)$ c) $\forall x\ (x^2 \neq 2)$

a) False.

If $x = 0$, $x^2 = x$

b) False.

If $x = 0$,

$|x| \not> 0$

c) False.

If $x = \sqrt{2}$,

$x^2 = 2$.