**BBM 205 - Discrete Structures: Quiz 4 - Solutions**
**Date: 7.11.2018**

**Name:**
**Student ID:**

1. (10 points) **Use the extended Euclid algorithm** to find integers $x$ and $y$ that satisfy

$$x \cdot 30 + y \cdot 22 = gcd(30, 22).$$

> **Solution:**
> $30 = 22 + 8$ $\quad rem(30, 22) = 8 = 30 - 22.$
> $22 = 2 \cdot 8 + 6$ $\quad rem(22, 8) = 6 = 22 - 2 \cdot (30 - 22) = -2 \cdot 30 + 3 \cdot 22$
> $8 = 6 + 2$ $\quad rem(8, 6) = 2 = 8 - 6 = (30 - 22) - (-2 \cdot 30 + 3 \cdot 22) = 3 \cdot 30 - 4 \cdot 22.$
> $6 = 3 \cdot 2$ $\quad rem(6, 2) = 0.$
> By this algorithm, $x = 3$ and $y = -4$ given in the question.

2. (10 points) Prove that $gcd(a^5, b^5) = (gcd(a, b))^5$ for every $a, b \in \mathbb{Z}$.

**Solution:**
The two claims below show that the statement is true.
**Claim 1:** $(gcd(a, b))^5 \leq gcd(a^5, b^5)$
*Proof of Claim 1:*
Let $k = gcd(a, b)$ such that $a = kx$ and $b = ky$. Since $a^5 = k^5x^5$ and $b^5 = k^5y^5$, we see that $k^5$ is a common divisor of both $a^5$ and $b^5$. Therefore, $k^5|gcd(a^5, b^5)$, so the claim is true.

**Claim 2:** $(gcd(a, b))^5 \geq gcd(a^5, b^5)$
*Proof of Claim 2: (by contradiction)*
Again, let $k = gcd(a, b)$ such that $a = kx$ and $b = ky$. By the observation above, $k^5|gcd(a^5, b^5)$. Assume that the negation of the claim is true, that is $(gcd(a, b))^5 = k^5 < gcd(a^5, b^5) = gcd(k^5x^5, k^5y^5)$. Since $k^5$ is a common divisor of $a^5$ and $b^5$, $gcd(a^5, b^5) = k^5 \cdot z$ for some integer $z > 1$. Let $p$ be a prime divisor of $z$. Since $p$ is prime and divides $z$, we have $p|x$ and $p|y$. However, this means $k \cdot p$ is a common divisor of $a$ and $b$ but greater that $gcd(a, b) = k$, a contradiction. The claim is true.