MOTION IN TWO AND THREE DIRECTIONS

Motion in three dimensions is not easy to understand, for example, you can probably drive a car easily along a freeway (? dimensional motion? – "1"), but would probably have a difficult time in landing a plane on a runway (3D motion).

POSITION AND DISPLACEMENT

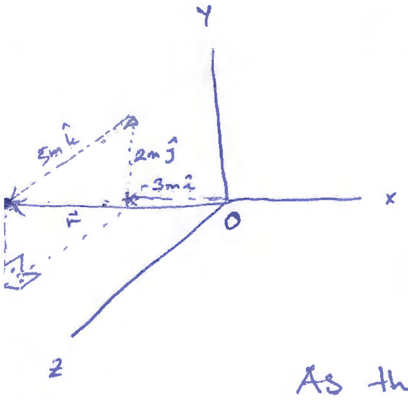
$$\vec{r} = \underbrace{x}\hat{i} + \underbrace{y}\hat{j} + \underbrace{z}\hat{k}$$

Vector components: $x\hat{i}$, $y\hat{j}$, $z\hat{k}$

Scalar components: x, y, z

Rectangular coordinates.

$$r = (-3m)\hat{i} + (2m)\hat{j} + (5m)\hat{k} \longleftrightarrow (-3m, 2m, 5m)$$



- Along the x-axis, the particle is 3m from the origin, in the $-\hat{i}$ direction
- " " y-axis, " " " 2m " " ", in the $+\hat{j}$ direction
- " " z-axis, " " " 5m " " ", in the $+\hat{k}$ direction.

As the particle moves, its position vector changes in such a way that the vector always extends to the particle from the reference point (the origin).

If the position vector changes from $\vec{r}_1$ to $\vec{r}_2$ during a certain time interval then, the particle's <u>displacement</u> $\Delta\vec{r}$ during that time interval is: $\boxed{\Delta\vec{r} = \vec{r}_2 - \vec{r}_1}$

$$\Delta\vec{r} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$= \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}$$

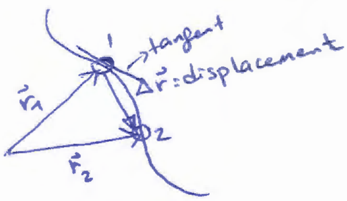
Average Velocity and Instantenous Velocity

Now we have <u>vectors</u>:

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k}$$

(instantenous) Velocity $\vec{v} = \frac{d\vec{r}}{dt}$

to find the instantenous velocity at t_1, we shrink the interval Δt to 0 about (around) t_1.



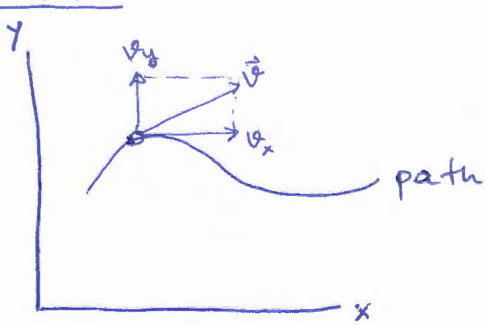
THREE things happen:

1) Position vector $\vec{r}_2$ moves toward $\vec{r}_1$ so $\Delta\vec{r} \to 0$
2) Direction of $\frac{\Delta\vec{r}}{\Delta t}$ (thus $\vec{v}_{avg}$) approaches the direction of the line tangent at position 1
3) $\vec{v}_{avg}$ approaches to the instantenous velocity $\vec{v}$ at t_1.

The direction of the instantenous velocity $\vec{v}$ of a particle is always tangent to the particle's path at the particle's position.

$$\vec{v} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$
$$= v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

REMARK:



Velocity vector is not from "Here" to "there". It shows the direction and its length can be drawn to any scale.

(WHY? → Check the units!)

AVERAGE ACCELERATION AND INSTANTENOUS ACCELERATION

$$a_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}$$

$\Delta t \to 0 \Rightarrow \vec{a} = \frac{d\vec{v}}{dt} \rightarrow$ if the velocity changes either in magnitude or direction (or both) the particle must have an acceleration.

$$\vec{a} = \frac{d}{dt}\left(v_x \hat{i} + v_y \hat{j} + v_z \hat{k}\right)$$

$$= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$= a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

Ex:

i) $x = -3t^2 + 4t - 2$; $y = 6t^2 - 4t$

ii) $x = -3t^3 - 4t$; $y = -5t^2 + 6$

iii) $\vec{r} = 2t^2 \hat{i} - (4t+3)\hat{j}$

iv) $\vec{r} = (4t^3 - 2t)\hat{i} + 3\hat{j}$

Are the x and y acceleration components constant?

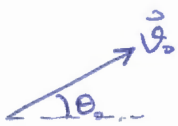
Is acceleration a constant?

PROJECTILE MOTION

A special case of 2-D motion: A particle moves in a vertical plane with some initial velocity $\vec{v}_0$ but its acceleration is always the free-fall acceleration $\vec{g}$ which is downward. Such a particle is called a <u>projectile</u> (projected/launched) and its motion is called projectile motion.

The projectile is launched with initial velocity $\vec{v}_0$

$$\vec{v}_0 = v_{0x}\,\hat{i} + v_{0y}\,\hat{j}$$



$$\rightarrow \quad v_{0x} = v_0 \cos\theta_0 \;, \quad v_{0y} = v_0 \sin\theta_0$$

During its motion, position vector $\vec{r}$ and velocity vector $\vec{v}$ change continously

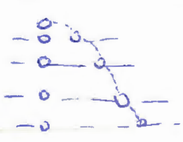
<u>But</u> its acceleration vector $\vec{a}$ is constant and always directed <u>vertically downward.</u>

<u>Also</u>, the projectile has <u>no</u> horizontal acceleration

The situation may look difficult at the first glance but,

!! In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

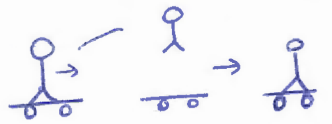
This allows us to break up a problem involving two-dimensional motion into two seperate and easier 1-D problems — one for the vertical motion (with a constant downward acceleration) and one for the horizontal motion (with zero acceleration).



PROJECTILE MOTION ANALYZED

* The HORIZONTAL motion

Because there is no acceleration in the horizontal direction, horizontal component v_x remains unchanged from its initial value v_{0x} throughout the motion.



skate border

$$x - x_0 = v_{0x} t$$
$$x - x_0 = (v_0 \cos\theta_0) t$$

* The Vertical Motion

$a \to -g$

$$y - y_0 = v_{0y} t - \frac{1}{2} g t^2$$
$$= (v_0 \sin\theta_0) t - \frac{1}{2} g t^2 \quad (i)$$

$$v_y = v_0 \sin\theta_0 - gt \quad (ii)$$

$$(i)+(ii) \to \boxed{v_y^2 = (v_0 \sin\theta_0)^2 - 2g(y - y_0)}$$

(ii): $v_y = v_0 \sin\theta - gt$

$\to t = -\dfrac{v_y - v_0 \sin\theta}{g}$

$\to (i)$: $y - y_0 = (v_0 \sin\theta)\left(\dfrac{v_0 \sin\theta - v_y}{g}\right) - \dfrac{1}{2} g \left(\dfrac{v_0 \sin\theta - v_y}{g}\right)^2$

$$2g(y - y_0) = 2 v_0^2 \sin^2\theta - 2 v_0 \sin\theta\, v_y - v_0^2 \sin^2\theta - v_y^2 + 2 v_0 \sin\theta\, v_y$$
$$= v_0^2 \sin^2\theta - v_y^2$$
$$\Rightarrow v_y^2 = (v_0 \sin\theta)^2 - 2g(y - y_0)$$

The vertical velocity is initially directed upward and its magnitude steadily decreases to zero, which marks the maximum height of the projectile.

THE Equation of the Path (Trajectory)

$$x - x_0 = (v_0 \cos\theta_0) t$$
$$y - y_0 = (v_0 \sin\theta_0) t - \frac{1}{2} g t^2$$

$$x_0 = y_0 = 0 \to x = (v_0 \cos\theta_0) t \to \boxed{t = \frac{x}{v_0 \cos\theta_0}}$$

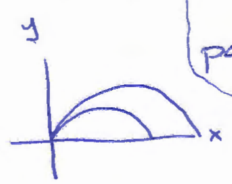
$$y = (v_0 \sin\theta_0)\left(\frac{x}{v_0 \cos\theta_0}\right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta_0}\right)^2$$

$$y = \underbrace{\tan\theta_0}_{\text{Const.}} \cdot x - \underbrace{\frac{g}{2(v_0 \cos\theta_0)^2}}_{\text{Const}} x^2 \quad \text{(trajectory)}$$

$a \equiv \tan\theta_0$

$b \equiv -\dfrac{g}{2(v_0 \cos\theta)^2}$

$$\Rightarrow y = a x + b x^2 \longrightarrow$$



parabola (directed downwards because of the negative sign)

THE HORIZONTAL RANGE

$X - X_0 = R \,, \qquad Y - Y_0 = 0$

$$\left.\begin{aligned} R &= (v_0 \cos\theta_0)t \\ 0 &= (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2 \end{aligned}\right\} \longrightarrow R = \frac{2v_0^2}{g}\sin\theta_0 \cos\theta_0$$

$$(\sin 2\theta_0 = 2\sin\theta\cos\theta) \longrightarrow R = \frac{v_0^2}{g}\sin 2\theta_0$$

$$R_{max} : \sin 2\theta_0 = 1 \rightarrow \theta_0 = 45^\circ$$

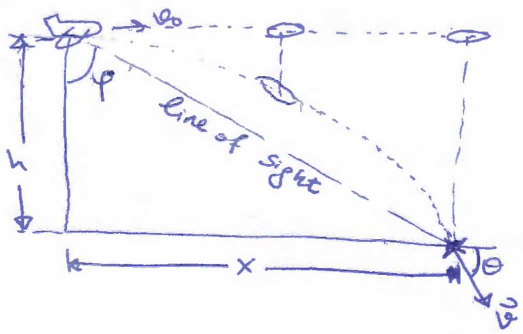
THE EFFECTS OF AIR

We have assumed that the air through which the projectile moves has no effect on its motion. However, the air resists (opposes) the motion.

Example: Projectile dropped from an airplane

$v_{plane} = 198\ km/h \ (55 m/s)$

$h = 500m$



a.) What should the angle φ of the pilot's Line of sight to the target be when the Release is made?

$$\varphi = \tan^{-1}\frac{x}{h}$$

$$X - X_0 = (v_0 \cos\theta_0)t \qquad (1)$$

but we don't know t

$$Y - Y_0 = v_0 \sin\theta_0 t - \tfrac{1}{2}gt^2 \quad (2)$$

$$-500m = (55 m/s)(\sin 0^\circ)t - \tfrac{1}{2}(9.8 m/s^2)t^2$$

the capsule moves downward

$$\Rightarrow t = 10.1\,s$$

$$(1) \rightarrow \quad x - 0 = (55 m/s)(\cos 0^\circ)(10.1 s)$$

$$\Rightarrow x = 555.5 m$$

$$\Rightarrow \varphi = \tan^{-1}\frac{555 m}{500 m} = \underline{\underline{48^\circ}}$$

b) when the capsule Reaches, what is the velocity $\vec{v}$ in vector & magnitude-angle notations?

$$v_x = v_0 \cos\theta_0 = 55\ m/s$$

$$v_y = v_0 \sin\theta_0 - gt = (55\,m/s)(\sin 0°) - (9.8\,m/s^2)(10.1s)$$

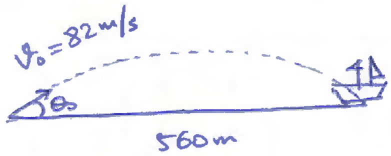
$$= -99\,m/s$$

$$\Rightarrow \vec{v} = (55\,m/s)\,\hat{i} - (99\,m/s)\,\hat{j}$$

$$v = \sqrt{55^2 + (-99)^2}\ m/s = 113.25\,m/s$$

$$\theta = \tan^{-1}\frac{-99}{55} = -60.945°$$

Example.



To hit the ship, $\theta_0 = ?$

(HORIZONTAL RANGE)

$$R = v_0 \cos\theta\, t \longrightarrow t = \frac{R}{v_0 \cos\theta}$$

$$y = 0 = (v_0 \sin\theta)\,t - \frac{1}{2}gt^2$$

$$v_0 \sin\theta \frac{R}{\cancel{v_0}\cos\theta} = \frac{1}{2} g \frac{R^2}{v_0^2 \cos^2\theta}$$

$$R = \frac{2v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}$$

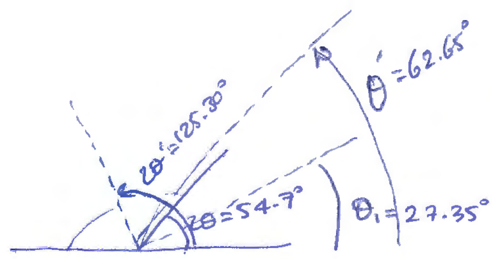
$$560\,m = \frac{(82\,m/s)^2 \cdot \sin 2\theta}{9.8\,m/s^2} \rightarrow \sin 2\theta = 0.81618$$

• $2\theta = 54.704°$

$\theta = 27.352° \sim 27°$

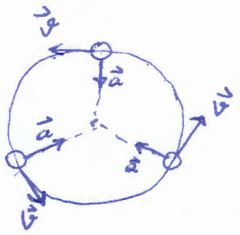
• $2\theta' = 125.30°$

$\theta' = 62.650 \sim 63°$ (also a solution)



Uniform Circular Motion

A particle is in uniform circular motion if it travels around a circle or a circular arc at a constant (uniform) speed. Although the speed does not vary, the particle is accelerating because the velocity changes direction.



$\vec{v}$ vector is always tangent to the path

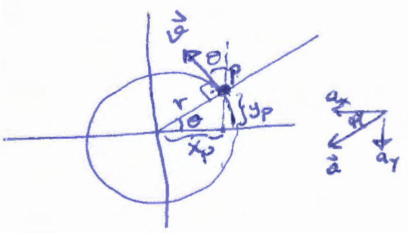
$\vec{a}$ vector always points to the center.

$$\boxed{a = \frac{v^2}{r}}$$

(Centripetal acceleration)
└ "center seeking"

$$2\pi r = vT \Rightarrow T = \frac{2\pi r}{v} \quad \left(\begin{array}{l}\text{period}\\ \text{of Revolution}\end{array}\right)$$

Proof of $\left(a = \frac{v^2}{r}\right)$



$$\vec{v} = v_x\hat{i} + v_y\hat{j} = (-v\sin\theta)\hat{i} + (v\cos\theta)\hat{j}$$

$$\sin\theta = \frac{y_p}{r}, \quad \cos\theta = \frac{x_p}{r}$$

$$\rightarrow \quad \vec{v} = \left(-\frac{v\,y_p}{r}\right)\hat{i} + \left(\frac{v\,x_p}{r}\right)\hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-\frac{v}{r}\underbrace{\frac{dy_p}{dt}}_{v_y}\right)\hat{i} + \left(\frac{v}{r}\underbrace{\frac{dx_p}{dt}}_{v_x}\right)\hat{j} \quad ; \; v, r : \text{constant}$$

$$v_x = -v\sin\theta$$

$$v_y = v\cos\theta$$

$$\Rightarrow \vec{a} = \underbrace{\left(-\frac{v^2}{r}\cos\theta\right)}_{a_x}\hat{i} + \underbrace{\left(-\frac{v^2}{r}\sin\theta\right)}_{a_y}\hat{j}$$

$$a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\underbrace{\sqrt{(-\cos\theta)^2 + (-\sin\theta)^2}}_{1} = \frac{v^2}{r}$$

$$\tan\varphi = \frac{a_y}{a_x} = \frac{-\frac{v^2}{r}\sin\theta}{-\frac{v^2}{r}\cos\theta} = \tan\theta \Rightarrow \varphi = \theta \Leftrightarrow$$ $\vec{a}$ is directed towards center

Example: Top Gun pilot in turns

$$\begin{cases} 2g, 3g : \text{feels heavy (Amusement Parks)} \\ 4g : \text{Vision switches to black \& white and narrows to "tunnel vision"} \\ \text{g-loc} : g \text{ induced loss of consciousness} \end{cases}$$

What is the magnitude of acceleration in g units of a pilot whose aircraft enters a horizontal circular turn with a velocity of

$\vec{v}_i = (400\hat{i} + 500\hat{j})\,m/s$ and 24s later

leaves the turn with a velocity of

$\vec{v}_f = (-400\hat{i} - 500\hat{j})\,m/s$?

$a = v^2/R$

$T = 2\pi R/v$

We don't have R, so let's substitute:

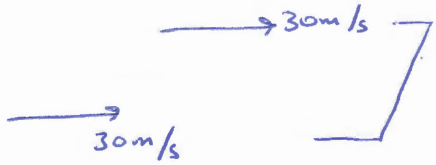
$$\frac{vT}{2\pi} = R \rightarrow a = v^2/R = \frac{v^2}{vT/2\pi} = \frac{2\pi v}{T}$$

$$|\vec{v}| = v = \text{Const.} : \sqrt{(400\,m/s)^2 + (500\,m/s)^2} = 640.31\,m/s$$

$180°$ in $24s \longrightarrow T = 48s$

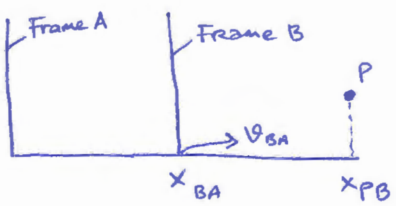
$$a = \frac{2\pi\,(640.31\,m/s)}{48s} = 83.81\,m/s^2 \simeq \underline{\underline{8.6g}}$$

Relative Motion in 1-D



stationary with respect to each other

Velocity Depends on Reference Frame



The coordinate x_{PA} of P as measured by A is equal to the coordinate x_{PB} of P as measured by B plus the coordinate x_{BA} of B as measured by A.

$$x_{PA} = x_{PB} + x_{BA}$$

$$X_{PA} = X_{PB} + X_{BA}$$

$$\underbrace{\frac{d}{dt}(x_{PA})}_{v_{PA}} = \underbrace{\frac{d}{dt}(x_{PB})}_{v_{PB}} + \underbrace{\frac{d}{dt}(x_{BA})}_{v_{BA}}$$

$$v_{PA} = v_{PB} + \underbrace{v_{BA}}_{\text{velocity of frame B relative to frame A}}$$

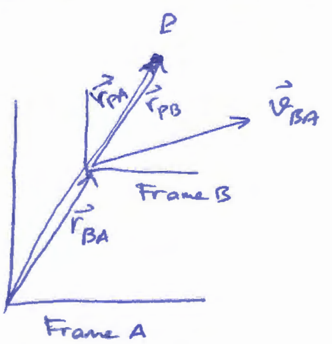
We consider only frames that move at constant velocity with respect to each other.

$$\frac{d}{dt}(v_{PA}) = \frac{d}{dt}(v_{PB}) + \frac{d}{dt}(\underbrace{v_{BA}}_{\text{const.}})$$

$$\Rightarrow \boxed{a_{PA} = a_{PB}}$$

observers in different frames of reference that move at constant velocity relative to each other will measure the same acceleration for a moving particle.

Relative Motion in 2D

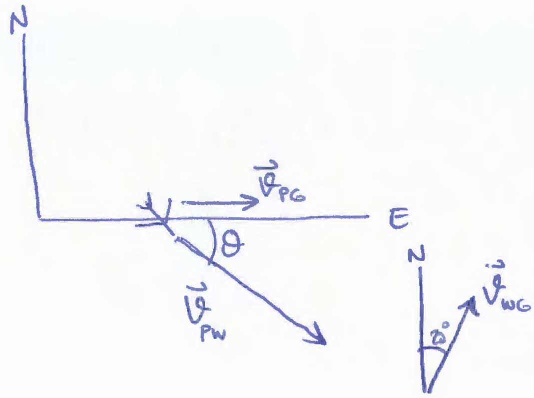


$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$$

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}$$

$$\vec{a}_{PA} = \vec{a}_{PB}$$

Example



A plane moves due east while the pilot points the plane somewhat south of east toward a steady wind that blows toward NE.

Plane has velocity $\vec{v}_{PW}$ relative to the wind, with an air speed (speed relative to the wind) of 215 km/h directed at an angle θ south of east.

The wind has $\vec{v}_{WG}$ relative to the ground with a speed of 65 km/h directed 20° East of North.

What is the magnitude of the velocity $\vec{v}_{PG}$ of the plane relative to the ground? What is θ?

$$\vec{v}_{PG} = \vec{v}_{PW} + \vec{v}_{WG}$$

$$v_{PG,y} = v_{PW,y} + v_{WG,y}$$

$$0 = (-215\,km/h)\sin\theta + (65\,km/h)(\cos 20^\circ) \Rightarrow \theta = 16.5^\circ$$

$$v_{PG,x} = v_{PW,x} + v_{WG,x}$$

$$v_{PG,x} = v_{PG}$$

$$v_{PG} = (215\,km/h)(\cos 16.5^\circ) + (65\,km/h)(\sin 20^\circ)$$

$$\Rightarrow v_{PG} = 228\,km/h$$