BBM 205 - Discrete Structures Midterm Date: 19.11.2015, Time: 10:00 - 11:45

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Sube /Section:

Question	1	2	3	4	5	6	Total
Points	15	15	10	20	20	20	100
Grade							

1. (15 points) There are 13 squares of side 1 positioned inside a circle of radius 2. Prove that at least 2 of the squares have a common point. (Let $\pi = 3.14$. The area inside a circle with radius r is πr^2 .)

Solution: If no 2 of the squares have a common point, then the total area they cover is equal to the sum of their individual areas, which is 13. On the other hand, a circle of radius 2 has area

$$\pi r^2 = \pi 2^2 = 4\pi \approx 12.56 < 13,$$

so it's impossible for the squares to cover an area of 13. Thus in fact some 2 of them must share a common point.

2. (15 points) Let f(n) = n + 1 and $g(n) = n^2$. Prove that $g(n) \neq O(f(n))$.

Solution: Suppose that

$$n^2 \le c(n+1)$$

eventually, that is, for all n greater than or equal to some number n_0 . Dividing by n, we have

$$n \le c(1 + \frac{1}{n}) \le 2c$$

for all $n \ge n_0$. This is impossible because 2c is a constant.

3. (10 points) Find the minimum number of ordered pairs of integers (a, b) that guarantees that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \equiv a_2 \pmod{5}$ and $b_1 \equiv b_2 \pmod{5}$. Explain your answer.

Solution: There are $k = 5 \cdot 5 = 25$ different (a, b) pairs. If we pick N pairs such that

$$\left\lceil \frac{N}{k} \right\rceil \ge 2,$$

then there will be at least two identical pairs. The smallest N satisfying this condition is 26.

4. (20 points) Find the solution of the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = 7$.

Solution: The characteristic polynomial is

$$r^2 - 8r + 16$$

Factoring gives us

$$r^2 - 8r + 16 = (r - 4)(r - 4),$$

so $r_0 = 4$.

Using the initial conditions, we get a system of equations

$$a_0 = 1 = \alpha_1$$

$$a_1 = 7 = 4\alpha_1 + 4\alpha_2.$$

Solving the second, we get $\alpha_2 = 3/4$. And the solution is

$$a_n = 4^n + \frac{3}{4}n4^n.$$

5. (20 points) Let x be any real number greater than -1. Prove that

$$(1+x)^n \ge 1 + nx$$

for every integer $n \geq 0$.

Solution: Let x be any real number greater than -1. Let $P(n): (1+x)^n \ge 1 + nx$. **Basis step:** To verify that P(0) is true: Notice that

$$(1+x)^0 = 1 \ge 1 + 0 \cdot x,$$

so P(0) is true.

Induction step: Assume that P(k) is true. We show that P(k+1) is true. By the inductive hypothesis, we have $(1+x)^k \ge 1 + kx$. Then,

$$(1+x)^{k+1} = (1+x)(1+x)^k \ge (1+x)(1+kx) = 1 + (k+1)x + kx^2 \ge 1 + (k+1)x.$$

Therefore, P(k+1) is also true. Thus by Mathematical Induction $(1+x)^n \ge 1 + nx$ for every integer $n \ge 0$.

- 6. (20 points) A bagel shop (simit dükkanı) has onion (Soğanlı) bagels, poppy seed (Haşhaşlı) bagels, egg (Yumurtalı) bagels, salty (Tuzlu) bagels, pumpernickel (Çavdarlı) bagels, sesame seed (Susamlı) bagels, raisin (Üzümlü) bagels, and plain (Sade) bagels. How many ways are there to choose
 - (a) (4 points) six bagels?

Solution:
$$\binom{6+7}{7} = \binom{13}{7}$$

(b) (4 points) a dozen bagels? (a dozen = 12)

Solution:
$$\binom{12+7}{7} = \binom{19}{7}$$

(c) (4 points) two dozen bagels?

Solution:
$$\binom{24+7}{7} = \binom{31}{7}$$

(d) (4 points) a dozen bagels with at least one of each kind?

Solution:
$$\binom{4+7}{7} = \binom{11}{7}$$

(e) (4 points) a dozen bagels with at least three egg bagels and no more than two salty bagels?

Solution:
$$\binom{6+9}{6} + \binom{6+8}{6} + \binom{6+7}{6} = \binom{15}{6} + \binom{14}{6} + \binom{13}{6}$$