Chapter 8
First Order Logic

BBM 405– Artificial Intelligence

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Slides are mostly adapted from AIMA and MIT Open Courseware

Pros and cons of propositional logic

- ☺ Propositional logic is declarative
- ☺ Propositional logic allows partial/disjunctive/negated information
  - (unlike most data structures and databases)
- ☺ Propositional logic is compositional:
  - meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- ☺ Meaning in propositional logic is context-independent
  - (unlike natural language, where meaning depends on context)
- ☹ Propositional logic has very limited expressive power
  - (unlike natural language)
  - E.g., cannot say "pits cause breezes in adjacent squares“
    - except by writing one sentence for each square

First Order Logic

- Propositional logic only deals with "facts", statements that may or may not be true of the world, e.g., "It is raining". But, one cannot have variables that stand for books or tables.
- In first-order logic, variables refer to things in the world and, furthermore, you can quantify over them: talk about all of them or some of them without having to name them explicitly.

FOL Motivation

Statements that cannot be made in propositional logic but can be made in FOL

- When you paint a block with green paint, it becomes green.
  - In propositional logic, one would need a statement about every single block, one cannot make the general statement about all blocks.
- When you sterilize a jar, all the bacteria are dead.
  - In FOL, we can talk about all the bacteria without naming them explicitly.
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First-order logic

- Whereas propositional logic assumes the world contains facts,
- first-order logic (like natural language) assumes the world contains
  - Objects: people, houses, numbers, colors, baseball games, wars, …
  - Relations: red, round, prime, brother of, bigger than, part of, comes between, …
  - Functions: father of, best friend, one more than, plus,
    - (relations in which there is only one value for a given input)

Syntax of FOL: Basic elements

- Constants : KingJohn, 2, ...
- Predicates : Brother, >,...
- Functions : Sqrt, LeftLegOf,...
- Variables x, y, a, b,...
- Connectives $\neg, \Rightarrow, \wedge, \vee, \Leftrightarrow$
- Equality $=$
- Quantifiers $\forall, \exists$

FOL Syntax

- Term
  - Constant symbols: Fred, Japan, Bacterium39
  - Variables: x, y, a
  - Function symbol applied to one or more terms: f(x), f(f(x)), mother-of(John)
- Sentence
  - A predicate symbol applied to zero or more terms: On(a,b), Sister(Jane, Joan), Sister(mother-of(John), Jane)
  - $t_1=t_2$
  - If v is a variable and Φ is a sentence, then $\forall v.\Phi$ and $\exists v.\Phi$ are sentences.
  - Closure under sentential operators: $\wedge$ $\vee$ $\neg$ $\rightarrow$ $\leftrightarrow$ ( )

Atomic sentences

Atomic sentence = *predicate* ($term_1$,...,$term_n$)
or $term_1 = term_2$

Term = *function* ($term_1$,...,$term_n$)
or *constant*
or *variable*

- E.g., *Brother(KingJohn,RichardTheLionheart)*
- *> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))*

Complex sentences

- Complex sentences are made from atomic sentences using connectives

$$\neg S,\ S_1 \wedge S_2,\ S_1 \vee S_2,\ S_1 \Rightarrow S_2,\ S_1 \Leftrightarrow S_2,$$

E.g. *Sibling(KingJohn,Richard)* $\Rightarrow$ *Sibling(Richard,KingJohn)*

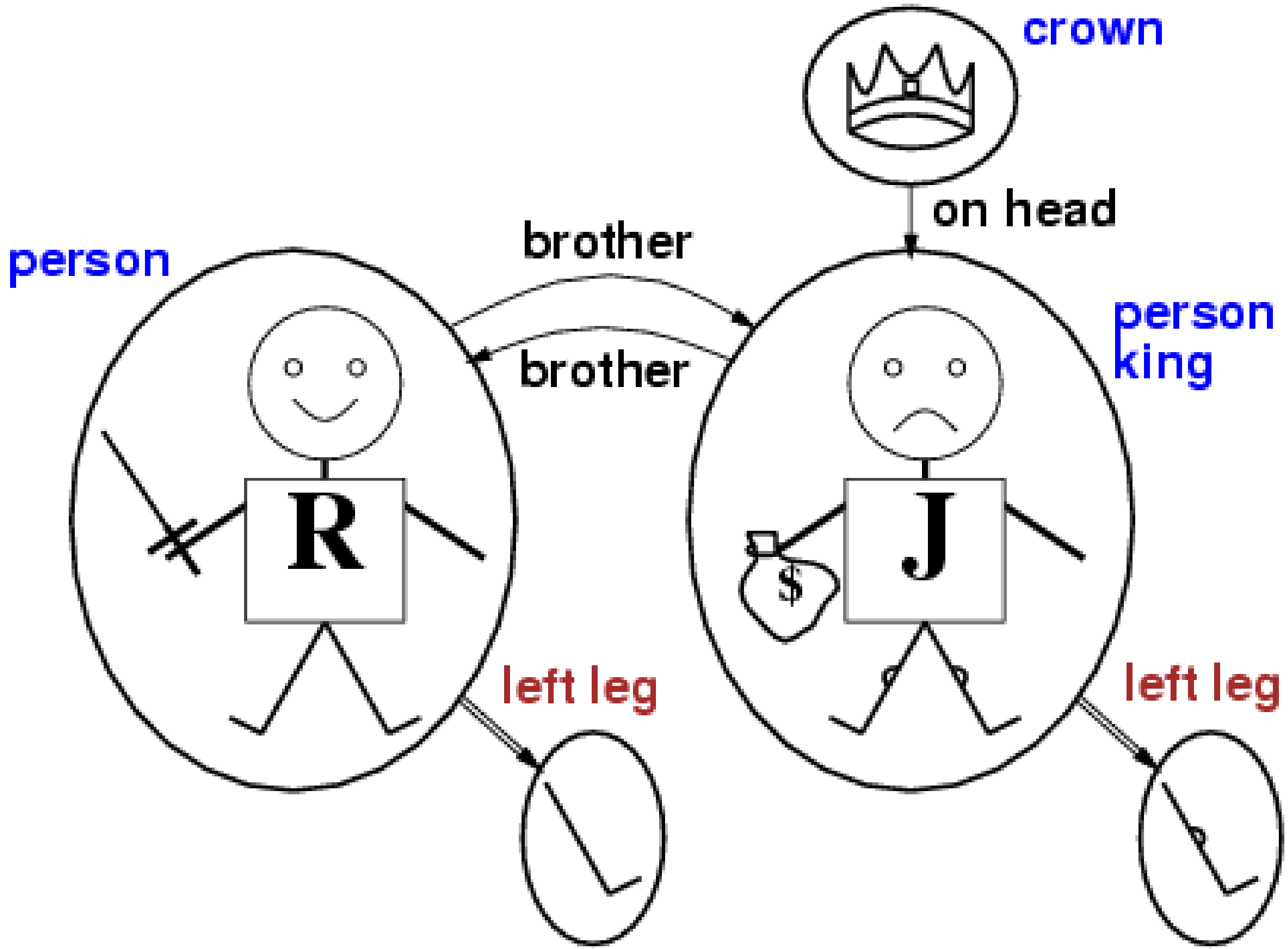
$>(1,2) \vee \leq (1,2)$

$>(1,2) \wedge \neg >(1,2)$

Truth in first-order logic

- Sentences are true with respect to a model and an interpretation
- Model contains objects (domain elements) and relations among them
- Interpretation specifies referents for
  constant symbols $\rightarrow$ objects
  predicate symbols $\rightarrow$ relations
  function symbols $\rightarrow$ functional relations
- An atomic sentence $predicate(term_1,\ldots,term_n)$ is true
  iff the objects referred to by $term_1,\ldots,term_n$
  are in the relation referred to by *predicate*

Models for FOL: Example



FOL Interpretations

- Interpretation I
  - U set of objects
    (called "domain of discourse" or "universe")
  - Maps constant symbols to elements of U
  - Maps predicate symbols to relations on U
    (binary relation is a set of pairs)
  - Maps function symbols to functions on U
    (function is a binary relation with a single pair for each element in U, whose first item is that element)

Holds

When does a sentence hold in an interpretation?

- P is a relation symbol
- $t_1, \ldots, t_n$ are terms

$\text{holds}(P(t_1, \ldots, t_n), I)$ iff $\langle I(t_1), \ldots, I(t_n)\rangle \in I(P)$

Brother(Jon, Joe)??

- I(Jon) =  [an element of U]
- I(Joe) =  [an element of U]
- I(Brother) = {<, >, <, >, <…,…>, … }

Semantic of Quantifiers

Extend an interpretation I to bind variable x to element $a \in U$: $I_{x/a}$

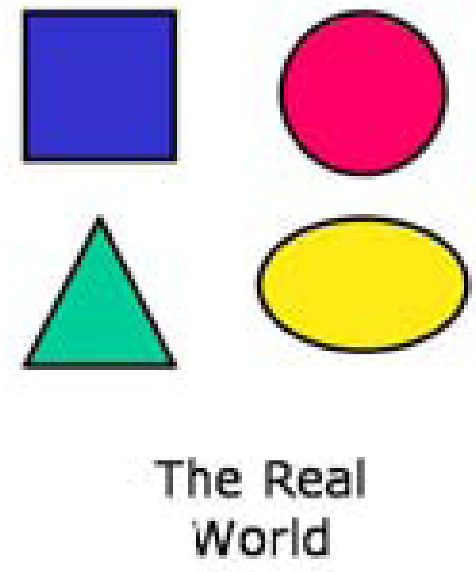
- holds($\forall x.\Phi$, I) iff holds(Φ, $I_{x/a}$) for all $a \in U$
- holds($\exists x.\Phi$, I) iff holds(Φ, $I_{x/a}$) for some $a \in U$

Quantifier applies to formula to right until an enclosing right parenthesis:

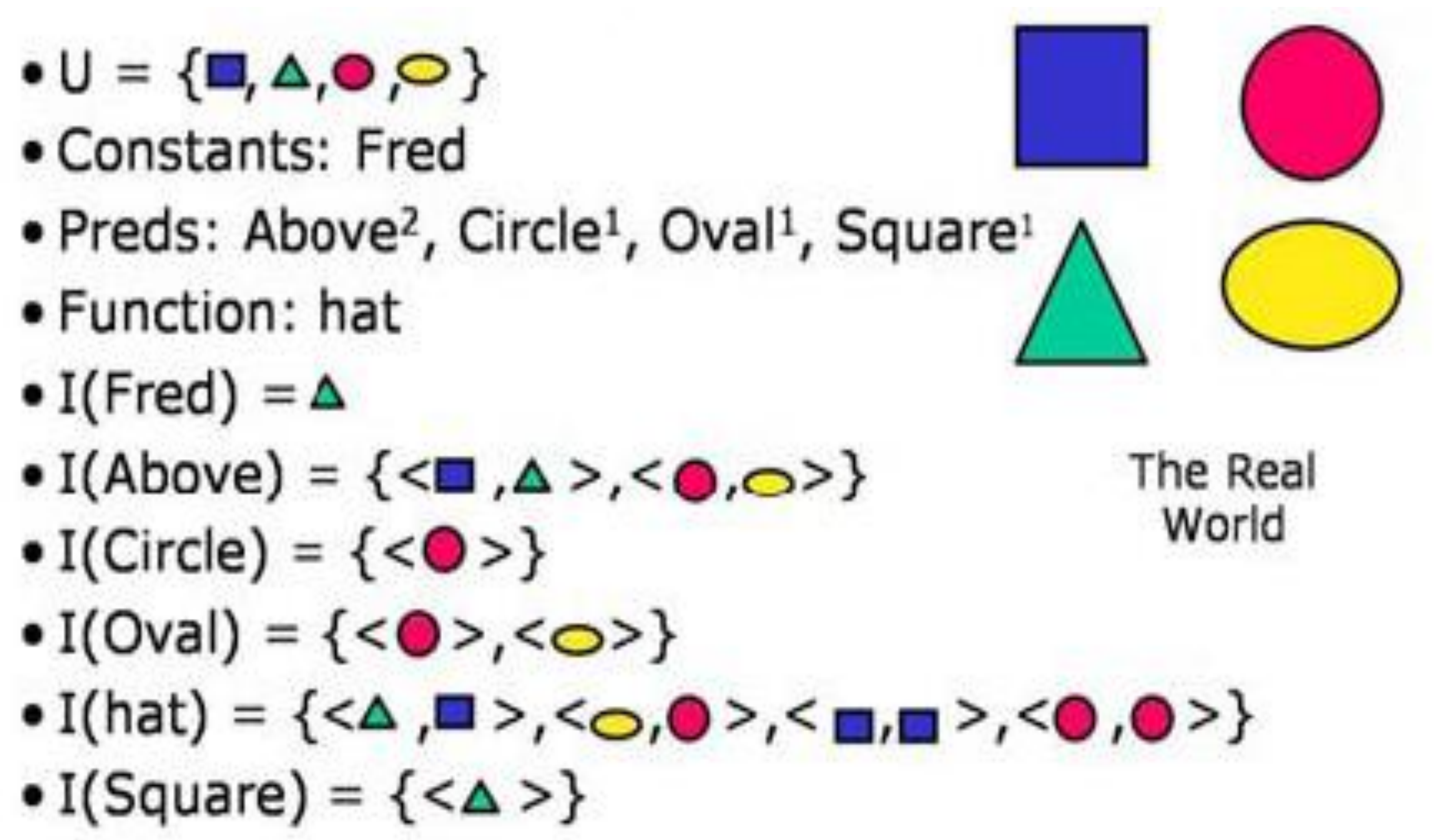
$$(\forall x.P(x) \vee Q(x)) \wedge \exists x.R(x) \rightarrow Q(x)$$

Example Domain

- U = 



Example Domain



Example Domain

- I(Fred) = △
- I(Above) = {<■,△>,<●,⬭>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬭>}
- I(hat) = {<△,■>,<⬭,●>
  <■,■>,<●,●>}
- I(Square) = {<△>}

- holds(Square(Fred), I) ?
  yes
- holds(Above(Fred, hat(Fred)), I) ? no
  - I(hat(Fred)) = ■
  - holds(Above(△, ■), I) ? no

- holds(∃x. Oval(x), I) ? yes
  - holds(Oval(x), Ix/●) ? yes

Example Domain

- I(Fred) = ▲
- I(Above) = {<■,▲>,<●,⬬>}
- I(Circle) = {<●>}
- I(Oval) = {<●>,<⬬>}
- I(hat) = {<▲,■>,<⬬,●>
  <■,■>,<●,●>}
- I(Square) = {<▲>}

- holds(∀x. ∃y. Above(x,y) v Above(y, x), I) ? yes
  - holds(∃y. Above(x,y) v Above(y,x), Ix/▲) ? yes
    holds(Above(x,y) v Above(y,x), Ix/▲,y/■) ? yes
  - verify for all other values of x

- holds(∀ x. ∀ y. Above(x,y) v Above(y,x), I) ? no
  - holds(Above(x,y) v Above(y,x), Ix/■,y/●) ? no

Writing FOL

- Cats are mammals [Cat^1, Mammal^1]
  - $\forall x.\ \text{Cat}(x) \rightarrow \text{Mammal}(x)$
- Jane is a tall surveyor [Tall^1, Surveyor^1, Jane]
  - $\text{Tall}(\text{Jane}) \wedge \text{Surveyor}(\text{Jane})$
- A nephew is a sibling's son [Nephew^2, Sibling^2, Son^2]
  - $\forall xy.\ [\text{Nephew}(x,y) \leftrightarrow \exists z\ .\ [\text{Sibling}(y,z) \wedge \text{Son}(x,z)]]$
- A maternal grandmother is a mother's mother [functions: mgm, mother-of]
  - $\forall xy.\ x=\text{mgm}(y) \leftrightarrow$
    $\exists z.\ x=\text{mother-of}(z) \wedge z=\text{mother-of}(y)$
- Everybody loves somebody [loves^2]
  - $\forall x.\ \exists y.\ \text{Loves}(x,y)$
  - $\exists y.\ \forall x.\ \text{Loves}(x,y)$ There is somebody who is loved by everybody

Writing FOL

- Nobody loves Jane
  - $\forall x. \neg Loves(x, Jane)$
  - $\neg\exists x. Loves(x, Jane)$
- Everybody has a father
  - $\forall x. \exists y. Father(y, x)$
- Everybody has a father and a mother
  - $\forall x. \exists yz. Father(y, x) \wedge Mother(z, x)$
- Whoever has a father, has a mother
  - $\forall x.[[\exists y. Father(y, x)] \rightarrow [\exists y. Mother(y, x)]]$

Universal quantification

$\forall$ *<variables> <sentence>*

All Kings are persons:

$\forall x\ King(x) \Rightarrow Person(x)$

$\forall x$ *P* is true in a model *m* iff *P* is true with *x* being each possible object in the model

- Roughly speaking, equivalent to the conjunction of instantiations of *P*

  Richard the Lionheart is a king $\Rightarrow$ Richard the Lionheart is a person
  $\wedge$ King John is a king $\Rightarrow$ King John is a person
  $\wedge$ Richard's left leg is a king $\Rightarrow$ Richard's left leg is a person
  $\wedge$ John's left leg is a king $\Rightarrow$ John's left leg is a person
  $\wedge$ The crown is a king $\Rightarrow$ The crown is a person

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
- Common mistake: using $\wedge$ as the main connective with $\forall$:

  $\forall x\ King(x) \wedge Person(x)$

  means "Everyone is a king and everyone is a person"

Existential quantification

∃<*variables*> <*sentence*>

- $\exists x$ Crown(x) $\wedge$ OnHead(x,John)

$\exists x\ P$ is true in a model m iff P is true with x being some possible object in the model

- Roughly speaking, equivalent to the disjunction of instantiations of P

  The crown is a crown $\wedge$ the crown is on John's head
  $\vee$ Richard the Lionheart is a crown $\wedge$ Richard the Lionheart is on John's head
  $\vee$ King John is a crown $\wedge$ King John is on John's head
  $\vee$ $\vee$ ...

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists x \text{ Crown}(x) \Rightarrow \text{OnHead}(x,\text{John})$$

is true even if there is anything which is not a crown

Properties of quantifiers

$\forall x\ \forall y$ is the same as $\forall y\ \forall x$

$\exists x\ \exists y$ is the same as $\exists y\ \exists x$

$\exists x\ \forall y$ is not the same as $\forall y\ \exists x$

$\exists x\ \forall y$ Loves(x,y)

- "There is a person who loves everyone in the world"

$\forall y\ \exists x$ Loves(x,y)

- "Everyone in the world is loved by at least one person"

- Quantifier duality: each can be expressed using the other

$\forall x$ Likes(x,IceCream) = $\neg\exists x\ \neg$Likes(x,IceCream)

$\exists x$ Likes(x,Broccoli) = $\neg\forall x\ \neg$Likes(x,Broccoli)

Equality

- $term_1$ = $term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object

- E.g., definition of *Sibling* in terms of *Parent*:

  $\forall x,y\ Sibling(x,y) \Leftrightarrow [\neg(x = y) \wedge \exists m,f\ \neg(m = f) \wedge$
  $Parent(m,x) \wedge Parent(f,x) \wedge Parent(m,y) \wedge Parent(f,y)]$

Using FOL

The kinship domain:

- Brothers are siblings

  $\forall x,y\ Brother(x,y) \Leftrightarrow Sibling(x,y)$

- One's mother is one's female parent

  $\forall m,c\ Mother(c) = m \Leftrightarrow (Female(m) \wedge Parent(m,c))$

- "Sibling" is symmetric

  $\forall x,y\ Sibling(x,y) \Leftrightarrow Sibling(y,x)$

Using FOL

The set domain:

$\forall s\ Set(s) \Leftrightarrow (s = \{\}\ ) \vee (\exists x,s_2\ Set(s_2) \wedge s = \{x|s_2\})$

$\neg\exists x,s\ \{x|s\} = \{\}$

$\forall x,s\ x \in s \Leftrightarrow s = \{x|s\}$

$\forall x,s\ x \in s \Leftrightarrow [\ \exists y,s_2\}\ (s = \{y|s_2\} \wedge (x = y \vee x \in s_2))]$

$\forall s_1,s_2\ s_1 \subseteq s_2 \Leftrightarrow (\forall x\ x \in s_1 \Rightarrow x \in s_2)$

$\forall s_1,s_2\ (s_1 = s_2) \Leftrightarrow (s_1 \subseteq s_2 \wedge s_2 \subseteq s_1)$

$\forall x,s_1,s_2\ x \in (s_1 \cap s_2) \Leftrightarrow (x \in s_1 \wedge x \in s_2)$

$\forall x,s_1,s_2\ x \in (s_1 \cup s_2) \Leftrightarrow (x \in s_1 \vee x \in s_2)$