QUESTIONS - I

1) Find all solutions of the system
$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 + 3x_3 &= -1 \\ 4x_1 - x_2 + 5x_3 &= 5 \end{aligned}$$
if they exist.

2) Consider the following system:
$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$$
What should k be if the system a) has no solution, b) infinitely many solution, c) exactly one solution?

3) It is known that the system
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 - 4x_3 + 7x_4 &= k+4 \\ x_1 + x_2 - x_3 + 2x_4 &= k-1 \end{aligned}$$
has a solution. What should k be? Find all solutions.

4) If $A = (a_{ij})$ is an $n \times n$ matrix, then the <u>trace</u> of A, $Tr(A)$, is defined the sum of all elements on the main diagonal of A, i.e. $Tr(A) = \sum_{i=1}^{n} a_{ii}$. Show the following:

a) $Tr(cA) = c\,Tr(A)$, $c \in \mathbb{R}$

b) $Tr(A+B) = Tr(A) + Tr(B)$

c) $Tr(AB) = Tr(BA)$

d) $Tr(A^T) = Tr(A)$

e) $Tr(A^TA) \geqslant 0$.

5) If $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$, then find A.

6) Let A be an $m \times n$ matrix. Show that $I_n + A^TA$ is symmetric.

7) If A, B are $n \times n$ matrices such that $A, B, A+B$ are all idempotent (i.e. satisfy $X^2 = X$). Prove that $AB = -BA$ and $AB = 0$. (Hint. Consider ABA)

8) Let A be an $n \times n$ matrix and I be the $n \times n$ identity matrix.

a) If $A^3 = 0$, verify that $(I-A)^{-1} = I + A + A^2$.

b) Find the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ by using (a).

c) If $A^n = 0$, find the formula for $(I-A)^{-1}$.

9) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that A is nonsingular if and only if $ad - bc \neq 0$

10) Let A be a diagonal matrix, $A = diag(a_1, a_2, \dots, a_n)$. If $a_i \neq 0$ for all i, show that A is invertible. What is its inverse?