Cutting planes for mixed-integer conic programming

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Conic duality 101

Conic program (CP) in primal-dual standard form:

$$\begin{aligned} \min_x \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \in \mathcal{K} \end{aligned} \qquad \qquad \begin{aligned} \max_{y,s} \quad & b^T y \\ s.t. \quad & A^T y + s = c \\ & s \in \mathcal{K}^* \end{aligned}$$

with $\mathcal{K}$ a closed, convex pointed cone, and its *dual cone*

$$\mathcal{K}^* = \{s \in \mathbb{R}^n \mid \forall x \in \mathcal{K},\ s^T x \geq 0\}.$$

Lemma 1 (Conic Farkas) *Assume* $\exists y : -A^T y \in int(\mathcal{K}^*)$. *Then exactly one of the following is non-empty:*

$$\{x \in \mathbb{R}^n \mid Ax = b, x \in \mathcal{K}\}$$
$$\{y \in \mathbb{R}^m \mid -A^T y \in \mathcal{K}^*,\ b^T y > 0\}$$

Conic has an LP-like flavour, BUT... **no basis information**, and **strong duality doesn't always hold**.

Mixed-integer conic programming

We consider MICP problems of the form

$$\begin{aligned} \min_x \quad & c^T x && (1.1) \\ s.t. \quad & Ax = b, && (1.2) \\ & x \in \mathcal{K}, && (1.3) \\ & x_i \in \mathbb{Z}, \quad i = 1, \dots, p, && (1.4) \end{aligned}$$

and denote $\mathcal{C}$ the domain of the continuous relaxation. Solved by non-linear branch-and-bound [5] or outer-approximation [3, 7].

Cutting planes & split cuts

Cuts are crucial to MILP, but **seldom used in MICP** (so far).

We focus on *split cuts*, a special case of disjunctive cuts [1, 2].

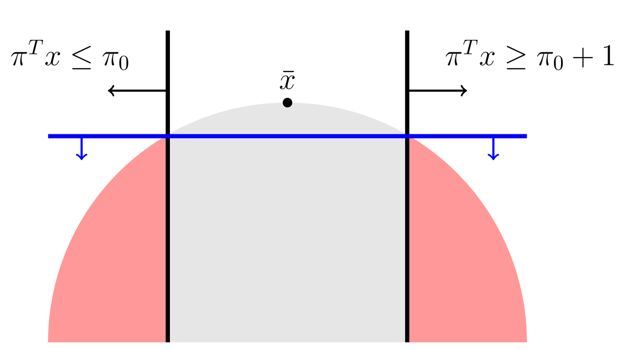


Fig. 1: Split disjunction (red) and split cut (blue)

Goal: extend existing LP frameworks to conic, and provide **geometric understanding** of the separation of split cuts.

Separation problem

Let $\bar{x} \in \mathbb{R}^n$, $\pi \in \mathbb{Z}^n$, $\pi_0 := \lfloor \pi^T \bar{x} \rfloor$, and $f := \pi^T \bar{x} - \pi_0$, $\bar{f} := 1 - f$.

The **cut-generating conic program** (CGCP) writes

$$\begin{aligned} \min \quad & \alpha^T \bar{x} - \beta && (2.1) \\ s.t. \quad & \alpha = A^T u + s - u_0 \pi, && (2.2) \\ & \alpha = A^T v + t + v_0 \pi, && (2.3) \\ & \beta = b^T u - u_0 \pi_0, && (2.4) \\ & \beta = b^T v + v_0(\pi_0 + 1), && (2.5) \\ & s, t \in \mathcal{K}^*, u_0, v_0 \geq 0. && (2.6) \end{aligned}$$

Its dual is the **membership conic program** (MCP)

$$\begin{aligned} \max_{y,z,y_0,z_0} \quad & 0 && (3a) \\ s.t. \quad & y + z = \bar{x}, && (3b) \\ & Ay - y_0 b = 0, && (3c) \\ & \pi^T y \leq y_0 \pi_0, && (3d) \\ & Az - z_0 b = 0, && (3e) \\ & \pi^T z \geq z_0(\pi_0 + 1), && (3f) \\ & y_0 + z_0 = 1, && (3g) \\ & y, z \in \mathcal{K}, y_0, z_0 \geq 0. && (3h) \end{aligned}$$

The domain of MCP describes exactly the convex hull of the union of the two split disjunctions.

Any feasible solution to CGCP with negative objective value yields a violated split cut. Any such solution is also an unbounded ray.

The role of normalization

Solving CGCP (2.1)-(2.6) only identifies a violated cut (if any), not a *most violated* one. Hence, we use a normalization condition.

Trivial normalization writes $u_0 + v_0 = 1$. It yields compact forms for $CGCP_{triv}$ and MCP_{triv}

$$\begin{aligned} \min_{\lambda,s,t} \quad & (\bar{f}s + ft)^T \bar{x} - f\bar{f} \\ s.t. \quad & A^T \lambda + (s - t) = \pi, \\ & s, t \in \mathcal{K}^*. \end{aligned} \qquad \qquad \begin{aligned} \max_y \quad & \pi^T y - f \cdot (\pi_0 + 1) \\ s.t. \quad & Ay = f \cdot b, \\ & 0 \leq_{\mathcal{K}} y \leq_{\mathcal{K}} \bar{x}. \end{aligned}$$

Lemma 2 *If* $\bar{x}$ *is an extreme ray of* $\mathcal{K}$, *then the feasible domain of* MCP_{triv} *has empty interior.*

Conic-SNC writes $\nu^T s + \nu^T t + u_0 + v_0 = 1$, with $\nu \in int(\mathcal{K})$. Generalizes the SNC normalization from LP [4].

Other normalizations include $\|\alpha\|_2 = 1$, which yields a deepest (but often dense) cut, or polyhedral normalizations such as $\|\alpha\|_1 = 1$ and $\|\alpha\|_\infty = 1$ (see e.g. [6]).

Advantages of conic-SNC normalization: it **better reflects the geometry** of $\mathcal{K}$, it consists of a **single linear equality**, and displays **better numerical properties**.

Practical example

Consider the MISOCP:

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ s.t. \quad & z = 1, \\ & (z, x_1, x_2) \in \mathcal{L}_3, \\ & x_1, x_2 \in \mathbb{Z}. \end{aligned}$$

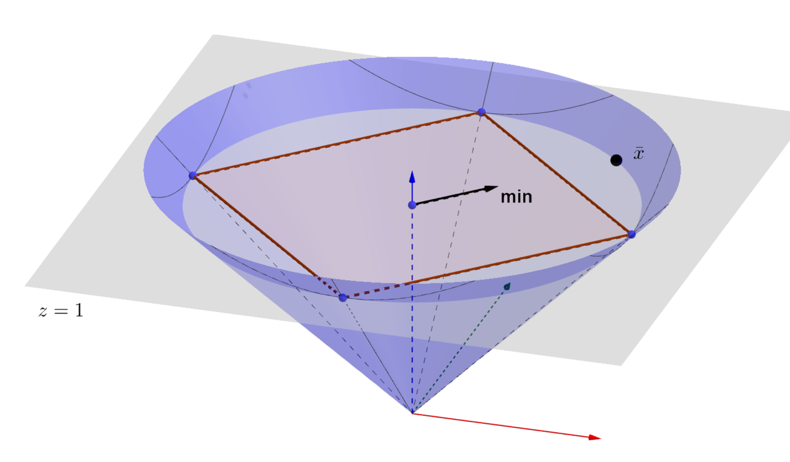
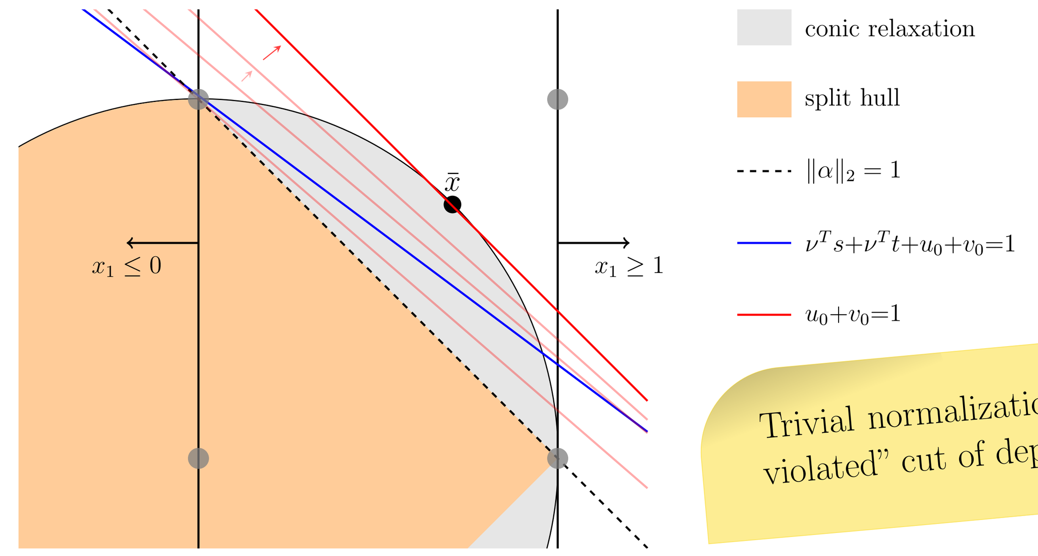


Fig. 2: Feasible region and fractional solution $\bar{x}$.

The fractional solution $\bar{x} = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is an extreme ray of $\mathcal{L}_3$.

Duality issues

Duality failure in $CGCP_{triv}$ is reflected in the obtained cuts.



Trivial normalization yields a "most violated" cut of depth zero!

Fig. 3: Split cuts obtained for various choices of normalization.

Explanation: although $CGCP_{triv}$ is bounded, it has no optimal solution (the optimum is not attained). As multipliers get close to optimal, their norm diverges which leads to numerical issues. Geometrically, the corresponding cuts converge towards a *non-violated* $\mathcal{K}^*$ cut.

Duality failure is resolved with conic-SNC normalization:

Theorem 1 *If* $\bar{x} \in \mathcal{C}$ *and* $\nu \in int(\mathcal{K})$, *then both* $CGCP_{SNC}$ *and* MCP_{SNC} *are strictly feasible. Therefore **strong duality holds**.*

What is NOT wrong with LP?

In LP, strong duality holds, and the CGLP with trivial normalization yields a GMI cut [4]. **What makes conic different?**

When using the trivial normalization, the optimal face of CGCP is a ray that is orthogonal to $\bar{x}$. When $\mathcal{K}$ is polyhedral (i.e., LP), that face is always attainable, and the simplex algorithm stops at an optimal vertex.

Numerical results

Testset: 93 MI-SOCP instances from CBLIB. Remove 29 instances with numerical trouble or root gap < 1%.

Gap closed measured as

$$g = \frac{z^* - z_{conic}}{\bar{z} - z_{conic}}$$

where $\bar{z}$ is the best known upper bound (given by Mosek).

		Gap closed (%)		
Type	# inst	SC	OA_{init}	OA_{SC}
clay	10	25.98	33.55	43.53
flay	10	46.95	57.96	80.91
slay	8	92.04	95.17	98.77
sssd	16	99.78	85.57	99.88
ufl	8	15.51	91.07	32.40
tls	6	12.70	28.42	41.37
fo	6	62.21	85.35	81.18
All	64	44.79	64.16	66.59

SC: Split cuts; OA_{init}: initial outer-approximation; OA_{SC}: final OA + split cuts. Geometric means with a shift of 1%. LPs and MILPs solved with Gurobi.

Conclusions & open questions

Take-away: using conic duality, we can extend most of existing LP tools. Greater care must be given to duality failure, though a good normalization helps. Still lots of numerical issues.

What's next: better understand duality failures; extend results to general disjunctive cuts. Computations: outer-approximation setting, efficient separation oracles.

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