Numerical Analysis [WI4014TU]
Assignment 3

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Consider the following boundary value problem:

$$-\nabla \cdot (k\nabla u) = f, \quad (x, y) \in \Omega = (0, 10) \times (0, 5),$$

$$u(x, y) = 0, \quad (x, y) \in \partial\Omega,$$

$$f(x, y) = \sum_{i=1}^{9} \sum_{j=1}^{4} e^{-\alpha(x-i)^2 - \alpha(y-j)^2}, \tag{1}$$

$$\text{with } \alpha = 40, \ (x, y) \in \bar{\Omega}.$$

where $\bar{\Omega} = [0, 10] \times [0, 5]$ is the rectangle with corners $(0, 0)$, $(10, 0)$, $(10, 5)$, and $(0, 5)$.

2. Finite Difference Method

For this method, we consider the homogeneous coefficient function

$$k(x, y) = 1 \quad (x, y) \in \bar{\Omega} \tag{2}$$

1. Write down the $\mathcal{O}(h^2)$ Finite-Difference (FD) approximation of $-\Delta u_{i,j}$ at a point (x_i, y_j) for a uniform, but not doubly-uniform mesh.

Solution: For a uniform, but not doubly-uniform mesh. The $\mathcal{O}(h^2)$ is given by:

$$\mathcal{O}(h_x^2) + \mathcal{O}(h_y^2) = \left| -\Delta u_{i,j} - \frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h_x^2} - \left(\frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{h_y^2}\right) \right|$$

2. For your mesh (see above), write down all the discrete FD equations of your problem, performing the substitution and elimination of the boundary points and values. The node values of the source function $f(x, y)$ may be written symbolically as $f_{i,j}$, without substituting the actual algebraic expression from 1.

Solution: For the sub-interval $N_x = N_y = 4$, we have grid steps $h_x = 2.5$ and $h_y = 1.25$. We will obtain 9 discrete FD equations for the inner grid points. These equations in the lexicographic order are:

$$
\begin{aligned}
-0.16u_{21} - 0.64u_{12} + 1.6u_{11} &= f_{11} \\
-0.16u_{11} - 0.16u_{31} - 0.64u_{22} + 1.6u_{21} &= f_{21} \\
-0.16u_{21} - 0.64u_{32} + 1.6u_{31} &= f_{31} \\
-0.16u_{22} - 0.64u_{11} - 0.64u_{13} + 1.6u_{12} &= f_{12} \\
-0.16u_{12} - 0.16u_{32} - 0.64u_{21} - 0.64u_{23} + 1.6u_{22} &= f_{22} \\
-0.16u_{22} - 0.64u_{31} - 0.64u_{33} + 1.6u_{32} &= f_{32} \\
-0.16u_{23} - 0.64u_{12} + 1.6u_{13} &= f_{13} \\
-0.16u_{13} - 0.16u_{33} - 0.64u_{22} + 1.6u_{23} &= f_{23} \\
-0.16u_{23} - 0.64u_{32} + 1.6u_{33} &= f_{33}
\end{aligned}
$$

3. The FD equations can be written as the linear algebraic problem of the form:

$$A\mathbf{u} = \mathbf{f} \tag{3}$$

Write down the system matrix A of the negative FD Laplacian, the vector $\mathbf{u}$ of unknowns and the right-hand-side (RHS) vector $\mathbf{f}$

Solution: In the lexicographic order, the equation $A\mathbf{u} = \mathbf{f}$ will be as follows:

$$
\begin{bmatrix}
1.6 & -0.16 & 0 & -0.64 & 0 & 0 & 0 & 0 & 0 \\
-0.16 & 1.6 & -0.16 & 0 & -0.64 & 0 & 0 & 0 & 0 \\
0 & -0.16 & 1.6 & 0 & 0 & -0.64 & 0 & 0 & 0 \\
-0.64 & 0 & 0 & 1.6 & -0.16 & 0 & -0.64 & 0 & 0 \\
0 & -0.64 & 0 & -0.16 & 1.6 & -0.16 & 0 & -0.64 & 0 \\
0 & 0 & -0.64 & 0 & -0.16 & 1.6 & 0 & 0 & -0.64 \\
0 & 0 & 0 & -0.64 & 0 & 0 & 1.6 & -0.16 & 0 \\
0 & 0 & 0 & 0 & -0.64 & 0 & -0.16 & 1.6 & -0.16 \\
0 & 0 & 0 & 0 & 0 & -0.64 & 0 & -0.16 & 1.6
\end{bmatrix}
\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{bmatrix}
=
\begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{13} \\ f_{23} \\ f_{33} \end{bmatrix}
$$

4. Show for your grid that, under lexicographic ordering, the FD matrix A of the negative 2D Laplacian operator can indeed be obtained as $A = I_y \bigotimes L_{xx} + L_{yy} \bigotimes I_x$, where L_{xx} and L_{yy} are the FD matrices of negative 1D Laplacians in the x- and y-direction, respectively, and $_x$, I_y are identity matrices of proper size. Also show that $L_{xx} = D_x^T D_x$ and $L_{yy} = D_y^T D_y$, where D_x and D_y are matrices representing one-sided FD approximations of the first derivatives. See slides of Lecture 5. You need to demonstrate that the Kronecker-product formula works.

Solution: Given $N_x = N_y = 4$. D_x and D_y are the backward difference matrix in the x- and y- direction. The required matrices are given as follows:

$$I_x = I_{N_x-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_y = I_{N_y-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the number of sub-grids is same, matrices D_x and D_y will also be the same but with different h_x and h_y values.

$$D_x = \frac{1}{h_x} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$D_y = \frac{1}{h_y} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Now, $L_{xx} = D_x^T D_x$

$$\implies L_{xx} = \frac{1}{h_x^2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \frac{1}{6.25} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Similarly, $L_{yy} = D_y^T D_y$

$$\implies L_{yy} = \frac{1}{h_y^2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \frac{1}{1.5625} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

L_{xx} and L_{yy} are same as the matrices that would be obtained by directly using the central-differencing scheme in the x- and y- direction. Now, $A = I_y \bigotimes L_{xx} + L_{yy} \bigotimes I_x$

$$I_y \bigotimes L_{xx} = \frac{1}{6.25} \begin{bmatrix} 1L_{xx} & 0 & 0 \\ 0 & 1L_{xx} & 0 \\ 0 & 0 & 1L_{xx} \end{bmatrix}$$

$$\implies I_y \bigotimes L_{xx} = \begin{bmatrix} 0.32 & -0.16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.16 & 0.32 & -0.16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.16 & 0.32 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.32 & -0.16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.16 & 0.32 & -0.16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.16 & 0.32 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.32 & -0.16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.16 & 0.32 & -0.16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.16 & 0.32 \end{bmatrix}$$

Similarly,

$$L_{yy} \bigotimes I_x = \frac{1}{1.5625} \begin{bmatrix} 2I_x & -I_x & 0 \\ -I_x & 2I_x & -I_x \\ 0 & -I_x & 2I_x \end{bmatrix}$$

$$\implies L_{yy} \bigotimes I_x = \begin{bmatrix} 1.28 & 0 & 0 & -0.64 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.28 & 0 & 0 & -0.64 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.28 & 0 & 0 & -0.64 & 0 & 0 & 0 \\ -0.64 & 0 & 0 & 1.28 & 0 & 0 & -0.64 & 0 & 0 \\ 0 & -0.64 & 0 & 0 & 1.28 & 0 & 0 & -0.64 & 0 \\ 0 & 0 & -0.64 & 0 & 0 & 1.28 & 0 & 0 & -0.64 \\ 0 & 0 & 0 & -0.64 & 0 & 0 & 1.28 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.64 & 0 & 0 & 1.28 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.64 & 0 & 0 & 1.28 \end{bmatrix}$$

Adding the two gives,

$$A = \begin{bmatrix} 1.6 & -0.16 & 0 & -0.64 & 0 & 0 & 0 & 0 & 0 \\ -0.16 & 1.6 & -0.16 & 0 & -0.64 & 0 & 0 & 0 & 0 \\ 0 & -0.16 & 1.6 & 0 & 0 & -0.64 & 0 & 0 & 0 \\ -0.64 & 0 & 0 & 1.6 & -0.16 & 0 & -0.64 & 0 & 0 \\ 0 & -0.64 & 0 & -0.16 & 1.6 & -0.16 & 0 & -0.64 & 0 \\ 0 & 0 & -0.64 & 0 & -0.16 & 1.6 & 0 & 0 & -0.64 \\ 0 & 0 & 0 & -0.64 & 0 & 0 & 1.6 & -0.16 & 0 \\ 0 & 0 & 0 & 0 & -0.64 & 0 & -0.16 & 1.6 & -0.16 \\ 0 & 0 & 0 & 0 & 0 & -0.64 & 0 & -0.16 & 1.6 \end{bmatrix}$$

The obtained matrix is similar to the one obtained earlier from the discrete FD equations for the inner points. Therefore, the Kronecker product works.

5. Visualize A with `plt.spy()` for $N_x = N_y = 4$.

Solution:

```
[[ 1.6  -0.16  0.   -0.64  0.    0.    0.    0.    0.  ]
 [-0.16  1.6  -0.16  0.   -0.64  0.    0.    0.    0.  ]
 [ 0.   -0.16  1.6   0.    0.   -0.64  0.    0.    0.  ]
 [-0.64  0.    0.    1.6  -0.16  0.   -0.64  0.    0.  ]
 [ 0.   -0.64  0.   -0.16  1.6  -0.16  0.   -0.64  0.  ]
 [ 0.    0.   -0.64  0.   -0.16  1.6   0.    0.   -0.64]
 [ 0.    0.    0.   -0.64  0.    0.    1.6  -0.16  0.  ]
 [ 0.    0.    0.    0.   -0.64  0.   -0.16  1.6  -0.16]
 [ 0.    0.    0.    0.    0.   -0.64  0.   -0.16  1.6 ]]
```

Figure 1: System matrix obtained for FDM.

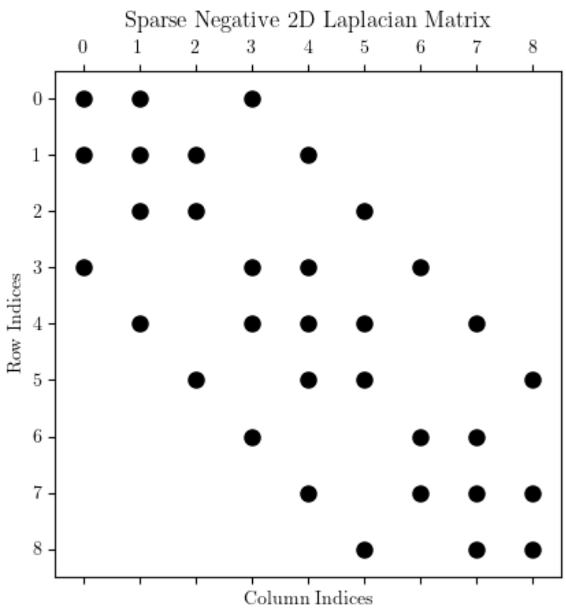

Figure 2: Visualization of the system matrix.

The system matrix obtained resembles the matrix derived earlier.

6. Explain the relation of the structure of `x` and `y` to the lexicographic ordering of the unknowns. Which array dimension corresponds to the x-direction and which to the y-direction along the coordinate axes?

Solution: The code `x,y = np.mgrid[...]` outputs two 2D arrays each for `x` and `y` corresponding to the x- and y- direction. `x` varies row-wise, while `y` varies column-wise; in essence, each 1D array in `x` corresponds to changing grid points in the y- direction.

7. Visualize the source function as heat map using the function `plt.imshow()`

Solution:

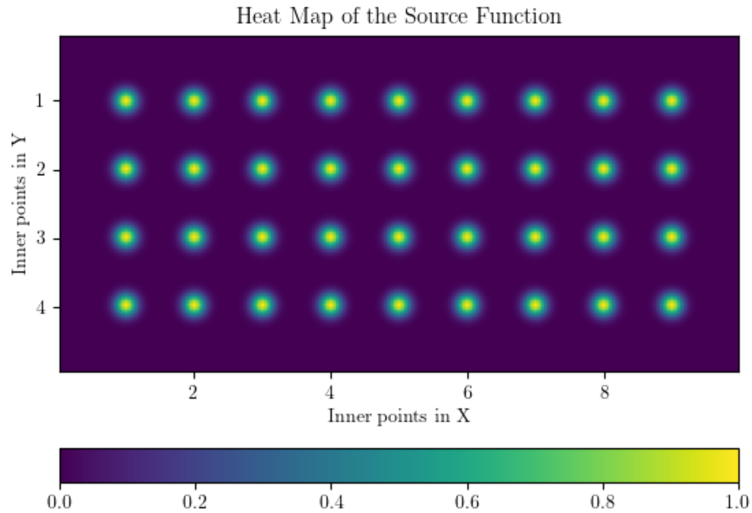

Figure 3: Heat map of the source function for $N_x = 200$ and $N_y = 100$.

8. Visualise the solution.

Solution:

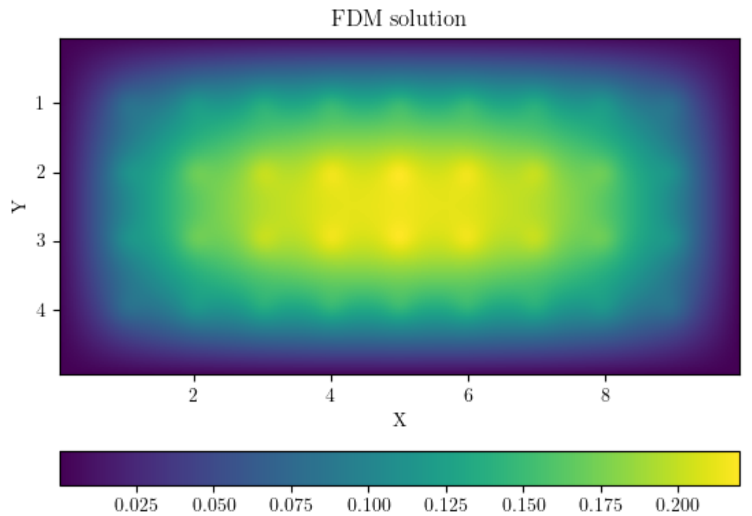

Figure 4: Solution for the FDM scheme with coefficient function $K = 1$.

3. Finite Volume Method

For this method we shall consider both the homogeneous coefficient function (2) and the following inhomogeneous coefficient function:

$$k(x, y) = 1 + 0.1(x + y + xy) \quad (x, y) \in \bar{\Omega} \tag{4}$$

1. Apply the vertex-centered FV Method and write down all discrete FV equations of the problem for the lexicographic ordering of the unknowns. The FV equations should be averaged over the area of the elementary cell. You may keep the general notation $k_{i,j}$, $f_{i,j}$ for the grid values of the known functions, as well as h_x, h_y for the grid steps. However, the actual values of the indices i and j should be written out explicitly.

Solution: The discrete FV equations using the vertex-centered FV method for the lexicographic ordering of the unknowns are as follows.

$$\left(\frac{k_{0.5,1}}{h_x^2} + \frac{k_{1.5,1}}{h_x^2} + \frac{k_{1,0.5}}{h_y^2} + \frac{k_{1,1.5}}{h_y^2}\right) u_{1,1} - \frac{k_{1.5,1}}{h_x^2} u_{2,1} - \frac{k_{1,1.5}}{h_y^2} u_{1,2} = f_{1,1}$$

$$-\frac{k_{1,1.5}}{h_y^2} u_{1,1} + \left(\frac{k_{0.5,2}}{h_x^2} + \frac{k_{1.5,2}}{h_x^2} + \frac{k_{1,1.5}}{h_y^2} + \frac{k_{1,2.5}}{h_y^2}\right) u_{1,2} - \frac{k_{1.5,2}}{h_x^2} u_{2,2} - \frac{k_{1,2.5}}{h_y^2} u_{1,3} = f_{1,2}$$

$$-\frac{k_{1,2.5}}{h_y^2} u_{1,2} + \left(\frac{k_{0.5,3}}{h_x^2} + \frac{k_{1.5,3}}{h_x^2} + \frac{k_{1,2.5}}{h_y^2} + \frac{k_{1,3.5}}{h_y^2}\right) u_{1,3} - \frac{k_{1.5,3}}{h_x^2} u_{2,3} = f_{1,3}$$

$$-\frac{k_{1.5,1}}{h_x^2} u_{1,1} + \left(\frac{k_{1.5,1}}{h_x^2} + \frac{k_{2.5,1}}{h_x^2} + \frac{k_{2,1.5}}{h_y^2} + \frac{k_{2,0.5}}{h_y^2}\right) u_{2,1} - \frac{k_{2.5,1}}{h_x^2} u_{3,1} - \frac{k_{2,1.5}}{h_y^2} u_{2,2} = f_{2,1}$$

$$-\frac{k_{1.5,2}}{h_x^2} u_{1,2} + \left(\frac{k_{1.5,2}}{h_x^2} + \frac{k_{2.5,2}}{h_x^2} + \frac{k_{2,2.5}}{h_y^2} + \frac{k_{2,1.5}}{h_y^2}\right) u_{2,2} - \frac{k_{2.5,2}}{h_x^2} u_{3,2} - \frac{k_{2,2.5}}{h_y^2} u_{2,3} - \frac{k_{2,1.5}}{h_y^2} u_{2,1} = f_{2,2}$$

$$-\frac{k_{1.5,3}}{h_x^2} u_{1,3} + \left(\frac{k_{1.5,3}}{h_x^2} + \frac{k_{2.5,3}}{h_x^2} + \frac{k_{2,3.5}}{h_y^2} + \frac{k_{2,2.5}}{h_y^2}\right) u_{2,3} - \frac{k_{2.5,3}}{h_x^2} u_{3,3} - \frac{k_{2,2.5}}{h_y^2} u_{2,2} = f_{2,3}$$

$$-\frac{k_{2.5,1}}{h_x^2} u_{2,1} + \left(\frac{k_{2.5,1}}{h_x^2} + \frac{k_{3.5,1}}{h_x^2} + \frac{k_{3,1.5}}{h_y^2} + \frac{k_{3,0.5}}{h_y^2}\right) u_{3,1} - \frac{k_{3,1.5}}{h_y^2} u_{3,2} = f_{3,1}$$

$$-\frac{k_{2.5,2}}{h_x^2} u_{2,2} + \left(\frac{k_{2.5,2}}{h_x^2} + \frac{k_{3.5,2}}{h_x^2} + \frac{k_{3,2.5}}{h_y^2} + \frac{k_{3,1.5}}{h_y^2}\right) u_{3,2} - \frac{k_{3,1.5}}{h_y^2} u_{3,1} - \frac{k_{3,2.5}}{h_y^2} u_{3,3} = f_{3,2}$$

$$-\frac{k_{2.5,3}}{h_x^2} u_{2,3} + \left(\frac{k_{2.5,3}}{h_x^2} + \frac{k_{3.5,3}}{h_x^2} + \frac{k_{3,3.5}}{h_y^2} + \frac{k_{3,2.5}}{h_y^2}\right) u_{3,3} - \frac{k_{3,2.5}}{h_y^2} u_{3,2} = f_{3,3}$$

2. For $k(x,y)$ given by (2), write down the numerical values of all system matrix elements and compare the result to the FD system matrix obtained in Section 2.1 (3).

Solution: Given $k(x,y) = 1$ and using $h_x = 2.5$, and $h_y = 1.25$, the system of equation obtained are

$$
\begin{aligned}
-0.16u_{21} - 0.64u_{12} + 1.6u_{11} &= f_{11} \\
-0.16u_{11} - 0.16u_{31} - 0.64u_{22} + 1.6u_{21} &= f_{21} \\
-0.16u_{21} - 0.64u_{32} + 1.6u_{31} &= f_{31} \\
-0.16u_{22} - 0.64u_{11} - 0.64u_{13} + 1.6u_{12} &= f_{12} \\
-0.16u_{12} - 0.16u_{32} - 0.64u_{21} - 0.64u_{23} + 1.6u_{22} &= f_{22} \\
-0.16u_{22} - 0.64u_{31} - 0.64u_{33} + 1.6u_{32} &= f_{32} \\
-0.16u_{23} - 0.64u_{12} + 1.6u_{13} &= f_{13} \\
-0.16u_{13} - 0.16u_{33} - 0.64u_{22} + 1.6u_{23} &= f_{23} \\
-0.16u_{23} - 0.64u_{32} + 1.6u_{33} &= f_{33}
\end{aligned}
$$

And the system matrix A for the lexicographic order is,

$$
A = \begin{bmatrix}
1.6 & -0.16 & 0 & -0.64 & 0 & 0 & 0 & 0 & 0 \\
-0.16 & 1.6 & -0.16 & 0 & -0.64 & 0 & 0 & 0 & 0 \\
0 & -0.16 & 1.6 & 0 & 0 & -0.64 & 0 & 0 & 0 \\
-0.64 & 0 & 0 & 1.6 & -0.16 & 0 & -0.64 & 0 & 0 \\
0 & -0.64 & 0 & -0.16 & 1.6 & -0.16 & 0 & -0.64 & 0 \\
0 & 0 & -0.64 & 0 & -0.16 & 1.6 & 0 & 0 & -0.64 \\
0 & 0 & 0 & -0.64 & 0 & 0 & 1.6 & -0.16 & 0 \\
0 & 0 & 0 & 0 & -0.64 & 0 & -0.16 & 1.6 & -0.16 \\
0 & 0 & 0 & 0 & 0 & -0.64 & 0 & -0.16 & 1.6
\end{bmatrix}
$$

The system matrix obtained is same as the one obtained for the Finite Difference Scheme.

3. Visualize both coefficient functions `K1` and `K2` on a 200 x 100 mesh in a single figure.

Solution:

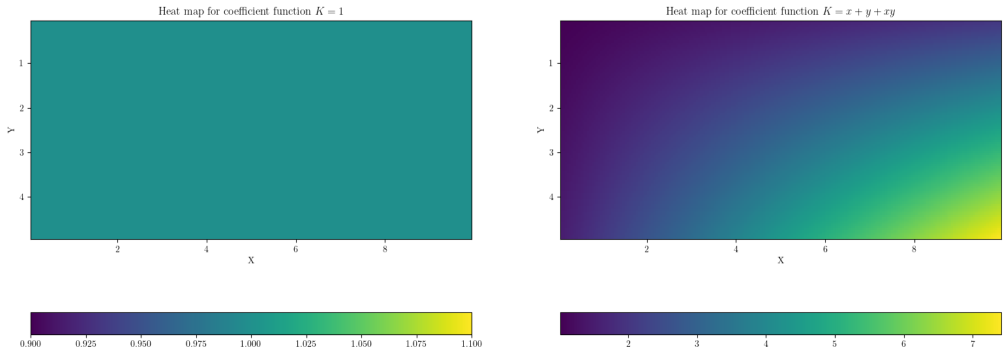

Figure 5: Subplots of coefficient function $K1$ and $K2$

4. Plot the obtained numerical solutions with `plt.imshow()` and insert the corresponding figure in your report.

Solution:

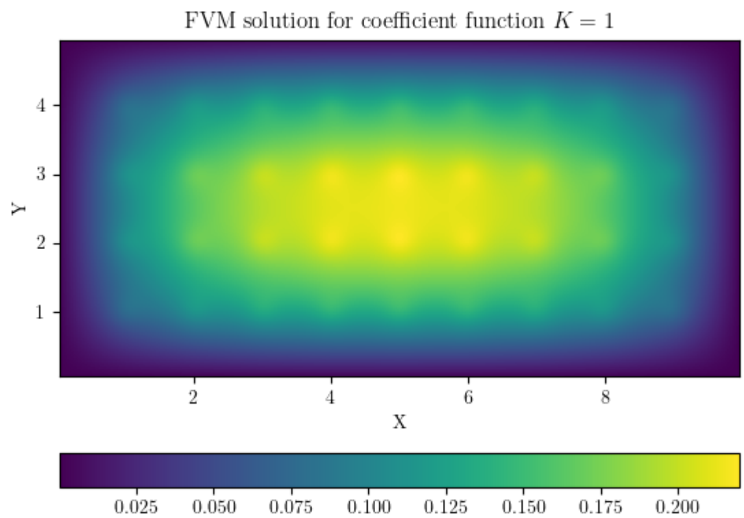

Figure 6: FVM solution for coefficient function $K = 1$

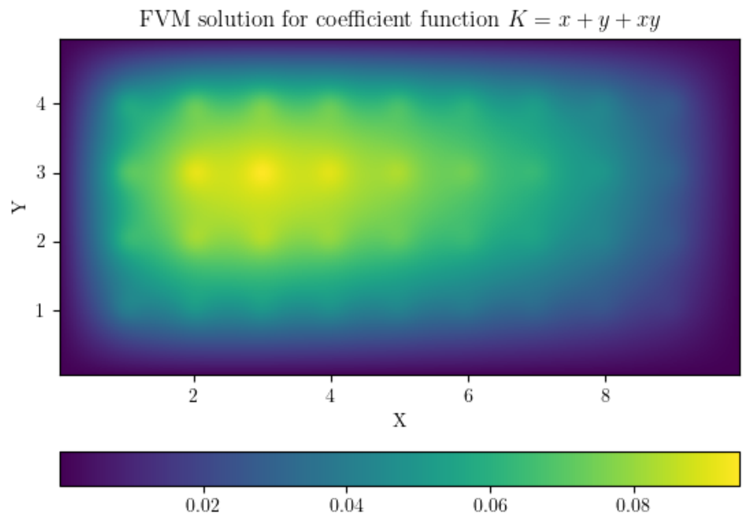

Figure 7: FVM solution for coefficient function $K = x + y + xy$

5. Explain why the obtained FDM and FVM solutions make sense from the physical and mathematical points of view.

Solution: Physically, this boundary value problem represents a heat transfer system with multiple source terms given by the function f. The coefficient function k describes the conductivity of the material. When $K = 1$, the temperature distribution throughout the rectangular domain is uniform and higher at the center and reduces as we move away from the center. Whereas, when conductivity is a function of the spatial terms, i.e. $K = x+y+xy$, we see higher temperatures away from the center and diffusion is then governed by $K(x, y)$ and higher temperatures are then observed away from the center.
Methematically as well, the PDE represents the physical system where for a given source function f, the temperature u inversely varies with the coefficient of conductivity K; where conductivity is higher, we expect lower temperatures, and vice versa, as is obtained in the solution plots (Fig. 4 (FDM solution), 6, and 7 (FVM solutions)).