Notes to a Math Puzzle Book

Martin Titz

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Contents

1 Introduction

I once have bought the book [2] by J.J.Clessa ”Math and Logic Puzzles for PC Enthusiasts” in the reprinted Dover edition and now found time to solve the majority of the so called micropuzzles. These are my collected notes.

Many puzzles can be best solved writing a computer program. In most cases I used the programming language Java, while I used the computer algebra system Mathematica for micropuzzles 25 and 55 to solve the minimization problem respective the polynomial equation of degree 4.

2 Errata

The following list is only what I have noticed myself, and is no official list.

- Micropuzzle 41: First example: Replace $321 = 19+2.1^2$ with $21 = 19+2\cdot1^2$
- Micropuzzle 55: Replace in drawing '20m' with '25m'
- Solution to micropuzzle 1: Replace 'perimeter of 144' with 'perimeter of 576'
- Solution to micropuzzle 5: Replace $A0 - 2$ with $A0 = 2$
- Solution to micropuzzle 26: Replace 'school' with 'local'
- Solution to micropuzzle 35: Replace $80 + D = 281$ with $80 + D = 28M$
- Solution to micropuzzle 54: Replace equation (1) with $x^2 + y^2 = 225$
- Solution to micropuzzle 54: The square is missing in $(x + y)^2 - 2xy = 225$
- Solution to micropuzzle 54: Rounding error in result, better 3.781555
- Solution to micropuzzle 55: Replace equation (2) with $\frac{1}{a} + \frac{1}{b} = \frac{1}{h}$
- Solution to micropuzzle 55: In the last sentence it should be 'distance between the walls'

3 Overview

Some of the Java programs require additional Java source files, either using Permutation.java for iterating through all permutations of a given size or using some number-theoretical algorithms like fast square detection, using an algorithm given in [3]. It is also mentioned when the internal Java class BigInteger is used.

In many cases the Java program solves a slightly more general problem, either by calculating a solution for a more general case than the special value given in the problem or by not only finding the first solution. In several cases the special puzzle problem is the default and other values can be given by command line parameters.

	Puzzle	Used	Remarks
1	Pythagoras for beginners	Java	uses NumberTheory.java
2	Flying the Glasgow shuttle	-	
3	A chessboard dilemma	Java	
4	A palindromic puzzle	Java	
5	Digital dexterity	Java	uses BigInteger
6	A palindromic square	Java	
7	More perfect squares	Java	uses NumberTheory.java
8	A question of logic	-	
9	The whole truth and nothing but the truth	-	
10	Talking turkey	Java	
11	The not-so-perfect square	Java	uses NumberTheory.java
12	Squaring the cube	Java	uses Permutation.java
13	A natural mistake	Java	
14	Ten-digit perfect squares	Java	

15	A bad connection		
16	A numerical traverse		
17	A problem of check-digits	Java	
18	Ceremonial rice pudding	Java	uses BigInteger
19	Who's who	Java	uses Permutation.java
20	Word frustration		
21	A positional problem	Java	
22	Generating a specific value	Java	uses Permutation.java
23	Sums of squares		
24	A very prime word		
25	Fieldcraft	Mathematica	
26	Opening day at the local		
27	A very charitable dilemma		
28	Cows, pigs and horses		
29	An exact number of factors		
30	Cubes and squares	Java	
31	A question of remainders	Java	uses NumberTheory.java
32	A problem of prime factors	Java	uses BigInteger
33	The ladies of the committee	Java	
34	An unusual number	Java	
35	Tadpoles, terrapins, tortoises, and turtles	Java	
36	More cubes and squares	Java	uses NumberTheory.java
37	Sums of cubes	Java	uses NumberTheory.java
38	Coconut galore		
39	More trouble with remainders	C++, Java	(as testcase)
40	Ball-bearing pyramids		
41	Sums of primes, etc.	Java	uses NumberTheory.java
42	The numerate marathon runner	Java	
43	Ten-digit primes	Java	uses NumberTheory.java
44	Approximations		
45	Palindromic cycles	Java	uses BigInteger
46	Mother and daughter		
47	A catastrophic puzzle		
48	Susan's perfect man		
49	An interesting pair of series	Java	
50	Another pyramid problem	Java	
51	A number and its square	Java	uses Permutation.java
52	A long-winded fraction	-	
53	An infernal triangle	-	
54	The ladder and the wall	Java	
55	More ladders and walls	Mathematica	
56	Reflections at a corner		
57	The census taker	Java	uses NumberTheory.java
58	A number crossword		
59	Another number crossword		
60	A series of primes	Java	
61	The hymn board	Java	
62	Three unusual digits	Java	
63	A recurring quotient	Java	
64	A rotating grid	Java	uses Permutation.java
65	The cocktail party		

4 Individual Puzzles

4.1 Quickie 16

From the book [2]: Here's a simple multiplication problem in which each letter represents a different digit. Can you solve it?

```
  IFx
  AT
-----
FIAT
-----
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The book gives only one solution, but as also 0 is a digit, I found three:

$$\begin{aligned} 21 * 60 &= 1260 \\ 41 * 35 &= 1435 \\ 51 * 30 &= 1530 \end{aligned}$$

4.2 Quickie 17

See also [4], pages 83-84 for a more general treatment. There the general formula for the total number of triangles is given when the big triangle sides each consist of n small triangles (sequence 0, 1, 5, 13, 27, 48, 78, 118, 170,..., M3827 in [6]).

$$\begin{array}{cc} \dfrac{n(n+2)(2n+1)}{8} & \text{for even n} \\ \dfrac{n(n+2)(2n+1)-1}{8} & \text{for odd n} \end{array}$$

It is derived in [4] from a difference pattern, but not proofed for general n.

4.3 Micropuzzle 5 – Digital dexterity

From the book [2]: A certain number ends in the digit 'a'. When the 'a' is taken from the end of the number and placed at the beginning, a new number is formed which is 'a' times the original number.

The problem can be solved more general for an arbitrary base B for the number system instead of only base 10.

A positive integer N can be represented in a number system with base $B \geq 2$ with n digits $0 \leq d_i \leq B-1$ for $i = 0, \ldots, n-1$

$$N = \sum_{i=0}^{n-1} d_i B^i \tag{1}$$

Let $d_0 \in \{2, 3, \cdots, B-1\}$. The case of $d_0 = 1$ is trivial, already $N = 1$ would be a solution.

Taking away the last digit d_0 of the number N can be expressed as first subtracting d_0 from N and as then the number ends with a 0 in base B, dividing by the base. Adding the digit d_0 then in front is just adding $d_0 B^{n-1}$ to the remaining number. As by condition of the puzzle this has to be equal d_0 times the original number N, the following equations results

$$d_0 \cdot N = d_0 B^{n-1} + \frac{N - d_0}{B} \tag{2}$$

Multiplying both sides with B

$$B d_0 N = d_0 B^n + N - d_0$$

and sorting all terms with N to the left side and the other ones to the right side yields

$$\left(B d_0 - 1\right) N = d_0 \left(B^n - 1\right)$$

As d_0 is an integer ≥ 2, the two integers $Bd_0 - 1$ and d_0 cannot have a common divisor, therefore $Bd_0 - 1$ must be a divisor of the other factor on the right side, therefore it must be

$$\left(B d_0 - 1\right) | \left(B^n - 1\right) \tag{3}$$

and if this condition is fulfilled for a given n then the original number N can be calculated by

$$N = \frac{d_0 \cdot \left(B^n - 1\right)}{B d_0 - 1} \tag{4}$$

To find the smallest n that condition (3) is fulfilled you can check

$$B^n \equiv 1 \pmod{B d_0 - 1} \tag{5}$$

as this can be effectively calculated, see for example [5], section 4.2.

With this, the solution of the puzzle can be done in the following two steps:

1. Find the smallest n that fulfills (5) using a powerMod algorithm.
2. Calculate N by the division (4), using high precision integer arithmetic as the powers B^n will soon become huge numbers well out of the range of normal integer arithmetic with int or long data types.

Example 1: $B = 10$, $d_0 = 4$: Here is $Bd_0 - 1 = 39$ and as the smallest exponent to be found is $n = 6$ for which $39 | \left(10^6 - 1\right)$, hence

$$N = \frac{4 \cdot \left(10^6 - 1\right)}{39} = \frac{4 \cdot 999999}{39} = 102564$$

Therefore

$$4 \cdot 102564 = 410256$$

Example 2: $B = 4, d_0 = 2$: Here is $Bd_0 - 1 = 7$ and $7 | \left(4^3 - 1\right)$, hence

$$N = \frac{2 \cdot \left(4^3 - 1\right)}{7} = \frac{2 \cdot 63}{7} = 18$$

Therefore

$$2 \cdot (102)_4 = (210)_4$$

For verification it is

$$\begin{aligned} (102)_4 &= 1 \cdot 4^2 + 0 \cdot 4^1 + 2 \cdot 4^0 = 16 + 2 = 18 \\ (210)_4 &= 2 \cdot 4^2 + 1 \cdot 4^1 + 0 \cdot 4^0 = 32 + 4 = 36 \end{aligned}$$

4.4 Micropuzzle 13 – A natural mistake

The British currency had many changes over time. The puzzles had been stated after the decimalization of the British coinage 1971, but at the time the puzzle had been first published, the halfpenny had still been legal tender. Therefore the variables a, b, c and d give the prices of the four items in halfpennies; a halfpenny is 1/200 of a pound.

We have to look for solutions in positive integers of the equations

$$\begin{aligned} \frac{a}{200} \cdot \frac{b}{200} \cdot \frac{c}{200} \cdot \frac{d}{200} &= \frac{711}{100} \\ \frac{a}{200} + \frac{b}{200} + \frac{c}{200} + \frac{d}{200} &= \frac{711}{100} \end{aligned}$$

Multiplying the first equation with 200 and the second one with 200^4 gives

$$\begin{aligned} a + b + c + d &= 1422 \\ a \cdot b \cdot c \cdot d &= 1422 \cdot 200^3 = 11376000000 \end{aligned}$$

Without loss of generality we can assume $a \geq b \geq c \geq d$.

4.5 Micropuzzle 14 – Ten-digit perfect squares

The program Micropuzzle14.java is more general than the exercise, allowing to give the number of digits as command line parameter. As the Java type long is used for the calculation, a maximum of 18 digits is possible.

Nice individual results:

$$\begin{aligned} 212^2 &= 44944 \\ 2538^2 &= 6441444 \\ 6888^2 &= 47444544 \end{aligned}$$

In the first result both numbers are additionally palindromic numbers.

4.6 Micropuzzle 25 – Fieldcraft

The fastest time is achieved when walking in straight lines across the different types of soil. Let s_1, s_2, s_3 be the distances on each kind of surface.

Let x_1 the horizontal distance to the point where the farmer leaves from bog to ploughed soil and x_2 the horizontal distance to the point where the farmer passes over from ploughed soil to turf.

Then by theorem of Pythagoras

$$\begin{aligned} s_1 &= \sqrt{x_1^2 + 100^2} \\ s_2 &= \sqrt{(x_2 - x_1)^2 + 200^2} \\ s_3 &= \sqrt{(600 - x_2)^2 + 300^2} \end{aligned}$$

Using $v = s/t$ in the form $t = s/v$ the total walking time becomes with $v_1 = 5/2$, $v_2 = 5$ and $v_3 = 10$

$$t(x_1, x_2) = \sum_{k=1}^{3} \frac{s_k(x_1, x_2)}{v_k} \tag{6}$$

Therefore the function

$$t(x_1, x_2) = \frac{2}{5}\sqrt{x_1^2 + 100^2} + \frac{1}{5}\sqrt{(x_2 - x_1)^2 + 200^2} + \frac{1}{10}\sqrt{(600 - x_2)^2 + 300^2} \tag{7}$$

has to be minimized with the constraints $0 \leq x_1 \leq x_2 \leq 600$.

The minimum calculated by the Mathematica function NMinimize is

$$\begin{aligned} x_1 &\approx 21.750059753139185663553649151236726394 89 \\ x_2 &\approx 115.669703582869959229191842239375824601 0 \\ t(x_1, x_2) &\approx 142.097659044197040067245550187723934325 0 \end{aligned}$$

The shortest time rounded to the nearest second is 2 minutes 22 seconds. The high precision is not needed for the puzzle answer, but could be useful as reference value for checking another implementation of this problem.

4.7 Micropuzzle 35 – Tadpoles, terrapins, tortoises, and turtles

To only use integers the prices are in pence for the calculation. Then using a, b, c and d for the numbers of each kind these two equations arise:

$$\begin{aligned} 59a + 199b + 287c + 344d &= 10000 \quad (8) \\ a + b + c + d &= 100 \quad (9) \end{aligned}$$

Multiplying (9) with 59 and then subtracting this from the equation (8) results in

$$(59 - 59)a + (199 - 59)b + (287 - 59)c + (344 - 59)d = 10000 - 59 \cdot 100$$

and simplified

$$140b + 228c + 285d = 4100 \tag{10}$$

The simple solution program Micropuzzle35.java varies the values of b and c in the expression given by the left side of (10) in the range from 0 to 4100. If then d would be an integer, d is calculated from (10) and finally $a = 100 - b - c - d$ can be calculated and a solution is found.

4.8 Micropuzzle 45 – Palindromic cycles

The actual puzzle solution for only two digit numbers could have been done within the range of the long Java data type. But the interesting cases are beyond this range.

It is still an open problem if these cycles ever terminate for the number 196 and likewise for many numbers above 196, while the record of 24 cycles for 89 and 98 for known terminating cycles is only first exceeded beyond 10000, as the number 10548 reaches after 30 cycles the palindrome 17858768886785871. Therefore the solution program Micropuzzle45.java follows cycles only until a given limit of 1000 cycles and lists these exceptional cases at the end of its output.

4.9 Micropuzzle 54 – The ladder and the wall

Let the distance from floor to top of the ladder be x feet.
Let the distance from wall to bottom of ladder be y feet.

Then by theorem of Pythagoras

$$x^2 + y^2 = 15^2 \tag{11}$$

and by the two similar triangles above and right of the box (all angles are equal)

$$\frac{x-3}{3} = \frac{3}{y-3} \tag{12}$$

Multiplying both sides of (12) with the denominators gives

$$(x-3)(y-3) = 9$$

$$xy - 3x - 3y + 9 = 9$$

$$xy = 3(x+y) \tag{13}$$

By the binomial theorem

$$(x+y)^2 = x^2 + 2xy + y^2$$

it is

$$x^2 + y^2 = (x+y)^2 - 2xy$$

Inserting this into (11) gives together with (13) an equation system

$$(x+y)^2 - 2xy = 15^2 \tag{14}$$

$$xy = 3(x+y) \tag{15}$$

where the variables x and y only appear in the forms $x+y$ and xy. This suggests the substitutions

$$u = x+y, \qquad v = xy \tag{16}$$

resulting in the equation system for u and v

$$u^2 - 2v = 15^2 \tag{17}$$

$$v = 3u \tag{18}$$

Inserting the expression for v from the second equation into the first one obtains the quadratic equation

$$u^2 - 6u - 15^2 = 0$$

with the two solutions

$$u_{1,2} = 3 \pm \sqrt{3^2 + 15^2} = 3 \pm 3\sqrt{1^2 + 5^2} = 3\left(1 \pm \sqrt{26}\right)$$

Because $u = x + y > 0$ only the positive solution matters, so it must be

$$u = 3\left(1 + \sqrt{26}\right)$$

and by (18)

$$v = 3u = 9\left(1 + \sqrt{26}\right)$$

Using as shortcut

$$R = 1 + \sqrt{26}$$

and inserting the values of u and v into (16) the equations for x and y become

$$\begin{aligned} x + y &= 3R \\ xy &= 9R \end{aligned}$$

By the second equation it is $y = 9R/x$, inserting this into the first equation obtains another quadratic equation, now for x

$$x + \frac{9R}{x} = 3R$$

or

$$x^2 - 3Rx + 9R = 0$$

with the solution

$$\begin{aligned} x_{1,2} &= \frac{3}{2}R \pm \sqrt{\frac{9}{4}R^2 - 9R} \\ &= \frac{3}{2}\left(R \pm \sqrt{R^2 - 4R}\right) \\ &= \frac{3}{2}\left(R \pm \sqrt{R(R-4)}\right) \end{aligned}$$

This gives the two possible solutions

$$\begin{aligned} x_1 &\approx 14.515503482069 \\ x_2 &\approx 3.781555058710 \end{aligned}$$

The distance between top of the wall and top of the ladder then becomes $15 - x_{1,2}$ giving the two solutions of 0.4844965 feet or 11.2184449 feet.

4.10 Micropuzzle 55 – More ladders and walls

Let a be the height at which the longer ladder of 30m rests against the right wall and b the height where the 25m ladder leans at the other wall, d the distance between the two walls we look for and for convenience let $h = \frac{504}{100}$ the height of the meeting point of the two ladders over the ground.

By the theorem of Pythagoras we have

$$\begin{aligned} a^2 + d^2 &= 30^2 && (19) \\ b^2 + d^2 &= 25^2 && (20) \end{aligned}$$

This gives $d^2 = 30^2 - a^2$ and $d^2 = 25^2 - b^2$, hence

$$30^2 - a^2 = 25^2 - b^2$$

or by adding $a^2 - 25^2$ on both sides

$$a^2 - b^2 = 30^2 - 25^2 \qquad (21)$$

Now let d_1 and d_2 be the distances from the left wall to the meeting point of the two ladders and the distance from there to the right wall, $d_1 + d_2 = d$.

The two right triangles with legs d_1, h and legs d, a are similar, thus

$$\frac{d_1}{d} = \frac{h}{a} \qquad (22)$$

and also by the same argument for the other ladder

$$\frac{d_2}{d} = \frac{h}{b} \tag{23}$$

Adding both sides of (22) and (23) gives

$$\frac{d_1 + d_2}{d} = \frac{h}{a} + \frac{h}{b}$$

or by using $d_1 + d_2 = d$, dividing by h and exchanging sides it is

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{h}$$

Resolving for b gives

$$b = \frac{ha}{a - h} \tag{24}$$

Inserting (24) into equation (21) then gives

$$a^2 - \frac{(ha)^2}{(a-h)^2} = 275$$

and by the steps

$$\begin{aligned} a^2(a-h)^2 - h^2a^2 &= 275(a-h)^2 \\ a^2(a^2 - 2ah + h^2) - h^2a^2 &= 275(a^2 - 2ah + h^2) \\ a^4 - 2ha^3 + h^2a^2 - h^2a^2 &= 275(a^2 - 2ah + h^2) \end{aligned}$$

the quartic equation for a (as h is a known constant value)

$$a^4 - 2ha^3 - 275a^2 + 550ha - 275h^2 = 0 \tag{25}$$

Inserting the value of $h = \frac{504}{100}$ into this equation yields

$$a^4 - \frac{252}{25}a^3 - 275a^2 + 2772a - \frac{174636}{25} = 0 \tag{26}$$

A quartic equation is still solvable in a closed form, see for example [7], section 2.2 about polynomials, or [1], section 3.8, but this is not a really practical way.

Alternatively you could look into the equations (19) and (20) and search for integer solutions. After some guessing you might deduce from $3^2 + 4^2 = 5^2$ that $(6 \cdot 3)^2 + (6 \cdot 4)^2 = (6 \cdot 5)^2$ and therefore $18^2 + 24^2 = 30^2$ and inserting 18 into (26) you see that you found a solution, then by extracting the linear factor $(a - 18)$

$$\frac{1}{25}(a - 18)(25a^3 + 198a^2 - 3311a + 9702) = 0$$

Or the equation (26) can be solved with computer help, either numerical or symbolic, resulting in the only positive real solution $a = 18$, a negative real and two conjugate complex solutions.

Only the positive real solution geometrically makes sense, and then

$$d = \sqrt{30^2 - 18^2} = \sqrt{900 - 324} = \sqrt{576} = 24$$

The walls are 24m apart.

References

[1] M. Abramowitz, I. A. Stegun, editors, *Handbook of Mathematical Functions.* United States Government Printing Office, 1964.

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[7] D. Zwillinger, Editor-in-Chief, *CRC Standard Mathematical Tables and Formulae*, 30th Edition. CRC Press, 1996.