Assignment 3

Recurrence Relations

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① $T(n) = \begin{cases} 1 & n=1 \\ 2T(\frac{n}{2})+n & n>1 \end{cases}$ Answer = $O(n \log n)$

② $T(n) = \begin{cases} 1 & n=1 \\ 8T(n/2)+n^2 & n>1 \end{cases}$

③ $T(n) = T(n/2) + T(n/3) + T(n/5) + c$

① $T(n) = \begin{cases} 1 & n=1 \\ 2T(n/2)+n & n>1 \end{cases}$

Substitution method

$T(n) = 2T(n/2)+n$ — 1st time

$\longrightarrow T(n/2) = 2T(n/2^2)+n/2$

$T(n) = 2[2T(n/2^2) + n/2] + n$ — 2nd time

$= 2^2T(n/2^2) + 2n/2 + n$

$= 2^2 T(n/2^2) + 2n$

$\longrightarrow T(n/2^2) = 2T(n/2^3) + n/2^2$

$T(n) = 2^2[2T(n/2^3) + n/2^2] + n^2 \longrightarrow$ 3rd times

$= 2^3T[(n/2^3) + \cancel{2}n/\cancel{2}] + n^2$

$= 2^3 T(n/2^3) + 3n$

⋮ k times

$T(n) = 2^k T(n/2^k) + kn$ — k times

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + kn \qquad n = 1$$

$$\frac{n}{2^k} = 1$$

$$= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n \cdot n \Rightarrow n = 2^k$$

$$\log_2 n = \log_2 2^k$$

$$\Rightarrow k = \log_2 n$$

$$= n^{\log_2 2} T\left(\frac{n}{n^{\log_2 2}}\right) + n\log_2 n$$

$$= n + n\log_2 n$$

$\therefore$ $O(n(\log_2 n))$ since $n < n\log_2 n$

② $T(n) \begin{cases} 1 & n = 1 \\ 8T(n/2) + n^2 & n > 1 \end{cases}$

Substitution method —

$$T(n) = 8T(n/2) + n^2 \longrightarrow 1^{st} \text{ times}$$

$$\longrightarrow T(n/2) = 8T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2$$

$$T(n) = 8\left[8T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right] + n^2 \text{ — } 2^{nd} \text{ times}$$

$$= 8^2 T\left(\frac{n}{2^2}\right) + \frac{\cancel{8}^2 \times n^2}{\cancel{4}} + n^2$$

$$= 8^2 T\left(\frac{n}{2^2}\right) + 2n^2 + n^2$$

$$= 8^2 T\left(\frac{n}{2^2}\right) + 3n^2$$

$$\longrightarrow T\left(\frac{n}{2^2}\right) = 8^2 T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2$$

$$T(n) = 8^2\left[8\,T\left(\frac{n}{2^2}\right) + 3\left(\frac{n}{2^2}\right)^2\right] + 3n^2$$

$$= 8^3\,T\left(\frac{n}{2^3}\right) + 8^2\left(\frac{n}{2^2}\right)^2 + 3n^2$$

$$= 8^3\,T\left(\frac{n}{2^3}\right) + \frac{\cancel{64}\,4\; n^2}{\cancel{16}\,4} + 3n^2$$

$$= 8^3\,T\left(\frac{n}{2^3}\right) + 7n^2$$

$\downarrow$ k times

$$T(n) = 8^k\,T\left(\frac{n}{2^k}\right) + (2^k - 1)\,n^2$$

$\frac{n}{2^k} = 1$

$k = \log_2 n$

$$T(n) = 8^{\log_2 n}\left(\frac{n}{2^{\log_2 n}}\right) + \left(2^{\log_2 n} - 1\right)n^2$$

$$= n^{\log_2 2^3}\left(\frac{\cancel{n}}{\cancel{n^{\log_2 2}}}\right) + \left(n^{\cancel{\log_2 2}} - 1\right)n^2$$

$$= n^3 + n^3 - n^2$$

$$\therefore O(n^3)$$